Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 52-16 (2019) 412–417

A Symmetric Group Method for Controllability Characterization of Bilinear Systems on the Special Euclidean Group ⋆

Wei Zhang * **Jr-Shin Li** **

* *Department of Electrical and Systems Engineering, Washington University, St. Louis, MO 63130 USA (e-mail: wei.zhang@wustl.edu).*
** *Department of Electrical and Systems Engineering, Washington University, St. Louis, MO 63130 USA (e-mail: jsli@wustl.edu)*

Abstract: Bilinear systems form models of wide-ranging applications in diverse areas of engineering and natural sciences. Investigating fundamental properties of such systems has been a prosperous subject of interest and remains essential toward the advancement of systems science and engineering. In this paper, we introduce an algebraic framework utilizing the theory of symmetric group to characterize controllability of bilinear systems evolving on special orthogonal and Euclidean groups. Our development is based on the most notable Lie algebra rank condition and offers an alternative to controllability analysis. The main idea of the developed approach lies in identifying the mapping of Lie brackets of vector fields governing the system dynamics to permutation multiplications on a symmetric group. Then, by leveraging the actions of the resulting permutations on a finite set, controllability and controllable submanifolds for bilinear systems evolving on the special Euclidean group can be explicitly characterized.

Keywords: Bilinear systems, controllability, Lie groups and Lie algebras, symmetric groups.

1. INTRODUCTION

Bilinear systems form a subclass of nonlinear systems, which emerge as natural models for various dynamical processes in science and engineering. Prominent examples include systems describing the evoluation of the bulk magnetization of a sample of nuclear spins immersed in a magnetic field in quantum physics (Glaser et al., 1998; Li and Khaneja, 2006; Li et al., 2011), the rotation and translation of a rigid body in the presence of external forces in mechanics and robotics (Baillieul, 1978; Isidori, 1995; Jurdjevic, 1996), and the movement of cells, molecules or radioisotope tracers between physiological compartments in biology (Mohler and Ruberti, 1978; Eisen, 1979; Mohler and Kolodziej, 1980). Since 1970s, tools from differential geometry and Lie theory have been extensively exploited to study nonlinear control systems from a geometric perspective, and the area of geometric control then took a sharp turn (Brockett, 2014, 1976; Baillieul, 1978). One of the most significant advancements in this field is the development of a necessary and sufficient controllability condition for control-affine systems, named the Lie algebra rank condition (LARC), by which controllability is characterized by the dimension of the Lie algebra generated by the vector fields governing the system dynamics (Brockett, 1972; Jurdjevic and Sussmann, 1972; Hermann and Krener, 1977; Huang et al., 1983).

In this paper, we present a new algebraic approach to analyzing controllability for right-invariant bilinear systems defined on the special orthogonal and Euclidean groups, denoted $\mathrm{SO}(n)$ and $\mathrm{SE}(n)$, respectively. This approach provides an alternative to the LARC and a new algebraic interpretation of controllability. In particular, for systems defined on $\mathrm{SO}(n)$, the key idea of the proposed approach is to map Lie bracket operations of the vector fields governing the system dynamics to permutation multiplications on the symmetric group S_n. Through this mapping, together with the LARC, controllability of systems on $\mathrm{SO}(n)$ can be characterized in terms of permutation cycles. Furthermore, by acting the resulting permutations on a set of n elements, e.g., $\{1,\ldots,n\}$, the orbits of the actions give rise to an explicit characterization of controllability, as well as the controllable submanifold in uncontrollable cases, of bilinear systems defined on $\mathrm{SE}(n)$ which are induced by the Lie group action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$.

The paper is organized as follows. In Section 2, we introduce an algebraic approach of analyzing controllability of bilinear systems defined on $\mathrm{SO}(n)$ through permutation cycles on S_n. Levaraged on this development and the decomposition of $\mathrm{SE}(n)$ into the rotational and translational components, in Section 3, we present the new method to characterize controllability and controllable submanifolds of bilinear systems defined on $\mathrm{SE}(n)$ in terms of the orbits resulting from the S_n action on a finite set containing n elements. In addition, the LARC for the bilinear system evolving on a compact and connected Lie group and some basics of the symmetric group theory are reviewed in Appendices A and B, respectively.

⋆ This work was supported in part by the National Science Foundation under the awards CMMI-1462796, ECCS-1509342, and ECCS-1810202, and by the Air Force Office of Scientific Research under the award FA9550-17-1-0166.

10.1016/j.ifacol.2019.11.815

2. SYMMETRIC GROUP METHOD FOR SYSTEMS ON SO(N)

In this section, we will introduce a new algebraic controllability condition for the right-invariant bilinear system defined on SO(n) of the form

$$\dot{X}(t) = B_0X(t) + \sum_{i=1}^{m} u_i(t)B_iX(t), \quad X(0) = I, \tag{1}$$

where $X(t) \in \mathrm{SO}(n)$ is the state, $u_i(t) \in \mathbb{R}$ are piecewise constant control inputs, and $B_0, \ldots, B_m$ are elements in the Lie algebra $\mathfrak{so}(n)$ of SO(n). The development of this condition is based on mapping Lie bracket operations of elements in $\mathfrak{so}(n)$ to permutation multiplications in S_n (Zhang and Li, 2017). To this end, we first briefly review some basics of the Lie algebra $\mathfrak{so}(n)$.

