Outage-Optimized Deployment of UAVs

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***Abstract*—We consider multiple unmanned aerial vehicles (UAVs) serving a density of ground terminals (GTs) as mobile base stations. The objective is to minimize the outage probability of GT-to-UAV transmissions. In this context, the optimal placement of UAVs under different UAV altitude constraints and GT densities is studied. First, using a random deployment argument, a general upper bound on the optimal outage probability is found for any density of GTs and any number of UAVs. Lower bounds on the performance of optimal deployments are also determined. The upper and lower bounds are combined to show that the optimal outage probability decays exponentially with the number of UAVs for GT densities with finite support. Next, the structure of optimal deployments are studied when the common altitude constraint is large. In this case, for a wide class of GT densities, it is shown that all UAVs should be placed to the same location in an optimal deployment. A design implication is that one can use a single multi-antenna UAV as opposed to multiple single-antenna UAVs without loss of optimality. Numerical optimization of UAV deployments are carried out using particle swarm optimization. Simulation results are also presented to confirm the analytical findings.**

***Index Terms*—UAV-aided communications, optimal placement, outage probability.**

I. Introduction

Unmanned aerial vehicles (UAVs) have recently been successively utilized in many diverse areas, including wireless communications [1]. Several applications have been mentioned for UAV-assisted communications including UAVs acting as base stations [2], [3], and relays [4]–[6]. The ability of UAVs acting as data collection units [7], [8] makes them particularly appealing for Internet of Things applications [1]. In fact, UAVs' anywhere, anytime relocation ability plays a significant role in their success in improving the performance of various wireless communications systems. In this context, UAVs can be utilized to improve network power-efficiency [9], [10].

Despite their effectiveness in wireless communications, there are still several fundamental challenges in UAV-based systems that have to be resolved. For example, trajectory optimization and optimal placement of UAVs is an important problem in designing UAV-aided wireless communication systems. The problem of optimal placement for UAVs, even in a scenario where the location of users is known and fixed, is a nonconvex optimization problem whose dimensionality increases with the number of UAVs.

There have been several studies on the placement/trajectory optimization of UAVs for different objectives. For example, static placement of UAVs as mobile base stations to maximize network coverage [11]–[13], and energy efficient placement of UAVs subject to ground terminal (GT) coverage or rate constraints are studied [14], [15]. The problem of interference-aware joint trajectory and power control of multiple UAVs have been explored [16]. In [17], the authors consider the UAV trajectory planning problem under communication security constraints. In [6], the authors propose a mobile UAV relaying method and jointly optimize throughput, trajectory and temporal power allocation. Power efficient deployment of UAVs as relays is studied in [18] for centralized and distributed UAV selection scenarios. Trajectory control of UAVs for maximizing the worst terrestrial user's spectral efficiency is studied in [19]. The deployment of UAVs for user-in-the-loop scenarios have also been considered [20]. In this framework, the UAV locations influence the ground user density as users will tend to congregate to high coverage or high rate areas.

Most of the above works consider optimizing the power or rate efficiency of the system or aim at maximizing the coverage. On the other hand, especially if the UAVs are tasked to collect data from low-power GTs, outage probability becomes an important performance measure. Also, most previous works consider a numerical approach to UAV location optimization, and formal analytical results on the optimal placement and the resulting performance is not available in general. In this work, we consider the optimal placement of UAVs serving as base stations to GTs with the goal of minimizing the outage probability of GT transmissions. We follow an analytical approach to find optimal placement of UAVs, and derive upper bound and lower bounds on the optimal outage probability. Using the outage probability as a performance metric results in a fundamentally different cost function as compared to earlier literature on UAVs. Correspondingly, we also obtain fundamentally different results. We also verify our analysis with numerical simulations conducted using the particle swarm optimization (PSO) algorithm [21].

A problem formulation that is similar to ours appears in the context of optimal control of mobile sensors [22], [23]. The objective of these two studies is to find control laws that maximize the cumulative event detection probabilities of the sensors. However, [22], [23] do not present analytical results on the structure and performance of optimal sensor deployments, and follow a gradient descent based numerical solution.

The rest of this paper is organized as follows: In

Section II, we introduce the system model. In Section III, we study the optimal outage probability for different number of UAVs at a fixed altitude. In Section IV, we study the optimal deployments of UAVs for different altitude constraints. In Section V, we present numerical simulation results. Finally, in Section VI, we draw our main conclusions and discuss future work. Some of the technical proofs are provided in the appendices.

II. System Model

We consider multiple GTs at zero altitude and multiple UAVs. Mathematically, we assume that all the GTs are located at $\mathbb{R}^d$, where $d \in \{1,2\}$. The case $d = 1$ is relevant when the GTs are located on a straight line on the ground, e.g. on a highway. We also assume that the GTs are distributed on $\mathbb{R}^d$ according to some probability density function $f : \mathbb{R}^d \to \mathbb{R}$.

In this paper, we consider a model where the UAVs are subject to a common minimum altitude constraint $h \geq 0$ due to governmental regulations or environmental constraints. In such a scenario, given that all the GTs are located at zero altitude, decreasing the altitude of any UAV decreases the GTs' access distance to the UAV, resulting in a better overall performance. We thus assume that all UAVs are located at the fixed elevation h.

