Secure Communication over 1-2-1 Networks

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***Abstract*—This paper starts by assuming a 1-2-1 network, the abstracted noiseless model of mmWave networks that was shown to closely approximate the Gaussian capacity in [1], and studies secure communication. First, the secure capacity is derived for 1-2-1 networks where a source is connected to a destination through a network of unit capacity links. Then, lower and upper bounds on the secure capacity are derived for the case when source and destination have more than one beam, which allow them to transmit and receive in multiple directions at a time. Finally, secure capacity results are presented for diamond 1-2-1 networks when edges have different capacities.**

I. Introduction

High-frequency communication, such as mmWave and Thz, can enable multi-gigabit communication, albeit at relatively short range, and with the help of beamforming to compensate for high path loss. To cover large areas, such as commercial buildings, requires deploying networks of relays that communicate through directional beams. In [1], the authors derived a model for high-frequency communication networks, that they termed Gaussian 1-2-1 networks, and presented capacity results as well as information flow algorithms. In this paper, we start by assuming a 1-2-1 network, namely the abstracted noiseless model of mmWave networks that was shown to closely approximate the Gaussian capacity in [1] and, study secure message communication over such networks.

The 1-2-1 model abstracts directivity: to establish a communication link, both the mmWave transmitter and receiver employ antenna arrays that they electronically steer to direct their beams towards each other - termed as 1-2-1 link, as both nodes need to focus their beams to face each other for the link to be active. Thus, in 1-2-1 networks, instead of broadcasting or interference, we have coordinated steering of transmit and receive beams to activate different links at each time.

We now review a fundamental result in network security, and then discuss how it changes over 1-2-1 networks. Consider a source, Alice, connected to a destination, Bob, through an arbitrary traditional network represented as a graph with unit capacity lossless links, and assume that the min-cut between the source and the destination equals H. That is, we can find H edge-disjoint unit capacity paths that connect the source to the destination. Assume that a passive eavesdropper, Eve, wiretaps any K links of the communication network. Alice can then securely (in the strong information theoretical sense) communicate at rate $H-K$ with the destination, by conveying linear combinations of K keys with $H-K$ information messages [2]. The rate $H-K$ is exactly the secure message capacity[1] - we cannot hope to do better.

In 1-2-1 networks of unit capacity edges, it turns out that even if the 1-2-1 min-cut is H, i.e., the maximum flow using mmWave communication is H, and Eve eavesdrops any K edges, we may be able to securely communicate at rates higher than $H-K$. Consider for example a diamond network with N relays shown in Fig. 1c with all edges of unit capacity: the unsecure communication capacity equals one - we cannot do better than rate one because Alice can beamform and transmit information at only one relay at each time, and it does not matter which relay she communicates with, since we assumed that all links have unit capacity. Assume that Eve wiretaps any one edge. That is, we have $H=1$ and $K=1$, which over traditional networks would result to a zero secure communication rate. However, Alice can vary which relay she communicates with over time; in fact, she can devote a fraction $\frac{1}{N}$ of her time to send information to Bob over any one out of the N unit capacity paths that connect them. Because Eve will only be observing one of these paths, as we formally show in Section III, Alice can securely communicate at rate of $1-\frac{1}{N}$. This is closer to the unsecure communication rate of one, than to zero. That is, for security over 1-2-1 networks, we can leverage the fact that we may have many possible choices of paths to achieve the unsecure capacity, to communicate at rates much higher than $H-K$.

Main Contributions. (a) We consider arbitrary 1-2-1 networks with unit capacity edges, where Eve wiretaps any K edges, and derive lower and upper bounds on the capacity, that are tight for some networks. (b) We derive the secure message capacity for the case where the source is connected to the destination through one layer of non-interfering relays (i.e., diamond network), where now each path from the source to the destination can have arbitrary capacity.

