Prospective High School Teachers' Understanding and Application of the Connection Between Congruence and Transformation in Congruence Proofs

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Undergraduate mathematics instructors are called by recent standards to promote prospective teachers' learning of a transformation approach in geometry and its proofs. The novelty of this situation means it is unclear what is involved in prospective teachers' learning of geometry from a transformation perspective, particularly if they learned geometry from an approach based on the Elements; hence undergraduate instructors may need support in this area. To begin to approach this problem, we analyze the prospective teachers' use of the conceptual link between congruence and transformation in the context of congruence. We identify several key actions involved in using the definition of congruence in congruence proofs, and we look at ways in which several of these actions are independent of each other, hence pointing to concepts and actions that may need to be specifically addressed in instruction.

Keywords: geometry, transformations, secondary teacher education

Instructors of undergraduate teacher preparation programs face a transition in geometry instruction. In the past several decades, geometry has been taught primarily from a perspective based on Euclid's *Elements* (Sinclair, 2008); in recent years, geometry from a *transformation perspective* has come to the fore in secondary standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) and guidelines (NCTM, 2018).

These changes in geometry standards have implications both mathematically and pedagogically. For instance, consider the well-known triangle congruence criterion "Angle-Side-Angle (ASA)": *If* ΔABC *and* ΔDEF *are triangles such that* $\overline{AB} \cong \overline{ED}$, $\measuredangle BAC \cong \measuredangle EDF$, *and* $\measuredangle ABC \cong \measuredangle DEF$, *then* $\Delta ABC \cong \Delta DEF$. In secondary and college geometry texts using an *Elements* approach, this criterion is often taken as a postulate: it is intended to be accepted as mathematical truth without proof (e.g., Education Development Center, 2009; Musser, Trimpe, & Maurer, 2008; Serra, 2008; Boyd, Cummins, Mallow, Carter, & Flores, 2005). These and other texts help students establish conviction in ASA through empirical exploration – a scheme for conviction, that taken by itself, can be unproductive when the objective is to construct a deductive proof (Harel & Sowder, 2007). In contrast, from a transformation approach, if a student is to show that two triangles ΔABC and ΔDEF in a plane are congruent, they must show that no matter the triangles' locations, there exists a sequence of translations, rotations, and rotations that map ΔABC to ΔDEF. (See Wu (2013) for a schematic for such a proof.) In the transformation approach, even if empirical exploration is beneficial, a teacher must also help students move toward deductive proof. In the *Elements* approach, a proof would be mathematically impossible.

It is critical for prospective and practicing teachers to understand not only the abstract notion that different axiom systems result in different proof approaches (Van Hiele-Geldof, 1957), but also that they may be teaching students from an axiomatic system different from the one they learned first. Consequently, teachers – including prospective teachers who are undergraduate students – may not be familiar with what can be proven, what cannot be proven, or how particular proofs operate. We address this problem from the perspective of developing knowledge for teaching prospective teachers, including understanding how prospective teachers learn. In this document, we report on a study guided by the question: *What concepts are entailed in prospective teachers' construction of congruence proofs?*

We focus this study on establishing congruence proofs because, as suggested by the example above, it is an area fundamental to the study of geometry at the secondary level where differences between *Elements* and transformation approaches are salient. We address our research question by analyzing data from prospective teachers for potential key developmental understandings (Simon, 2006) related to constructing congruence proofs.

Conceptual perspective

Transformation approaches to school geometry, though only recently sanctioned in standards documents such as that of the Common Core, are not new. Following Usiskin and Coxford (1972), we take a *transformation approach* to geometry as one that features:

- Postulation of preservation properties of transformations:
    - in particular, reflections, rotations, and translations are assumed without proof to preserve geometric properties such as length and angles; and
    - these transformations are defined as maps from the plane to the plane;
- Definition of congruence in terms of transformations: two subsets X and Y of the plane (e.g., two triangles) are said to be congruent if there exists a reflection, rotation, or translation, or sequence of these transformations[1], that maps X to Y;
- Definition of similarity in a corresponding way, via transformations.

The details of these features may differ across texts, for instance, different statements of postulates of transformations may be taken, but they have in common that the postulates are about transformations, rather than congruence criteria for particular objects such as triangles.