Let us now consider a GT at location $x \in \mathbb{R}^d$. Also, let $u_i \in \mathbb{R}^d$, $i = 1, \ldots, n$ denote the locations of the UAVs projected to the ground (Hence, the actual location of UAV i is given by $[u_i\ h] \in \mathbb{R}^{d+1}$). The GT can communicate with any one of the UAVs, all of which act as service providers. Fig. 1 illustrates the system model for the special case of $n = d = 2$.

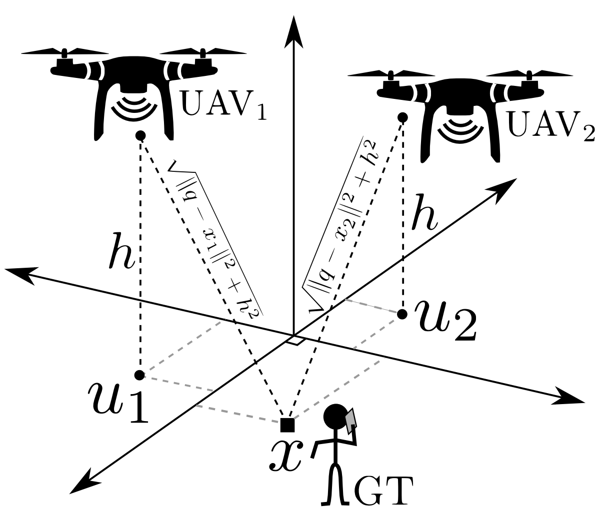

Fig. 1. A network of two UAVs serving a GT.

Suppose that the GT wishes to communicate with rate ρ bits/sec/Hz and transmits with fixed power P. Note that the distance between the GT at x and UAV i is given by $(\|x - u_i\|^2 + h^2)^{\frac{1}{2}}$. Assuming Gaussian noise with unit power at the receiver, the capacity of the channel between the GT and UAV i is thus $\log_2(1 + (\|x - u_i\|^2 + h^2)^{-\frac{r}{2}}|h_i|^2 P)$ bits/sec/Hz, where r is the path loss exponent, and h_i denotes the fading coefficient between the GT and UAV i. We assume $h_1, \ldots, h_n$ are independent and identically distributed as circularly-symmetric complex Gaussian random variable with unit variance. The GT transmission will be successful if the channel capacity is at least ρ. The probability of a failed transmission from a GT at x to UAV i is then given by the outage probability

$$P_{O_i}(x,u_i) \triangleq \mathrm{P}\left(\log_2\left(1+\frac{|h_i|^2 P}{(\|x-u_i\|^2+h^2)^{\frac{r}{2}}}\right) \leq \rho\right) \quad (1)$$

$$= \mathrm{P}\left(|h_i|^2 \leq \frac{2^\rho-1}{P}(\|x-u_i\|^2+h^2)^{\frac{r}{2}}\right). \quad (2)$$

Form now on, we set $\rho = P = 1$ without loss of generality. Also, since $|h_1|^2$ is well-known to be an exponential random variable with cumulative distribution function $F_{|h_i|^2}(x) = 1 - e^{-x}$, we obtain

$$P_{O_i}(x,u_i) = 1 - g(x,u_i), \quad (3)$$

where

$$g(x,u_i) \triangleq \exp\left(-(\|x-u_i\|^2 + h^2)^{\frac{r}{2}}\right). \quad (4)$$

The transmitted GT data cannot be received by any one of the UAVs if an outage event occurs at all UAVs. The overall outage probability of the system given the deployment $U = [u_1 \cdots u_n]$ of UAVs is thus

$$P_O(x,U) = \prod_{i=1}^{n} [1 - g(x,u_i)]. \quad (5)$$

Averaging out the GT density function f, we obtain our objective function

$$P_O(U) = \int \prod_{i=1}^{n} [1 - g(x,u_i)] f(x) \mathrm{d}x. \quad (6)$$

Throughout the paper, all unspecified integration domains are $\mathbb{R}^d$. The problem is then to find optimal UAV locations $U^\star \triangleq \arg\min_U P_O(U)$ that minimize the outage probability, and the corresponding minimum-possible outage probability $P_O(U^\star)$. In the following, we first consider the behavior of $P_O(U^\star)$ for different values of the number of UAVs n for a fixed UAV altitude h. We then consider the behavior of $P_O(U^\star)$ as a function of the constraint h on the UAV altitudes.

III. Optimal Outage Probability for Different Number of UAVs

In this section, we study how the optimal outage probability varies with respect to the number of UAVs for a fixed h, and the corresponding optimal deployments. In this context, finding the exact minimizers of (6) appears to be a hopeless task for a general n. Even the case of a single UAV $n = 1$ results in a non-trivial optimization problem, for which there appears to be a no closed-form solution in general. Still, one can find the optimal positioning of a single UAV for unimodal densities that are defined in the following.