Related Work. In our work, we essentially leverage directivity and multipath for security, over a "lossless" network model. The fact that directivity can help with security has been observed in the context of MIMO beamforming, see [3] and later work [4]; in these works, the main observation is that, by creating a narrow beam, we limit the locations where the adversary Eve can collect useful information - or at least,

[1]This holds under some standard assumptions in the literature [2], in particular under the assumption that only Alice can generate randomness.

significantly weaken her channel, so as to utilize wiretapping coding. However, to the best of our knowledge, these ideas have not been extended to networks. Exploiting multipath for security over lossless networks with unit capacity links has notably been used in secure network coding [2]. This was followed by a number of works such as [5], [6], [7], [8]. For edges with non-uniform capacities, Cui et al. [9] designed a secure achievable scheme. These results however, consider only the "traditional network", where a node can communicate to other nodes using all the edges it is connected with, and not 1-2-1 networks where a node with one beam can only transmit to one among its neighbors at each point in time.

Paper Organization. Section II presents the 1-2-1 network model and results on the unsecure capacity for arbitrary networks with unit edge capacities. Section III contains secure capacity results for arbitrary networks with unit edge capacities and Section IV presents our secure capacity results for diamond networks with arbitrary edge capacities. Section V concludes the paper.

II. System Model And Unsecure Capacity

Notation. $[m] := \{1,\ 2,\ \ldots,\ m\}$, $[a:b]$ is the set of integers from a to $b \geq a$ and $A_{[m]} = \{A_1,\ A_2,\ A_3,\ \ldots,\ A_m\}$.

1-2-1 Model. The work in [1] examined capacity characterizations for unsecure communication, and showed that we can approximately achieve the capacity of a Gaussian mmWave network within a constant gap, by considering instead of the underlying Gaussian network, a lossless network that was termed 1-2-1 network model. In this paper, we examine security over such 1-2-1 networks, that we describe next. We consider a source connected to a destination through a directed acyclic graph $G = (V, E)$ with edges of fixed finite capacities, where each link can be activated according to the 1-2-1 constraints. That is, at any particular time, an intermediate node can simultaneously receive and transmit but it can at most listen to one node (one incoming edge) and direct its transmission to one node (one outgoing edge) in the network. The source (respectively, destination) can transmit to (respectively, receive from) M other nodes i.e., on M outgoing edges (respectively, on M incoming edges), simultaneously with no interference.

Adversary Model and Security. We assume that the source wishes to communicate a message W of entropy rate R securely from a passive external adversary Eve who can wiretap any K edges of her choice.

If Eve wiretaps edges in the set $S \subseteq E,\ |S| = K$, and the symbols transmitted on these edges over n network uses are denoted by $\{T_e^n,\ e \in S\}$, then we require that:

$$I(W; \{T_e^n,\ e \in S\}) \leq \epsilon,\ \forall S \subseteq E,\ |S| = K. \tag{1}$$

We are interested in characterizing the secure message capacity C, using the standard definition of the maximum rate at which the source can communicate with the destination under (1).

Unsecure Capacity: Here, we derive the capacity in the absence of the eavesdropper Eve. 1-2-1 networks with arbitrary edge capacities, and $M = 1$, under Gaussian channel models are analyzed in [1], where the main result is that over such networks, one can **approximately** (i.e., up to a gap that only depends on N) achieve the capacity by routing information across paths; moreover, out of an exponential number (in N) of paths that potentially connect the source to the destination, capacity can be achieved by utilizing at most a linear number (in N) of them. In this section, we derive an additional results, namely the exact capacity for any M when all the edges are of unit capacity.

Theorem 1. *For arbitrary 1-2-1 networks with unit capacity edges, the capacity in absence of Eve is given by,*

$$C_u = \min(M, H_v), \tag{2}$$

where H_v is the maximum number of vertex disjoint paths in the network.

