Faithful Simulation of Distributed Quantum Measurements with Applications in Distributed Rate-Distortion Theory

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***Abstract*—We investigate faithful simulation of distributed quantum measurements as an extension of Winter's measurement compression theorem. We characterize a set of communication and common randomness rates needed to provide faithful simulation of distributed measurements. To achieve this, we introduce binning and mutual packing lemma for distributed quantum measurements. These techniques can be viewed as the quantum counterpart of their classical analogues. Finally, using these results, we develop a distributed quantum-to-classical rate distortion theory and characterize a rate region analogous to Berger-Tung's in terms of single-letter quantum mutual information quantities.**

I. Introduction

Measurements are the interface between the intricate quantum world and the perceivable macroscopic classical world. A measurement associates to a quantum state a classical attribute. However, quantum phenomena, such as superposition, entanglement and non-commutativity contribute to uncertainty in the measurement outcomes. A key concern, from an information-theoretic standpoint, is to quantify the amount of "relevant information" conveyed by a measurement about a quantum state.

Winter's measurement compression theorem (as elaborated in [1]) quantifies the "relevant information" as the amount of resources needed to simulate the output of a quantum measurement applied to a given state. Imagine that an agent (Alice) performs a measurement M on a quantum state ρ and sends a set of classical bits to a receiver (Bob). Bob intends to *faithfully* recover the outcomes of Alice's measurements without having access to ρ. The measurement compression theorem states that at least quantum mutual information ($I(X;R)$) amount of classical information and conditional entropy ($S(X|R)$) amount of common shared randomness are needed to obtain a *faithful simulation*.

The measurement compression theorem finds its applications in several paradigms including local purity distillation [1] and private classical communication over quantum channels [2]. This theorem was later used by Datta, et al. [3] to develop a quantum-to-classical rate-distortion theory. The problem involved lossy compression of a quantum information source into classical bits, with the task of compression performed by applying a measurement on the source. In essence, the objective of the problem was to minimize the storage of the classical outputs resulting from the measurement while ensuring sufficient reliability so as to be able to recover the quantum state (from classical bits) within a fixed level of distortion from the original quantum source. To achieve this, the authors in [4] advocated the use of measurement compression protocol and subsequently characterized the so called rate-distortion function in terms of single-letter quantum mutual information quantities. The authors further established that by employing a naive approach of measuring individual output of the quantum source, and then applying Shannon's rate-distortion theory to compress the classical data obtained is insufficient to achieve optimal rates.

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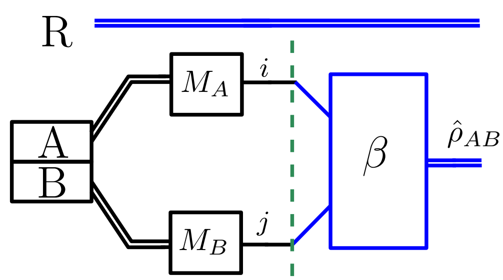

Fig. 1. The diagram of a distributed quantum measurement applied to a bipartite quantum system AB. A tensor product measurement $M_A \otimes M_B$ is performed on many copies of the observed quantum state. The outcomes of the measurements are given by two classical bits. The receiver functions as a classical-to-quantum channel β mapping the classical data to a quantum state.

In this work, we seek to quantify "relevant information" for quantum measurements performed in a distributed fashion. In this setting, as shown in Fig. 1, a composite bipartite quantum system AB is made available at two separate agents, named Alice and Bob. Alice and Bob have access only to sub-systems A and B, respectively. Two separate measurements, one for each sub-system, are performed in a distributed fashion with no communication taking place between Alice and Bob. Imagine that there is a third party (named Eve) who tries to simulate the action of the measurements without any access to the quantum systems. To achieve this objective, Alice and Bob send classical bits to Eve at rate r_1 and r_2, respectively. Eve on receiving these pairs of classical bits from Alice and

Bob wishes to reconstruct the joint quantum state ρ_{AB} using a classical-to-quantum channel. The reconstruction has to satisfy a fidelity constraint characterized using a distortion observable or a trace norm.