Definition 1. *We call a univariate density function $f : \mathbb{R} \to \mathbb{R}$ unimodal with center $\mu \in \mathbb{R}$ if f is non-decreasing on $(-\infty, \mu]$, and f is non-increasing on $[\mu, \infty)$. We call a bivariate density function $f : \mathbb{R}^2 \to \mathbb{R}$ unimodal with center $\mu = \left[\begin{smallmatrix}\mu_1 \\ \mu_2\end{smallmatrix}\right] \in \mathbb{R}^2$ if for all y, the function $f_1(x) = f(x, y)$ is unimodal with center μ_1, and for all x, the function $f_2(y) = f(x, y)$ is unimodal with center μ_1.*

A circularly-symmetric Gaussian random vector or uniform random vectors are both examples of unimodal densities. We have the following theorem.

Theorem 1. *Let $n = 1$, and suppose that f is unimodal with center μ. Then, for any $h \geq 0$, an optimal deployment of the single UAV is given by $U^\star = \mu$.*

Proof. See Appendix A. $\square$

This verifies the intuition that if the density is "symmetric" around μ, then the optimal deployment of the single UAV should be μ.

For the case of a general $n \geq 1$, we provide general upper and lower bounds on the outage probability that hold for any number of UAVs and different GT densities. The bounds allow us to obtain estimates on the achievable outage probability without going through the time-consuming numerical optimization methods. They also allow us to obtain the asymptotic behavior of the outage probability as $n \to \infty$.

We first present a general upper bound via the following theorem. As defined in Section II, let $P_O(U^\star)$ denote the minimum possible outage probability provided by an optimal deployment. Let X represent a random variable whose probability density equals the GT density f.

Theorem 2. *The following upper bound holds for arbitrary random variables $U_1, \ldots, U_n$:*

$$P_O(U^\star) \leq \mathrm{E}_{X,U_1,\ldots,U_n}\left[\prod_{i=1}^{n}(1 - g(X, U_i))\right]. \quad (7)$$

In particular, suppose $U_1, \ldots, U_n$ have the same density as some random variable U, are mutually independent, and also independent of X. Then, we have

$$P_O(U^\star) \leq \int \left(\mathrm{E}_U[1 - g(x, U)]\right)^n f(x)\mathrm{d}x. \quad (8)$$

Proof. The proof (7) relies on the following random deployment argument: We assume that the location of UAV i is the random variable U_i. The expected outage probability with this random deployment scenario is the right side of (7). It is guaranteed that there exists at least one deployment that achieves the expected performance, and this proves (7). Result (8) follows immediately from (7) by independence of random variables. $\square$

In particular, the theorem shows via (8) that in general, the outage probability decays at least exponentially with the number of available UAVs provided that the support of f is finite.[1] On the other hand, the upper bound in (8) is not tight in general for a given finite n. An important question in this context that we shall leave as future work is the optimization of the upper bound (7) with respect to the densities $U_1, \ldots, U_n, X$ for tighter upper bounds.

We now provide lower bounds. A very simple lower bound that holds for arbitrary densities is the following:

Theorem 3. *We have*

$$P_O(U^\star) \geq (1 - e^{-h^r})^n. \quad (9)$$

Proof. The bound follows once we use the estimate $g(x, u_i) = \exp(-(\|x - u_i\|^2 + h^2)^{\frac{r}{2}}) \leq \exp(-h^r)$ in (6). $\square$

The bound (9) is unfortunately not useful for the case of unmanned ground vehicles (UGVs) $h = 0$. For this scenario, using somewhat more sophisticated methods, one can still obtain a lower bound with an exponential rate of decay.

Theorem 4. *Let $h = 0$. There is a constant $c \geq 0$ such that for any n, we have $P_O(U^\star) \geq c^n$.*

Proof. See Appendix B. $\square$

Theorems 2, 3, and 4 together show that for any fixed h, the best-possible decay of the outage probability is exponential with respect to the number of UAVs.

IV. Optimal Deployments for Different UAV Altitude Constraints

We now consider the behavior of the optimal UAV deployments and the corresponding optimal outage probabilities for different UAV altitude constraints h. By Theorem 1, we have shown that, for one UAV and a unimodal GT density, the optimal UAV location is at the center of the GT density. For example, for a uniform distribution on $[0, 1]$, the optimal UAV location is at 0.5 for every UAV altitude. With more UAVs, one would expect the UAVs to be evenly distributed to $[0, 1]$ to provide an even coverage of the GT domain $[0, 1]$. This is the case, indeed, if one wishes to place the UAVs so as to minimize the average GT power consumption

[1] In fact, setting $U = 0$ to be deterministic, we have $P_O(U^\star) \leq \mathrm{E}[(1 - g(X, 0))^n] \leq \mathrm{E}[(1 - e^{-\|X\|^r})^n]$. Given that the support of f is contained on a ball with radius R, we have $P_O(U^\star) \leq (1 - e^{-R^r})^n$. A proof of the exponential decay of the outage probability for arbitrary densities will be discussed elsewhere.

to guarantee zero-outage transmission [10], or in many existing studies on UAV location optimization [1].

A counterintuitive phenomenon occurs in the case of our objective function of minimizing the outage probability. In fact, during our numerical experiments, we have observed that after a certain finite altitude, all UAVs collapse to a single location in an optimal deployment. Here, we prove a slightly weaker claim that, in an optimal deployment, all UAVs converge to a common location as $h \to \infty$. We write $f(x) \sim g(x)$ as a shorthand notation for the asymptotic equality $\lim_{x\to\infty} f(x)/g(x) = 1$.