Proof: **Achievability:** Let $p_{[H_v]}$ be the H_v vertex disjoint paths. The fact that paths are vertex disjoint is crucial under the 1-2-1 constraints. This is because intermediate nodes can transmit and receive from only one node each, and this ensures that multiple paths (depending on the number of source and destination beams) can be simultaneously operated at each time. We pick $\min(M, H_v)$ such paths and use these for the transmission and thus achieve a rate of $\min(M, H_v)$.

Outer Bound: Whenever there are direct edges from the source to the destination, we add a virtual node in between, so that a direct edge turns into a two-hop path. This does not change the transmission rate as if there was a transmission on the direct edge in G, it can also be performed using the added virtual node with no extra resources. Thus, we can assume that there are no direct edges from the source to the destination.

Now, we consider the minimum vertex cut of the network, i.e., the minimum number of vertices (excluding the source and the destination), such that when we remove them there is no path from the source to the destination. This minimum number of vertices is equal to the maximum number of vertex disjoint paths, i.e., H_v. We denote these vertices as $V_1,\ V_2,\ ,\ldots,\ V_{H_v}$. Each of these intermediate nodes can transmit only on one of its outgoing edges. We denote the symbols transmitted on the outgoing edges of these nodes over n network uses as $T^n_{V_{[H_v]}}$, where $T^n_{V_i}$ denotes the symbols transmitted by vertex V_i. We represent the symbols received by the destination as T^n_D.

By Fano's inequality, we obtain

$$\begin{aligned} nR &\leq H(W) \overset{(a)}{=} H(W) - H(W|T^n_D) \\ &\overset{(b)}{\leq} H(W) - H(W|T^n_{V_{[H_v]}}) \\ &= I(W; T^n_{V_{[H_v]}}) \leq H(T^n_{V_{[H_v]}}) \overset{(c)}{\leq} nH_v. \\ nR &\leq H(W) - H(W|T^n_D) = I(W; T^n_D) \\ &\leq H(T^n_D) \overset{(d)}{\leq} Mn. \\ R &\leq \min(M, H_v), \end{aligned}$$

where (a) is due to the reliable decoding constraint; (b) follows from the 'conditioning does not increase the entropy' principle and since $V_{[H_v]}$ is a vertex cut and thus all the information going to the destination passes through these vertices (i.e., T_D^n is a deterministic function of $T^n_{V_{[H_v]}}$); (c) is because there are H_v symbols for every instance and there are n such instances; and (d) holds because the destination can receive only on M incoming edges from M nodes. ■

III. Arbitrary Networks with Unit Link Capacity

In this section, we prove lower and upper bounds on the secure capacity.

Theorem 2. *Consider an arbitrary 1-2-1 network with unit capacity edges.*

(a) *For $M = 1$: If H_e is the maximum number of **edge disjoint** paths connecting the source to the destination on the underlying graph, then the 1-2-1 secure capacity C can be lower bounded as follows:*

$$C \geq \left(1 - \frac{K}{H_e}\right). \quad (3)$$

(b) *For $M > 1$: If H_v is the maximum number of **vertex disjoint** paths connecting the source to the destination on the underlying graph, then the 1-2-1 secure capacity C can be lower bounded as follows:*

$$C \geq \min(M, H_v)\left(1 - \frac{K}{H_v}\right). \quad (4)$$

Proof. The main intuition behind the proof is that we can apply the optimal secure communication scheme we would have used on the underlying graph if we did not have the 1-2-1 constraints, and then use this scheme under the 1-2-1 constraints, as described in what follows.

(a) **For** $M = 1$: Let $p_{[H_e]}$ be the edge disjoint paths. We start by generating K random packets and make H_e linear combinations of these using an MDS code matrix of size $K \times H_e$. We refer to these packets as $X_{[H_e]}$. Any K of these combinations are mutually independent. Next, we take $H_e - K$ message packets, and add (i.e., encode) these with the first $H_e - K$ random packets. In other words, after this coding operation we obtain

$$T_i = \begin{cases} W_i + X_i & \text{if } i \leq H_e - K \\ X_i & \text{else} \end{cases},$$

where $W_{[H_e-K]}$ are message packets.

