First graders outwit a famous mathematician

Find zero minus four? Pascal argued it was impossible! Twenty-first-century students, given the right tools, can solve counterintuitive problems.

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We often associate the terms *integers* and *negative numbers* with difficulty, frustration, and confusion. Many of us probably still remember our initial struggles to understand negative numbers and how to operate with them. We find ourselves in good company, though, with the many great mathematicians who also struggled with the idea of numbers less than zero. In the third century, Diophantus, the "Father of Algebra" no less, described equations of the form $x + 20 = 4$ as "absurd." The absurdity stemmed from the fact that the result of four is obviously less than the addend of twenty. And more than 1300 years later, Pascal argued that subtracting four from zero leaves zero because of the impossibility of taking something from nothing. Surely, then, ideas this challenging are too complex for first graders—or are they?

Recent research shows that children as young as six years of age can, in fact, reason about negative numbers and even perform basic calculations using them (Behrend and Mohs 2006; Wilcox 2008). Our goal was to build on

BOY: HOLOS/GETTY IMAGES; PASCAL: GEORGIOS KOLLIDAS/ISTOCKPHOTO.COM this research to further explore young children's ideas about negative numbers. The remainder of the article describes the background of our study, provides an overview of the tasks we used, discusses children's responses to these tasks, and identifies two ways that students in our study reasoned about and approached problems involving negative numbers.

Background

Because we wanted to understand the variety of ways in which first graders make sense of negative numbers, we asked a first-grade teacher with whom we had previously done classroom research to help us select a group of her students with a wide range of mathematical understanding. This teacher had not previously introduced or discussed negative numbers with her students. We interviewed seven first graders, giving them multiple activities and posing problems designed to elicit their thinking about negative numbers during a single thirty- to forty-minute session. The same researcher interviewed each child individually. Although identical basic tasks were posed to all students, follow-up questions were based on individual children's responses. These questions were designed to uncover each child's thinking, so they necessarily differed. During the problem-solving interview, students first played the number-line game described in Wilcox (2008). (See also Schifter, Bastable, and Russell's Developing Mathematical Ideas series [2007] for a slightly different version of the game.) In this game, children draw two 2 × 3-inch cards—one an action card (labeled with a plus or a minus sign) showing the direction to move on the number line and the other a magnitude card (1, 2, 3, … 8) that shows how far to move.

Our number line was labeled starting at zero and extending to the right (0, 1, 2, 3, etc.) but had unlabeled tick marks to the left of zero. The interviewer and the child took turns playing the game; the interviewer extended the game for at least three turns after the child first landed to the left of zero.

We chose the game for two reasons:

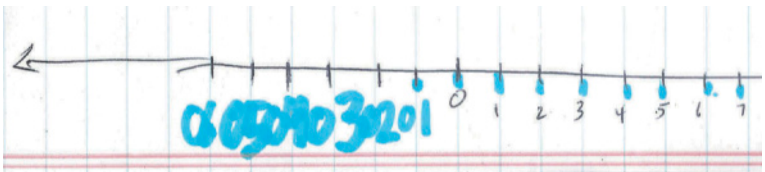
1. To help students conceive of addition and subtraction as moving forward and backward
2. To expose children's ideas about numbers to the left of zero

On the basis of previous work by Wilcox (2008) and Behrend and Mohs (2006), we expected that children could begin to interpret addition and subtraction as moving backward and forward on a number line and to identify where negative numbers are located on the line.

After playing the number line game, students received a set of equations (see **table 1**) that they were unlikely to have seen before, inasmuch as the problems have negative solutions. Moreover, these solutions are inconsistent with common interpretations of addition as making bigger and subtraction as making smaller. For example, 4 + □ = 3 may seem nonsensical to children, given that the result of three is smaller than the addend of four. We were curious to see how young children face such counterintuitive situations.

FIGURE 1

Lucy called "numbers under zero" *zero one, zero two, zero three,* and so on.



First graders surprised us with their initial ideas about negative numbers.

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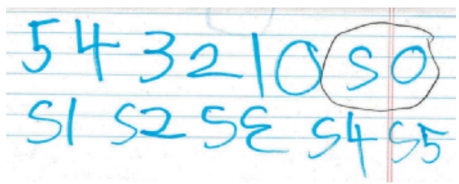
FIGURE 2

Nola used the letter *S* as a symbol for the *something*, her invented word for *negative*.