One strategy is to apply Winter's measurement theorem [5] to compress each individual measurements M_A and M_B separately into $\tilde{M}_A$ and $\tilde{M}_B$. As a result, faithful simulation of M_A by $\tilde{M}_A$ is possible when at least $nI(X;R)$ classical bits of communication and $nS(X|R)$ bits of common randomness are available between Alice and Eve. Similarly, a faithful simulation of M_B by $\tilde{M}_B$ is possible with $nI(X;R)$ classical bits of communication and $nS(Y|R)$ bits of common randomness between Eve and Bob. The challenge here is that the direct use of single-POVM compression theorem for each individual POVMs, M_A and M_B, does not necessarily ensure a "distributed" faithful simulation for the overall measurement, $M_A \otimes M_B$.

One can further reduce the amount of classical communication by exploiting the statistical correlations between Alice's and Bob's measurement outcomes. The challenge here is that the classical outputs of the approximating POVMs (operating on n copies of the source) are not IID sequences — rather they are codewords generated from random coding. Therefore, standard classical source coding techniques are not applicable here. This issue also arises in classical distributed source coding problem which was addressed by Wyner-Ahlswede-Körner [6] by developing Markov Lemma and Mutual Packing Lemma.

Building upon these ideas, we developed a quantum-classical counterpart of these lemmas for the multi-user quantum measurement simulation problem. We characterize a set of sufficient communication and common randomness rates in terms of single-letter quantum information quantities (Theorem 2). To prove this theorem, we develop binning of quantum measurements. This technique can be viewed as the quantum counterpart of its classical analogues. The idea of binning in quantum setting has been used in [7] and [8] for quantum data compression involving side information. However, in this paper we introduce a novel binning technique for measurements which is different from these works. The binning in this work is used to construct measurements for Alice and Bob with fewer outcomes compared to the above individual measurements, i.e., $\tilde{M}_A$ and $\tilde{M}_B$.

Secondly, we use our results on the simulation of distributed measurements to develop a distributed quantum-to-classical rate distortion theory (Theorem 3). For the achievability part, we characterize a rate region analogous to Berger-Tung's [6] in terms of single-letter quantum mutual information quantities.

II. Preliminaries

We here establish all our notations, briefly state few necessary definitions, and also provide Winter's theorem on measurement compression. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a finite dimensional Hilbert space $\mathcal{H}$. Further, let $\mathcal{D}(\mathcal{H})$ denote the set of positive operators of unit trace acting on $\mathcal{H}$. By I denote the identity operator. The trace distance between two operators A and B is defined as $\|A-B\|_1 \triangleq \text{tr}|A-B|$, where for any operator Λ we define $|\Lambda| \triangleq \sqrt{\Lambda^\dagger \Lambda}$. The von Neumann entropy of a density operator $\rho \in \mathcal{D}(\mathcal{H})$ is denoted by $S(\rho)$. The quantum mutual information and conditional entropy for a bipartite density operator $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ are defined, respectively, as

$$I(A;B)_\rho \triangleq S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$
$$S(A|B)_\rho \triangleq S(\rho_{AB}) - S(\rho_B).$$

A positive-operator valued measure (POVM) acting on a Hilbert space $\mathcal{H}$ is a collection of $M \triangleq \{\Lambda_x\}$ of positive operators in $\mathcal{B}(\mathcal{H})$ that form a resolution of the identity:

$$\Lambda_x \geqslant 0, \forall x, \qquad \sum_x \Lambda_x = I.$$

Let Ψ_ρ^{RA} denote a purification of a density operator $\rho \in D(\mathcal{H}_A)$. Given a POVM $M \triangleq \{\Lambda_x^A\}$ acting on ρ, the post-measurement state of the reference together with the classical outputs is represented by

$$(\text{id} \otimes M)(\Psi_{RA}^\rho) \triangleq \sum_x |x\rangle\langle x| \otimes \text{tr}_A\{(I^R \otimes \Lambda_x^A)\Psi_{RA}^\rho\}, \quad (1)$$

where Ψ_{RA}^ρ is a purification of ρ. Consider two POVMs $M_A = \{\Lambda_x^A\}$ and $M_B = \{\Lambda_x^B\}$ acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Define $M_A \otimes M_B$ as a the collection of all operators of the form $\Lambda_x^A \otimes \Lambda_y^B$, for all x, y. With this definition, $M_A \otimes M_B$ is a POVM acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. By $M^{\otimes n}$ denote the n-fold tensor product of the POVM M with itself. Consider a POVM $M = \{\Lambda_x\}_{x \in \mathcal{X}}$ with classical outputs $\mathcal{X}$. Given a mapping $\beta : \mathcal{X} \mapsto \mathcal{Y}$, define $\beta(M)$ as a new POVM with operators $\Gamma_y \triangleq \sum_{x:\beta(x)=y} \Lambda_x$ for all $y \in \mathcal{Y}$. For this POVM equation (1) can be written as