We use the network H_e times, and in each instance we use one of the paths from $p_{[H_e]}$. Thus, we would be able to communicate all encoded symbols in H_e time instances. Moreover, the destination will be able to cancel out the keys and thereby decode $H_e - K$ messages, as there are K symbols $T_{[H_e-K+1:T_{H_e}]}$, which are just independent combinations of the K random packets we started with.

Moreover, in each instance, Eve will receive a symbol if the edges she eavesdrops are part of the path that is used in that particular instance. Since her K edges can at most be part of K paths, Eve will receive at most K symbols, all of which are encoded with independent keys. Thus, the scheme securely transmits $H_e - K$ message packets in H_e uses of the network. Hence, we get a rate $R = \frac{H_e-K}{H_e} = 1 - \frac{K}{H_e}$, which is precisely the one in (3). Note that security follows from the security of the underlying scheme, that is a standard scheme for multipath security.

(b) For $M > 1$**:** Let $p_{[H_v]}$ be the vertex disjoint paths. Again, the fact that paths are vertex disjoint is crucial under the 1-2-1 constraints. This is because intermediate nodes can transmit and receive from only one node each, and this ensures that M paths can be simultaneously operated at each time (note that having vertex disjoint paths is a sufficient but not a necessary condition).

Let $\hat{M} = \min(M, H_v)$. We start by generating $K\binom{H_v-1}{\hat{M}-1}$ random packets and extend them to $\hat{M}\binom{H_v}{\hat{M}}$ packets using an MDS code matrix. Then, similar to the case $M = 1$, we take the first $\hat{M}\binom{H_v}{\hat{M}} - K\binom{H_v-1}{\hat{M}-1}$ of these random packets and add (i.e., encode) them with the same amount of message packets. More formally, if $\left\{X_i, i \in \left[\hat{M}\binom{H_v}{\hat{M}}\right]\right\}$ are the random packets after the extension using the MDS code matrix, and $\left\{W_i, i \in \left[\hat{M}\binom{H_v}{\hat{M}} - K\binom{H_v-1}{\hat{M}-1}\right]\right\}$ are the message packets, then

$$T_i = \begin{cases} X_i + W_i & \text{if } i \leq \hat{M}\binom{H_v}{\hat{M}} - K\binom{H_v-1}{\hat{M}-1} \\ X_i & \text{else} \end{cases}.$$

We use this network $\binom{H_v}{\hat{M}}$ times, and in each instance we use a different choice of $\hat{M}$ paths to communicate. It is not difficult to see that each of the K edges eavesdropped by the adversary will intersect with $\binom{H_v-1}{\hat{M}-1}$ such network uses. This is because, for a fixed choice of edge, there are $\binom{H_v-1}{\hat{M}-1}$ network instances where a symbol in carried via this edge. Hence, in total the adversary will receive only $K\binom{H_v-1}{\hat{M}-1}$ symbols, which are encoded with independent keys. The receiver, after the $\binom{H_v}{\hat{M}}$ network uses will be able to cancel out the keys. Thus, we can securely communicate $\hat{M}\binom{H_v}{\hat{M}} - K\binom{H_v-1}{\hat{M}-1}$ over $\binom{H_v}{\hat{M}}$ instances of the network, and achieve a rate R equal to

$$\begin{aligned} R &= \frac{\hat{M}\binom{H_v}{\hat{M}} - K\binom{H_v-1}{\hat{M}-1}}{\binom{H_v}{\hat{M}}} \\ &= \hat{M} - \frac{K\hat{M}}{H_v} \\ &= \min(M, H_v)\left(1 - \frac{K}{H_v}\right), \end{aligned}$$

which is precisely the one in (4). This concludes the proof of Theorem 2. □

Theorem 3. *Let H_e be the maximum number of **edge disjoint** paths connecting the source to the destination on the underlying directed graph, then the 1-2-1 secure capacity C can be upper bounded as follows:*

$$C \leq \min(M, H_e)\left(1 - \frac{K}{H_e}\right).$$

Proof. From the min-cut, max-flow theorem there are H_e edges such that, when removed, the source gets disconnected from the destination. Let $e_1, e_2, \ldots, e_{H_e}$ denote these edges. Assume that the network is used n times, and let