2.1 Basics of the Lie Algebra $\mathfrak{so}(n)$

The Lie algebra $\mathfrak{so}(n)$ is an $n(n-1)/2$-dimensional vector space consisting of all n-by-n skew symmetric matrices. As a Lie algebra, it is also equipped with a binary operation called the Lie bracket and defined as $[A, B] = AB - BA$ for all $A, B \in \mathfrak{so}(n)$.

Let $\Omega_{ij} \in \mathfrak{so}(n)$ denote the matrix with 1 in the ij^{th}-entry, -1 in the ji^{th}- entry, and 0 elsewhere, then the set $\mathcal{B} = \{\Omega_{ij} : 1 \le i < j \le n\}$ forms a basis of $\mathfrak{so}(n)$. The following lemma then characterizes the Lie bracket relations among these basis elements in $\mathcal{B}$.

Lemma 1. The Lie bracket of Ω_{ij} and Ω_{kl} satisfies $[\Omega_{ij}, \Omega_{kl}] = \delta_{jk}\Omega_{il} + \delta_{il}\Omega_{jk} + \delta_{jl}\Omega_{ki} + \delta_{ik}\Omega_{lj}$, where

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

is the Kronecker delta function.

Proof. The proof directly follows from calculation. □

Notice that, by Lemma 1, $[\Omega_{ij}, \Omega_{kl}] \neq 0$ if and only if there exists a bridging index $j = k$, $i = l$, $j = l$, or $i = k$. This observation plays an important role in the establishment of the map between Lie bracket operations and permutation multiplications in the next section.

2.2 Mapping Lie Brackets to Permutations

In this section, we introduce the approach to characterizing controllability of bilinear systems defined on SO(n) by permutation cycles. In particular, systems governed by vector fields in $\mathcal{B}$ serve as suitable examples to illustrate this algebraic framework due to the nice structure of $\mathcal{B}$ shown in Lemma 1. Specifically, we will define a map that transforms Lie bracket operations among elements in $\mathcal{B}$ to permutation multiplications in the symmetric group S_n so that the image of the control vector fields of a given system under this map represents controllability of the system.

Formally, let $\mathcal{P}(\mathcal{B})$ denote the power set of $\mathcal{B}$, then we define a map $\iota : \mathcal{P}(\mathcal{B}) \to S_n$ by $\{\Omega_{i_1j_1}, \Omega_{i_2j_2}, \ldots, \Omega_{i_mj_m}\} \mapsto (i_l, j_l) \cdots (i_2, j_2) \cdot (i_1, j_1)$, where (i_k, j_k) is the cyclic notation of the permutation

$$\begin{pmatrix} 1 & \cdots & i_k & \cdots & j_k & \cdots & n \\ 1 & \cdots & j_k & \cdots & i_k & \cdots & n \end{pmatrix}$$

for each $k = 1, \ldots, m$.

On the Lie algebra level, if we pick $\Omega_{ij}, \Omega_{jk} \in \mathcal{B}$, then Lemma 1 implies $[\Omega_{ij}, \Omega_{jk}] = \Omega_{ik} \neq 0$, which is due to the existence of the bridging index j, and consequently, $\mathrm{Lie}(\{\Omega_{ij}, \Omega_{jk}\}) = \mathrm{span}\{\Omega_{ij}, \Omega_{jk}, \Omega_{ik}\}$ holds. In this case, on the symmetric group level, we have $\iota(\{\Omega_{ij}, \Omega_{jk}\}) = (i, j)(j, k) = (i, j, k)$, i.e., the bridging index j bridges i to k and hence increases the cycle length by 1 ((i, j) and (j, k) have length 2 and (i, j, k) has length 3). Inductively, given a subset $\mathcal{S} \subset \mathcal{B}$ satisfying the following two conditions

(i) for any $\Omega_{ij} \in \mathcal{S}$, there exists some $\Omega_{i'j'} \in \mathcal{S}$ such that $[\Omega_{ij}, \Omega_{i'j'}] \neq 0$,
(ii) if $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathrm{Lie}(\mathcal{S}') = \mathrm{Lie}(\mathcal{S})$, then $\mathcal{S}' = \mathcal{S}$,

then $\iota(\mathcal{S})$ is a cycle of length $|\mathcal{S}| + 1$, where $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. Moreover, any subset of $\mathcal{B}$ satisfying the condition in (ii) is called a set with no *redundant element*. This bridging property immediately suggests a characterization of controllability of bilinear systems defined on SO(n) by permutation cycles as shown in the following theorem.

Theorem 2. A bilinear control system defined on SO(n) in the form of

$$\dot{X}(t) = \left[\sum_{k=1}^{m} u_k(t)\Omega_{i_kj_k}\right] X(t), \quad X(0) = I, \tag{2}$$

where $\Omega_{i_kj_k} \in \mathcal{B}$ for all $k = 1, \ldots, m$ and I is the identity matrix, is controllable if and only if there is a subset $\mathcal{S}$ of $\{\Omega_{i_1j_1}, \ldots, \Omega_{i_mj_m}\}$ such that $\iota(\mathcal{S})$ is an n-cycle.