Theorem 5. *Let*

$$u^\star(h) \triangleq \arg\max_u \int g(x,u)f(x)\mathrm{d}x. \tag{10}$$

As $h \to \infty$, we have the asymptotic equality

$$(1 - P_O(U^\star)) \sim n \int g(x, u^\star(h))f(x)\mathrm{d}x. \tag{11}$$

An optimal deployment $U^\star$ that achieves (11) is where all the n UAVs are located at $u^\star(h)$ for a given h.

Proof. See Appendix C. □

The theorem provides a precise asymptotic formula for the optimal outage probability as $h \to \infty$ and the optimal UAV locations that achieve this outage probability. In particular, placing all UAVs to the point $\lim_{h\to\infty} u^\star(h)$ is asymptotically optimal. The important design implication is that, for a large UAV altitude constraint, one can just use a single UAV with multiple antennas (and use selection diversity), as compared to multiple UAVs with a single antenna. In fact, since it is physically impossible to place multiple single-antenna UAVs to a single location, one *must* use a multi-antenna UAV to minimize the outage probability for a given maximum number of selectable UAV antennas.

A special case is when the density f is unimodal with center μ. In such a scenario, we have $u^\star(h) = \mu$, $\forall h$, and the UAV locations converge to μ as $h \to \infty$ by Theorem 5. For example, if f is one-dimensional Gaussian with mean μ and variance σ^2, the outage probability when all UAVs are located at μ is given by

$$\begin{aligned} &P_O([\mu \cdots \mu]) \\ &= \int (1 - g(x,\mu))^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x \qquad (12) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k e^{-kh^2}}{\sqrt{1+2k\sigma^2}} \mathrm{erf}\left(\sqrt{\frac{1+2k\sigma^2}{2\sigma^2}}\right). \qquad (13) \end{aligned}$$

For large h, we expect (13) to be a good approximation on the optimal outage probability. As a two-dimensional example, for a uniform distribution on $[0,1]^2$, we have

$$P_O(\left[\begin{smallmatrix} 0.5 & \cdots & 0.5 \\ 0.5 & \cdots & 0.5 \end{smallmatrix}\right]) = 1 + \frac{\pi}{4} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k e^{kh^2}} \mathrm{erf}^2(\sqrt{k}) \tag{14}$$

to approximate the optimal outage probability at large h.

V. Numerical Results

In this section, we provide numerical simulation results that confirm our analytical findings. In order to optimize the UAV locations, we have used the PSO method, which is a population-based iterative algorithm for solving non-linear optimization problems [21]. In general, population-based optimization algorithms such as PSO are known to outperform the simpler gradient descent like approaches. In fact, the existence of multiple candidate solutions (population members) help to avoid locally optimal solutions. In the following, we compare the optimal placement of UAVs and the corresponding outage probabilities obtained using the PSO method with those provided by our analytical results.

In Fig. 2, we show the optimal outage probability provided by the PSO algorithm in comparison with the analytical formula (14) for different values of the UAV altitudes h. We consider the choice of path loss exponent $r = 2$. The horizontal axis represents the number of UAVs, and the vertical axis represents the logarithm of the optimal outage probability. We can observe that the logarithm of the outage probability decays linearly with the number of UAVs. Hence, the outage probability decays exponentially with n, verifying Theorems 2, 3, and 4. We can also observe that, as h increases, the analytical formula in (14) provides a very good approximation on the optimal outage probability. In fact, for $h \in \{0.5, 1\}$, the analysis is almost indistinguishable from the simulations.

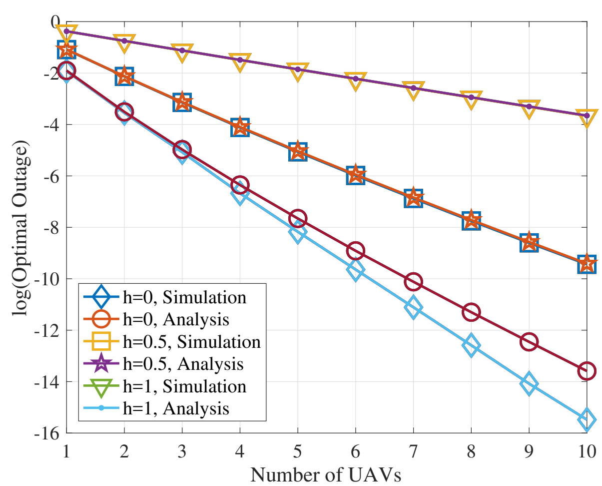

Fig. 2. Outage probabilities for a uniform distribution on $[0,1]^2$.

In Fig. 3, we consider a standard normal distribution for the GT density, and compare the outage probabilities provided by the PSO algorithm with the analytical formula in (13) for $r = 3$. We can similarly observe the exponential decay of the outage probability with respect to the number of UAVs. Also, for any number of UAVs, the analysis matches the simulations perfectly when $h^2 = 2$. On the other hand, the mismatch for lower altitudes is more pronounced compared to the case of the uniform distribution in Fig. 2. As a result, we have also included the upper bounds derived using (8). We chose the random variable U to be a standard normal distribution, which provided the best upper bound among all other choices for U that we have considered for large n. We can observe that the upper bound in (8) provides a better prediction of the optimal performance when n is large. Yet, there is still a gap between the estimated and the actual performance. This shows the necessity of better bounds on the outage probability, a topic that will be considered as future work.

