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Stability Analysis for Uncertain Chains of Integrators Driven by Nested Nonlinear Feedbacks

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***Abstract*— It is proved that the linear uncertain systems described by chains of integrators with unknown positive parameters can be stabilized by a kind of nested nonlinear feedback controllers with any positive gains. A Lyapunov/Chetaev function is constructed for the stability analysis of the closed-loop system, and a necessary and sufficient condition for the asymptotic stability is derived by using the technique of homogeneous domination.**

I. INTRODUCTION

This paper considers the local stability problem of the general uncertain chain of integrators

$$\begin{aligned} \dot{x}_1 &= a_1 x_2, \\ \dot{x}_2 &= a_2 x_3, \\ &\vdots \\ \dot{x}_{n-1} &= a_{n-1} x_n, \\ \dot{x}_n &= a_n u, \end{aligned} \tag{1}$$

derived by a nested nonlinear controller with the form

$$\begin{aligned} u = -((\cdots((k_1 x_1^{\frac{r_2}{r_1}} + k_2 x_2)^{\frac{r_3}{r_2}} + k_3 x_3)^{\frac{r_4}{r_3}} + \cdots \\ + k_{n-1} x_{n-1})^{\frac{r_n}{r_{n-1}}} + k_n x_n), \end{aligned} \tag{2}$$

where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$ is the system state vector, $u \in \mathbf{R}$ is the control input, a_i's are uncertain positive constant parameters, and r_i's are ratios of odd positive integers. The structure of the controller (2) is proposed in [1] for global stabilization. However, in [1], the bounds of a_i's are known and the gains k_i's are consequently designed by using the bounds. In this paper, we focus on the stability analysis of the closed-loop system instead of the controller design. We assume that all the a_i's are arbitrary positive unknown parameters and all the gains k_i are any non-zero constants. For example, simulations show that the nested nonlinear controller

$$u = -(((k_1 x_1)^{5/3} + k_2 x_2)^{33/25} + k_3 x_3) \tag{3}$$

with any given positive gains k_i's can stabilize (1) with $n = 3$ for any possible uncertain positive parameters a_i's. This motivates us to present a general theoretical result on the asymptotic stability of (1) with a nested nonlinear feedback strategy (2).

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It should be noted that, generally speaking, it is impossible to design a linear feedback controller

$$u = -(k_1 x_1 + k_1 x_2 + \cdots + k_n x_n) \tag{4}$$

to stabilize (1) for all the possible unknown positive parameters a_i's. For example, as $n = 3$, the characteristic polynomial of the closed-loop system of (1) with (2) is

$$\lambda^3 + a_3 k_3 \lambda^2 + a_2 a_3 k_2 \lambda + a_1 a_2 a_3 k_1. \tag{5}$$

For any given $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$, letting $a_1 = \frac{2k_2}{k_1}$, $a_2 = \frac{k_3}{k_2}$ and $a_3 = \frac{1}{k_3}$ results a non-Hurwitz polynomial $\lambda^3 + \lambda^2 + \lambda + 2$, which implies that the closed-loop system is unstable.

The stabilization problem of chains of integrators has been extensively investigated in the community of control theory. Many control systems such as single-input controllable linear systems and feedback linearizable nonlinear systems can be equivalently transformed into chains of integrators. The stabilization problem of chains of integrators using bounded input with delays [2] or without delays [3] was already treated. In [4], nested saturations were used for chains of integrators. Delayed feedback control can also be used to stabilize chains of integrators [5]. In [6], the application of model (1) to planar vertical take off and landing (PVTOL) was presented. The robust control strategy was designed for chains of integrators with external disturbances [7]. A nonlinear PI controller was proposed for (1) as $a_i = 1$, $i = 1, 2, \dots, n-1$ and the unknown parameter $a_n \neq 0$ [8]. In [1], as the unknown a_i's have known bounds, the gains of the nested nonlinear controller (2) are designed by using the technique of adding a power integrator (AAPI) proposed in [9]. The AAPI technique is a very powerful tool to design global stabilization controllers, which has been widely applied to many control systems [7], [10], [11]. However, the AAPI technique usually results a high-gain feedback controller, which may cause implementation issue in practice. In order to easily implement the nested nonlinear controller (2), we try to analyze the stability of the closed-loop system under the controller (2) with free feedback gains.