$$\begin{aligned}(\text{id} \otimes \beta(M))(\Psi_{RA}^\rho) &= \sum_{y \in \mathcal{Y}} |y\rangle\langle y| \otimes \text{tr}_A\{(I^R \otimes \sum_{x:\beta(x)=y} \Lambda_x)\Psi_{RA}^\rho\} \\ &= \sum_{x \in \mathcal{X}} |\beta(x)\rangle\langle\beta(x)| \otimes \text{tr}_A\{(I^R \otimes \Lambda_x)\Psi_{RA}^\rho\}\end{aligned}$$

A. *Quantum Information Source*

Consider a family of quantum states $\rho_i, i \in [1, m]$ acting on a Hilbert space $\mathcal{H}$. For each state assign a priori probability p_i. We denote such a setup by the ensemble $\{p_i, \rho_i, i \in [1:m]\}$. For such an ensemble, a quantum source is a sequence of states each equal to ρ_i with probability $p_i, i \in [1, m]$. Each realization of the source, after n generations of states, is represented by $\rho_{x^n} \triangleq \bigotimes_{j=1}^n \rho_{x_i}$, where x^n is a vector with elements in $[1, m]$. Let $\rho \triangleq \sum_i p_i \rho_i$, then the average density operator of the source after n generations is $\rho^{\bigotimes n}$.

B. *Measurement Compression Theorem*

Here, we provide a brief overview of the measurement compression theorem [5].

Definition 1 (Faithful simulation [1])**.** *Given a POVM* $M \triangleq \{\Lambda_x\}_{x \in \mathcal{X}}$ *acting on a Hilbert space* $\mathcal{H}_A$ *and a density operator*

$\rho \in \mathcal{D}(\mathcal{H}_A)$*, a POVM* $\tilde{M}$ *acting on* $\mathcal{H}_A^{\otimes n}$ *is said to be* ϵ*-faithful for* $\epsilon > 0$*, if the following holds:*

$$\|(id \otimes M_A^{\otimes n})(\Psi_{R^n A^n}^{\rho}) - (id \otimes \tilde{M})(\Psi_{R^n A^n}^{\rho})\|_1 \leqslant \epsilon, \quad (2)$$

where $\Psi_{R^n A^n}^{\rho}$ *is the* n*-fold tensor product of the state* Ψ_{RA}^{ρ}*, which is a purification for* ρ*.*

Theorem 1. *[5] For any* $\epsilon > 0$*, any density operator* $\rho \in \mathcal{D}(\mathcal{H}_A)$ *and any POVM* M *acting on the Hilbert space* $\mathcal{H}_A$*, there exist a collection of POVMs* $\tilde{M}^{(\mu)}$ *for* $\mu \in [1, N]$*, each acting on* $\mathcal{H}_A^{\otimes n}$*, and having at most* 2^{nR} *outcomes where*

$$R \geqslant I(U;R)_\sigma + \delta(\epsilon), \quad \textit{and} \quad \frac{1}{n}\log_2 N \geqslant S(U|R)_\sigma + \delta(\epsilon)$$

such that $\tilde{M} \triangleq \frac{1}{N}\sum_\mu \tilde{M}^{(\mu)}$ *is* ϵ*-faithful, where* $\sigma_{UR} \triangleq (id \otimes M)(\Psi_{RA}^{\rho})$*, and* $\delta(\epsilon) \searrow 0$ *as* $\epsilon \searrow 0$*.*