$T^n_{e_i},\ i \in \{1,2,\ldots,H_e\}$ be the symbols transmitted on these H_e edges over n uses of the network. By denoting the symbols transmitted by the source on n network instances by T^n_S, then,

$$\begin{aligned} nM &\geq H(T^n_S) \overset{(a)}{=} H(T^n_S, \{T^n_{e_i},\ i \in [H_e]\}) \\ &\geq H(\{T^n_{e_i},\ i \in [H_e]\}), \end{aligned}$$

where (a) follows because $\{T^n_{e_i},\ i \in [H_e]\}$ is a deterministic function of T^n_S. Moreover, $H(\{T^n_{e_i},\ i \in [H_e]\}) \leq nH_e$. Thus,

$$H(\{T^n_{e_i},\ i \in [H_e]\}) \leq \min(nH_e, nM). \tag{5}$$

In the remaining part of the proof, we use the result in the following lemma, which is proved in the Appendix.

Lemma 1. *$\forall m$, there exists a set $S \subset [L], |S| = m$, such that $H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) \leq \frac{L-m}{L} H(\{X_i,\ i \in [L]\})$.*

Without loss of generality, for $m = K$, we assume $S = [K] \subset [H_e]$ in Lemma 1. Then, by Fano's inequality, we have

$$\begin{aligned} nR &\leq H(W) = H(W) - H(W|\{T^n_{e_i},\ i \in [H_e]\}) \\ &= I(W; \{T^n_{e_i},\ i \in [H_e]\}) \\ &= I(W; \{T^n_{e_i},\ i \in [K]\}) + \\ &\quad I(W; \{T^n_{e_i},\ i \in [H_e] \setminus [K]\}|\{T^n_{e_i},\ i \in [K]\}) \\ &\overset{(a)}{\leq} \epsilon + I(W; \{T^n_{e_i},\ i \in [H_e] \setminus [K]\}|\{T^n_{e_i},\ i \in [K]\}) \\ &\leq \epsilon + H(\{T^n_{e_i},\ i \in [H_e] \setminus [K]\}|\{T^n_{e_i},\ i \in [K]\}) \\ &\overset{(b)}{\leq} \epsilon + \frac{H_e - K}{H_e} \min(nH_e, nM) \\ &\implies R \leq \min(M, H_e)\left(1 - \frac{K}{H_e}\right), \end{aligned}$$

where (a) follows since, for security, $I(W; \{T^n_{e_i},\ i \in [K]\}) \leq \epsilon$ and (b) is because of Lemma 1 and (5). This concludes the proof of Theorem 3. □

A. Discussion

For some special cases, we can exactly characterize the capacity (i.e., the upper and lower bounds previously derived coincide). In particular, these include:

- Networks where the number of edge disjoint paths is equal to the number of vertex disjoint paths. For these networks, the capacity if given by $C = \min(M, H_e)(1 - \frac{K}{H_e})$.
- For networks where the source and the destination have one transmit and one receive beam each, i.e., $M = 1$. For these networks, the capacity is given by $C = 1 - \frac{K}{H_e}$.

We next provide two different network examples where: 1) the upper bound is tight (Example 1) and 2) the outer bound is not tight, but the lower bound is tight (Example 2).