For homogeneous systems, it is well-known that an asymptotically stable homogeneous system admits a homogeneous Lyapunov function. A natural idea is to generalize the concept of homogeneity and apply the generalized homogeneity to the stability analysis of a class of nonlinear systems. Actually, in [12]-[15], the concept homogeneity with monotone degrees (HMD) is proposed and has been successfully applied to the design of stabilizing controllers for inherently nonlinear systems. Note that even if a nonlinear system admits HMD, it may not be homogeneous. So it is interesting to analyze the asymptotic stability for an inherently nonlinear system with HMD.

This paper proposes a special HMD called homogeneity with strictly decreasing degrees (HSDD), which plays an important role in the construction of the Lyapunov/Chetaev function. Then by using the Lyapunov second method and Chetaev instability theorem, a necessary and sufficient condition for the asymptotic stability of the closed-loop system is obtained.

II. PRELIMINARIES

This section presents some fundamental theorem and some useful inequalities which will play important roles in obtaining the main results of this paper.

Theorem 1: (Lyapunov Stability Theorem) [16], [17] Consider a nonlinear system

$$\dot{x} = f(x), \quad x \in \Re^n, \tag{6}$$

where $f(x)$ is Lipschitz continuous with respect to x, $f(0) = 0$. If there exists a locally positive definite function $V(x)$ such that

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) \tag{7}$$

is locally negative definite, then system (6) is asymptotically stable.

Theorem 2: (Chetaev Instability Theorem [18]) If there exists a continuously differentiable function $V(x)$ such that (i) the origin is a boundary point of the set $G = \{x \in \Re^n \mid V(x) > 0\}$; (ii) there exists a neighborhood U of the $x = 0$ such that $\dot{V}(x) > 0 \ \ \forall\, x \in U \cap G$, then $x = 0$ is an unstable equilibrium point of the system.

Lemma 1: (Jensen's inequality) [19] For $p \geq 1$ and $x_i \in \Re$, $i = 1, \cdots, n$, the following holds

$$|x_1+x_2+\cdots+x_n|^p \leq n^{p-1}(|x_1|^p+|x_2|^p+\cdots+|x_n|^p). \tag{8}$$

Lemma 2: [20] For $p \geq 1$ which is a ratio of positive odd integers, the following holds

$$x(x+a)^p \geq 2^{1-p}x^{p+1} + xa^p, \ \forall\ x,\ a \in \Re.$$

Lemma 3: [12] Let c and d be positive constants. Given any number $\gamma > 0$, the following inequality holds

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}, \forall\ x,\ y \in \Re.$$

Definition 1: [13] A continuous vector field $v : \mathrm{R}^n \to \mathrm{R}^n$ with $v = [v_1, \cdots, v_n]^{\mathrm{T}}$ is said to satisfy homogeneity with monotone degrees (HMD), if we can find positive real numbers $(r_1, \cdots, r_n)$ and real numbers $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ such that

$$v_i(\epsilon^{r_1}x_1, \cdots, \epsilon^{r_n}x_n) = \epsilon^{r_i+\tau_i}v_i(x) \tag{9}$$

for all $x \in \Re^n$, $\epsilon > 0$ and $i = 1, 2, \cdots, n$. The constants r_i's and τ_i's are called homogeneous weights and degrees, respectively.

Definition 2: A continuous vector field $v : \mathrm{R}^n \to \mathrm{R}^n$ with $v = [v_1, \cdots, v_n]^{\mathrm{T}}$ is said to satisfy homogeneity with strictly decreasing degrees (HSDD), if it has HMD defined in Definition 1 and the homogeneous degrees satisfy

$$\tau_1 > \tau_2 > \cdots > \tau_n.$$

III. MAIN RESULTS

The nested controller (2) can be rewritten as the recursive form as follows:

$$\begin{aligned} f_1(x_1) &= k_1x_1, \\ f_2(x_1, x_2) &= f_1^{r_2/r_1} + k_2x_2, \\ &\vdots \\ f_{i+1}(x_1, \cdots, x_{i+1}) &= f_i^{r_{i+1}/r_i} + k_{i+1}x_{i+1}, \\ &\vdots \\ f_n(x_1, \cdots, x_n) &= f_{n-1}^{r_n/r_{n-1}} + k_nx_n, \end{aligned} \tag{10}$$

where every $r_i \geq 1$ is a ratio of two positive odd integers.

