Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-9 (2018) 337–342

A Stochastic Formulation of the Optimal Boundary Control Problem Involving the Lighthill Whitham Richards Model

Hao Liu * **Christian Claudel** ** **Randy B. Machemehl** ***

* *Department of Civil, Architectural and Environmental Engineering, University of Texas Austin, USA (e-mail: haoliu@utexas.edu).*
** *Department of Civil, Architectural and Environmental Engineering, University of Texas Austin, USA (e-mail: christian.claudel@utexas.edu)*
*** *Department of Civil, Architectural and Environmental Engineering, University of Texas Austin, USA (e-mail: rbm@mail.utexas.edu)*

Abstract: It has been previously shown that the traffic control problem can be formulated as a Linear Programming (LP) problem when the corresponding initial conditions are fixed while they can be uncertain in actual control problems. This paper gives a stochastic programming formulation of the control problem, involving chance constraints to capture the uncertainty associated with the initial conditions. Different objective functions are explored using this framework and the solutions to the control problems agree well with the Monte Carlo simulation based control. To the authors' best knowledge, this is the first time that the influence of initial condition uncertainty on traffic control is investigated through stochastic programming with chance constraints.

Keywords: Traffic control, linear programming, stochastic programming, chance constraints, optimal control.

1. INTRODUCTION

At present, the number of vehicles all over the world is enormous and continues to grow. As the number of vehicles increases, both people and the environment are affected badly in terms of congestion and pollution, which translates into a waste of time and money. Therefore, reducing traffic congestion is a critical issue to our human society. There are a number of ways to help reduce congestion, such as increasing the road capacity and decreasing user demand. However, these methods are expensive or sometimes even impractical.

The traffic flow is usually modeled as various partial defferential equations (PDEs) (See Lighthill and Whitham (1955); Richards (1956)). Some promising techniques used to mitigate traffic congestion, such as traffic congestion forecasting and traffic flow control, are based on the set of PDE equations. The Kalman Filter (Zhang et al. (2012)) and Autoregressive Integrated Moving Average model (Tan et al. (2007)) are the most frequently used methods for the congestion forecasting in the past decade. Numerous traffic control methodologies (Pasquale et al. (2015); Pisarski and Canudas-de Wit (2016); Li et al. (2014); Bekiaris-Liberis and Bayen (2015)) have been developed in the past decades. For example, ramp metering, which is a device that regulates the flow of traffic entering freeways based on the current traffic condition, is a widespread traffic control method. ALINEA (Papageorgiou et al. (1991)) is one of the ramp metering strategies that has been used around the world. However, the PDEs used in the references (Lu et al. (2009); Carlson et al. (2011)) were discretized to obtain ODEs to use methods in De Wit (2011) to find the optimal solution. A new control method in Li et al. (2014) doesn't require the discretization or approximation. This method shows that the traffic control problem can be posed as a Linear Programming (LP) problem under the triangular fundamental diagram for the traffic flow modeled by the LWR PDE. This framework significantly reduces computational complexity over standard traffic control computational methods.

There are no reliable strategies to capture densities on a highway link, and the measurment of the traffic density depends on indirect measurements of other variables such as speed and flow. Both deterministic approaches (Kurzhanskiy and Varaiya (2012)) and stochastic approaches (Tampère and Immers (2007)) have been proposed to make the density estimation more dependable. Unexpected consequences may be caused by the neglect of the uncertainty associated with each of these quantities. Therefore, robust control which considers the uncertainty associated with each of these quantities is necessary for the traffic control problem.

In this paper, we introduced uncertainty into the LP problem and changed it into a Stochastic Programming problem based upon on the model in Li et al. (2014). In the Stochastic Programming problem, chance constraints were used to formulate the uncertainty in the initial conditions, which were assumed to have normal distributions.

Section 2 reviews the framework for the LP formulation for the traffic control problem. Section 3 presents the stochastic programming that takes into consideration the

2405-8963 © 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.07.055

uncertainty of the initial densities. In section 4, to verify that the relaxation made in section 3 was rational, solutions from Monte Carlo simulation was compared to that from Section 3. Section 5 shows potential future work.

2. LP MODEL DEFINITION

In this section, a comprehensive framework used to derive an LP formula for the traffic control problem is explained.

2.1 Traffic flow models

Lighthill-Whitham-Richards (LWR) PDE model (Lighthill and Whitham (1955)) is one of the most commonly used models to depict the evolution of traffic flow,

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} = 0 \tag{1}$$

where $\rho(t,x)$ is the density of the point x away from a reference point at time t, ψ is the concave Hamiltonian, which is used to denote the experimental relationship between flow and density. For simplicity, a triangular fundamental diagram is used to present the relationship between flow and density such that,

$$\psi(\rho) = \begin{cases} v_f \rho & \rho \in [0, \rho_c] \\ w(\rho - \rho_m) & \rho \in [\rho_c, \rho_m] \end{cases} \tag{2}$$

where v_f is the free flow speed, w is the congestion speed, ρ_c is the critical density where the flow is maximum, ρ_m is the jam density, where the flow is zero due to the total congestion.