Example 1: In Fig. 1a, there are four edge disjoint paths from the source to the destination, i.e., $H_e = 4$. Assume that $M = 2$, i.e., both the source and the destination can transmit and receive from two nodes and $K = 1$, i.e., Eve wiretaps any one edge of her choice. From Fig. 1a, we refer to these four paths as $p_1,\ p_2,\ p_3$ and p_4, ordered from top to bottom. To achieve the outer bound, one can first use p_1 and p_4 and then use p_2 and p_3 to communicate two symbols in each instance of network use. Thus, on two time instances, one can communicate 4 messages (3 securely since $K = 1$). This gives a secure rate of $\frac{3}{2}$, which matches the outer bound.

Example 2: Fig. 1b has also $H_e = 4$. However, for $M = 2$ and $K = 1$, it can be shown that the secure capacity is 1, whereas our outer bound in Theorem 3 is still $\frac{3}{2}$. In order to achieve a secure rate of one, we can select the two paths on the top and on the bottom, which are node disjoint, and use them to communicate. We next derive an outer bound for the network in Fig. 1b that is tighter than the one in Theorem 3. Assume that, at any time instant t, node 1 transmits symbol $X_1^{(t)}$ (it can transmit only one symbol even though it has three outgoing edges) and node 2, transmits $X_2^{(t)}$. Suppose the network is used n times, then by Fano's inequality,

$$\begin{aligned} nR \leq H(W) &= H(W) - H\left(W|\{X_i^{(t)},\ i \in [2],\ t \in [n]\}\right) \\ &= I(W; \{X_i^{(t)},\ i \in [2],\ t \in [n]\}) \\ &= I(W; \{X_2^{(t)},\ t \in [n]\}) + \\ &\quad I(W; \{X_1^{(t)},\ t \in [n]\}|\{X_2^{(t)},\ t \in [n]\}) \\ &\overset{(a)}{\leq} \epsilon + n \implies R \leq 1, \end{aligned}$$

where (a) is because, if Eve wiretaps the edge outgoing from node 2, then $I(W; \{X_2^{(t)},\ t \in [n]\}) \leq \epsilon$ and there are only n symbols in $\{X_1^{(t)},\ t \in [n]\}$.

IV. Diamond Networks with Non-Uniform Path Capacities

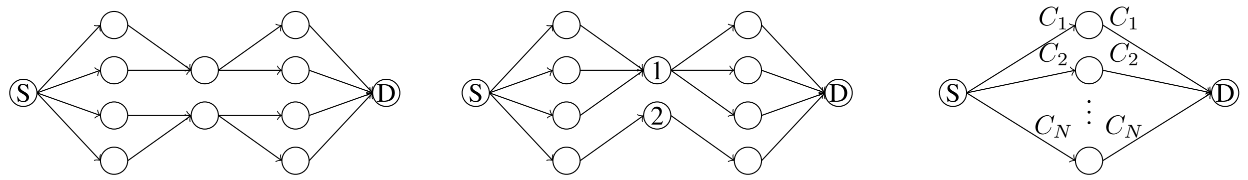
For the N-relay diamond network (shown in Fig. 1c) with unit edge capacities, the lower and upper bounds in Theorem 2 and Theorem 3 match (since all the N edge disjoint paths are also vertex disjoint, namely $H_e = H_v = N$), and thus the secure capacity equals $C = \min(M, N)(1 - \frac{K}{N})$.

We next consider the case where the edges have non-uniform capacities. In particular, we assume that path $i \in [N]$ that connects the source to the destination through relay i has capacity C_i, as depicted in Fig. 1c. In general, even over traditional networks, the problem of security over unequal capacity edges is everything but easily solvable [9]. The main reason is that we need to consider all possible subsets of edges that Eve may wiretap.