We are interested in the stability analysis of the closed-loop system:

$$\begin{aligned} \dot{x}_1 &= a_1x_2, \\ \dot{x}_2 &= a_2x_3, \\ &\vdots \\ \dot{x}_{n-1} &= a_{n-1}x_n, \\ \dot{x}_n &= -a_nf_n(x_1, x_2, \cdots, x_n), \end{aligned} \tag{11}$$

where r_i's are ratios of odd positive integers satisfy

$$\begin{aligned} &1 \leq r_1 < r_2, \\ &0 < r_{i+2} - r_{i+1} < r_{i+1} - r_i,\ i = 1, \ldots, n-2. \end{aligned} \tag{12}$$

Remark 1: Given r_i and r_{i+1}, one can determine r_{i+2} by (12). So it is easy to construct $\{r_i\}_{i=1}^n$ satisfying (12). Let $\tau_i = r_{i+1} - r_i$ for $i = 1, 2, \ldots, n-1$ and $\tau_n = 0$. Then one can easily check that the closed-loop system (11) has the HSDD

$$\tau_1 > \tau_2 > \cdots > \tau_n$$

with respect to $(r_1, r_2, \ldots, r_n)$.

Lemma 4: Let r_1, r_2, $\cdots$, r_s and n_1, n_2, $\cdots$, n_s $(s \geq 2)$ be any given positive constants. For any $\varepsilon > 0$, there exists a positive number A such that

$$|x_1|^{n_1}\cdots|x_s|^{n_s} \leq \varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_sr_s}{r_1}} + A\sum_{i=2}^{s}|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}} \tag{13}$$

for all x_1, x_2, $\cdots$, $x_s \in \Re$.

Proof: (Mathematical Induction) As $s = 2$, by Lemma 3, we have that, for any $\varepsilon > 0$, there exists $\tilde{A} > 0$ such that

$$\begin{aligned} |x_1|^{n_1}|x_2|^{n_2} &= (|x_1|^{\frac{1}{r_1}})^{r_1n_1}(|x_2|^{\frac{1}{r_2}})^{r_2n_2} \\ &\leq \varepsilon|x_1|^{\frac{r_1n_1+r_2n_2}{r_1}} + \tilde{A}|x_2|^{\frac{r_1n_1+r_2n_2}{r_2}}. \end{aligned} \tag{14}$$

Suppose that Lemma 4 holds for the case of s, i.e. assume that (13) holds. In the rest of this proof, we consider the case of $s+1$. By (13), we have that

$$\begin{aligned}&|x_1|^{n_1}\cdots|x_s|^{n_s}|x_{s+1}|^{n_{s+1}}\\ &\le (\varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_sr_s}{r_1}}+A\sum_{i=2}^{s}|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}})|x_{s+1}|^{n_{s+1}}. \end{aligned}\tag{15}$$

By Lemma 3, for each $i\ge 1$, there exists $B_i>0$ such that

$$\begin{aligned}&|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}}|x_{s+1}|^{n_{s+1}}\\ &\le |x_i|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_i}}+B_i|x_{s+1}|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_{s+1}}}. \end{aligned}\tag{16}$$

Applying (16) to (17) yields that there exists $\hat{A}$ such that

$$\begin{aligned}&|x_1|^{n_1}\cdots|x_s|^{n_s}|x_{s+1}|^{n_{s+1}}\\ &\le\varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_1}}+\hat{A}\sum_{i=2}^{s+1}|x_i|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_i}}. \end{aligned}\tag{17}$$

