Calculus II Students' Understanding of the Univalence Requirement of Function

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A robust conceptual understanding of function is essential for students studying calculus and higher levels of mathematics as they continue to pursue the learning of mathematics. In this study, we investigated the ways in which students in a Calculus II course understand functions by examining student engagement with a vending machine applet. Specifically, we considered how these students made sense of the univalence requirement of functions in the context of a vending machine in which a single input produces an output of two cans. We identify and discuss in detail several themes that emerged in students' categorization of machines as functions or non-functions when encountering this two-can scenario.

Keywords: Functions, Calculus, Univalence

Functional relationships are an essential construct in undergraduate students' mathematical learning (Cooney, Beckmann, & Lloyd, 2010; Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990). However, research has shown that undergraduate students often display incomplete conceptions regarding the concept of function (e.g., Oehrtman, Carlson, & Thompson, 2008), including an incomplete conceptual understanding of domain and range (Dorko & Weber, 2014). These conceptions or other difficulties that students have may be due to a lack of understanding of the nature of connections between the different representations of functions (e.g., Clement, 2001; Stylianou, 2011), the abstract nature of the function concept (Steele, Hiller, Smith, & 2013), or lack of a fully developed definition of function (Clement, 2001). Additionally, without a robust understanding of function, students may struggle with the function concept when moving from two to three dimensions in multivariable calculus (Dorko & Weber, 2014).

One's concept of function depends on his or her previous experiences with function, including the definitions to which they have been introduced (Thompson & Carlson, 2017). The most commonly used definition of function in schools is a variation of Dirichlet's definition (e.g., a function is a relation between two sets in which every element in the domain is mapped to exactly one element in the range) (Cooney & Wilson, 1993; Thompson & Carlson, 2017; Vinner & Dreyfus, 1989). This definition attends primarily to the relationship between two sets of elements (i.e., domain and range). As a result, students' difficulties are often related to the univalence requirement of the definition of function (Dubinsky & Wilson, 2013). To address this issue, we designed an applet in the form of a vending machine to problematize univalence. We used this applet to examine how Calculus II students make sense of the univalence requirement of functions situated in a vending machine context in which an output of two cans is produced by a single input. In this study, we seek to identify the themes that arose through this two-can scenario. We attempt to answer the question: In what ways do Calculus II students make sense of a vending machine applet that produces two cans from a single input?

Background Literature

Much of the research on student understanding of function has occurred in the context of college algebra, precalculus, or calculus classes. Through these studies there has been a careful identification of common understandings that students develop related to the concept of function. Common student understandings include that functions are defined by an algebraic formula or two expressions separated by an equal sign, and that functions are represented by graphs (that pass the vertical line test) (Carlson, 1998; Clement, 2001; Breidenbach et al., 1992; Thompson & Carlson, 2017). All of these conceptions are limited and can be problematic when distinguishing functions from non-functions, especially in non-algebraic settings (Steele et al., 2013).

An important aspect of the identification of functions is the univalence requirement (i.e., a function maps each element in the domain to exactly one element in the range). When using a graphical view of function, students often satisfy the univalence condition by using the vertical line test; however, the arbitrary nature of what a function can represent is lost within this narrow view (Clement, 2001; Steele et al., 2013). In addition, research has shown that a common incomplete conception regarding the univalence requirement is believing that it is synonymous to saying that the function has a one-to-one correspondence (Dubinsky & Wilson, 2013).

Due to the concern that calculus students may have developed a weak understanding of the concept of function (Moore, Carlson, & Oehrtman, 2009), researchers have suggested that students be engaged in activities that require using various representations (Zeytun, Cetinkaya, & Erbas, 2010; Moore et al., 2009). One way in which this can be accomplished is by using interactive applets that do not make use of any type of algebraic representations. The use of technology in this way can cause a cognitive conflict and require students to reflect and reassess their current understanding of function (Pea, 1987). For instance, a student whose understanding of function is only related to input-output relationships or reliance on the vertical line test may have trouble when encountering non-algebraic functions in a novel context (Steele et al., 2013). Sherman, Lovett, McCulloch, Edgington, Dick, and Casey (2018) found that the use of an online applet in the context of a vending machine, designed to support calculus students' opportunities to consider functions in a novel environment, improved student understanding of the definition of function and strengthened their ability to distinguish functions from non-functions. Through analysis of 105 undergraduate students' pre- and post-definitions, they found that students' interaction with the applet resulted in improved attention to the univalence requirement in their stated post-definitions (pre-definitions 36.6%; post-definitions 85.3%). However, by attending only to the students' pre- and post-definitions, student thinking in regard to univalence in the context of differentiating between function and non-function relationships remains unclear.