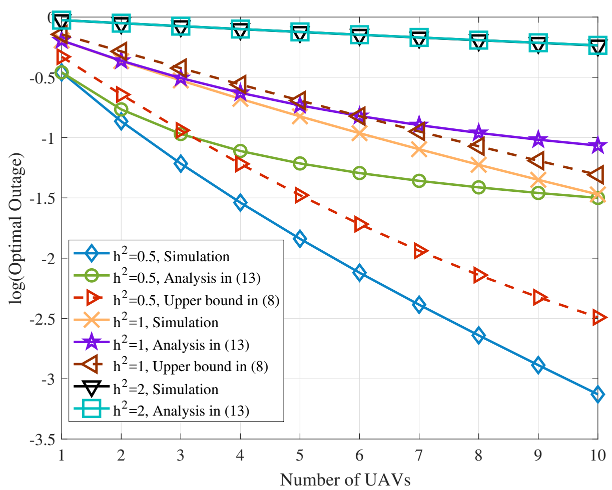

Fig. 3. Outage probabilities for a standard normal GT density.

In Fig. 4, we illustrate the collapse of the optimal UAV locations to a single location as the UAV altitude constraint h increases. We have considered the choice $r = 2$. The horizontal axis represents h, and the vertical axis represents the optimal UAV locations. We can observe that at $h = 0$, the optimal deployment is roughly given by $U^\star = [0.08\ 0.33\ 0.66\ 0.92]$. As h is increased to around 0.15, the four UAVs first collapse to two distinct locations. After $h \geq 0.4$, the optimal locations for all four UAVs is equal to 0.5, as Theorem 5 suggests. Note that Theorem 5 shows that the UAV locations will converge to 0.5 as $h \to \infty$. Here, we observe the stronger phenomenon that the UAVs should be located at 0.5 for all sufficiently large altitudes.

VI. Conclusions

We have studied the optimal placement of UAVs serving as mobile base stations to several GTs. Our objective has been to minimize the outage probability

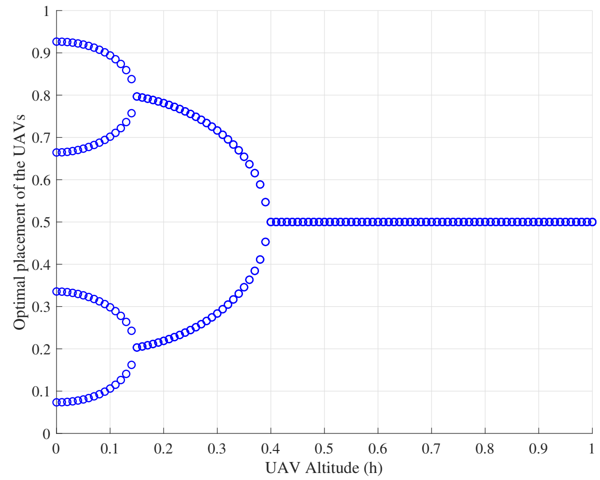

Fig. 4. Optimal locations of 4 UAVs for a uniform density on $[0, 1]$.

of the system. We have shown the optimal outage probability decays exponentially with the number of UAVs. We have also proved that in an optimal deployment, all UAVs collapse to a unique location at large altitudes. We have used particle swarm optimization to numerically optimize the UAV locations. As future work, we aim to provide better bounds for general GT densities and UAV altitudes. Also, we have only considered a static deployment of UAVs; we plan to study the movement of UAVs according to a time-varying GT density. Moreover, our model has only utilized selection diversity for reception of the GT data by one of the UAVs. In principle, it is possible to apply maximum ratio combining for a better performance. Optimal placement of UAVs for this scenario is an interesting direction for future work.

Acknowledgement

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Appendix A
Proof of Theorem 1

We need to show that a global maximum of

$$h(u) \triangleq \int g(x,u) f(x) \mathrm{d}x \tag{15}$$

is located at $u = \mu$. We first provide a proof for the case $d = 1$. We need the following properties of a unimodal function. If f is a univariate unimodal density with center μ, we have

$$f(\mu + c) = f(\mu - c),\ \forall c \in \mathbb{R}, \tag{16}$$
$$f(\mu + c) \geq f(\mu + d),\ d \geq c \geq 0, \tag{17}$$
$$f(\mu - c) \geq f(\mu - d),\ d \geq c \geq 0. \tag{18}$$

These properties follow immediately from Definition 1.