■

Proposition 1: By a diffeomorphism transformation, (11) is equivalent to the system as follows:

$$\begin{aligned}\dot{e}_1&=\frac{k_1a_1}{k_2}(e_2-e_1^{\frac{r_2}{r_1}})=:g_1(e_1,e_2),\\ \dot{e}_i&=\frac{k_ia_i}{k_{i+1}}(e_{i+1}-e_i^{\frac{r_{i+1}}{r_i}})+\frac{r_i}{r_{i-1}}e_{i-1}^{\frac{r_i}{r_{i-1}}-1}g_{i-1}(e_1,\cdots,e_i)\\ &=:g_i(e_1,\cdots,e_{i+1}),\qquad i=2,3,\cdots,n-1,\\ \dot{e}_n&=-k_na_ne_n+\frac{r_n}{r_{n-1}}e_{n-1}^{\frac{r_n}{r_{n-1}}-1}g_{n-1}(e_1,\cdots,e_n).\end{aligned}\tag{18}$$

Proof: Construct a nonlinear transformation

$$e_i=f_i(x_1,\cdots,x_i),\quad i=1,2,\cdots,n,\tag{19}$$

where each f_i is defined by (10). It is easy to check that the inverse mapping of (19) is

$$\begin{aligned}x_1&=k_1^{-1}e_1,\\ x_2&=k_2^{-1}(e_2-e_1^{r_2/r_1}),\\ &\vdots\\ x_i&=k_i^{-1}(e_i-e_{i-1}^{r_i/r_{i-1}}),\\ &\vdots\\ x_n&=k_n^{-1}(e_n-e_{n-1}^{r_n/r_{n-1}}).\end{aligned}\tag{20}$$

Since $r_i>r_{i-1}$, both the transformation (19) and its inverse mapping (20) are smooth, which implies that the transformation described by (19) is a diffeomorphism. A straightforward computation shows that system (11) is equivalently transformed into (18). ■

Lemma 5: Each function $g_i(e_1,\cdots,e_{i+1})$ defined by the right hand side of (18) satisfies

$$|g_i(e_1,\cdots,e_{i+1})|\le C_i(\sum_{k=1}^{i}\sum_{j=k}^{i}|e_j|^{\frac{\tau_k+r_i}{r_j}}+|e_{i+1}|^{\frac{\tau_i+r_i}{r_{i+1}}}),\tag{21}$$

where $\tau_k=r_{k+1}-r_k$ and each C_i is a constant dependent on $g_i(\cdot)$.

Proof: (Mathematical Induction) For the case of $i=1$, from (8) it follows that

$$\begin{aligned}|g_1(e_1,e_2)|&\le C_1(|e_1|^{\frac{r_2}{r_1}}+|e_2|)\\ &=C_1(|e_1|^{\frac{\tau_1+r_1}{r_1}}+|e_2|),\end{aligned}\tag{22}$$

where C_1 is dependent on g_1. Suppose that the lemma holds for the case of i, i.e. (21) holds. In the following, let us estimate $|g_{i+1}|$. From (18), Lemma 1 and Assumption 1, it follows that there exists a constant $A>0$ such that

$$\begin{aligned}|g_{i+1}|&=\left|\frac{k_{i+1}}{k_{i+2}}(e_{i+2}-e_{i+1}^{\frac{r_{i+2}}{r_{i+1}}})+\frac{r_{i+1}}{r_i}e_i^{\frac{r_{i+1}}{r_i}-1}g_i\right|\\ &\le A(|e_{i+1}|^{\frac{r_{i+2}}{r_{i+1}}}+|e_{i+2}|+|e_i|^{\frac{r_{i+1}}{r_i}-1}|g_i|)\\ &=A(|e_{i+1}|^{\frac{\tau_{i+1}+r_{i+1}}{r_{i+1}}}+|e_{i+2}|)+|e_i|^{\frac{r_{i+1}}{r_i}-1}|g_i|).\end{aligned}\tag{23}$$