Theoretical Perspective/Conceptual Framework

In considering undergraduate students' learning related to function, we adopted a theoretical lens of transformation theory (Mezirow, 2009). Transformation theory is consistent with constructivist assumptions, specifically in that meaning resides within each person and is constructed through experiences (Confrey, 1990). Mezirow (2009) describes four forms of learning that lie at the heart of this theory: elaborating upon existing meaning schemes, learning new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. Meaning schemes are the specific expectations, knowledge, beliefs, attitudes or feelings that are used to interpret experiences (Cranton, 2006; Peters, 2014).

Learning by transforming meaning schemes often begins with a *disorienting dilemma*. This stimulus requires one to question his or her current understandings that have been formed from previous experiences (Mezirow, 2009). It is this type of learning experience that we are particularly interested. Given the evidence that undergraduates often have a view of function that is limited to algebraic expressions and the associated graphs (e.g., Carlson, 1998; Even, 1990) and that such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998), we designed an experience that would problematize these understandings, thereby creating a stimulus for transformation.

One strategy that has been suggested for diminishing common misunderstandings related to function is the use of a function machine as a cognitive root. The idea of a *cognitive root* was introduced by Tall, McGowen, and DeMarois (2000) as an "anchoring concept which the learner finds easy to comprehend yet forms a basis on which a theory may be built" (p.497). As an example of a cognitive root for the function concept, Tall et al. (2000) suggest the use of a *function machine*, typically referring to a type of "guess my rule" activity in which inputs and associated outputs are provided, challenging students to determine the pattern (i.e., identify the function rule). The use of such machines proved quite promising as a cognitive root for function, yet some students still struggled with connecting representations and determining what is and is not a function (McGowen, DeMarois, & Tall, 2000). Given the potential of using a machine metaphor as a cognitive root for function, as well as our desire to present a disorienting dilemma for undergraduate students, we designed an applet to provide students with a learning experience.

Context of this Study: Vending Machine Applet

The vending machine applet (McCulloch, Lovett, & Edgington, 2017) was designed to provide an opportunity for students to reexamine their definition of function by interacting with a non-algebraic representation. The applet was built using a GeoGebra workbook and uses a vending machine metaphor to represent functions and non-functions. The first three pages of the applet each contain two soda vending machines (Machines A-F), each with buttons for Red Cola, Diet Blue, Silver Mist, and Green Dew. When the user clicks a button (input), one or more cans (red, blue, silver, and/or green) appear in the bottom of the machine (output). To remove the can(s) the user clicks the "take can" button. Students are asked to compare the different machines and determine which of the two represent a function. The non-function machines have at least one button that produces at least one random can when clicked (i.e., the resulting can is not predictable based upon the button that is pressed). The fourth page of the applet contains an additional six vending machines (Machines G-L); students are asked to consider whether or not each machine could represent a function. Student work on Machines D, E, I, and K (see Table 1) are the focus of this study, as these are the machines that result in a two can output.

Table 1. Machine output for each button clicked

	Button Clicked			
	Red Cola	Diet Blue	Silver Mist	Green Dew
Machine D	random pair	blue can	silver can	green can
Machine E	red can	blue & random can	silver can	green can
Machine I	two silver cans	green can	red can	blue can
Machine K	red can	blue can	silver can	red & green can

Research Methods

The purpose of this qualitative study was to investigate the ways in which Calculus II students make sense of functions in a vending machine context. We attempt to answer the question: In what ways do Calculus II students make sense of a vending machine applet that produces two cans from a single input?

Participants and Data Collection

A total of 40 students from one centrally located U.S. university participated in the study. At the time of data collection, all participants were enrolled in a Calculus II class. Each student recorded a screencast of themselves working through the applet while noting decisions leading to the classification of each machine as a function or non-function. Students also wrote their rationale for each decision on an accompanying worksheet. Students were asked to use a *think aloud* protocol to explain their reasoning while interacting with the applet. The data used for this study include the video-recorded screencasts and the accompanying worksheets. Upon review of the data, four students were eliminated from analysis as their screencasts lacked audio or were incomplete. A total of 36 students' data were analyzed.

Analysis

The first phase of data analysis consisted of creating descriptions of students' engagement with the applet that included timestamps with direct student quotes. Next, we coded for articulated dilemmas and the triggers for those dilemmas. This study is focused on one trigger, one input mapped to an output of two cans, as such descriptive transcriptions were created for the portions of the video in which this specific situation occurred. While transcribing the screencasts, we created memos related to student thinking about the two-can scenario.