Hence, from a transformation perspective:

- **[T-to-C]** *To establish a proof of congruence of two objects in the plane, such as two triangles, one constructs a sequence of assertions that show that there exists a single one of or a sequence of reflections, rotations, or translations that maps one object to the other*,

where the assertions can be justified with reasoning and represented in ways that the community learning these concepts understands (Stylianides, 2007). Moreover,

- **[C-to-T]** *When two objects are congruent*, *the transformation perspective provides that there then exists a single one of or a sequence of reflections, rotations, or translations that maps the first object to the other.*

We emphasize and name the "T-to-C" (transformations are used to establish congruence) and "C-to-T" (congruence provides a sequence of transformations) statements for two reasons. First, they represent an unpacking of the two directions of the definition of congruence from a transformation approach, when the definition is taken as an if-and-only-if statement. Second, they are essential to the tasks used in the reported study.

We take an *Elements approach* to be one that features the postulation of at least one triangle congruence criterion (e.g., SSS, ASA, or SAS), and definition of congruence similarity in terms of individual geometric objects (e.g., congruence for triangles is defined separately from congruence of circles).

[1] Note that glide reflections can be expressed as compositions of reflections and translations.

As Jones and Tzekaki (2016) reviewed, there is "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139). To our knowledge, there have been few studies on teachers' conceptions of congruence *proofs* from a transformation perspective. One exception is Hegg, Papadopoulos, Katz, and Fukawa-Connelly (2018), who examined how teachers managed their prior knowledge of congruence criteria when showing the congruence of two triangles. They found that teachers preferred to use triangle congruence criteria rather than transformations, but could, when asked, successfully complete proofs using transformations. However, their study did not examine the case of proving congruence of figures that are not triangles.

Hence, because of the novel nature of this study, we pursue an inductive analytic design, and we present related literature in the discussion section rather than in the introduction. This structure is "most suitable for the inductive process of qualitative research" and allows related literature to be "a basis for comparing and contrasting findings of the qualitative study" (Creswell & Creswell, 2017, p. 27).

Data and Method

Data

A post-hoc analysis was conducted of 20 prospective secondary teachers' responses to two congruence proof tasks, the Line Point Task and the Boomerang Task (below). The tasks were distributed as part of an in-class midterm examination in a mathematics course taught by one of the authors in Fall 2017.

- **Line Point Task.** Let ℓ, m be lines. Among all the points that are a unit distance from ℓ, choose one point P. Among all the points that are a unit distance from m, choose one point Q. Prove that no matter what points P and Q you chose, it is always true that $\ell \cup P \cong m \cup Q$.
- **Boomerang Task.** Let ΔABC and ΔDEF with congruences marked as shown. Let O be a point on the inside of ΔABC and P be a point on the inside of ΔDEF so that the angle measures $\alpha = \gamma$ and $\beta = \delta$ as shown. Given the all the above, prove that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ (Figure 1).

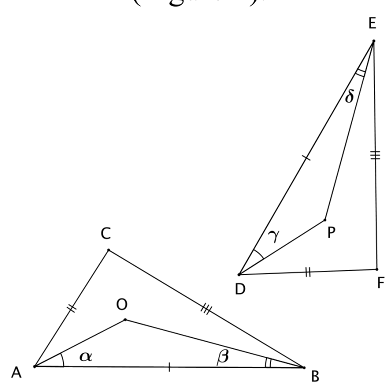

Figure 1: The Boomerang Task was distributed with this representation of $\Delta AOB \cup \Delta ABC$ *and* $\Delta DPE \cup \Delta DEF$

Analysis

The analysis focused on identifying potential *key developmental understandings* (KDU: Simon, 2006) used in constructing congruence proofs. A full conceptualization of KDU is beyond the scope of this brief report, but we emphasize that a KDU affords a learner a different way of thinking about mathematical relationships (Simon, 2006). For our analysis, this meant that to determine whether something may be a KDU, we must be able to identify how having or not having the KDU could make a difference in learners' capacity to construct congruence proofs. We proceeded by coming to consensus about the logic of each prospective teacher's response to the tasks, then generating potential descriptions of ways of thinking about congruence and proof that account for differences among responses. These descriptions became provisional codes. We consolidated or distinguished codes based on how and whether the use of the definition of congruence changed what was possible mathematically later in the argument.