By the induction assumption, applying (21) to (23), we have that

$$\begin{aligned}|g_{i+1}|\le{}& A(|e_{i+1}|^{\frac{\tau_{i+1}+r_{i+1}}{r_{i+1}}}+|e_{i+2}|)\\ &+AC_i\sum_{k=1}^{i}\sum_{j=k}^{i}(|e_j|^{\frac{\tau_k+r_i}{r_j}}+|e_{i+1}|)|e_i|^{\frac{r_{i+1}}{r_i}-1}.\end{aligned}\tag{24}$$

Using Lemma 4 to the last term of (24), we obtain that there exists $B>0$ such that

$$\begin{aligned}&(|e_j|^{\frac{\tau_k+r_i}{r_j}}+|e_{i+1}|)|e_i|^{\frac{r_{i+1}}{r_i}-1}\\ &\le B(|e_j|^{\frac{\tau_k+r_{i+1}}{r_j}}+|e_i|^{\frac{\tau_k+r_{i+1}}{r_i}}+|e_{i+1}|^{\frac{\tau_i+r_{i+1}}{r_{i+1}}}+|e_i|^{\frac{\tau_i+r_{i+1}}{r_i}}).\end{aligned}\tag{25}$$

From (24) and (25), it follows the conclusion of the case of $i+1$. So, by Mathematical Induction, the proof is complete. ■

Theorem 3: Consider the closed-loop (11) of the uncertain chain of integrators (1) under the nested nonlinear controller (1). Assume that the unknown parameters a_i's are positive, the feedback gains k_i's are nonzero, and r_i's satisfy (12). Then system (11) is asymptotically stable if and only if $k_i>0$.

Proof: Construct the following Lyapunov/Chetaev function

$$V(e)=\sum_{i=1}^{n}\frac{l_i}{\alpha_i}e_i^{\alpha_i},\tag{26}$$

where

$$l_i=-k_{i+1}k_i^{-1}a_i^{-1},\ i=1,2,\cdots,n-1,\ l_n=-k_n^{-1}a_n^{-1}\tag{27}$$

and

$$\alpha_i=2r_nr_i^{-1}\ge 2,\ i=1,2,\cdots,n.\tag{28}$$

The derivative of (26) along (18) can be easily computed as

$$\begin{aligned}\dot{V}={}&\sum_{i=1}^{n-1}e_i^{\alpha_i-1}(e_i^{\frac{r_{i+1}}{r_i}}-e_{i+1})+e_n^2\\ &+\sum_{i=2}^{n}\frac{l_ir_i}{r_{i-1}}e_i^{\alpha_i-1}e_{i-1}^{\frac{r_i}{r_{i-1}}-1}g_{i-1}(e_1,\cdots,e_i).\end{aligned}\tag{29}$$

From (29), Lemma 2 and Assumption 1, it follows that

$$\dot V(x) \geq \sum_{i=1}^{n-1} e_i^{\alpha_i-1+\frac{r_{i+1}}{r_i}} - \sum_{i=1}^{n-1} |e_i^{\alpha_i-1} e_{i+1}| + e_n^2 - \sum_{i=2}^{n} \frac{l_i r_i}{r_{i-1}} |e_i^{\alpha_i-1} e_{i-1}^{\frac{r_i}{r_{i-1}}-1} g_{i-1}(e_1,\cdots,e_i)|$$
$$= \sum_{i=1}^{n-1} |e_i|^{\mu_i} + |e_n|^2 - \sum_{i=1}^{n-1} |e_i^{\alpha_i-1} e_{i+1}| - \sum_{i=2}^{n} \frac{l_i r_i}{r_{i-1}} |e_i^{\alpha_i-1} e_{i-1}^{\frac{r_i}{r_{i-1}}-1} g_{i-1}(e_1,\cdots,e_i)|, \quad (30)$$

where

$$\mu_i = \alpha_i - 1 + \frac{r_{i+1}}{r_i} = \frac{\tau_i + \alpha_i r_i}{r_i}, \quad i = 1, 2, \cdots, n-1,$$