A preliminary codebook was developed based on themes that emerged in the memos related to the ways in which students made sense of the two can dilemma. The researchers then used this codebook along with open coding and a constant comparative method (Strauss & Corbin, 1998) to develop a final set of codes. For example, the final codebook included: function - consistent, function - corresponding color, non-function - two outputs, and non-function - random. Once the codebook was finalized and inter-rater reliability achieved, all remaining data was double coded with any differences discussed until agreement was reached. Finally, the researchers looked within the data for each code to identify themes across and within machines.

Results

Analyses illuminated several themes that directly address the purpose of this study – namely, to examine the ways in which students identify functions and non-functions when faced with the dilemma of an output consisting of two elements (see Table 2). Looking across all machines that produced a two can output, nearly a quarter of the students (8 of 36) indicated that a machine with an output of two cans is never a function regardless of the consistency of the output. For example, one student commented "Machine K is consistent with what it's giving out, but it's giving out two cans, and I feel like it shouldn't be able to do that. Like a function shouldn't allow it to have two outcomes for one input." This student's decision was based exclusively on the number of cans produced in the output. Students who decided that machines producing two cans automatically qualified as a non-function often attempted to make sense of the applet by viewing the cans as numbers or coordinates. One student stated, "Oh, it's not a function, because the Green Dew produces two y-values." This statement indicates that the student was considering the idea of univalence and viewing the two green cans as two separate outputs. Many of these students incorrectly used the language "one to one" to refer to univalence while relying on procedural knowledge to make a decision.

While some students focused on the number of cans, others made decisions based upon predictability of the output or lack thereof. Many students (56%) commented on consistency or randomness for at least one of the two can machines, however only 22% *always* used the idea of consistency or randomness to decide whether the machine was or was not a function. Students who decided that both Machines I (Red Cola → two silver) and K (Green Dew → red and green) were functions tended to focus on a consistent two can output. For example, one student remarked, "So even though the green button dispenses two different cans, it does generate the same outcome each time, as well as the other button, so Machine K is also a function." This student was unconcerned with the output quantity and was attending to the predictability of the machine. The remaining 78% of students did not reliably consider consistency or randomness. The following sections detail the emerging themes that arose from the two can dilemmas.

Table 2. Overarching Themes regarding the Two-Can Scenario

Overarching Themes	
Rationale	Percentage of Students (N=36)
Based decision on number of cans every time	22% (8)
Based decision on consistency and randomness of output every time	22% (8)
Based decisions on both consistency/randomness and number of cans	22% (8)
Various other reasons	34% (12)

Dilemma: Two cans with at least one being random

Machines D (Red Cola → random pair) and E (Diet Blue → blue and random) not only had a two can output but also included an element of randomness in the output. When confronted with both randomness and a two can output, 22% of the students described the lack of consistency of the output in their justifications. For example, when one student clicked the Red Cola button on Machine D the first time, two red cans were given as the output. When Red Cola was clicked a second time, two blue cans were dispensed. The student commented,

> Okay so it looks like the Red Cola is different, see it's moving between the different colors for the two cans; so, it was red, now it's blue. So, because of that, I feel that Machine D is not a function because, because it's creating different outputs for the Red Cola and it's not consistent.

Similarly, another student used randomness to justify a decision,

> However, red is the odd one out here as it is a different, it's giving off two of a random color drink. Because Red Cola has a random, has a random effect. Every time, there's no rhyme or reason as to why it does it, it's just, possibly, random number generator.

Both students commented on the random or inconsistent output of the Red Cola button and decided that the lack of predictability make Machine D a non-function. When assessing Machine E, another student stated "The blue always does random, while the other ones keep clicking their same color can. So, I think in this case, F is a function because blue is always a constant silver. While in E blue is a random." The predictability of Machine F and the randomness of Machine E seemed to inform this student's decision.

In contrast, eight of the 36 students attended to both randomness and the number of cans in the output when justifying that Machines D and E were non-functions. For example, one student commented that "Machine E, however, can't be, uh, multiple outputs, especially multiple different outputs. So, E should not be a function." This student noticed that Machine E both produces two cans ("multiple") and produces random colored output ("different outputs"). In the cases of these students who attended to both of these factors, it was unclear in both the screencasts and worksheets which reason predominated their decision-making process.

Dilemma: Two Consistent Cans

Interacting with Machines I (Red Cola → two silver) and K (Green Dew → red and green) presented the students with the situation of a button having a consistent output of two cans, yet eight of the 36 students treated these machines differently from one another, labeling one as a function and the other as a non-function (see Table 3).