Rationale for Task Design

The Line Point Task and Boomerang Task were part of a sequence of tasks intended for developing prospective teachers' understanding of using definition of congruence from a transformation perspective to prove the congruence (or non-congruence) of given figures, especially when the proof requires showing the extension of transformations from a proper subset of figures to entire figures. The prospective teachers' responses to these tasks suggest that there are KDUs underlying the doing of the tasks; responses to the tasks indicated different understandings of the role of the definition of congruence and the need for showing extensions of transformations. Moreover, in-class discussions indicate that understandings were more likely to develop as a result of reflection and multiple experiences than through direct instruction.

The second author selected and designed this sequence using variation theory; in brief, this theory holds that knowledge of a particular idea develops from tasks that keep constant the use of the idea while varying other aspects of tasks (Lo, 2012). The sequence included tasks co-designed by teachers, mathematics educators, and mathematicians to support this goal (Park City Mathematics Institute, 2016), beginning with prospective teachers' discovering that, from a transformation perspective, the statement that "two line segments of equal length are congruent" required proof. Building on the transformations used in a proof of this statement, prospective teachers then used extensions of these transformations for proofs involving triangles and other unions of line segments during class and for homework. Prospective teachers were then asked to prove that two rectangles of equal dimensions are congruent, which requires showing that a candidate sequence of transformations can extend from mapping parts of a figure to mapping entire figures as desired. Two of the authors designed the Line Point and Boomerang Tasks as variations of the rectangle task.

Results

Decomposition of using the definition of congruence in congruence proofs

Using the prospective teachers' responses, we first decomposed the definition of congruence into the concepts C-to-T and T-to-C, and then decomposed each of these concepts. In particular:

- Using C-to-T involves prospective teachers explicitly using known congruence between two figures, known theorems, or axioms to infer the existence of a sequence of rigid motions mapping one figure to a second figure.
- Using T-to-C involves two actions:
  - the teacher consistently states that in order to establish congruence one must establish a sequence of rigid motions to map one figure to the other *and*
  - the teacher establishes rigid motions or a sequence of rigid motions to map one figure to another to show congruence between the figures.

Using these criteria, we found that using C-to-T does not predict using T-to-C, or vice versa. With this independence of C-to-T and T-to-C in mind, we then analyzed how prospective teachers' responses invoked C-to-T and T-to-C. Our analysis resulted in two potential KDUs. Due to space limitations we only describe illustrative examples for the first result; we elaborate upon the results in the presentation.

Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one <u>entire</u> figure to the other <u>entire</u> figure.

Prospective teachers without this KDU may know that rigid motions are involved in congruence proof, but they may not understand that figures remain fundamentally un-altered with every motion. For instance, we found responses that established rigid motions and thus congruence between parts that compose a whole (such as between ℓ and m as well as P and Q, or ΔAOB and ΔDPE as well as ΔABC and ΔDEF) but that did not necessarily establish congruence of entire wholes ($\ell \cup P$ and $m \cup$ Q, or $\Delta AOB \cup \Delta ABC$ and $\Delta DPE \cup \Delta DEF$).

To illustrate, in the Boomerang Task, some responses used the premise that $\overline{AB} \cong \overline{DE}$ to claim abstractly the existence of a transformation mapping $\overline{AB}$ to $\overline{DE}$, but then the responses concluded that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ because $\Delta AOB \cong \Delta DPE$ and $\Delta ABC \cong \Delta DEF$ – and not because the transformations could extend to the unions. (See Figure 2 for an example.)