III. Approximation of Distributed POVMs

We provide our extension to the Winter's measurement compression protocol for a distributed setting. Consider a bipartite composite quantum system (A, B) represented by Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let ρ_{AB} be a quantum information source on $\mathcal{H}_A \otimes \mathcal{H}_B$. Imagine that three parties, named Alice, Bob and Eve, are trying to collectively implement two measurements, one applied to each sub-system. Eve has no access to the quantum system; while Alice and Bob have access to sub-system A and B, respectively. Alice and Bob perform a measurement M_A and M_B on sub-systems A and B, respectively. The measurements are performed in a distributed fashion with no communication taking place between Alice and Bob. In this context, the overall measurement is characterized by the tensor product measurement $M_A \otimes M_B$. The objective of Eve is to reconstruct an asymptotically faithful simulation of $M_A \otimes M_B$ when it is performed on ρ_{AB}. For that, Alice and Bob send a number of classical bits to Eve. Then, Eve applies a decoding map to the received bits and reconstructs the original measurement outcomes. The design objective is to minimize the amount of the classical bits that Eve needs to simulate the measurements. The problem is formally defined as in the following.

Definition 2. *For a given Hilbert space* $\mathcal{H}_A \otimes \mathcal{H}_B$*, a distributed protocol with parameters* $(n, \Theta_1, \Theta_2, N)$ *is characterized by a collections of POVM-pairs* $\tilde{M}_A^{(\mu)}$ *and* $\tilde{M}_B^{(\mu)}, \mu \in [1, N]$*, each 1) acting on* $\mathcal{H}_A^{\otimes n}$ *and* $\mathcal{H}_B^{\otimes n}$*, and 2) having at most* Θ_1 *and* Θ_2 *outcomes, respectively.*

In the above definition, (Θ_1, Θ_2) determines the amount of classical bits communicated from Alice and Bob to Eve. The amount of common randomness is determined by N, and μ can be viewed as the common randomness bits distributed among the parties. In the following, we define a measure for faithful simulation.

Definition 3. *Given a POVM* $M_A \otimes M_B$ *acting on* $\mathcal{H}_A \otimes \mathcal{H}_B$ *and a density operator* $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$*, a distributed protocol with POVM-pairs* $(\tilde{M}_A^{(\mu)}, \tilde{M}_B^{(\mu)}), \mu \in [1, N]$ *is* ϵ*-faithful, if there exist a collection of mappings* $\beta^{(\mu)}$ *such that the average POVM* $\tilde{M}_{AB} \triangleq \frac{1}{N}\sum_{\mu=1}^{N} \beta^{(\mu)}(\tilde{M}_A^{(\mu)} \otimes \tilde{M}_B^{(\mu)})$ *is* ϵ*-faithful according to Definition 1.*

In the above definition, the mappings $\beta^{(\mu)}$ represent the action of Eve on the received classical bits.

Definition 4. *Given a POVM* $M_A \otimes M_B$ *acting on* $\mathcal{H}_A \otimes \mathcal{H}_B$*, and a density operator* $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$*, a triplet* (R_1, R_2, C) *is said to be achievable, if for all* $\epsilon > 0$ *and for all sufficiently large* n*, there exists an* ϵ*-faithful distributed protocol with parameters* $(n, \Theta_1, \Theta_2, N)$ *satisfying*

$$\frac{1}{n}\log_2 \Theta_i \leqslant R_i + \epsilon, i = 1, 2$$
$$\frac{1}{n}\log_2 N \leqslant C + \epsilon.$$

Theorem 2. *Given a POVM* $M_A \otimes M_B$ *acting on* $\mathcal{H}_A \otimes \mathcal{H}_B$ *and a density operator* $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$*, the following triplet* (R_1, R_2, C) *is achievable*

$$R_1 \geqslant I(U;RB)_\sigma - I(U;V)_\sigma, \quad (3a)$$
$$R_2 \geqslant I(V;RA)_\sigma - I(U;V)_\sigma, \quad (3b)$$
$$R_1 + R_2 \geqslant I(U;RB)_\sigma + I(V;RA)_\sigma - I(U;V)_\sigma, \quad (3c)$$
$$C \geqslant \max\{S(U|RA)_\sigma, S(V|RB)_\sigma\}, \quad (3d)$$

where the information quantities are computed for the auxiliary state

$$\sigma_{UVRAB} \triangleq \sum_{u,v} |u,v\rangle\langle u,v| \otimes \left((I^R \otimes \Lambda_u^A \otimes \Lambda_v^B)\Psi_{RAB}^{\rho_{AB}}\right),$$

where $\Psi_{RAB}^{\rho_{AB}}$ *is a purification of* ρ_{AB}*, and* (U, V) *represents the output of* $M_A \otimes M_B$*.*

Proof. The proof is provided in a more detailed version of the paper [9]. ☐

Fig. 2 demonstrates the region in Theorem 2 in terms of the quantum information quantities. It also shows the gains achieved by employing such an approach as opposed to independently compressing the two sources ρ_A and ρ_B.

