Developing Algebraic Conceptual Understanding: Can procedural knowledge get in the way?

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*In this study we use latent class analysis, distractor analysis, and qualitative analysis of cognitive interviews of student responses to questions on an algebra concept inventory, in order to generate theories about how students' selections of specific answer choices may reflect different stages or types of algebraic conceptual understanding. Our analysis reveals three groups of students in elementary algebra courses, which we label as "mostly random guessing", "some procedural fluency with key misconceptions", and "procedural fluency with emergent conceptual understanding". Student responses also revealed high rates of misconceptions that stem from misuse or misunderstanding of procedures, and whose prevalence often correlates with higher levels of procedural fluency.*

*Keywords:* elementary algebra, conceptual understanding, concept inventory

Elementary algebra and other developmental courses have consistently been identified as barriers to student degree progress and completion. Only as few as one fifth of students who are placed into developmental mathematics ever successfully complete a credit-bearing math course in college (see e.g., Bailey, Jeong, & Cho, 2010). At the same time, elementary algebra has higher enrollments than any other mathematics course at US community colleges (Blair, Kirkman, & Maxwell, 2010).

There is evidence that students struggle in these courses because they do not understand fundamental algebraic concepts (see e.g., Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010). Conceptual understanding has been identified as one of the critical components of mathematical proficiency (see e.g., National Council of Teachers of Mathematics (NCTM), 2000; National Research Council, 2001), and many research studies have documented the negative consequences of learning algebraic procedures without any connection to the underlying concepts (see e.g., J. C. Hiebert & Grouws, 2007). However, developmental mathematics classes currently focus heavily on recall and procedural skills without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003). This focus on procedural skills in isolation may actually be counter-productive, in that students may often attempt to use procedures inappropriately because they lack understanding of when and why the procedures work (e.g., Givvin et al., 2011; Stigler et al., 2010).

In this paper we explore student response to conceptual questions at the end of an elementary algebra course in college. We combine quantitative analysis of responses (using latent class analysis and distractor analysis) with qualitative analysis of cognitive interviews in order to better understand different typologies of student reasoning around some basic conceptual questions in algebra, and to explore the relationship between conceptual understanding and procedural fluency in this context.

## Conceptual understanding

The definition of conceptual understanding (and its relationship with other dimensions of mathematical knowledge, particularly procedural fluency) has been much debated and discussed (e.g., Baroody, Feil, & Johnson, 2007; Star, 2005), with as yet no clear consensus. This study recognizes the interrelatedness of conceptual understanding with other mathematical skills (e.g., Hiebert & Lefevre, 1986; National Research Council, 2001), and defines it this way: An item tests *conceptual understanding* if a student must use *logical reasoning* grounded in mathematical definitions to answer correctly, and it is *not* possible to arrive at a correct response *solely* by carrying out a procedure or restating memorized facts. We define a *procedure* as a sequence of algebraic actions and/or criteria for implementing those actions that could be memorized and correctly applied with or without a deeper understanding of the mathematical justification. Using this definition, no question is wholly conceptual or procedural, but rather falls on a spectrum, with more conceptual questions at one end, and more procedural questions at the other. For more details about how conceptual understanding was operationalized during the creation of the questions analyzed here see (Wladis, Offenholley, Licwinko, Dawes, & Lee, 2018).

## Methods

This study focuses on student responses to the multiple choice questions on the Elementary Algebra Concept Inventory (EACI). For details on the development and validation of the EACI, see Wladis et al., (2018). In this paper we focus on 698 students who took the inventory at the end of the semester of their elementary algebra class in 2016-2017. Ninety-one percent were students of color (half black and a third Hispanic), and roughly two-thirds were women. Roughly half were first-semester freshmen, and one-third were repeating the course because they failed or dropped it previously. The mean GPA for returning students was 2.47. Participants earned a mean score on the (entirely procedural) university final exam of 58%, and roughly one-third passed the course. In order to supplement quantitative data, 10 cognitive interviews conducted towards the end of the semester with students who were enrolled in an elementary algebra class were also analyzed using grounded theory (Glaser & Strauss, 1967), although a full qualitative analysis is not presented here due to space constraints.