We are now ready to prove the theorem for $d = 1$. We will show that for any $\alpha \in \mathbb{R}$, we have $h(\mu) \geq h(\mu+\alpha)$. Equivalently, defining $H \triangleq h(\mu) - h(\mu+\alpha)$, we will show that $H \geq 0$. First, we consider $\alpha \geq 0$. We have

$$H = \int (g(x,\mu) - g(x,\mu+\alpha)) f(x) \mathrm{d}x \tag{19}$$

Applying a change of variables $t = x - \mu - \frac{\alpha}{2}$, we obtain

$$H = \int \zeta(t) f(t + \mu + \tfrac{\alpha}{2}) \mathrm{d}t, \tag{20}$$

where $\zeta(t) \triangleq g(t, -\frac{\alpha}{2}) - g(t, \frac{\alpha}{2})$. It can easily be verified that $\zeta(t) = -\zeta(-t)$ so that

$$H = \int_{-\infty}^{0} \zeta(t) f(t+\mu+\tfrac{\alpha}{2}) \mathrm{d}t + \int_{0}^{\infty} \zeta(t) f(t+\mu+\tfrac{\alpha}{2}) \mathrm{d}t \tag{21}$$

$$= \int_{-\infty}^{0} \zeta(t)(f(t+\mu+\tfrac{\alpha}{2}) - f(-t+\mu+\tfrac{\alpha}{2})) \mathrm{d}t. \tag{22}$$

The equality follows from a change of variables $t \leftarrow -t$, and the fact that $\zeta(\cdot)$ is an odd function.

Let us now partition the integration domain in (22) to two subsets, namely, $(-\infty, -\frac{\alpha}{2})$ and $[-\frac{\alpha}{2}, 0]$, and call the resulting integrals H_1 and H_2, respectively. We have

$$H_1 = \int_{-\infty}^{-\frac{\alpha}{2}} \zeta(t)(f(t+\mu+\tfrac{\alpha}{2}) - f(-t+\mu+\tfrac{\alpha}{2})) \mathrm{d}t \tag{23}$$

$$= \int_{-\infty}^{-\frac{\alpha}{2}} \zeta(t)(f(-t+\mu-\tfrac{\alpha}{2}) - f(-t+\mu+\tfrac{\alpha}{2})) \mathrm{d}t. \tag{24}$$

The equality follows from the symmetry (16) of f around μ. Now, given $t \leq -\frac{\alpha}{2}$, we have $0 \leq -\frac{\alpha}{2} - t \leq \frac{\alpha}{2} - t$. According to (17), we obtain

$$f(-t + \mu - \tfrac{\alpha}{2}) \geq f(-t + \mu + \tfrac{\alpha}{2}),\ t \leq -\frac{\alpha}{2} \tag{25}$$

It can also be easily verified that

$$\zeta(t) \geq 0,\ \forall t \leq 0. \tag{26}$$

Applying the bounds (25) and (26) to (24), we obtain $H_1 \geq 0$. Similarly, we can show the non-negativity of

$$H_2 = \int_{-\frac{\alpha}{2}}^{0} \zeta(t)(f(t+\mu+\tfrac{\alpha}{2}) - f(-t+\mu+\tfrac{\alpha}{2})) \mathrm{d}t. \tag{27}$$

In fact, given $t \in [-\frac{\alpha}{2}, 0]$, we have the chain of inequalities $0 \leq t + \frac{\alpha}{2} \leq -t + \frac{\alpha}{2}$. By (17), we obtain $f(\mu+t+\frac{\alpha}{2}) \geq f(\mu-t+\frac{\alpha}{2})$. Applying this bound to (27) together with (26), we obtain $H_2 \geq 0$. Since $H_1 \geq 0$ as already shown, we have $H = H_1 + H_2 \geq 0$. Using the same arguments, one can also show $H \geq 0$ for $\alpha \leq 0$. This concludes the proof for $d = 1$.

Now, suppose $d = 2$, $\mu = [\begin{smallmatrix}\mu_1 \\ \mu_2\end{smallmatrix}]$. To prove the theorem, it is sufficient to show that (i) for every x and $\alpha \geq 0$, we have $h([\begin{smallmatrix}x \\ \mu_2\end{smallmatrix}]) \geq h([\begin{smallmatrix}x \\ \mu_2+\alpha\end{smallmatrix}])$, and (ii) for every y and $\alpha \geq 0$, we have $h([\begin{smallmatrix}\mu_1 \\ y\end{smallmatrix}]) \geq h([\begin{smallmatrix}\mu_1+\alpha \\ y\end{smallmatrix}])$. Both claims can be verified using the same arguments that we have used for $d = 1$. This concludes the proof of the theorem.

APPENDIX B
PROOF OF THEOREM 4

We first consider the case $d = 1$ and a uniform distribution on $[0, M]$. Without loss of generality, assume $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq M$. Given $j \in \{1, \ldots, n+1\}$, let $I_j \triangleq [u_{j-1}, u_j]$, with the convention that $u_0 = 0$ and $u_{n+1} = 1$. We have

$$P_O(U) = \sum_{j=1}^{n} \int_{I_j} \prod_{i=1}^{n} (1 - \exp(-(x-u_i)^r)) \frac{\mathrm{d}x}{M} \tag{28}$$

For the jth term of the sum, instead of integrating over all $I_j = [u_{j-1}, u_j]$, we integrate over

$$I_j' \triangleq \left[u_{j-1} + \frac{|I_j|}{3}, u_j - \frac{|I_j|}{3} \right],\ j \in \{1, \ldots, n\}. \tag{29}$$