Proof. See (Zhang and Li, 2017). □

3. CONTROLLABILITY ANALYSIS OF BILINEAR SYSTEMS ON SE(N)

In this section, we will extend the algebraic approach introduced in Section 2 to develop a method for analyzing controllability of bilinear systems evolving on the non-compact Lie group SE(n). To this end, we first briefly review some basics of the Lie group SE(n) and its Lie algebra $\mathfrak{se}(n)$.

3.1 Basics of SE(n) and $\mathfrak{se}(n)$

Consider the Euclidean space $\mathbb{R}^n$ as a Lie group under addition, then its semidirect product with SO(n), denoted by $\mathrm{SE}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$, is called the special Euclidean group. Every element in SE(n) can be represented by a 2-tuple (x, X) with $x \in \mathbb{R}^n$ and $X \in \mathrm{SO}(n)$, and this also reveals that SE(n) contains SO(n) and $\mathbb{R}^n$ as Lie subgroups. Algebraically, the group multiplication on SE(n) is given by $(x, X)(y, Y) = (x + Xy, XY)$ for any $x, y \in \mathbb{R}^n$ and $X, Y \in \mathrm{SO}(n)$, which also indicates that $(0, I)$ is the identity element of SE(n). Topologically, due to the non-compactness of $\mathbb{R}^n$, SE(n) is also a non-compact Lie group. In addition, SE(n) can be smoothly embedded into $\mathrm{GL}(n+1, \mathbb{R})$, the general linear group consisting of all $(n+1)$-by-$(n+1)$ invertible matrices, and this embedding yields a matrix representation for each $(x, X) \in \mathrm{SE}(n)$ as

$$(x, X) = \begin{bmatrix} X & x \\ 0 & 1 \end{bmatrix},$$

where $X \in \mathrm{SO}(n)$ and $x \in \mathbb{R}^n$.

From the geometric aspect, let $\gamma(t) = (x(t), X(t))$ be a smooth curve in $\mathrm{SE}(n)$ with $\gamma(0) = (0, I)$, then its time derivative at $t = 0$, i.e.,

$$\dot{\gamma}(0) = (\dot{x}(0), \dot{X}(0)) = \begin{bmatrix} \dot{X}(0) & \dot{x}(0) \\ 0 & 0 \end{bmatrix},$$

gives rise to an element in the Lie algebra $\mathfrak{se}(n)$ by identifying $\mathfrak{se}(n)$ with $T_{(0,I)}\mathrm{SE}(n)$, the tangent space of $\mathrm{SE}(n)$ at the identity element $(0, I)$. Note that $X(t)$ is a curve in $\mathrm{SO}(n)$ with $X(0) = I$, and hence we have $\dot{X}(0) \in \mathfrak{so}(n)$. Therefore, every element $(v, \Omega) \in \mathfrak{se}(n)$ also admits a matrix representation as

$$(v, \Omega) = \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix},$$

where $\Omega \in \mathfrak{so}(n)$ and $v \in \mathbb{R}^n$.

Similar to $\mathfrak{so}(n)$, $\mathfrak{se}(n)$ is also a finite-dimensional vector space, and hence has a basis. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^n$, and define $\mathcal{R} = \{R_{ij} \in \mathfrak{se}(n) : R_{ij} = (0, \Omega_{ij}), 1 \le i < j \le n\}$ and $\mathcal{T} = \{T_k \in \mathfrak{se}(n) : T_k = (e_k, 0), 1 \le k \le n\}$, then the set $\mathcal{R} \cup \mathcal{T}$ forms a basis of $\mathfrak{se}(n)$. Analogous to Lemma 1, the following lemma characterizes the Lie bracket relations among the basis elements of $\mathfrak{se}(n)$.

Lemma 3. The Lie brackets among the basis elements of $\mathfrak{se}(n)$ satisfy $[R_{ij}, R_{kl}] = \delta_{jk}R_{il} + \delta_{il}R_{jk} + \delta_{jl}R_{ki} + \delta_{ik}R_{lj}$, $[R_{ij}, T_k] = \delta_{jk}T_i - \delta_{ik}T_j$ and $[T_k, T_l] = 0$ for all $1 \le i, j, k, l \le n$.

Proof. The proof directly follows from the computation of Lie brackets by using the matrix representations of R_{ij}, R_{kl}, T_k and T_l. □

Notice that Lie brackets of the elements in $\mathcal{R} = \{R_{ij} : 1 \le i < j \le n\}$ follow the same relation as those elements in $\mathcal{B} = \{\Omega_{ij} : 1 \le i < j \le n\}$ shown in Lemma 1. This indicates that the Lie algebra $\mathfrak{se}(n)$ contains $\mathfrak{so}(n)$ as a Lie subalgebra, and hence, together with the fact that $\mathrm{SE}(n)$ contains $\mathrm{SO}(n)$ as a Lie subgroup, a system defined on $\mathrm{SE}(n)$ also contains a system on $\mathrm{SO}(n)$ as a subsystem. This observation provides a main tool for controllability analysis of systems on $\mathrm{SE}(n)$ in the next section.