A. Proof Techniques

Binning for POVMs: We introduce a quantum-counterpart of the classical binning technique used to prove Theorem 2. Here, we describe this technique.

Consider a POVM M with observables $\{\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, .., \Lambda_{\alpha_N}\}$. Given K for which N is divisible, partition $[1, N]$ into K equal bins and for each $i \in [1, K]$, let $B(i)$ denote the i^{th} bin. The binned POVM $\tilde{M}$ is given by the collection of operators $\{\tilde{\Lambda}_{\beta_1}, \tilde{\Lambda}_{\beta_2}, .., \tilde{\Lambda}_{\beta_K}\}$ where $\tilde{\Lambda}_{\beta_i}$ is defined as

$$\tilde{\Lambda}_{\beta_i} = \sum_{j \in \mathcal{B}(i)} \Lambda_{\alpha_j}.$$

Using the fact that Λ_{α_i} are self-adjoint and positive $\forall i \in [1, N]$ and $\sum_{i=1}^{N} \Lambda_{\alpha_i} = I$, (which is because M is a POVM); it follows that $\tilde{M}$ is a valid POVM.

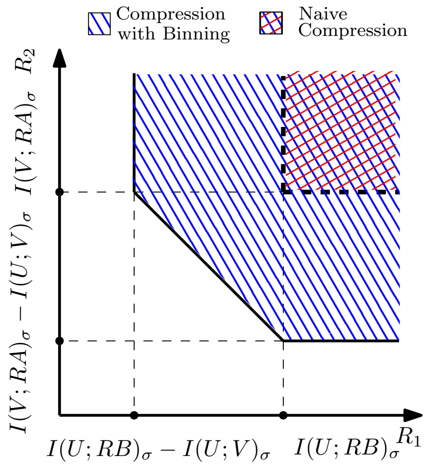

Fig. 2. Figure shows the achievable rate region with two different schemes. The Naive compression scheme is where each quantum source is independently compressed, while the other scheme, in order to exploit the correlation among the measurement outcomes, bins the POVMs before applying the measurements. As a result, the rate achieved by the latter is lower than the naive compression which translates into a larger rate region.

Mutual Packing Lemma for POVMs: Another technique used to prove Theorem 2 is a quantum version of mutual packing lemma. In what follows, we describe the mutual packing lemma for quantum measurements. For a Hilbert Space $\mathcal{H}_{AB}$ consider a POVM of the form $M_A \otimes M_B$, where (M_A, M_B) are two POVMs each acting on one sub-system. The observables for M_A and M_B are denoted, respectively, by $\Lambda^A_u \in \mathcal{B}(\mathcal{H}_\mathcal{A}), u \in \mathcal{U}$ and $\Lambda^B_v \in \mathcal{B}(\mathcal{H}_\mathcal{B}), v \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are finite sets. Fix a joint-distribution P_{UV} on the set of all outcomes $\mathcal{U} \times \mathcal{V}$. For each $l \in [1, 2^{nr_1}]$, let $U^n(l)$ be a random sequence generated according to $\prod_{i=1}^n P_U$. Similarly, let $V^n(k)$ be a random sequence distributed according to $\prod_{i=1}^n P_V$, where $k \in [1, 2^{nr_2}]$. Suppose $U^n(l)$'s and $V^n(k)$'s are independent. Define the following random observables:

$$A_{u^n} \triangleq |l : U^n(l) = u^n|\Lambda^A_{u^n}, \quad B_{v^n} \triangleq |k : V^n(k) = v^n|\Lambda^B_{v^n}$$

where $\Lambda^A_{u^n} = \bigotimes_i \Lambda^A_{u_i}$ and $\Lambda^B_{v^n} = \bigotimes_i \Lambda^B_{v_i}$.

Lemma 1. *For any $\epsilon > 0$ and sufficiently large n, with high probability*

$$\| \sum_{(u^n,v^n)\in\mathcal{T}_\delta^{(n)}(U,V)} A_{u^n} \otimes B_{v^n} \|_\infty \leqslant \epsilon \tag{4}$$

provided that $r_1 + r_2 < I(U;V) - \delta(\epsilon)$.