In this paper we pursued latent class analysis (LCA) of the binary scored (right/wrong) multiple-choice items on the inventory. LCA is a latent variable model that is based on the principle of local independence but presumes that the items to be locally independent conditional on a discrete nominal latent variable (e.g., Collins & Lanza, 2010). As such, it does not assume an underlying continuous latent trait. We used Stata 15.1 (StataCorp, 2017) to fit the analysis via the EM algorithm using random starts to protect against local optima. No convergence problems were observed during the process of fitting. Both a two-class and a three-class analysis were explored. The three-class analysis fit the data better (AIC 7122 and 7114, respectively). For the three class versus the saturated model, $G^2(482) = 8.79$, with $p < 0.001$, suggesting that it fit the data well. We did not use covariates or the nominal item responses in LCA models to ensure that we could use the classes to examine the relationship with external variables (e.g., end-of-class standardized test scores, course outcome) and distractor responses.

## Description of the classes

The latent class analysis revealed three groups that we characterize in the following way:

- C1 (27%): Students whose answers to most items are indistinguishable from random guessing, likely due to low procedural/conceptual knowledge and/or low motivation.

- C2 (28%): Students who likely have some good procedural skills but limited conceptual understanding.
- C3 (45%): Students who likely have good procedural skills and emergent conceptual understanding.

These class descriptions emerged from looking at the data in a number of different ways. Firstly, we consider the response patterns of students from each of the three classes, and we see some clear trends (see Figure 1).

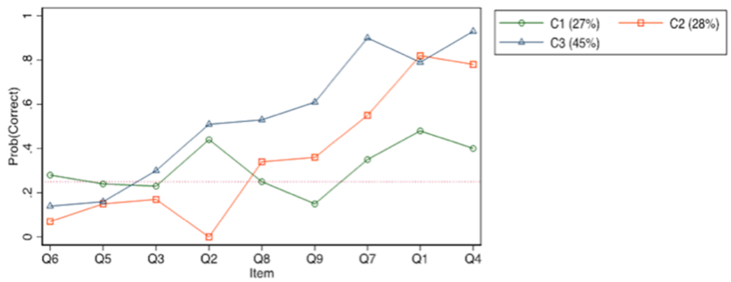



*Figure 1. LCA profiles of student responses in each class*[1]

Student responses in class 1 do not vary much from what would be expected for random guessing on four-option multiple choice items. These students only answer correctly at rates that are higher than chance on questions 1, 2, 4, and 7, with 1 and 4 being the two easiest questions of these nine for all classes. Their performance is never better than 50% on even their best item. While all of the items on the test were designed to test conceptual understanding, some of them are closer to traditional procedural questions or ways of thinking than others. Questions 1, 4, 7, 8 and 9 are more similar to standard problems and procedures than questions 2, 3, 5, and 6, which use more abstract or non-standard formulations of algebraic ideas. Responses on these more procedural questions are precisely what primarily distinguish class 2 from class 1. Class 2 answers significantly worse than chance on questions 2 and 6 because of the presence of attractive distractors that likely tap into misconceptions related to the misuse of procedures. Classes 2 and 3 are distinguished by improved performance on the items overall as well as different proportions of key misconceptions. We see also that students who passed the class were most likely to be in class 3, then class 2, and least likely to be in class 1 (see Figure 2). An end-of-course standardized assessment that measures procedural fluency showed a similar outcome.

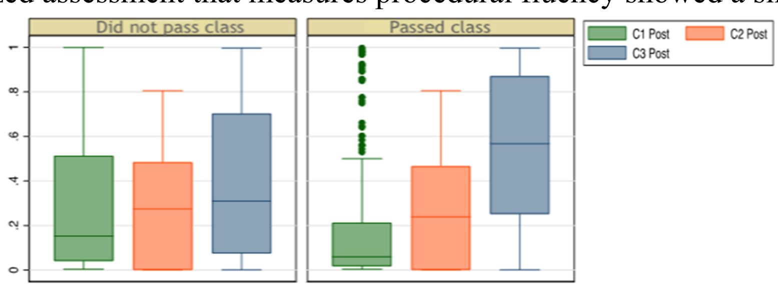



*Figure 2. Posterior probabilities of class membership by course outcome (passing the class)*

In order to illustrate how different response patterns might distinguish these three classes, we performed a distractor analysis and analyzed cognitive interviews for three exemplars: items 2, 4,

[1] For reference, in the plot above a somewhat conservative approximate 95% margin of error for a given point is .15. That is to say, points that differ by .15 are outside the 95% confidence interval.

and 6. We used the Bayes modal assignment to determine class membership. The median of the modal membership probabilities was 0.73. Examining the normalized entropy within each class suggested that class 2 was the best distinguished although no class was so poorly distinguished as to make classification useless.