3.2 Decomposition of Systems on SE(n) *for Controllability Analysis*

After the review of the Lie group $\mathrm{SE}(n)$ and its Lie algebra $\mathfrak{se}(n)$ in the previous section, we are well prepared for the investigation into controllability of systems defined on $\mathrm{SE}(n)$. In particular, we focus on bilinear systems governed by vector fields in $\mathcal{R} \cup \mathcal{T}$ of the form

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X(t) & x(t) \\ 0 & 1 \end{bmatrix} &= \Bigg(\sum_{s=1}^{m_1} u_s(t) \begin{bmatrix} \Omega_{i_s j_s} & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \sum_{l=1}^{m_2} v_l(t) \begin{bmatrix} 0 & e_{k_l} \\ 0 & 0 \end{bmatrix} \Bigg) \begin{bmatrix} X(t) & x(t) \\ 0 & 1 \end{bmatrix}, \\ &(x(0), X(0)) = (0, I), \end{aligned} \tag{3}$$

where $\Omega_{i_s j_s} \in \mathcal{B}$ is a basis element of $\mathfrak{so}(n)$, e_{k_l} is the k_l-th standard basis vector of $\mathbb{R}^n$, and $u_s(t), v_l(t) \in \mathbb{R}$ are piecewise constant control functions for all $s = 1, \ldots, m_1$ and $l = 1, \ldots, m_2$. Furthermore, as discussed in the previous section, because $\mathrm{SE}(n)$ contains $\mathrm{SO}(n)$ and $\mathbb{R}^n$ as Lie subgroups, the system in (3) can be decomposed into two subsystems as

$$\dot{X}(t) = \left[\sum_{s=1}^{m_1} u_s(t) \Omega_{i_s j_s} \right] X(t), \quad X(0) = I, \tag{4}$$

$$\dot{x}(t) = \left[\sum_{s=1}^{m_1} u_s(t) \Omega_{i_s j_s} \right] x(t) + \sum_{l=1}^{m_2} v_l(t) e_{k_l}, \quad x(0) = 0, \tag{5}$$

so that the systems in (4) and (5) are on $\mathrm{SO}(n)$ and $\mathbb{R}^n$, repectively.

In practice, systems on $\mathrm{SE}(n)$ are widely used to describe the motion of rigid bodies whose dynamics are composed of rotations and translations. The decomposition of the system in (3) into the two subsystems in (4) and (5) exactly reveals the corresponding rotational and translational dynamics of the system in (3), respectively. The next theorem further explores that the controllability analysis of a system on $\mathrm{SE}(n)$ can also be equivalently carried over to its corresponding rotational component on $\mathrm{SO}(n)$ and translational component on $\mathbb{R}^n$.

Theorem 4. A system defined on $\mathrm{SE}(n)$ in the form of (3) is controllable if and only if its rotational component in (4) and translational component in (5) are controllable on $\mathrm{SO}(n)$ and $\mathbb{R}^n$, respectively.

Proof. (Necessity): Geometrically, the Euclidean group $\mathrm{SE}(n)$ is trivially diffeomorphic to $\mathbb{R}^n \times \mathrm{SO}(n)$ through the identity map $(x, X) \mapsto (x, X)$. Therefore, if the system in (3) is controllable on $\mathrm{SE}(n)$, then the direct product of the controllable submanifolds of its subsystems in (5) and (4) must be $\mathbb{R}^n \times \mathrm{SO}(n)$, and hence, they are controllable on $\mathrm{SO}(n)$ and $\mathbb{R}^n$, respectively.

(Sufficiency): Given any $X_F \in \mathrm{SO}(n)$ and $x_F \in \mathbb{R}^n$, it suffices to show that there exist piecewise constant control functions $u_1, \ldots, u_{m_1}, v_1, \ldots, v_{m_2}$ simultaneously steering the systems in (4) from I to X_F and (5) from 0 to x_F.

At first, we claim that $m_2 \ge 1$ must hold if the system in (5) is controllable on $\mathbb{R}^n$. Otherwise, the system reduces to

$$\dot{x}(t) = \left[\sum_{s=1}^{m_1} u_s(t) \Omega_{i_s j_s} \right] x(t), \tag{6}$$

which describes the dynamics of the system on $\mathrm{SO}(n)$ in (4) acting on $\mathbb{R}^n$. However, the homogeneous spaces of the Lie group action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ are the spheres centered at the origin (Lee, 2003). Consequently, the controllable submanifold of the system in (6) must be contained in a sphere, which contradicts the controllability of the translational component on $\mathbb{R}^n$.

Now, let $\mathbb{S}^{n-1}_{\|x_F\|}$ denote the sphere centered at the origin with radius $\|x_F\|$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, and V be the subspace of $\mathbb{R}^n$ spanned by $e_{k_1}, \ldots, e_{k_{m_2}}$, then $V \cap \mathbb{S}^{n-1}_{\|x_F\|} \ne \varnothing$ holds. Pick a point $z \in V \cap \mathbb{S}^{n-1}_{\|x_F\|}$, because $\mathrm{SO}(n)$ acts on $\mathbb{S}^{n-1}_{\|x_F\|}$ transitively (Lee, 2003), there exists $A \in \mathrm{SO}(n)$ such that $x_F = Az$.