Proof. From the triangle-inequality and the definition of A_{u^n} and B_{v^n}, the norm in the lemma does not exceed the following

$$\sum_{l,k} \sum_{(u^n,v^n)\in\mathcal{T}_\delta^{(n)}(U,V)} \mathbb{1}\{U^n(l) = u^n, V^n(k) = v^n\} \|\Lambda^A_{u^n} \otimes \Lambda^B_{v^n}\|_\infty$$
$$\leqslant \sum_{l,k} \mathbb{P}\{(U^n(l), V^n(k)) \in \mathcal{T}_\delta^{(n)}(U,V)\}$$

where the last inequality holds since $\Lambda^A_{u^n} \otimes \Lambda^B_{v^n} \leqslant I$. The proof completes from the classical mutual packing lemma. $\square$

IV. Quantum-to-Classical (q-c) Distributed Rate Distortion Theory

As an application to the above theorem on faithful simulation of distributed measurements (Theorem 2), we investigate the distributed extension of quantum-to-classical (q-c) rate distortion coding [3]. This problem is a quantum counterpart of the classical distributed source coding. In this setting, many copies of a bipartite quantum information source $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ are generated. Alice and Bob have access to the partial trace of the copies denoted by ρ_A and ρ_B, respectively; each performs a measurement on their copies and sends the classical outputs to Eve. The objective of Eve is to produce a reconstruction of the source ρ_{AB} within a targeted distortion threshold which is measured by a given distortion observable. To this end, upon receiving the classical bits sent by Alice and Bob, a reconstruction state is produced by Eve.

We first formulate this problem as follows. For any quantum information source $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, denote its purification by Φ^ρ_{RAB}.

Definition 5. *A q-c source coding setup is characterized by a purified quantum information source $\Phi^{\rho_{AB}}_{RAB} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A \otimes \mathcal{H}_B)$, a reconstruction Hilbert space $\mathcal{H}_{\hat{X}}$, and a distortion observable $\Delta \in \mathcal{B}(\mathcal{H}_R \otimes \mathcal{H}_{\hat{X}})$ which satisfies $\Delta \geqslant 0$.*

Next, we formulate the action of Alice, Bob and Eve by the following definition.

Definition 6. *An (n, Θ_1, Θ_2) q-c protocol for a given input and reconstruction Hilbert spaces $(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_{\hat{X}})$ is defined by POVMs $M^{(n)}_A$ and $M^{(n)}_B$ acting on $\mathcal{H}^{\otimes n}_A$ and $\mathcal{H}^{\otimes n}_B$ with Θ_1 and Θ_2 number of outcomes, respectively, and a set of reconstruction states $S_{i,j} \in \mathcal{D}(\mathcal{H}^{\otimes n}_{\hat{X}})$ for all $i \in [1:\Theta_1], j \in [1:\Theta_2]$.*

The overall action of Alice, Bob and Eve, as a q-c protocol, on a quantum source ρ_{AB} is denoted by the following operation

$$\mathcal{N}_{A^nB^n \mapsto \hat{X}^n} : \rho^{\otimes n}_{AB} \mapsto \sum_{i,j} \mathrm{tr}\{(\Lambda^A_i \otimes \Lambda^B_j)\rho^{\otimes n}_{AB}\}\ S_{i,j}, \tag{5}$$

where $\{\Lambda^A_i\}$ and $\{\Lambda^B_j\}$ are, respectively, the operators of the POVMs $M^{(n)}_A$ and $M^{(n)}_B$. With this notation and given a q-c source coding setup as in Definition 5, the distortion of a $(n = 1, \Theta_1, \Theta_2)$ q-c protocol is measured as

$$d(\rho_{AB}, \mathcal{N}_{AB\mapsto\hat{X}}) \triangleq \mathrm{tr}\left\{\Delta\left((\mathrm{id}_R \otimes \mathcal{N}_{AB\mapsto\hat{X}})(\Psi^{\rho_{AB}}_{RAB})\right)\right\}.$$