Table 3. Machine I and K Inconsistencies

Rationale	Inconsistency	
	I is a function; K is not a function (N=6)	K is a function; I is not a function (N=2)
Lack of Coherent Explanation	1	
Machine I produces 2 cans of same color; Machine K produces 2 different colors	5	
Machine K: Cola button *does* match output can color; Machine I: Cola button *does not* match output can color		2

The majority of the students who classified I and K differently were attending to the color of the two can output instead of the consistency of those outputs. The attention to color manifested in two ways: students either identified the machine as a function 1) if the two cans produced were the same color (Machine I: Red Cola → two silver), or 2) if the button color matched at least one of the output cans (Machine K: Green Dew → red and green). The two students who labeled Machine K as a function and Machine I as a non-function were looking for colors of the output cans to correspond with the color of the pressed button. For example, one student said, "Machine K is a function because for all of the buttons I do get what I want, but even though I click Green Dew I get something else, I still get what I want right." This student's explanation included that the Green Dew button output both a green can ("I still get what I want") and a red can ("something else"). Students are specifically examining whether the button colors correspond to the color of the output can(s). This reasoning was unique to these two students.

The remaining five students who gave a coherent explanation regarding their attention to color labeled Machine I as a function and Machine K as a non-function. These students identified Machine I as a function because the output created two cans of *the same color*. For example, one student commented that "I guess it's still a function, but if they had two separate color cans, then I think that would imply different y, values for y. So I'm gonna say that Machine I is definitely a function." These students also indicated that Machine K was not a function since the output consisted of two cans of *different colors*. For example, one student wrote on the worksheet, "Although the 'Green Dew' button always gives the same outcome it releases two different cans unlike all the other buttons on this machine." Some students elaborated further and commented that different buttons produce the same color can, "It appears that multiple input buttons, like green and red, both produce red cans as their output which makes them not a function." In an attempt to make sense of this problem, one student tried to connect the cans to numbers and make use of the vertical line test,

> Um, I think that it is not a function because every, every input should only have one output, and this one, it has two. So, I just, I picture it on a graph and I don't think that would pass a vertical line test and I think that is something, um, that a function needs to pass, so I don't think Machine K is a function.

This student is viewing the two cans as two separate outputs because they are different colors. Lastly, one student did not provide a clear enough think aloud or written rationale to ascertain why Machine I was labelled as a function and Machine K as a non-function.

Discussion and Conclusion

Sherman et al. (2018) found that students' definitions of function showed increased attention to univalence after engaging with the vending machine applet. This study builds on that work by attending to students' engagement with the applet as it relates to the two-can scenario. Our results revealed that while many students justified their decisions by referencing consistency or randomness, it was uncommon for students to do so reliably. One concern is that 78% of students in this study steadfastly focused on irrelevant elements or unreliable rationale when presented with machines producing a two can output. For example, some students focused on whether the button color matched the output can color (irrelevant elements). Other students switched reasoning from machine to machine focusing on predictability one time and on the number of cans the next time (unreliable rationale). This may be due to students lacking a fully developed definition of a function (Moore et al., 2009), or that students' understandings of function are too narrow or include erroneous assumptions (Clement, 2001). This suggests that Calculus II instructors need to help students develop a strong definition of function which can be applied to a variety of representations.

The attention that some students placed on attempting to connect the vending machine context to numbers or coordinates confirms the known difficulties students have with univalence (Dubinsky & Wilson, 2013) and their over reliance on procedures (Steele et al., 2013). It was evident from the screencasts that these students were linking the two can output to two y-values and confusing univalence with one-to-one correspondence. The prevalence of this confusion and the incorrect use of language regarding one-to-one correspondence suggests that some students do not understand that one-to-one correspondence is a special case and not a requirement. This is an area that warrants further research.

One limitation of our study was that our analysis only included transcriptions of videos of students' interactions with the applet, in that we did not utilize students' personal definition of function in tandem with their interactions. Future studies with this applet should analyze the reliability of student rationale in conjunction with their definitions to determine if weak or narrow function definitions are related to the inconsistent classification of machines. Moreover, as this study focused on only one dilemma trigger, future studies should explore other triggers.

Calculus II students have had many experiences with functions, yet the analysis of their interactions with the vending machine applet revealed the possibility that students have underdeveloped definitions of function or consider functions too narrowly. Further research is needed to explain the unreliable rationales when determining function from non-function in non-algebraic settings to better understand why students have difficulties with univalence. With the concept of function permeating mathematics past Calculus II, the results of this study demonstrate the need to allow students to reexamine their conceptual understanding of function in advanced classes, where these topics are not necessarily in the scope of the class.

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