In the following, we will develop a control strategy to simultaneously steer the system in (4) from I to X_F and the system in (5) from 0 to x_F in three steps. First, because the system in (4) is controllable on $\mathrm{SO}(n)$, the control inputs $u_1, \ldots, u_{m_1}$ can be appropriately designed to steer the system in (4) from I to $A^{-1}X_F$, and simultaneously, the system in (5) stays at the origin by setting $v_1 = \cdots = v_{m_2} = 0$. Then, we set $u_1 = \cdots = u_{m_1} = 0$ and apply $v_1, \ldots, v_{m_2}$ to steer the system in (5) from the origin to z. In this step, the system in (4) stays at $A^{-1}X_F$. At last, $u_1, \ldots, u_{m_2}$ can be turned on again to steer the system in (4) from $A^{-1}X_F$ to X_F. Since $x_F = Az$, the system in (5) will be simultaneously steered to x_F from z, and then the proof is done. □

The proof of Theorem 4 provides a systematic control design procedure to simultaneously accomplish transitions of the systems in (4) and (5) between desired states, which concludes the controllability of the system in (3). Alternatively, the proof can also be carried out algebraically by computing the Lie algebras generated by the control vector fields of these systems, which will be the main focus of the next section through a symmetric group approach.

3.3 Controllability Characterization of Systems on SE(n) *via Symmetric Group Actions*

In this section, we aim to illustrate the application of the symmetric group approach of characterizing controllability to bilinear systems defined on the non-compact Lie group $\mathrm{SE}(n)$. At first, recall that, for such a system, its rotational component is exactly governed by a system on $\mathrm{SO}(n)$ in the form of (2). As a result, the algebraic controllability condition represented in terms of the length of permutation cycles shown in Theorem 2 can be directly adopted to evaluate the controllability of the rotational component in (4). Now, according to Theorem 4, in order to analyze the controllability of the whole system on $\mathrm{SE}(n)$, we only need to focus on the translational component in (5). As shown in the proof of Theorem 4, the translational component in (5) contains the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, which motivates the leverage of the action of S_n on $Z_n = \{1, \ldots, n\}$ to study its controllability. To this end, let's first briefly recap some basics of the group action of S_n on Z_n.

For any $\sigma \in S_n$, let $\langle\sigma\rangle = \{\sigma^r \in S_n : r = 0, \pm 1, \ldots\}$ denote the cyclic subgroup of S_n generated by σ and $\sigma = \sigma_1 \cdots \sigma_l$ be its decomposition into a product of disjoint cycles. By the definition of S_n, every element $\sigma^r \in \langle\sigma\rangle \subseteq S_n$ is a bijective function on Z_n. Then, for any $k \in Z_n$, we define the orbit of k under the group action of $\langle\sigma\rangle$ on Z_n to be the set of images of k under all of the functions in $\langle\sigma\rangle$, denoted by $\langle\sigma\rangle \cdot k = \{\pi(k) : \pi \in \langle\sigma\rangle\} = \{\sigma^r(k) : r = 0, \pm 1, \ldots\}$. It can be checked that $\langle\sigma\rangle \cdot k = \mathcal{O}_{\sigma_i}$ for an unique $i = 1, \ldots, l$, where $\mathcal{O}_{\sigma_i}$ denotes the nontrivial orbit of the cycle σ_i.

With the brief survey of the group action of S_n on Z_n above, we are able to explore the relationship between the actions of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ and S_n on Z_n. Corresponding to the Lie bracket relation $[R_{ij}, T_k] = \delta_{jk}T_i - \delta_{ik}T_j$ shown in Lemma 3, it can be checked that $[\Omega_{ij}x, e_k] = \delta_{jk}e_i - \delta_{ik}e_j$ holds for any $x \in \mathbb{R}^n$. Specifically, $[\Omega_{ij}x, e_k]$, as well as $[R_{ij}, T_k]$, vanishes unless $k = i$ or $k = j$. On the other hand, for the action of S_n on Z_n, $\langle(i, j)\rangle \cdot k = \{i, j\}$ holds if $k = i$ or $k = j$, and $\langle(i, j)\rangle \cdot k = \{k\}$ is the trivial action otherwise. Therefore, the Lie bracket $[\Omega_{ij}x, e_k]$ can be represented by using ι as

$$[\Omega_{ij}x, e_k] = \begin{cases} \mp e_{\iota(\Omega_{ij})\cdot k}, & \text{if } k = i \text{ or } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Following from this observation, if $\iota(\mathcal{S}) = (a_1, \ldots, a_l)$ is an l-cycle for some $\mathcal{S} \subset \mathcal{B}$ with no redundant element, then, because of $\mathrm{Lie}(\mathcal{S}) = \mathrm{span}\{\Omega_{ij} : i, j = a_1, \ldots, a_l\}$ (Zhang and Li, 2017), for any $\Omega_{ij} \in \mathcal{S}$, $[\Omega_{ij}x, e_k] \neq 0$ if and only if $k \in \mathcal{O}_{\iota(\mathcal{S})}$, and consequently, $\mathrm{Lie}(\mathcal{S} \cdot x) = \mathrm{span}\{e_k : k \in \mathcal{O}_{\iota(\mathcal{S})}\}$, where $\mathcal{O}_{\iota(\mathcal{S})} = \{a_1, \ldots, a_l\}$ denotes the nontrivial orbit of $\iota(\mathcal{S})$ and $\mathcal{S} \cdot x = \{\Omega_{ij}x : \Omega_{ij} \in \mathcal{S}\}$ is the set of vector fields on $\mathbb{R}^n$ induced by the action of $\mathcal{S}$ on $\mathbb{R}^n$. On the symmetric group level, we have