For an n-letter protocol, we use symbol-wise average distortion observable defined as

$$\Delta^{(n)} = \frac{1}{n}\sum_{i=1}^n \Delta_{R_i\hat{X}_i} \otimes I^{\otimes[n]\setminus i}_{R\hat{X}}, \tag{6}$$

where $\Delta_{R_i\hat{X}_i}$ is understood as the observable Δ acting on the ith instance space $\mathcal{H}_{R_i} \otimes \mathcal{H}_{\hat{X}_i}$ of the n-letter space $\mathcal{H}^{\otimes n}_R \otimes \mathcal{H}^{\otimes n}_{\hat{X}}$. With this notation, the distortion for an (n, Θ_1, Θ_2) q-c protocol is given by

$$\bar{d}(\rho^{\otimes n}, \mathcal{N}_{A^nB^n\mapsto\hat{X}^n}) \triangleq \mathrm{tr}\Big\{\Delta^{(n)}(\mathrm{id}\otimes\mathcal{N}_{A^nB^n\mapsto\hat{X}^n})(\Psi^{\rho_{AB}}_{R^nA^nB^n})\Big\}$$

where $\Psi^{\rho_{AB}}_{R^nA^nB^n}$ is the n-fold tensor product of $\Psi^{\rho_{AB}}_{RAB}$ which is the given purification of the source.

The authors in [3] studied the point-to-point version of the above formulation. They considered a special distortion observable of the form $\Delta = \sum_{\hat{x}\in\hat{\mathcal{X}}} \Delta_{\hat{x}} \otimes |\hat{x}\rangle\langle\hat{x}|$, where $\Delta_{\hat{x}} \geqslant 0$ acts on the reference Hilbert space and $\hat{\mathcal{X}}$ is the reconstruction alphabet. In this paper, we allow Δ to be any non-negative and bounded operator acting on the appropriate Hilbert spaces. Moreover, we allow for the use of any c-q reconstruction mapping as the action of Eve.

Definition 7. *For a q-c source coding setup, a rate-distortion triplet (R_1, R_2, D) is said to be achievable, if for all $\epsilon > 0$ and all sufficiently large n, there exists an (n, Θ_1, Θ_2) q-c protocol satisfying*

$$\frac{1}{n}\log_2 \Theta_i \leqslant R_i + \epsilon, \quad i = 1, 2,$$
$$\bar{d}(\rho^{\otimes n}, \mathcal{N}_{A^nB^n\mapsto\hat{X}^n}) \leqslant D + \epsilon,$$

where $\mathcal{N}_{A^nB^n\mapsto\hat{X}^n}$ is defined as in (6).

Theorem 3. *For a q-c source coding setup with a purified source $\Phi^{\rho_{AB}}_{RAB} \in \mathcal{D}(\mathcal{H}_R\otimes\mathcal{H}_A\otimes\mathcal{H}_B)$, and distortion observable Δ acting on $\mathcal{H}_R\otimes\mathcal{H}_{\hat{X}}$, any rate-distortion triplet (R_1, R_2, D) satisfying the following inequalities is achievable*

$$R_1 \geqslant I(U; RB)_\sigma - I(U; V)_\sigma,$$
$$R_2 \geqslant I(V; RA)_\sigma - I(U; V)_\sigma,$$
$$R_1 + R_2 \geqslant I(U; RB)_\sigma + I(V; RA)_\sigma - I(U; V)_\sigma$$

for some POVMs $\{\Lambda^A_u\}$, $\{\Lambda^B_v\}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, and reconstruction states $\{S_{u,v}\}$ with each state in $\mathcal{D}(\mathcal{H}_{\hat{X}})$ such that $d(\rho_{AB}, \mathcal{N}_{AB\mapsto\hat{X}}) \leqslant D$. The quantum mutual information quantities are computed according to the state

$$\sigma_{UVRAB} \triangleq \sum_{u,v} |u, v\rangle\langle u, v| \otimes \left((I^R \otimes \Lambda^A_u \otimes \Lambda^B_v)\Psi^{\rho_{AB}}_{RAB}\right),$$

where (U, V) represents the output of $M_A \otimes M_B$.