By construction, for all $j \in \{1, \ldots, n\}$ and every $x \in I_j'$, we have $|x - u_i| \leq \frac{1}{3}|I_j|$, $\forall i \in \{1, \ldots, n\}$. It follows that

$$P_O(U) \geq \sum_{j=1}^{n} \int_{I_j'} \prod_{i=1}^{n} (1 - \exp(-(x-u_i)^r)) \frac{\mathrm{d}x}{M} \tag{30}$$

$$\geq \sum_{j=1}^{n} \int_{I_j'} \prod_{i=1}^{n} \left(1 - \exp\left(-|I_j|^r / 3^r\right)\right) \frac{\mathrm{d}x}{M} \tag{31}$$

$$= \sum_{j=1}^{n} \frac{|I_j'|}{M} \left(1 - \exp\left(-|I_j|^r / 3^r\right)\right)^n \tag{32}$$

$$= \sum_{j=1}^{n} \frac{|I_j|}{3M} \left(1 - \exp\left(-|I_j|^r / 3^r\right)\right)^n \tag{33}$$

The second derivative of the function $g(x) \triangleq 1 - \exp(-x^r/3^r)$ can be calculated to be

$$g''(x) = r 9^{-r} \exp(-x^r/3^r) x^{r-2} (3^r(r-1) - r x^r), \tag{34}$$

which means that g is convex whenever $0 \leq x \leq 3(1 - \frac{1}{r})^{1/r}$. By the second derivative test, it is straightforward to show that the product of two non-decreasing convex functions is convex. It follows that, with regards to the second sum in (33), the function $x \mapsto x(1 - \exp(-x^r/3^r))^n$ is convex whenever $x \leq 3(1 - \frac{1}{r})^{1/r}$.

Now, suppose there exists $\ell \in \{1, \ldots, n\}$ such that $|I_\ell| \geq 3(1 - \frac{1}{r})^{1/r}$. We then have

$$P_O(U) \geq \frac{|I_\ell|}{3M} \left(1 - \exp\left(-|I_\ell|^r / 3^r\right)\right)^n \tag{35}$$

$$\geq \frac{(1 - \frac{1}{r})^{1/r}}{M} \left(1 - \exp\left(-1 + \tfrac{1}{r}\right)\right)^n, \tag{36}$$

and Theorem 4 follows. We can therefore assume $|I_j| \leq 3(1 - \frac{1}{r})^{1/r}$, $\forall j \in \{1, \ldots, n\}$. We now recall Jensen's inequality: For an arbitrary convex function h, and arbitrary real numbers $a_1, \ldots, a_n$, we have $\frac{1}{n}\sum_{i=1}^{n} h(a_i) \geq h(\frac{1}{n}\sum_{i=1}^{n} a_i)$. Applying to (33), we obtain

$$P_O(U) \geq \frac{1}{3} \left(1 - \exp\left(-1/3^r\right)\right)^n. \tag{37}$$

This concludes the proof for $d = 1$ and a uniform distribution. The case of a uniform distribution on $[0, M]^2$ in $d = 2$ dimensions can be handled similarly by choosing I_js to be a sequence of disks that cover $[0, M]^2$. Correspondingly, for each j, we choose I'_j to be the disk that is concentric to I_j, but with one-third the radius. The case of a non-uniform density can be handled by considering a box where the density is bounded from below by a positive constant and applying the results already found for a uniform density. This concludes the proof of the theorem in the general case.

Appendix C
Proof of Theorem 5

We have the expansion

$$P_O(U) = \int \prod_{i=1}^{n} (1 - g(x, u_i)) f(x) \mathrm{d}x \tag{38}$$

$$= \int \Bigg[1 - \sum_{i=1}^{n} g(x, u_i) + \sum_{k=2}^{n} (-1)^k \sum_{J \in C_{n,k}} \prod_{j \in J} g(x, u_j) \Bigg] f(x) \mathrm{d}x, \tag{39}$$

where $C_{n,k}$ is the collection of all k-combinations of the set $\{1, \ldots, n\}$ (For example, $C_{3,2} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$). Now, note that

$$g(x, u) = \exp\left(-(\|x-u\|^2 + h^2)^{\frac{r}{2}}\right) \leq e^{-h^r}. \tag{40}$$

Therefore, for any $J \in C_{n.k}$, we have

$$\int \prod_{j \in J} g(x, u_j) f(x) \mathrm{d}x \leq e^{-|J| h^r}. \tag{41}$$

Using (41) and the bound $(-1)^k \leq 1$ in (39), we obtain

$$P_O(U) \leq 1 - \sum_{i=1}^{n} \int g(x, u_i) f(x) \mathrm{d}x + \sum_{k=2}^{n} \binom{n}{k} e^{-kh^r} \tag{42}$$

$$\leq 1 - \sum_{i=1}^{n} \int g(x, u_i) f(x) \mathrm{d}x + 2^n e^{-2h^r} \tag{43}$$

Taking the minimum over all deployments, we have

$$P_O(U^\star) \leq 1 - n \int g(x, u^\star(h)) f(x) \mathrm{d}x + 2^n e^{-2h^r}. \tag{44}$$

A similar argument results in the converse estimate

$$P_O(U^\star) \geq 1 - n \int g(x, u^\star(h)) f(x) \mathrm{d}x - 2^n e^{-2h^r}. \tag{45}$$

The statement of the theorem follows from (44), (45), and the dominated convergence theorem.