$$\langle\iota(\mathcal{S})\rangle \cdot k = \begin{cases} \mathcal{O}_{\iota(\mathcal{S})}, & \text{if } k \in \{a_1, \ldots, a_l\}, \\ \{k\}, & \text{otherwise,} \end{cases}$$

i.e., $\langle\iota(\mathcal{S})\rangle \cdot k$ is nontrivial if and only if $k \in \mathcal{O}_{\iota(\mathcal{S})}$. Similar to the case of systems defined on $\mathrm{SO}(n)$ shown in Theorem 2, the above relationship between the actions of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ and S_n on Z_n immediately gives rise to a symmetric group approach to characterizing controllability of systems induced by the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ in the form of (5).

Theorem 5. A bilinear system defined on $\mathbb{R}^n$ in the form of (5), i.e.,

$$\frac{d}{dt}x(t) = \left[\sum_{s=1}^{m_1} u_s(t)\Omega_{i_s j_s}\right] x(t) + \sum_{l=1}^{m_2} v_l(t)e_{k_l}, \quad x(0) = 0,$$

is controllable if and only if there exists $\mathcal{S} \subseteq \{\Omega_{i_s j_s} : s = 1, \ldots, m_1\}$ such that $\bigcup_{l=1}^{m_2}(\langle\iota(\mathcal{S})\rangle \cdot k_l) = Z_n$.

Proof. (Necessity): Under the condition that the system in (5) is controllable on $\mathbb{R}^n$, it suffices to show $Z_n \subseteq \bigcup_{l=1}^{m_2}(\langle\iota(\mathcal{S})\rangle \cdot k_l)$, or equivalently, any $r \in Z_n$ satisfies $r \in \langle\iota(\mathcal{S})\rangle \cdot k_l$ for some k_l.

Let $\mathcal{S}$ be a subset of $\{\Omega_{i_1 j_1}, \ldots, \Omega_{i_{m_1} j_{m_1}}\}$ such that it does not contain redundant element and satisfies $\mathrm{Lie}(\mathcal{S}) = \mathrm{Lie}\{\Omega_{i_1 j_1}, \ldots, \Omega_{i_{m_1} j_{m_1}}\}$. Pick any $r \in Z_n$, if $r = k_l$ for some $l = 1, \ldots, m_2$, then $r \in \langle\iota(\mathcal{S})\rangle \cdot k_l$ holds trivially. Otherwise, because the system in (5) is controllable on $\mathbb{R}^n$, there must exist $\Omega_{\alpha\beta} \in \mathrm{Lie}(\mathcal{S})$ and e_{k_γ} for some $\gamma = 1, \ldots, m_2$ such that $[\Omega_{\alpha\beta}x, e_{k_\gamma}] = e_r$, which then implies $r \in \langle\iota(\mathcal{S})\rangle \cdot k_\gamma$.

(Sufficiency): We need to show that any standard basis element e_r of $\mathbb{R}^n$ can be generated by iterated Lie brackets of the control vector fields of the system in (5).

If $\{k_1, \ldots, k_{m_2}\} = Z_n$, then $\{e_{k_1}, \ldots, e_{k_{m_2}}\} = \{e_1, \ldots, e_n\}$ holds, which immediately implies the controllability of the system in (5) on $\mathbb{R}^n$. Otherwise, pick any $r \in Z_n \backslash \{k_1, \ldots, k_{m_2}\}$, by the assumption that $\bigcup_{l=1}^{m_2}(\langle\iota(\mathcal{S})\rangle \cdot k_l) = Z_n$ for some $\mathcal{S} \subseteq \{\Omega_{i_1 j_1}, \ldots, \Omega_{i_{m_1} j_{m_1}}\}$, there exists some $l \in \{1, \ldots, m_2\}$ such that $r \in \langle\iota(\mathcal{S})\rangle \cdot k_l$. Because $r \neq k_l$ by the choice of r, $\langle\iota(\mathcal{S})\rangle \cdot k_l$ contains at least two elements r and k_l, or equivalently, $\iota(\mathcal{S})$ has an orbit containing both of r and k_l, which indicates $\Omega_{rk_l} \in \mathrm{Lie}(\mathcal{S})$. Since $[\Omega_{rk_l}x, e_{k_l}] = e_r$, the proof is completed. □

In addition to the controllability examination, if a system in the form of (5) is not controllable on $\mathbb{R}^n$, the orbits of the action of $\iota(\mathcal{S})$ on Z_n also characterizes the controllable submanifold of the system as shown in the following corollary.

Corollary 6. The controllable submanifold of a bilinear system defined on $\mathbb{R}^n$ in the form of (5) is the linear subspace of $\mathbb{R}^n$ spanned by $\{e_r : r \in \bigcup_{l=1}^{m_2}(\langle\iota(\mathcal{S})\rangle \cdot k_l)\}$ for some $\mathcal{S} \subseteq \{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$.