Numbers less than zero

Because ways of locating and naming points on the number line can influence one's reasoning and solutions to counterintuitive problems, we begin with a brief discussion of the children's representations and identification of numbers less than zero. During the course of the number line game, all students eventually landed on a place to the left of zero. Their responses when asked to name these places were telling. Brad, Lucy, and Teddy initially called each position *zero,* but each later revised his or her thinking. Teddy, for example, renamed each position to the left of zero as *negative*, describing the entire left side of the number line as the *negative hallway*. Others, like Jackson and Brian, correctly used the conventional notation of –1, –2, and so on, whereas two children invented their own notation for *numbers under zero*. Lucy did not remember what to call numbers less than zero or how to write them. She conjectured that such numbers might "start with zero," eventually naming them *zero one*, *zero two*, *zero three,* and so on. She was also asked to write those numbers (see **fig. 1**).

Similarly, Nola discussed numbers less than zero: "There is some, but I forgot the word for them. Like *something 1*, *something 2*, *something 3*, *something 4*." When Nola was asked to count backward from five, she started with the 5 on the left, used an *S* to stand for *something* (see **fig. 2**), and added a zero (the number that is circled), which she called *something zero*.

Five of the seven children were eventually able to name places to the left of zero either explicitly as negative numbers or by using some form of invented notation that enabled them to compare and order negative numbers on the basis of magnitude. The other two children, Teddy and Brad, did not *numerically* distinguish unique places to the left of zero, but they did see these places as locations and even gave them names, such as *none*, *no numbers*, *the negative classroom*, *the negative cafeteria*, and *the negative principal's office*. In this sense, these two children saw the places as distinct and different, but their system of naming did not afford them ways to easily count, order, or perform such operations as addition and subtraction on them. Overall, we found that first graders have many intuitive ideas about numbers to the left of zero that could be leveraged to further explore operations with negative numbers.

Counterintuitive problems

Before posing the counterintuitive problems (see **table 1**), we started with more familiar equations, such as 5 + 3 = □ and 4 + □ = 7,

TABLE 1

First graders exhibited various responses to counterintuitive problems such as these.

Problem	Rationale for choosing the problem
3 – 5 = □	When viewed as *take away* and tied to contexts of having and giving, this subtraction problem requires one to *take away more than one has*.
4 + □ = 3	When addition is understood as joining, this problem is inconsistent with the generalization that *addition makes larger*.
6 – □ = 8	When subtraction is viewed as separating or taking away, this problem is inconsistent with the idea that *subtraction makes smaller*.

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Negative numbers seemed unproblematic for some children.

followed by a problem in the form of 3 – 5 = □, the first of the counterintuitive problems. In the following section, we describe three broad groups of increasingly sophisticated student responses to this set of problems.

Three types of thinking

We encountered a variety of responses to the set of counterintuitive problems.

1. Solving such problems is impossible. The most common initial response to the first counterintuitive problem was, "That doesn't make sense." Four of the seven children eventually changed their answers to this problem. However, Nola, Brad, and Teddy, not unlike many early mathematicians, steadfastly maintained that solving this problem is impossible. They argued their cases: "You can't do it, because you can't take five from three," explained Teddy.

Nola stated, "Three minus four doesn't make sense, because three is less than four."

The three seven-year-olds had similar responses for the next counterintuitive problem. Brad claimed, "Four plus blank equals three is not a real problem. It's not true." He then crossed out the problem and explained, "Four *minus* [*with emphasis*] one would equal three."