References

[1] Y. Zeng, R. Zhang, and T. J. Lim, "Wireless communications with unmanned aerial vehicles: Opportunities and challenges," *IEEE Commun. Mag.*, vol. 54, no. 5, pp. 36–42, May 2016.

[2] R. I. Bor-Yaliniz, A. El-Keyi, and H. Yanikomeroglu, "Efficient 3-D placement of an aerial base station in next generation cellular networks," *IEEE Intl. Conf. Commun.*, May 2016.

[3] J. Lyu, Y. Zeng, R. Zhang, and T. J. Lim, "Placement optimization of UAV-mounted mobile base stations," *IEEE Commun. Lett.*, vol. 21, no. 3, pp. 604–607, Mar. 2017.

[4] W. Zhao, M. Ammar, and E. Zegura, "A message ferrying approach for data delivery in sparse mobile ad hoc networks," *ACM MobiHoc*, May 2004.

[5] P. Zhan, K. Yu, and A. L. Swindlehurst, "Wireless relay communications with unmanned aerial vehicles: Performance and optimization," *IEEE Trans. Aerospace Elect. Sys.*, vol. 47, no. 3, pp. 2068–2085, July 2011.

[6] Y. Zeng, R. Zhang, and T. J. Lim, "Throughput maximization for UAV-enabled mobile relaying systems," *IEEE Trans. Commun.*, vol. 64, no. 12, pp. 4983–4996, Dec. 2016.

[7] B. Pearre and T. X. Brown, "Model-free trajectory optimization for wireless data ferries among multiple sources," *IEEE GLOBECOM*, Dec. 2010.

[8] J. Lyu, Y. Zeng, and R. Zhang, "Cyclical multiple access in UAV-aided communications: A throughput-delay tradeoff," *IEEE Wireless Commun. Lett.*, vol. 5, no. 6, pp. 600–603, Dec. 2016.

[9] E. Koyuncu, R. Khodabakhsh, N. Surya, and H. Seferoglu, "Deployment and trajectory optimization for UAVs: A quantization theory approach," *IEEE WCNC*, Apr. 2018.

[10] E. Koyuncu, M. Shabanighazikelayeh, and H. Seferoglu, "Deployment and trajectory optimization of UAVs: A quantization theory approach," *IEEE Transactions on Wireless Communications*, vol. 17, no. 12, pp. 8531–8546, Nov. 2018.

[11] M. Mozaffari, W. Saad, M. Bennis, and M. Debbah, "Wireless communication using unmanned aerial vehicles (UAVs): Optimal transport theory for hover time minimization," *IEEE Trans. Wireless Commun.*, vol. 16, no. 12, pp. 8052?8066, Sept. 2017.

[12] ———, "Efficient deployment of multiple unmanned aerial vehicles for optimal wireless coverage," *IEEE Commun. Letters*, vol. 20, no. 8, pp. 1647–1650, Aug. 2016.

[13] H. Shakhatreh, A. Khreishah, and B. Ji, "Providing wireless coverage to high-rise buildings using UAVs," *IEEE ICC*, May. 2017.

[14] Y. Zeng and R. Zhang, "Energy-efficient UAV communication with trajectory optimization," *IEEE Trans. Wireless Commun.*, vol. 16, no. 6, pp. 3747–3760, June 2017.

[15] Y. Xu, L. Xiao, D. Yang, L. Cuthbert, and Y. Wang, "Energy-efficient UAV communication with multiple GTs based on trajectory optimization," *Mobile Information Systems*, 2018.

[16] C. Shen, T.-H. Chang, J. Gong, Y. Zeng, and R. Zhang, "Multi-UAV interference coordination via joint trajectory and power control," Sept. 2018. [Online] Available: https://arxiv.org/abs/1809.05697

[17] Y. Cai, Z. Wei, R. Li, D. W. K. Ng, and J. Yuan, "Energy-efficient resource allocation for secure UAV communication systems," Jan. 2019. [Online] Available: https://arxiv.org/abs/1901.09308

[18] E. Koyuncu, "Power-efficient deployment of UAVs as relays," *IEEE SPAWC*, June 2018.

[19] S. Roth, A. Kariminezhad, and A. Sezgin, "Base-stations up in the air: multi-UAV trajectory control for min-rate maximization in uplink C-RAN," Nov. 2018. [Online] Available: https://arxiv.org/abs/1811.10585

[20] I. Bor-Yaliniz, A. El-Keyi, and H. Yanikomeroglu, "Spatial configuration of agile wireless networks with drone-BSs and user-in-the-loop," *IEEE Trans. Wireless Commun.*, vol. 18, no. 2, pp. 753–768, Oct. 2018.

[21] J. Kennedy, "Particle swarm optimization," *Encyclopedia of Machine Learning*, 2011.

[22] W. Li and C. G. Cassandras, "Distributed cooperative coverage control of sensor networks," *IEEE CDC*, Dec. 2005.

[23] M. Zhong and C. G. Cassandras, "Distributed coverage control and data collection with mobile sensor networks," *IEEE Trans. Automatic Control*, vol. 56, no. 10, pp. 2445–2455, Oct. 2011.