Proof. At first, we pick $\mathcal{S} \subseteq \{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$ such that it does not contain redundant element and satisfies $\text{Lie}(\mathcal{S}) = \text{Lie}\{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$. Furthermore, let $\iota(\mathcal{S}) = \sigma_1 \cdots \sigma_s$ with σ_i pairwise disjoint cycles and $\mathcal{O}_i$ denote the nontrivial orbit of σ_i for each $i = 1, \ldots, s$. Then, the Lie subalgebra of $\mathfrak{so}(n)$ generated by $\mathcal{S}$ can be decomposed as $\text{Lie}(\mathcal{S}) = \oplus_{i=1}^{s}\text{span}\{\Omega_{\alpha\beta} : \alpha, \beta \in \mathcal{O}_i\}$ (Zhang and Li, 2017), where $\oplus$ denotes the direct sum of vector spaces. For each component $V_i = \text{span}\{\Omega_{\alpha\beta} : \alpha, \beta \in \mathcal{O}_i\}$, Theorem 5 implies that the Lie algebra generated by $e_1, \ldots, e_{m_2}$ and $V_i \cdot x$ is $\text{span}\{e_r : r \in \bigcup_{l=1}^{m_2}(\langle\sigma_i\rangle \cdot k_l)\}$. On the other hand, because σ_i are pairwise disjoint, the subgroups $\langle\sigma_i\rangle$ of S_n have trivial intersection. Consequently, for each $k_l \in Z_n$, there is an unique σ_i such that $\langle\iota(\mathcal{S})\rangle \cdot k_l = \langle\sigma_i\rangle \cdot k_l$. Therefore, the controllable submanifold of the system in (5) is the linear subspace of $\mathbb{R}^n$ spanned by $\bigcup_{i=1}^{s}\{e_r : r \in \bigcup_{l=1}^{m_2}(\langle\sigma_i\rangle \cdot k_l)\} = \{e_r : r \in \bigcup_{i=1}^{s}\bigcup_{l=1}^{m_2}(\langle\sigma_i\rangle \cdot k_l)\} = \{e_r : r \in \bigcup_{l=1}^{m_2}(\langle\iota(\mathcal{S})\rangle \cdot k_l)\}$. □

Now, we will integrate the controllability characterization of the translational component on $\mathbb{R}^n$ shown in Theorem 5 with that of the rotational component on $\text{SO}(n)$ shown in Lemma 1 to yield an algebraic controllability condition of the whole system on $\text{SE}(n)$.

Corollary 7. A bilinear system on $\text{SE}(n)$ in the form of (3), i.e.,

$$\frac{d}{dt}\begin{bmatrix} X(t) & x(t) \\ 0 & 1 \end{bmatrix} = \left(\sum_{s=1}^{m_1} u_s(t)\begin{bmatrix} \Omega_{i_sj_s} & 0 \\ 0 & 0 \end{bmatrix} + \sum_{l=1}^{m_2} v_l(t)\begin{bmatrix} 0 & e_{k_l} \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} X(t) & x(t) \\ 0 & 1 \end{bmatrix},$$
$$(x(0), X(0)) = (0, I),$$

is controllable if and only if there exists a subset $\mathcal{S}$ of $\{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$ such that $\iota(\mathcal{S})$ is an n-cycle and $m_2 \geq 1$.

Proof. (Necessity): It is equivalent to showing that if one of the two conditions, i.e., $\iota(\mathcal{S})$ is an n-cycle for some $\mathcal{S} \subseteq \{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$ and $m_2 \geq 1$, violates, then the system in (3) is not controllable on $\text{SE}(n)$.

By Lemma 1, if there does not exist any subset $\mathcal{S} \subseteq \{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$ such that $\iota(\mathcal{S})$ is an n-cycle, then the rotational component of the system in (3), i.e., the system in (4), fails to be controllable on $\text{SO}(n)$. On the other hand, $m_2 = 0$ results in uncontrollability of the translational component in (5) on $\mathbb{R}^n$ as discussed in the proof of Theorem 4. Both of these two situations will lead to uncontrollability of the system in (3) on $\text{SE}(n)$ according to Theorem 4, which then proves the necessity of the two conditions.

(Sufficiency): By Theorem 4, it suffices to show that both of the systems in (4) and (5) are controllable providing that $\iota(\mathcal{S})$ is an n-cycle for some $\mathcal{S} \subseteq \{\Omega_{i_1j_1}, \ldots, \Omega_{i_{m_1}j_{m_1}}\}$ and $m_2 \geq 1$. Moreover, Lemma 1 has revealed that $\iota(\mathcal{S})$ being an n-cycle guarantees the controllability of the rotational component in (4) on $\text{SO}(n)$. Then, it remains to show that the translational component in (5) is controllable on $\mathbb{R}^n$ under the additional condition $m_2 \geq 1$.