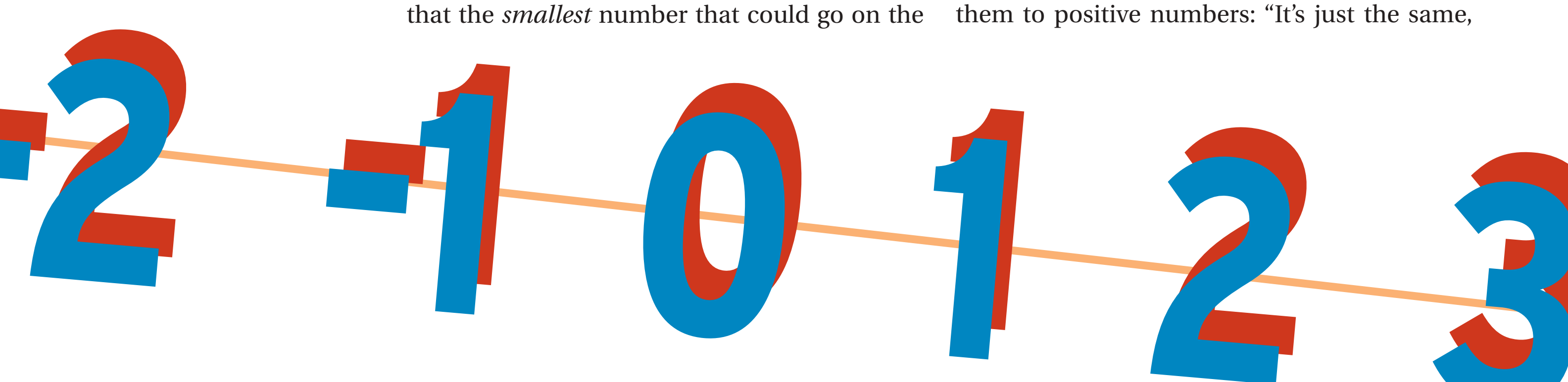
To further examine this idea, the interviewer presented both Nola and Brad with the problem 4 + □ = ____ and asked them to put a number on the blank line that would make the problem "real" or "true." Both children said that the *smallest* number that could go on the line would be five. Nola justified her response: "It [*gesturing to the number on the line*] has to be bigger [than four] if it's a plus right here."

At the end of the interview, Nola made up a "hard" math problem for the interviewer: 6 + □ = 1. When given the answer of "something five" (or S5, using her invented notation for negative numbers), Nola wrinkled her nose, shook her head no, and laughed. She responded, "That's not right. This problem doesn't make sense."

Like most first graders, Nola, Brad, and Teddy had experience understanding and operating with numbers in contexts that were concrete, tangible, and connected to the existence of objects. In their worlds, where numbers are almost exclusively positive, these kinds of responses make sense and are to be expected. Thus we grouped the three children's responses to the counterintuitive problems together as characterizing one way of thinking about negative numbers; for them, numbers to the left of zero do not exist in ways that enable these children to operate with them.

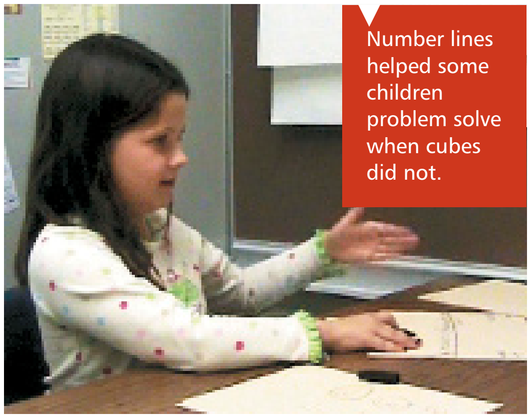
2. Counting back strategies extend to negative numbers. Other students encountered the same problems but were able to draw upon alternative models and tools to engage differently with the counterintuitive problems. Negative numbers seemed to be unproblematic for Jackson, who solved 3 – 5 = □ using a counting strategy. He counted backward starting at three: Three, two [*raising one finger*], one [*a second finger*], zero [*a third finger*], negative one [*a fourth finger*], negative two [*a fifth finger*].

Counting below zero seemed to present no special challenge for Jackson inasmuch as he was able to easily extend his counting-back strategy to the negative numbers. Earlier, he had explained negative numbers by comparing them to positive numbers: "It's just the same,



except it's negative; … it goes in the same order, one, two, three—it's just smaller."