Theorem 4. *For the diamond network with $M = 1$ and N relays as shown in Fig. 1c, the secure capacity equals*

$$C = \max_{\substack{f_i \geq 0, \forall i \\ \sum_i f_i = 1}} \left[\sum_{i=1}^{N} f_i C_i - \max_{\substack{S \subseteq [N] \\ |S| = K}} \sum_{i \in S} f_i C_i \right]. \tag{6}$$

Proof. **Achievability:** It is clear that we can transmit $\sum_{i=1}^{N} f_i C_i$ symbols from the source to the destination, by using for a fraction f_i of time the path with capacity C_i. Thus, each



(a) (b) (c)

Fig. 1: (a) Network example $H_e = 4$ for which the outer bound is tight for $M = 2$. (b) Network example with $H_e = 4$ for which the outer bound is not tight for $M = 2$.(c) Diamond network with non-uniform path capacities.

of the N outgoing edges from the source (and similarly each of the N incoming edges to the destination) will carry $f_1C_1,\ f_2C_2,\ \dots\ f_NC_N$ packets, respectively. The adversary, in the worst case wiretaps K edges, which carry the maximum number of packets. Using a similar encryption scheme as we designed in Section III, ensures a secure rate $\left[\sum_{i=1}^{N} f_iC_i - \max_{S} \sum_{i\in S} f_iC_i\right]$, where $S \subseteq [N],\ |S| = K$. By optimizing over the f_i's we get that C in (6) is achievable.

Outer Bound: Since $M = 1$, at any time instant, the source can transmit on at most one of its N outgoing edges. We let $\{T^t_{e_{i_t}},\ t \in [n]\}$ be the symbols transmitted over n such instances, where e_{i_t} denotes the edge used in the t-th instance. Some of these symbols will flow through e_1, some through e_2, and similarly some through e_N, where e_i is the edge of capacity C_i outgoing from the source. Let T_{e_i} denote the symbols transmitted on e_i in all such instances. Thus, $\{T^t_{e_{i_t}}, t \in [n]\} = \{T_{e_i}, i \in [N]\}$. Let $|T_{e_i}| = n_i,\ i \in [N]$ such that $\sum_i n_i = n$. Because of the edge capacity constraints we have $H(T_{e_i}) \le n_iC_i,\ \forall i \in [N]$. Now, by Fano's inequality,

$$\begin{aligned}
nC &\le H(W) = H(W) - H(W|\{T^t_{e_{i_t}},\ t \in [n]\}) \\
&= I(W;\{T_{e_i}, i \in S\}) + I(W;\{T_{e_i}, i \notin S\}|\{T_{e_i}, i \in S\}) \\
&\overset{(a)}{\le} \epsilon + \min_{S\subseteq[N],|S|=K} I(W;\{T_{e_i}, i \notin S\}|\{T_{e_i}, i \in S\}) \\
&= \epsilon + \min_{S\subseteq[N],|S|=K} H(\{T_{e_i}, i \notin S\}|\{T_{e_i}, i \in S\}) \\
&\le \epsilon + \min_{\substack{S \subseteq [N] \\ |S| = K}} \sum_{i\notin S} n_iC_i \\
&= \epsilon + \sum_{i\in[N]} n_iC_i - \max_{\substack{S \subseteq [N] \\ |S| = K}} \sum_{i\in S} n_iC_i
\end{aligned}$$

$$C \le \frac{\sum_{i\in[N]} n_iC_i - \max_{\substack{S \subseteq [N] \\ |S| = K}} \sum_{i\in S} n_iC_i}{\sum_{i\in[N]} n_i}$$

$$\implies C \le \sum_{i\in[N]} f_iC_i - \max_{\substack{S \subseteq [N] \\ |S| = K}} \sum_{i\in S} f_iC_i,$$

where (a) follows from the security condition and the choice of S to have the tightest bound, and $f_i = \frac{n_i}{\sum_{i\in[N]} n_i} \ge 0, \sum_{i\in[N]} f_i = 1$. Optimizing over all such choices of $n_i,\ i \in [N]$, we get that C in (6) is an outer bound on the secure capacity. This concludes the proof of Theorem 4. □

Example 3: Consider a diamond network with $N = 4$, and $C_1 = 3$, $C_2 = 2$, $C_3 = 2$ and $C_4 = 1$ and assume $K = 1$. If we were to use each path the same number of times, we would get a secure rate of $\frac{5}{4}$. In constrast, the optimal scheme from Theorem 4 uses the first path twice, the second and third three times each, and does not use the last path, achieving a secure rate of $\frac{3}{2}$. Thus, we see that different from the traditional network, here we might need to discard some of the resources.