**Three example questions: illustrating different class response patterns**
**Item 4:** First we consider Item 4:

Which of the following is a result of correctly substituting $x-4$ for $y$ in the equation $3y-2=y^2+1$?

a. $3x-4-2=x-4^2+1$
b. $3x-4-2=x^2-4^2+1$
c. $3(x-4)-2=(x-4)^2+1$
d. $3x-3\cdot 4-2=x^2(-4)^2+1$

The correct answer is c. We would expect students who understand that substitution means to substitute $x-4$ in for $y$, but who do not completely understand how the underlying structure of substitution works would select options a and b with high frequency.

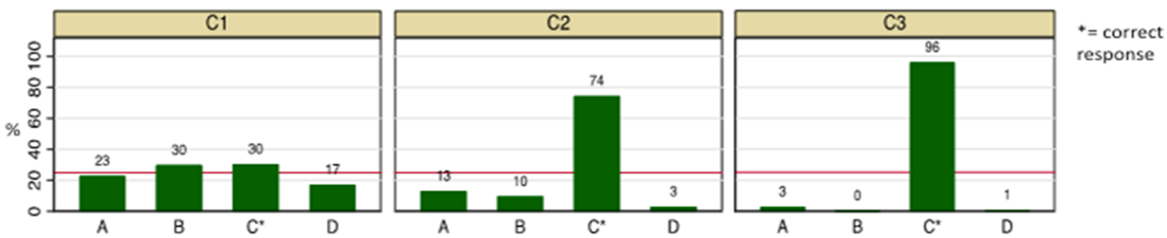


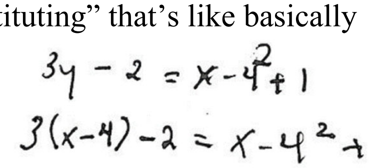

*Figure 3. Item 4 Distractor Analysis*

C1's option selection is clearly scattered in a pattern consistent with random guessing (see Figure 3). By contrast, classes 2 and 3 have a high probability of choosing the correct response, with C3's probability being significantly higher than that of C2. Selecting option c is highly correlated with student scores on the procedural exam, corresponding to a score that is higher by 10.8 percentage points ($p=0.000$).

Looking at student interview responses reinforces our interpretation of the three classes.

**C1 (chose B):** It says $x-4$ for $y$, this is what I think like because $y^2$. It could be like changed to a $4^2$. I put together like $3x-4-2=x^2-4^2+1$. [I didn't pick c or d because] they [pointing to the $x-4$ in the item stem] didn't have no bracket around them. [I picked B with the $x^2$ in it instead of A, which doesn't have the $x^2$] because $x$ equals $y^2$ so it has to have an $x^2$ in it because the $y$ is squared there.

**C2 (chose C):** So usually like when a math question says, "substituting" that's like basically putting the numbers that they give you into $x$ or $y$ that they say to put it. And then I automatically substituted it in, and my correct answer was $3(x-4)-2$… I didn't pick any other answer, because I didn't see the parentheses.




**C3 (chose C):** I didn't choose A because when trying to multiply the $y$, which is $x-4$, you have to put the parenthesis behind 3, unless you already multiplied 3 times $x-4$… it [answer choice D] does have the parenthesis on $-4$, but then, it will be missing the complete equation for $y$ because $-4$ is not the only equation that equals to $y$ is $x-4$.

In these examples, the C1 student shows a conception of substitution that involves putting $x-4$ in where the $y$ is in the equation, but does not show an awareness of the equation structure (e.g. that the $x-4$ needs to be treated as a single unit when substituting). The C2 student shows an awareness of the procedure of putting in a set of parentheses around whatever is being substituted, but doesn't execute this procedure completely correctly on both sides, and doesn't demonstrate any awareness of why the parentheses are necessary. In contrast, the C3 student shows both an awareness of the need for the parentheses and an understanding of *why* the parentheses are necessary—because without them, the structure of the equation will be altered.

**Item 6:** Now we consider item 6, which shows a different pattern of responses:

A student is trying to simplify two different expressions:

i. $(x^2y^3)^2$
ii. $(x^2 + y^3)^2$

Which one of the following steps could the student perform to correctly simplify each expression?

a. For both expressions, the student can distribute the exponent.
b. The student can distribute the exponent in the first expression, but not in the second expression.
c. The student can distribute the exponent in the second expression, but not in the first expression.
d. The student cannot distribute the exponent in either expression.