Note that an n-cycle in S_n has only one orbit Z_n. Hence, $\langle\iota(\mathcal{S})\rangle \cdot k_l = Z_n$ holds for any $l = 1, \ldots, m_2$. This implies that the system in (5) is controllable on $\mathbb{R}^n$ by Theorem 5, which also concludes the proof. □

4. CONCLUSION

In this paper, we introduce a novel algebraic approach to characterizing controllability of bilinear systems evolving on Lie groups. Specifically, we leverage the structural properties of the system on the compact and connected Lie group $\text{SO}(n)$ and the symmetric group to establish a transparent controllability analysis through permutation orbits. For systems evolving on the non-compact Lie group $\text{SE}(n)$ induced by the Lie group action of $\text{SO}(n)$ on $\mathbb{R}^n$, we explore the relationship between the action of $\mathfrak{so}(n)$ on $\mathbb{R}^n$ and S_n on a set of n elements, which leads to an algebraic characterization of controllability and controllable submanifolds of such systems in terms of the orbits of the symmetric group action. The established methodology for analyzing controllability not only provides an alternative to the canonical controllability condition LARC, but also takes an important step toward understanding geometric control from an algebraic perspective.

Appendix A. LIE ALGEBRA RANK CONDITION

Theorem 8. Let G be a compact and connected Lie group and $\mathfrak{g}$ be its Lie algebra, then a bilinear system on G of the form

$$\dot{X}(t) = B_0X(t) + \left(\sum_{i=1}^{m} u_i(t)B_i\right)X(t), \quad X(0) = I,$$

where $X(t) \in G$, $B_i \in \mathfrak{g}$, $u_i(t) \in \mathbb{R}$ are piecewise constant, and I is the identity element of G, is controllable if and only if $\text{Lie}\{B_0, \ldots, B_m\} = \mathfrak{g}$.

Proof. See (Brockett, 1972) and (Jurdjevic and Sussmann, 1972).

Appendix B. BASICS OF THE SYMMETRIC GROUP THEORY

A symmetric group on n letters, denoted by S_n, is the group of all bijective maps defined on a set containing n elements, in which the group operation is the composition of functions and elements of S_n are called permutations. Conventionally, the set of n elements is denoted by $Z_n = \{1, \ldots, n\}$. For any $\sigma \in S_n$, we further define an equivalence relation on Z_n by $a \sim b$ if and only if $b = \sigma^k(a)$ for $a, b \in Z_n$, and then the equivalence classes determined by this equivalence relation are called the orbits of σ. A permutation $\sigma \in S_n$ is said to be a cycle if it has at most one orbit containing more than one element, and its length is defined to be the number of elements in its nontrivial orbit. A cycle of length k is also named as a k-cycle, and, in particular, a 2-cycle is also called a transposition. It can be shown that every permutation is a product of disjoint cycles, i.e., cycles whose nontrivial orbits have empty intersection, and a k-cycle can be decomposed into a product of no less than $k-1$ transpositions (Lang, 2002).

REFERENCES

Baillieul, J. (1978). Geometric methods for nonlinear optimal control problems. *Journal of Optimization Theory and Applications*, 25(4), 519–548.

Brockett, R. (1972). System theory on group manifolds and coset spaces. *SIAM Journal on Control*, 10(2), 265–284.

Brockett, R. (1976). Nonlinear systems and differential geometry. *Proceedings of the IEEE*, 64(1), 61–72.

Brockett, R. (2014). The early days of geometric nonlinear control. *Automatica*, 50(9), 2203 – 2224.

Eisen, M. (1979). *Mathematical Models in Cell Biology and Cancer Chemotherapy*, volume 30 of *Lecture Notes in Biomathematics*. Springer-Verlag Berlin Heidelberg.

Glaser, S.J., Schulte-Herbrüggen, T., Sieveking, M., O. Schedletzky, N.C.N., Sørensen, O.W., and Griesinger, C. (1998). Unitary control in quantum ensembles, maximizing signal intensity in coherent spectroscopy. *Science*, 280, 421–424.

Hermann, R. and Krener, A. (1977). Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5), 728–740.

Huang, G.M., Tarn, T.J., and Clark, J.W. (1983). On the controllability of quantum-mechanical systems. *Journal of Mathematical Physics*, 24(11), 2608–2618.

Isidori, A. (1995). *Nonlinear Control Systems*. Communications and Control Engineering. Springer-Verlag London, 3rd edition.

Jurdjevic, V. (1996). *Geometric Control Theory*. Cambridge University Press, New York.

Jurdjevic, V. and Sussmann, H. (1972). Control systems on lie groups. *Journal of Differential Equations*, 12(2), 313 – 329.

Lang, S. (2002). *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag New York, New York, 3rd edition.

Lee, J.M. (2003). *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag New York, New York, 2nd edition.

Li, J.S. and Khaneja, N. (2006). Control of inhomogeneous quantum ensembles. *Physical Review A*, 73, 030302.

Li, J.S., Ruths, J., Yu, T.Y., Arthanari, H., and Wagner, G. (2011). Optimal pulse design in quantum control: A unified computational method. *Proceedings of the National Academy of Sciences*, 108(5), 1879–1884.

Mohler, R.R. and Kolodziej, W. (1980). An overview of bilinear system theory and applications. *IEEE Transactions on Systems, Man, and Cybernetics*, 10(10), 683 – 688.

Mohler, R.R. and Ruberti, A. (1978). *Recent Developments in Variable Structure Systems, Economics and Biology*, volume 162 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag Berlin Heidelberg.

Zhang, W. and Li, J.S. (2017). Analyzing controllability of bilinear systems on symmetric groups: Mapping lie brackets to permutations. *arXiv:1708.02332v1*.