V. Conclusions

We explored security over 1-2-1 networks where, since we need to use beamforming and align beams to activate links, we cannot use all the underlying graph links simultaneously, but instead we can use each link for a fraction of time that we can decide. Over such networks, we have shown that we can achieve a secure capacity that in some cases can be very close to the unsecure capacity; we have derived upper and lower bounds for arbitrary unit capacity networks, and exact capacity characterizations for some special classes of networks.

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APPENDIX
PROOF OF LEMMA 1

Lemma 1. *$\forall m$, there exists a set $S \subset [L], |S| = m$, such that $H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) \leq \frac{L-m}{L} H(\{X_i,\ i \in [L]\})$.*

Proof: Assume for all choices of $S \subset [L]$,$|S| = m$, $H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) > \frac{L-m}{L} H(\{X_i,\ i \in [L]\})$. Then,

$$\begin{aligned}
\binom{L}{m} H(\{X_i,\ i \in [L]\}) &\overset{(a)}{=} \sum_{\substack{S \subset [L] \\ |S| = m}} \left(H(\{X_i,\ i \in S\}) + H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\})\right) \\
&\overset{(b)}{\geq} \sum_{\substack{S \subset [L] \\ |S| = m}} \left(\left(\sum_{i \in S} H(X_i|\{X_j,\ j < i\}) \right) + H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) \right) \\
&\overset{(c)}{=} \binom{L-1}{m-1} \left(\sum_{i \in [L]} H(X_i|\{X_j,\ j < i\}) \right) + \sum_{\substack{S \subset [L] \\ |S| = m}} H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) \\
&\overset{(d)}{=} \binom{L-1}{m-1} H(\{X_i,\ i \in [L]\}) + \sum_{\substack{S \subset [L] \\ |S| = m}} H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) \\
&\overset{(e)}{>} \binom{L-1}{m-1} H(\{X_i,\ i \in [L]\}) + \sum_{\substack{S \subset [L] \\ |S| = m}} \frac{L-m}{L} H(\{X_i,\ i \in [L]\}) \\
&= \binom{L-1}{m-1} H(\{X_i,\ i \in [L]\}) + \binom{L}{m} \frac{L-m}{L} H(\{X_i,\ i \in [L]\}) \\
&= \binom{L}{m} \left(\frac{m}{L} H(\{X_i,\ i \in [L]\}) + \frac{L-m}{L} H(\{X_i,\ i \in [L]\}) \right) \\
&= \binom{L}{m} H(\{X_i,\ i \in [L]\}),
\end{aligned}$$

and hence we get a contradiction. Here (a) is because there are $\binom{L}{m}$ ways of breaking $\{X_i,\ i \in [L]\}$ into two sets of size m and $L-m$, and then it follows from the chain rule of entropy; (b) follows because for any $S \subset [L]$, we can order $\{X_i,\ i \in S\}$ according to their index, and then we use the chain rule of entropy followed by the condition reduces entropy principle; (c) follows because for each $i \in [L]$, there will be $\binom{L-1}{m-1}$ choices of S where this i will be part of S; (d) follows again from the chain rule of entropy; and (e) follows because of the assumption in the proof that for all choices of $S \subset [n], |S| = m$, $H(\{X_i,\ i \in S^c\}|\{X_i,\ i \in S\}) > \frac{L-m}{L} H(\{X_i,\ i \in [L]\})$. ■