The correct answer to this question is b. Classes 2 and 3 were strongly attracted to option a (see Figure 4), likely because they are familiar with procedures associated with the distributive properties but do not recognize the critical difference between distributing multiplication versus exponents—likely because they lack a deeper conceptual understanding of why these properties work. We note that all three classes were strongly attracted to this distractor, likely for similar reasons. Examinees in class 1 do best on this item. Unlike item 4, selecting the correct answer for this item was negatively correlated (and selecting the distractor a was positively correlated) with scores on the procedural exam, with students who selected this distractor on average scoring 7.1 percentage points higher ($p < 0.000$). This suggests that in this context (where procedures are typically taught in isolation from concepts) procedural fluency in standard problem contexts is inversely related to conceptual understanding of the distributive properties.

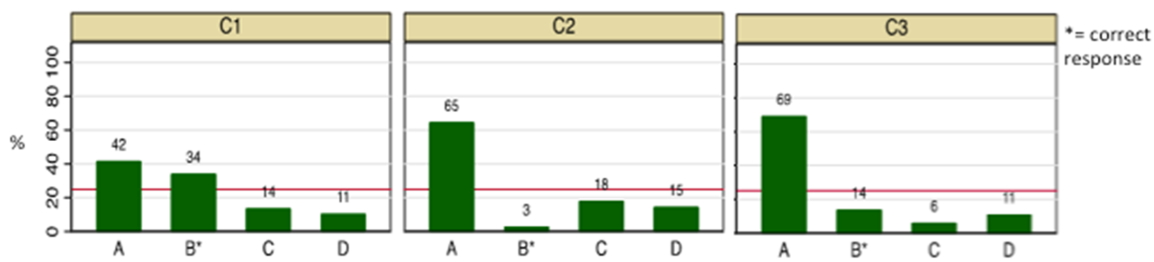



*Figure 4. Item 6 distractor analysis*

Looking at student interview responses reinforces our interpretation of the three classes.

**C1 (chose B):** [The difference between the first and second equation] is that there's a plus right there [pointing to the second equation]. I think for this one [pointing to the second equation], you have to add and for this one [pointing to the first equation] you don't…. Actually, I think like over here [pointing to the second equation] you add a 3. 3 plus 2. [For the first one] you do $x^2$ times $x^2$ and $y^3$ times $y^3$.

**C2 (chose A):** I feel like that's correct because in order to solve $x^2$ and $y$, you have to distribute…. Because I've seen problems like this before and it's like you have to solve it, there is no not solving it because there is no … there is no solution.

i. $(x^2y^3)^2 = x^4y^5$
ii. $(x^2 + y^3)^2 = x^4 + y^6$

**C3, but close to C2 (chose A):** Since [both equations] are in parenthesis and they have exponents, the first thing that came into my head was PEMDAS…so after parenthesis will be exponents. So with the exponent, I know you would have to distribute and then you'll be able to solve the rest.

**C3 (chose A):** Both expressions the student can distribute the exponents because for the parenthesis you do multiply.

**C3 (chose A):** That's how you kind of get rid of the parenthesis and get rid of the outer exponents by distributing it in the inside. Whether it's with another exponent or with a number… You want to add or multiply that exponent [outside the parentheses] to the ones inside the parentheses but I can't remember whether you add or multiply…

In these examples, the C1 student notices that there is a difference between the two equations and suggests that it is important, but doesn't actually know how to perform the distribution correctly. For the C2 and C3 students, we see a number of ways in which students are incorrectly employing procedures or experience with procedures—we have only listed a few of them here, but every interviewee cited a different, procedural explanation for why the exponent could be distributed, including: reciting a procedure for distributing; citing standard problem contexts based on surface structure; stating that parentheses always mean that one should multiply; citing the order of operations. None of the students we interviewed in any class showed a deeper understanding of what distributing means or when it is possible. While C2 and C3 students may have shown evidence of understanding what exponents mean, they did not provide any evidence that they understood what distributing means in this case, beyond a basic citing of procedures (often inaccurately) that they had learned in class.

**Item 2:** Next we consider item 2, which reveals another interesting pattern of responses:

Consider the equation $x + y = 10$. Which of the following statements must be true?

a. There is only one possible solution to this equation, a single point on the line $x + y = 10$.
b. There are an infinite number of possible solutions, all points on the line $x + y = 10$
c. This equation has no solution.
d. There are exactly two possible solutions to this equation: one for $x$ and one for $y$.

For this question, the correct answer is b, which was the most popular answer chosen by students in classes 1 and 3, but no examinee in class 2 chose it (see Figure 5). They were strongly attracted to option d, which was also the second most popular choice of students in both other classes, although at a much lower rate. Answer option d is a common response from students who are used to finding a solution to two linear equations in two variables; thus many students may select this answer because of an inappropriate application of procedural knowledge based on surface features of the equation. Interestingly, both the correct answer b (+4.0 percentage points, $p = 0.039$) and the popular distractor d (+5.5 percentage points, $p = 0.017$) are correlated with higher scores on the standardized procedural exam, although choosing the distractor is more strongly correlated with higher procedural skills as measured by the exam.