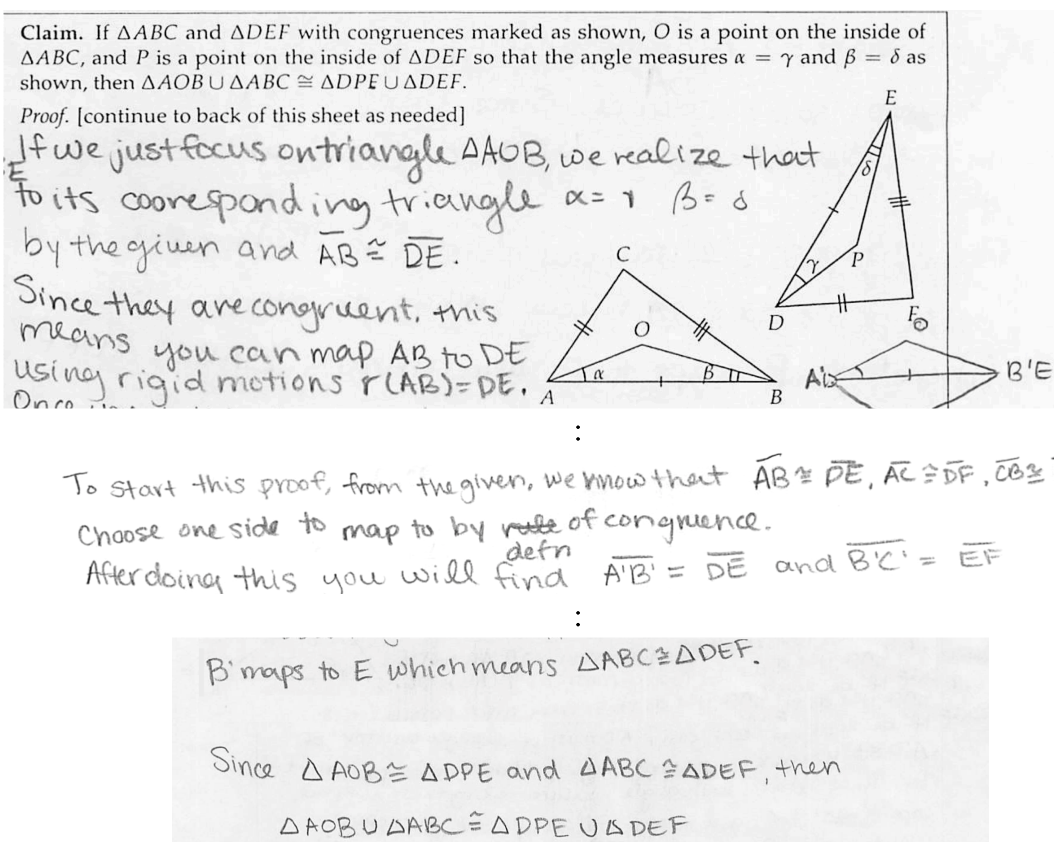

Figure 2: These show key steps of one teacher's work on the Boomerang Problem. In the first part the teacher used C-to-T. Just before the second part above the teacher concluded using these rigid motions that ΔAOB maps to ΔDPE. In the third part we see that the teacher did not use T-to-C to conclude congruence of the unions.

Additionally, some prospective teachers' responses described rigid motions that mapped some or all corresponding parts of the first figure to the second, but the rigid motions constructed did not extend to the entire figures – in this case, the responses exhibited different rigid motions for different components that could not extend. Other responses constructed rigid motions that did extend to the entire figure, but this extension was not recognized explicitly in the responses. Furthermore, some prospective teachers defined a transformation that did "double-duty", that is the teacher noted that two parts of the figures are congruent and therefore claimed the existence of a single transformation that mapped both pieces to their corresponding parts at the same time.

Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure.

To understand the necessity of proving that two figures need to be superimposed, one must conceive of the possibility that they may *not* be superimposed. Being able to conceive of this possibility allows for a learner to realize that there is more to show than identifying a candidate sequence of transformations.

Teachers without this KDU may declare the proof complete after defining the transformations or providing minimal justification. For instance, on the Line Point Task, some prospective teachers defined a sequence of rigid motions and claimed that $\ell \cup P$ had been mapped to $m \cup Q$ without further justification. Several other prospective teachers minimally attempted to justify superposition by stating that rigid motions preserve distance. We note that in this case, prospective teachers showed evidence of potential KDU 1 but not potential KDU 2.

Discussion/Conclusion

In this study, we analyzed prospective teachers' responses to tasks, designed using variation theory, for underlying understandings that support constructing congruence proofs. Based on this analysis, we proposed an empirically-based decomposition of and two potential KDUs for the use of the definition of congruence in congruence proofs. We now discuss our findings in relation to previous results in the literature. We highlight two such results; our findings corroborate one result and add nuance to the other.