It seems that for Jackson, there was nothing particularly special about negative numbers; they just happened to be on the other side of zero. Counting strategies like Jackson's comprise a second category of student reasoning about negative numbers. For children who reason in this way, negative numbers not only exist but also seem well-ordered and sequential. This numeric ordering provides opportunities for children to operate with negative numbers but does not guarantee that they will do so.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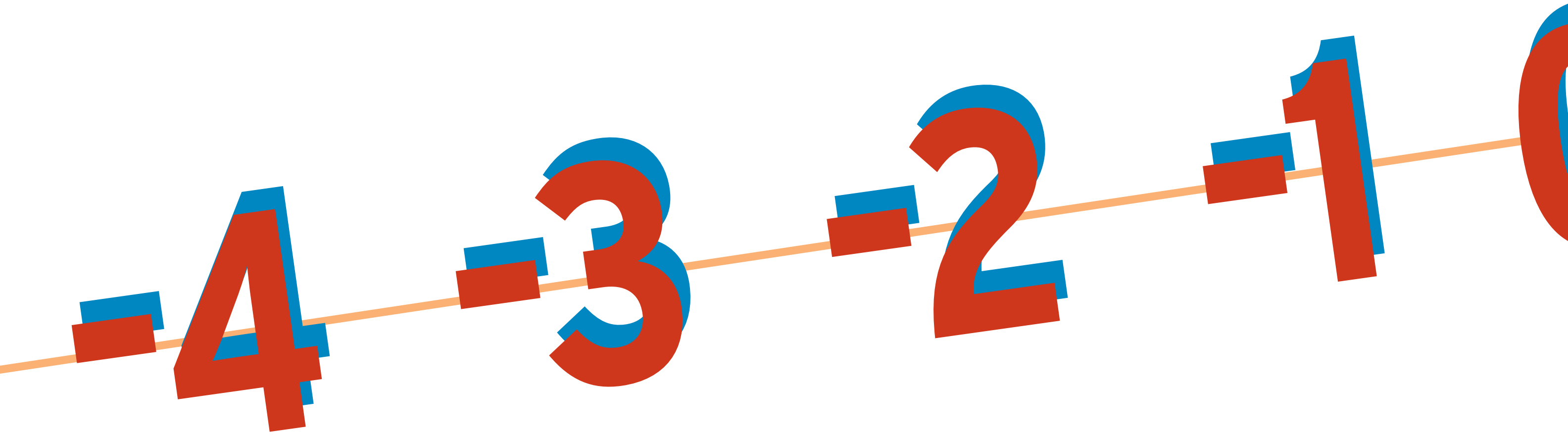
Number lines helped some children problem solve when cubes did not.

3. Motion on the number line supports operating with negative numbers. Children who engage in a third type of reasoning for solving counterintuitive problems again view negative numbers as sequential and well ordered, but in their problem-solving strategies, they make use of number lines and contexts of motion. We highlight this kind of strategy because we believe that movement is a particularly useful way of interpreting addition and subtraction. Both Brian and Lucy, the remaining two students, spontaneously chose the number line to help them correctly solve 3 – 5 = □ and other counterintuitive problems.



Although Lucy initially made stacks of three and five Unifix® cubes, she quickly abandoned the use of the cubes and used the number line. She put a pointer at the three and moved it five spots to the left, answering, "Zero two." (Recall from **figure 1** that Lucy represented negative two as 02, using her own invented notation.) "That number line actually helped me a lot."

When the interviewer pointed to the cubes and asked, "Did these help you?" Lucy responded, "When I used cubes, I mean, what could they help me with this? How am I gonna do it?"

When faced with the need to take "something from nothing," Lucy was able to move beyond the more familiar interpretation of subtraction—having and removing objects—through her use of the number line.

Brian's answers to the interviewer's questions about 6 + □ = 4 were also revealing. Brian asked, "What is that plus for? Isn't that supposed to be a minus?"

"Brian, is there any kind of number you can add to make it smaller?"

"A negative number! I think, hmmm. Oh, it's a negative number [*moving from the six to the four on the number line*]. Two; I mean negative two."

"How are you getting that?"

"Well, six minus two would equal four, so six plus negative two would equal four."

"Those sound different; one is six minus and one is six plus. How do you know they are equal?"

"Plussing a number, plussing a number, a negative number, would be minusing instead of plussing."

Brian's responses were surprising because we had not anticipated that a six-year-old would state the generalization that adding an inverse is the same as subtraction, a connection many middle—and even high school—students struggle to make. We suspect that this tool and the underlying idea of motion helped Brian progress beyond the notion that addition *always* makes larger numbers to consider what adding a negative number might mean. The number line seemed to provide Lucy and Brian a new way of viewing addition and subtraction, helping them solve problems they previously could not solve.

In summary, three of the seven students solved these counterintuitive problems correctly and a fourth child had an emerging idea that the answers would be negative but was unsure of the exact solutions. All seven children approached the task from a sense-making perspective, and their strategies ranged from expressing ideas that "there are no such things as numbers less than zero" (true enough for young children, given their experiences with numbers and counting); to counting back by ones with and without the number line; and, finally, to expressing a generalization that adding a negative is the same as subtracting.