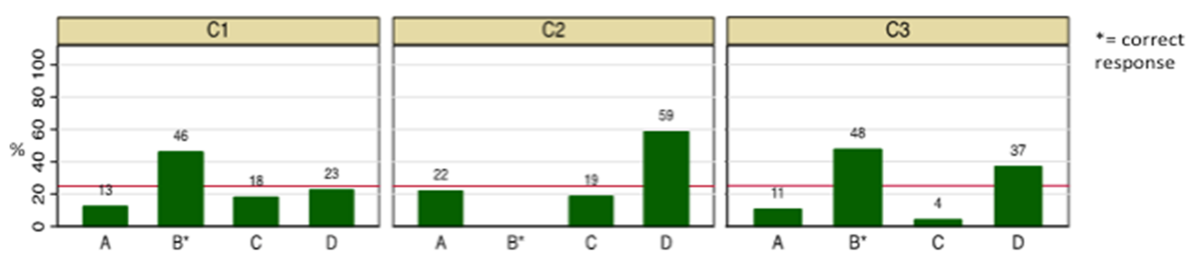


*Figure 5. Item 2 distractor analysis*

Looking at student interview responses reinforces our interpretation of the three classes.

**C1 (originally chose C, but drifted towards B in the interview):** $x + y$ equals nothing so it can't be 10. Right?... [Maybe infinite means] what could be like possible? I don't know. Like equal number maybe? $x + y = 10$. It could be possible like it equals 10. [Option D isn't correct] maybe because $x$ and $y$ could be equal to anything?

**C2 (chose D):** What I assumed was the $x$ term and the $y$ term, you would have to substitute. And I know there are certain numbers that will add up to ten, so there could be two solutions, since there's only a $x$ term and a $y$ term… Like $x$ could equal 5, $y$ could equal 5… since it is two terms, so you could say two different solutions.

**C3 (chose B):** Ten could equal to many things. Like five plus five could equal ten. Nine plus one could equal ten. Seven plus three. That's why I chose that, because it could be any number that will equal to ten. It's not just one certain number.

In these examples, the C1 student chose "no solution" because they didn't know what $x$ and $y$ could be, but as they explained more, they started to relate this to the idea that $x$ and $y$ could be "anything". While their reasoning is not strictly correct, they are beginning to explore the idea that $x$ and $y$ may have many possible values. The C2 student seems to understand enough about what the equation means to find a single solution, but once they find one solution they stop there, not exploring whether there might be others. Further, they confuse the number of solutions with the number of variables in the solution set, showing that they do not understand that a solution set is a collection of all possible combinations of variables that make the statement true. The student from C3 describes how this equation could have multiple solutions, demonstrating some conceptual understanding of how solution sets for equations work. They also demonstrate understanding that the values for both variables are related and that the solution set describes this. Other students in C3 cited the graphical representation of a line to describe similar ideas.

## Discussion and Limitations

The patterns of student responses and explanations in cognitive interviews suggest that many students, including those who pass the course, are consistently using procedures inappropriately and without understanding; thus, these results suggest that instruction which stresses procedures divorced from conceptual understanding likely worsens a number of misconceptions. As we saw on two of the more conceptual questions, higher procedural fluency on standard problems actually corresponded to lower conceptual understanding of certain concepts. This suggests that the widespread use of instruction through repeated procedural practice, when isolated from any systematic attempts to practice interpreting and understanding these procedures, may actually be worsening fundamental algebraic conceptual understanding.

We note that this study did not attempt to directly link student responses to specific types of instruction—there is a pressing need for future research to examine the relationship between instructional characteristics and pre-post response patterns on validated concept inventories, in order to determine which kinds of instruction have the most positive or negative impact on student growth in conceptual understanding. In the meantime, this research reinforces existing research that suggests that teaching procedures in isolation, without concomitant conceptual understanding, may have negative consequences (Givvin et al., 2011; Stigler et al., 2010).

One limitation of this analysis is that we have only examined traditional four-option multiple choice items. Other items were administered that use a "choose all the apply" format, but are not analyzed here. This format may be very helpful in providing more information about student thinking than can be obtained from single option multiple choice; however, scoring these items and managing the local dependence is complex and is thus left for further research.

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