By (30) and Lemma 4, for any $\varepsilon > 0$, there is $\bar A$ such that

$$|e_i^{\alpha_i-1} e_{i+1}| \leq \varepsilon |e_i|^{\frac{\alpha_i r_i - r_i + r_{i+1}}{r_i}} + \bar A |e_{i+1}|^{\frac{\alpha_i r_i - r_i + r_{i+1}}{r_{i+1}}}$$
$$= \varepsilon |e_i|^{\frac{\tau_i + \alpha_i r_i}{r_i}} + \bar A |e_{i+1}|^{\frac{\tau_i+\alpha_i r_i}{r_{i+1}}}$$
$$= \varepsilon |e_i|^{\mu_i} + \bar A |e_{i+1}|^{\hat\mu_{i+1}}, \quad (31)$$

where

$$\hat\mu_{i+1} = \frac{\tau_i + \alpha_i r_i}{r_{i+1}} > \frac{\tau_{i+1} + \alpha_{i+1} r_{i+1}}{r_{i+1}} = \mu_{i+1}$$

due to $\tau_i > \tau_{i+1}$ and (28). Moreover, by Lemma 5, we have

$$|e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} |g_{i-1}| \leq C_{i-1} \sum_{k=1}^{i-1} \sum_{j=k}^{i-1} |e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} (|e_j|^{\frac{\tau_k + r_{i-1}}{r_j}} + |e_i|^{\frac{\tau_{i-1}+r_{i-1}}{r_i}}). \quad (32)$$

Now, let us estimate the terms of the right-hand side of (32). As $k = i-1$, we have that $j = i-1$ and

$$|e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} |e_j|^{\frac{\tau_k + r_{i-1}}{r_j}}$$
$$= |e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{\tau_{i-1}+r_i}{r_{i-1}}}$$
$$\leq \varepsilon |e_{i-1}|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}}} + \hat A |e_i|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_i}} \quad (33)$$

where

$$\frac{\tau_{i-1} + \alpha_i r_i}{r_{i-1}} = \frac{\tau_{i-1} + \alpha_{i-1} r_{i-1}}{r_{i-1}} = \mu_{i-1}$$

and

$$\frac{\tau_{i-1} + \alpha_i r_i}{r_i} > \frac{\tau_i + \alpha_i r_i}{r_i} = \mu_i.$$

As $k < i-1$, we have that

$$|e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} |e_j|^{\frac{\tau_k + r_{i-1}}{r_j}}$$
$$\leq \varepsilon |e_j|^{\frac{\tau_k+\alpha_i r_i}{r_j}} + A(|e_i|^{\frac{\tau_k+\alpha_i r_i}{r_i}} + |e_{i-1}|^{\frac{\tau_k+\alpha_i r_i}{r_{i-1}}}).$$

Considering $k < i-1$ and $k \leq j < i$, we have

$$\frac{\tau_k + \alpha_i r_i}{r_j} \geq \frac{\tau_j + \alpha_j r_j}{r_j} = \mu_j,$$

$$\frac{\tau_k + \alpha_i r_i}{r_i} > \frac{\tau_i + \alpha_i r_i}{r_i} = \mu_i$$

and

$$\frac{\tau_k + \alpha_i r_i}{r_{i-1}} > \frac{\tau_{i-1} + \alpha_{i-1} r_{i-1}}{r_{i-1}} = \mu_{i-1}.$$

Moreover,

$$|e_i|^{\alpha_i-1} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} |e_i|^{\frac{\tau_{i-1}+r_{i-1}}{r_i}}$$
$$= |e_i|^{\frac{\alpha_i r_i - r_i + \tau_{i-1} + r_{i-1}}{r_i}} |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1}$$
$$\leq \varepsilon |e_{i-1}|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}}} + A |e_i|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_i}}$$

where

$$\frac{\tau_{i-1} + \alpha_i r_i}{r_{i-1}} = \frac{\tau_{i-1} + \alpha_{i-1} r_{i-1}}{r_{i-1}} = \mu_{i-1},$$

$$\frac{\tau_{i-1} + \alpha_i r_i}{r_i} > \frac{\tau_i + \alpha_i r_i}{r_i} = \mu_i.$$