Discussion and implications

Throughout our interviews, we found that six- and seven-year-olds were quite ready to grapple with negative numbers and displayed a wide range of understanding. On the basis of our interviews with these children, we found it useful to distinguish two ways of reasoning about numbers in general, which seem to have implications for understanding how children make sense of negative numbers:

1. Number as a position or location, viewed in relation to other numbers
2. Number as a tangible quantity, amount, or object

This distinction in reasoning is clearly illustrated in the first graders' differing approaches to the counterintuitive problems. Jackson, Brian, and Lucy seemed able, at times, to approach numbers more flexibly as locations or positions unrelated to amounts. This is not to say that these children did not also see numbers as representing quantities; in fact, they understood numbers as *both* positional and countable. In contrast, Nola, Brad, and Teddy approached numbers as tangible quantities only, a view that seemed to limit their frames of reference for making sense of the counterintuitive problems. For example, consider the problem 3 – 5 = □. Some children approached this problem in a context of motion or a number line, in which negative two was a *position*; it is the place one lands when starting at three and moving left five tick marks (or starting at negative eight and moving right six tick marks). Specifically, Brian's and Lucy's understanding of subtraction and addition included a motion interpretation: movement to the left or right. This new interpretation of subtracting and adding as *moving* likely helped these children grapple with, and eventually move beyond, questions of whether addition always makes larger numbers or subtraction always makes smaller numbers. Although Jackson did not directly use ideas of movement, he leveraged the sequencing and ordering of the number system to successfully engage with and solve counterintuitive problems using counting strategies. In this way of reasoning, whether one uses counting strategies or motion on a number line, negative numbers are seen as ordinal and sequentially related to other numbers but not necessarily as representing an amount or quantity.

Some children (like Nola, Brad, and Teddy), though, thought of numbers as related to a quantity of countable objects, explaining that numbers tell you "how many." In relation to the example 3 – 5 = □, they could think of a context in which they had three of something and promised five to a friend. After giving the three they had, they would still owe two. Although this idea can be represented symbolically as –2, it cannot be represented in a *tangible* way. What meaning can be attached to having –2 cubes or to taking something (two, in this case) from nothing (zero)? When reasoning about number solely as a quantity, these kinds of counter-

"There are other numbers under zero. I know about them, but I don't know what they're called." *—Lucy, a first grader*

intuitive problems really do not make sense. We emphasize that not only is this understanding of number valid but also, for these children, this approach to number is conceptually coherent, consistently applied, and, by a standard that existed for more than a thousand years, mathematically correct.

The difficulty representing negative numbers as quantities was also highlighted by Lucy, who, when trying to use cubes to think about 3 – 5 = □, asked, "What could they help me with this? How am I gonna do it?" However, presented with the number line representation, Lucy commented that the "number line actually helped me a lot." We find this an important outcome of our study. Historically, mathematicians did not have the number line representation until relatively recently (Barnabas Hughes, personal communication, May 13, 2010), and it came about only after much grappling with various views of number. However, once the number line became available, it seemed to support the number-as-position-or-location reasoning used by people of all ages. We who grew up with and rely on the number line model *underestimate* how powerful this representation is, but this study highlights the important role this tool plays in the reasoning of even young children.

We found that—given the right tools and opportunities—young children can reason in powerful ways about negative numbers. Counterintuitive problems are similar to the fair-sharing problems used to introduce fractions (Empson 2001), wherein the problem statements use natural numbers (1, 2, 3, …) but the solutions use fractions (e.g., Four children share six cookies. How much does each child get?). Counterintuitive problems also have only natural numbers in the problem statement but have negative numbers as solutions (e.g., 4 + □ = 3). We suspect that these types of problems, as well as the number line game, are particularly rich sites for children's initial learning about negative numbers. Moreover, such problems can provide opportunities for children to construct new views of numbers as positions or locations so that they can move flexibly between *both* ways of reasoning about numbers. Although integers are not part of the first-grade curriculum, we would like teachers to be aware of ways in which they can easily enrich and extend children's mathematical thinking by building on their ideas about negative numbers when they arise naturally in the classroom.

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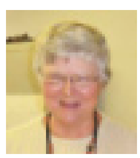


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