First, as Edwards (2003) described, students at middle school, secondary, and undergraduate levels predominately hold a *motion view* of transformations. From this perspective, a transformation is conceptualized as the movement of a geometric object, which sits "on top" of the plane, from one location to the next. This contrasts with a *map view* (Hegg et al., 2018) in which objects are perceived to be subsets of the plane, and transformations to be maps of the plane. Multiple subsequent studies suggest that prospective middle school and secondary mathematics teachers may also hold a motion view (Portnoy et al., 2006; Hegg et al., 2018; Yanik, 2011), and that this view may make it difficult to construct proofs of congruence from a transformation perspective.

Our analysis corroborated the "motion-versus-map" findings of previous studies, instantiated as expressed conflation of pre-images and images. For instance, after applying a transformation to $\ell \cup P$ in the Line Point Task, some prospective teachers continued to refer to the image as $\ell \cup P$. We interpreted this notational usage as the consequence of a movement conception of transformation rather than a map conception. In contrast, when teachers used notation such as $r(\ell \cup P)$ or $(\ell \cup P)'$, we interpreted this as the consequence of a map conception. However, some teachers who used notation consistent with a movement conception nonetheless otherwise produced valid arguments for congruence, suggesting that this conception is not necessarily a barrier to understanding the structure of congruence proofs.

Second, as far as the ability to construct congruence proofs, Hegg et al. (2018) found that, after participating in a course which incorporated transformational geometry content, prospective teachers could successfully use transformations to establish congruence between two triangles. In our findings we also found this to be true; however, our data suggest that prospective teachers may not be as successful in establishing congruence for other objects, and that they encounter difficulties in applying the definition of congruence. The design of our study allowed us to examine prospective teachers' capabilities for writing congruence proofs beyond standard triangle congruence proofs. These tasks required not only finding sequences of transformations between familiar objects, but showing that a sequence could simultaneously map the objects in a *union* of these objects to another *union.* Furthermore, our data included working with lines and points—objects which, though familiar—are not often discussed in the context of congruence proofs.

We now make some points about the relation of our proposed KDUs to successful completion of congruence proofs from a transformation perspective. First, these potential KDUs are necessary but not sufficient for teachers to successfully complete congruence proofs. For instance, a teacher who has attained potential KDU 2 may know that further justification is necessary after defining a sequence of transformations but be unsure as to what justification to use. It also appears possible that a teacher may have one of the above KDUs but not the other, as with responses demonstrating KDU 1 in the Line Point Task but not KDU 2.

Additionally, we note that the conceptual link between transformations and congruence in the context of congruence proofs involves understanding C-to-T (the fact that the congruence of two figures implies that there exists a sequence of transformations carrying one figure to another) and T-to-C (the fact that the existence of a sequence of transformations carrying one figure to another implies that the two figures are congruent). A teacher who applies C-to-T in a mathematically valid way will use known congruences between two figures to infer existence of rigid motions mapping one figure to a second figure. A teacher who applies T-to-C in a mathematically valid way will both (a) consistently state that in order to establish congruence one must establish a sequence of rigid motions to map one figure to the other and (b) construct or declare rigid motions that carry one entire figure to another. A few additional ways of thinking related to the above concepts have also been noted. As the above actions are all teacher actions that appear to be prerequisites to the creation of mathematically valid and complete congruence proof construction, these are skills that instructors will likely need to address.

While the above actions may be conceptually related, they appear in this data set to be independently adopted by prospective teachers, with prospective teachers sometimes engaging in only one or two of the corresponding actions at a time. As a result, an instructor may need to keep in mind that successfully addressing only one or two of these concepts and actions may not be sufficient in helping prospective teachers create mathematically valid and complete congruence proofs.

Applications of this work may include the construction of lessons, assignments, and assessments that directly address each above potential KDUs and conceptual links. Such materials may help instructors as they attempt to help prospective teachers learn the subtle concepts listed above in addition to those involved in notation. Future work is needed to interrogate the accuracy of these KDUs.

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