Proof. The proof follows from Theorem 2. Fix POVMs (M_A, M_B) and reconstruction states $S_{u,v}$ as in the statement of the theorem. Let $\mathcal{N}_{AB\mapsto\hat{X}}$ be the mapping corresponding to these POVMs and the reconstruction states. Then, $d(\rho_{AB}, \mathcal{N}_{AB\mapsto\hat{X}}) \leqslant D + \epsilon$. According to Theorem 2, for any $\epsilon > 0$, there exists an $(n, 2^{nR_1}, 2^{nR_2}, N)$ distributed protocol for ε-faithful simulation of $M_A \otimes M_B$ on ρ_{AB} such that (R_1, R_2) satisfies the inequalities in (3). For each $\mu \in [1 : N]$, we use the q-c protocol with parameters $\Theta_i = 2^{nR_i}, i = 1, 2$ and POVMs $\tilde{M}^{(\mu)}_A, \tilde{M}^{(\mu)}_B$, and reconstruction states $S_{\beta(i,j)}$ and the corresponding mappings $\mathcal{N}^{(\mu)}_{A^nB^n\mapsto\hat{X}^n}$. With this notation, for the averaged random protocols the following bounds hold

$$\begin{aligned}
&\frac{1}{N}\sum_\mu \bar{d}(\rho^{\otimes n}, \tilde{\mathcal{N}}^{(\mu)}_{A^nB^n\mapsto\hat{X}^n}) \\
&= \frac{1}{N}\sum_\mu \operatorname{tr}\Big\{\Delta^{(n)}(\mathrm{id}\otimes\tilde{\mathcal{N}}^{(\mu)}_{A^nB^n\mapsto\hat{X}^n})\Psi^\rho_{R^nA^nB^n}\Big\} \\
&= \operatorname{tr}\Big\{\Delta^{(n)}(\mathrm{id}\otimes\mathcal{N}^{\otimes n}_{AB\mapsto\hat{X}})\Psi^\rho_{R^nA^nB^n}\Big\} \\
&\quad + \operatorname{tr}\Big\{\Delta^{(n)}(\mathrm{id}\otimes(\mathcal{N}^{\otimes n}_{AB\mapsto\hat{X}} - \tilde{\mathcal{N}}_{A^nB^n\mapsto\hat{X}^n}))\Psi^\rho_{R^nA^nB^n}\Big\} \\
&\leqslant \operatorname{tr}\left\{\Delta\left((\mathrm{id}_R\otimes\mathcal{N}_{AB\mapsto\hat{X}})(\Psi^{\rho_{AB}}_{RAB})\right)\right\} \\
&\quad + \|\Delta^{(n)}(\mathrm{id}\otimes(\mathcal{N}^{\otimes n}_{AB\mapsto\hat{X}} - \tilde{\mathcal{N}}_{A^nB^n\mapsto\hat{X}^n}))\Psi^\rho_{R^nA^nB^n}\|_1 \\
&\leqslant D \\
&\quad + \|\Delta^{(n)}\|_\infty\|(\mathrm{id}\otimes(\mathcal{N}^{\otimes n}_{AB\mapsto\hat{X}} - \tilde{\mathcal{N}}_{A^nB^n\mapsto\hat{X}^n}))\Psi^\rho_{R^nA^nB^n}\|_1 \\
&\leqslant D + \varepsilon\|\Delta\|_\infty,
\end{aligned}$$

where $\tilde{\mathcal{N}}_{AB\mapsto\hat{X}}$ is the average of $\tilde{\mathcal{N}}^{(\mu)}_{AB\mapsto\hat{X}}$, and the second inequality follows by the following lemma, and the last inequality follows by Theorem 2.

Lemma 2. *For any operator A and B acting on a Hilbert space $\mathcal{H}$ the following inequalities hold.*

$$\|BA\|_1 \leqslant \|B\|_\infty\|A\|_1, \qquad \textit{and} \qquad \|AB\|_1 \leqslant \|B\|_\infty\|A\|_1.$$

□

One can observe that the rate-region in Theorem 3 matches exactly with the classical Berger-Tung region when ρ_{AB} is a mixed state of a collection of orthogonal pure states. Note that the rate-region is an inner bound for the set of all achievable rates. The single-letter characterization of the set of achievable rates is still an open problem even in the classical setting.

V. Conclusion

We established a distributed extension of Winter's measurement compression theory. A set of communication rate-pairs and common randomness rate is characterized for faithful simulation of distributed measurements. We further investigated distributed quantum-to-classical rate-distortion theory and derived a quantum counterpart of Berger-Tung rate-region.

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