By the discussion below (30), letting $\varepsilon > 0$ be sufficiently small, we conclude that there exist constants K_i ($i = 1, 2, \cdots, n$) such that

$$\dot V(x) \geq \sum_{i=1}^{n} K_i |e_i|^{\mu_i} + h(e_1, e_2, \cdots, e_n),$$

where $h(e_1, e_2, \cdots, e_n)$ is composed of higher order terms. Therefore, if ε is sufficiently small, there exists a domain $D \subset \Re^n$ such that $\dot V(e)$ is positive definite on D, that is,

$$\dot V(e) > 0, \quad \forall\, e \in D \backslash \{0\}. \quad (34)$$

From (27), it is easily seen that

$$l_i < 0 \ (i = 1, 2, \cdots, n) \ \Leftrightarrow k_i > 0 \ (i = 1, 2, \cdots, n). \quad (35)$$

If $k_i > 0$ ($i = 1, 2, \cdots, n$), it is clear that $V(e)$ is negative definite due to (35) and (26). This, together with (34), implies that the zero solution of (11) is asymptotically stable by Lyapunov Stability Theorem. Therefore the positivity of k_i's is sufficient for the asymptotic stability of (11) .

On the other hand, if there exists a $k_i < 0$, by (35) there exists an $l_j > 0$. In this case, by (26) we know that the set $G := \{e \in \Re^n \mid V(e) > 0\}$ is not empty and $e = 0$ is a boundary point of G. Therefore, from (34) and Chetaev Instability Theorem, it follows that the zero solution of (11) is unstable. This implies that the positivity of k_i's is also necessary for the asymptotic stability of (11) . ■

Remark 2: In our recent papers [21] and [22], the idea of HSDD has been used to stabilize a kind of nonlinear systems via a linear feedback control. However, in this paper, we use a nonlinear feedback to stabilize a class of linear systems.

Remark 3: One interesting question is whether the main result of this paper can be generalized to the p-power integrator system as follows:

$$\begin{aligned} \dot x_1 &= a_1 x_2^{p_1}, \\ \dot x_2 &= a_2 x_3^{p_2}, \\ &\vdots \\ \dot x_{n-1} &= a_{n-1} x_n^{p_{n-1}}, \\ \dot x_n &= a_n u^{p_n}. \end{aligned} \quad (36)$$

Actually, one can design a similar nested nonlinear controller. If the closed-loop system admits HSDD, the asymptotic stability can be obtained in a similar way. However, how to ensure the existence of HSDD for (36) seems to be a difficult problem.

IV. Simulations

Consider the nonlinear system (1) with $n = 3$. Let

$$r_1 = 1, \quad r_2 = \frac{5}{3}, \quad r_3 = \frac{11}{5}. \tag{37}$$

It is easy to see that

$$r_2 - r_1 = \frac{2}{3} > \frac{8}{15} = r_3 - r_2. \tag{38}$$

Therefore, we conclude that the closed-loop system

$$\begin{aligned} \dot{x}_1 &= a_1 x_2, \\ \dot{x}_2 &= a_2 x_3, \\ \dot{x}_3 &= -a_3(((k_1 x_1)^{5/3} + k_2 x_2)^{33/25} + k_3 x_3) \end{aligned} \tag{39}$$

is asymptotically stable as long as $a_i > 0$ and $k_i > 0$, $i = 1, 2, 3$.

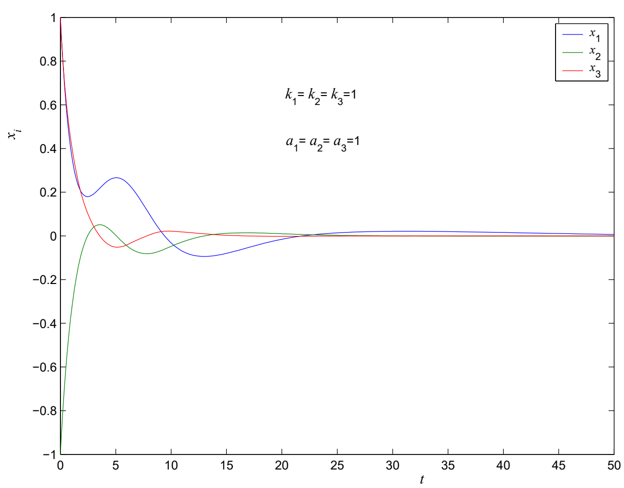

Fig. 1. The time response curves as $k_i = a_i = 1$, $i = 1, 2, 3$.

Simulations show that as the parameters and the gains vary, the stability of the closed-loop system is not destroyed.

V. Conclusion

For a class of uncertain linear systems described by multiple integrators controlled by a nested nonlinear feedback, the asymptotic stability is proved by using the concept homogeneity with strictly decreasing degrees (HSDD) and the technique of homogeneous domination. In the future work, we will investigate the general nonlinear systems admitting HSDD.

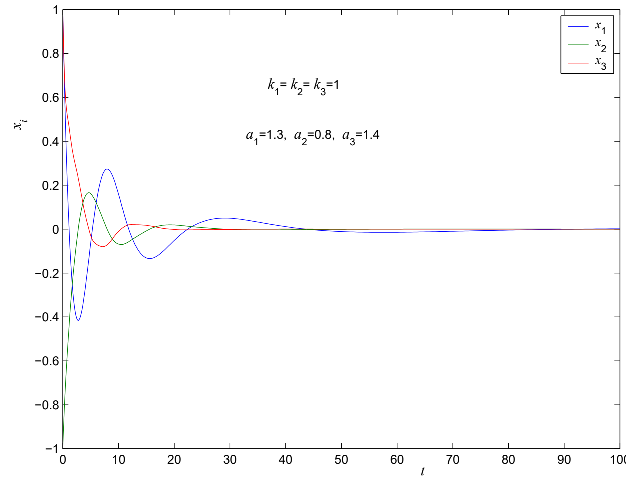

Fig. 2. The time response curves as a_i's are disturbed.

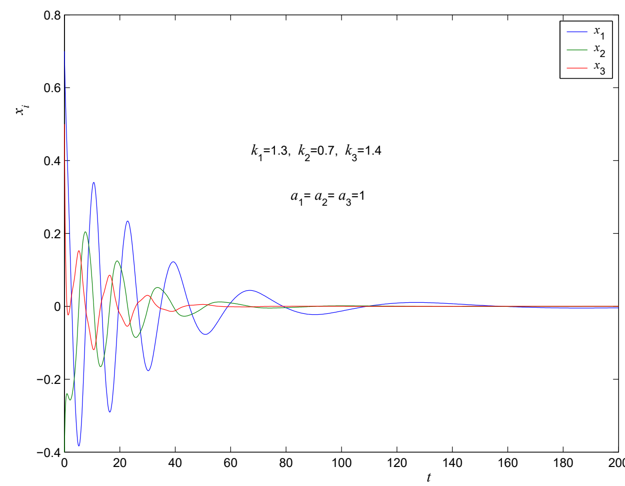

Fig. 3. The time response curves as k_i's are disturbed.

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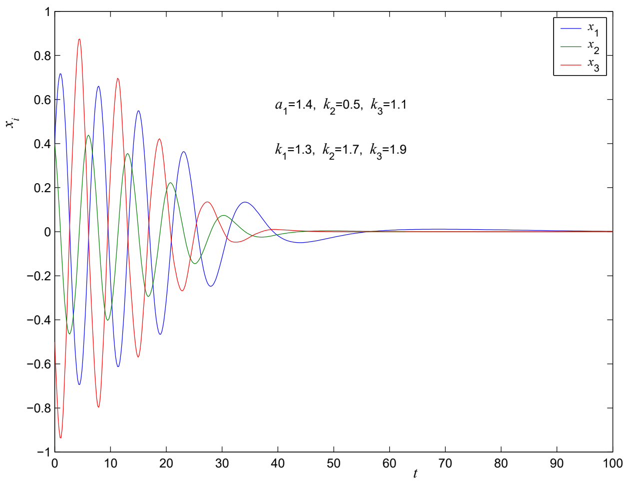

Fig. 4. The time response curves as all k_i's and a_i's are disturbed.

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