Alternatively, the traffic flow can be modeled by a scalar function $M(t,x)$, known as Moskowitz function (Moskowitz (1965)), which represents the index of the vehicle at $(t,\, x)$. The relationship between the Moskowitz function and density and flow can be expressed as,

$$\rho(t,x) = -\frac{\partial M}{\partial x}, \quad q(t,x) = \frac{\partial M}{\partial t} \tag{3}$$

Therefore, another traffic flow model, Hamilton-Jacobi (H-J) PDE, can be obtained from the integration of the LWR PDE model (1) in space,

$$\frac{\partial M(t,x)}{\partial t} - \psi(-\frac{\partial M(t,x)}{\partial x}) = 0 \tag{4}$$

In this paper, the spatial domain $[\xi, \chi]$, where ξ is the upstream boundary and χ is the downstream boundary, and time domain $[0, t_{max}]$, where t_{max} is the simulation time, for a highway link were divided evenly into k_{max} and n_{max} equal segments, respectively. Also, we defined $K = \{1, ..., k_{max}\}$ and $N = \{1, ..., n_{max}\}$. The expression for piecewise affine initial, upstream boundary, and downstream boundary conditions can be found in the reference(Li et al. (2014)). Let X and T be the length for the spatial segment and time segment, separately. $\rho(i)$ is the initial density for the ith spatial segment, $q_{in}(i)$ and $q_{out}(i)$ are the inflow and outflow, respectively, for the ith time segment at boundaries. In these conditions, we chose the appropriate initial and boundary segment length and assumed the initial density and boundary flow conditions in the corresponding segments are constants. To ensure the consistency with the physics of the problem, the segment length needs to satisfy the Courant-Friedrichs-Lewy (CFL) condition (Courant et al. (1928)), $|v_f T/X| < 1$.

2.2 Moskowitz solutions

In this paper, we used the Barron-Jensen/Frankowska (B-J/F) solution (Barron and Jensen (1990); Frankowska (1993)) to solve the H-J equation. The B-J/F solutions are fully characterized by the Lax-Hopf formula.

Definition 1 (Value Condition): A value condition $c(\cdot,\cdot)$ is a lower semicontinuous function defined on a subset of $[0, t_{max}] \times [\xi, \chi]$.

In the following, all of the initial conditions and boundary conditions are regarded as value conditions.

Proposition 1 (Lax-Hopf Formula): Let $\psi(\cdot)$ be a concave and continuous Hamiltonian, and let $c(\cdot,\cdot)$ be a value condition. The B-J/F solution $M_c(\cdot,\cdot)$ to (4) associated with $c(\cdot,\cdot)$ is defined Aubin et al. (2008); Claudel and Bayen (2010a,b) by

$$M_c(t,x) = \inf_{(u,T)\in(\varphi^*)\times R_+} (c(t-T, x+Tu) + T\varphi^*(u)) \tag{5}$$

where $\varphi^*(\cdot)$ is the Legendre-Fenchel transform of an upper semicontinuous Hamiltonian $\psi(\cdot)$, which is given by,

$$\varphi^*(u) := \sup_{p\in Dom(\psi)} [p \cdot u + \psi(p)] \tag{6}$$

Using Lax-Hopf formula, the Moskowitz solution from each value condition can be obtained explicitly. We refer the readers to (Claudel and Bayen (2011)) for a detailed discussion of the solution.

2.3 Linear Constraints

So far, we have obtained the Moskowitz solutions from the given conditions. However, the solution may not be incompatible with the value condition, *i.e.* $M_c(t,x)$ may not equal to $c(t,x)$ at some points in the domain of $c(\cdot,\cdot)$. The Lax-Hopf formula (5) leads to the inf-morphism property (Aubin et al. (2008)).

Proposition 2 (Inf-morphism property): Let the value condition $\boldsymbol{c(\cdot,\cdot)}$ be minimum of a finite number of lower semicontinuous functions:

$$\forall (t,x) \in [0, t_{max}] \times [\xi, \chi], \quad \boldsymbol{c}(t,x) := \min_{j\in J} \boldsymbol{c}_j(t,x) \tag{7}$$

The corresponding solution $\boldsymbol{M_c(\cdot,\cdot)}$ can be decomposed Aubin et al. (2008); Claudel and Bayen (2010a) as

$$\forall (t,x) \in [0, t_{max}] \times [\xi, \chi], \quad \boldsymbol{M_c}(t,x) := \min_{j\in J} \boldsymbol{M_{c_j}}(t,x) \tag{8}$$

Based on the *Inf-morphism* property, the Moskowitz solutions have to satisfy the compatibility conditions (Claudel and Bayen (2011)).

Proposition 3 (Compatibility Conditions): Use the value condition $c(t,x)$ and the corresponding solution in *Proposition 2*. The equality $\forall (t,x) \in Dom(\boldsymbol{c}), \boldsymbol{M_c}(t,x) = \boldsymbol{c}(t,x)$ is valid if and only if the inequalities below are satisfied,

$$\boldsymbol{M_{c_j}}(t,x) \geq \boldsymbol{c}_i(t,x), \quad \forall (t,x) \in Dom(\boldsymbol{c}_i), \forall (i,j) \in J^2 \tag{9}$$

These constraints are linear in terms of initial and boundary conditions and can be expanded as (Canepa and Claudel (2012, 2013)).

$$\begin{cases} M_{M_k}(0,x_p) \geq M_p(0,x_p) & \forall (k,p) \in K^2 \\ M_{M_k}(pT,\chi) \geq \beta_p(pT,\chi) & \forall k \in K, \quad \forall p \in N \\ M_{M_k}(\frac{\chi - x_k}{v_f},\chi) \geq \beta_p(\frac{\chi - x_k}{v_f},\chi) & \forall k \in K, \quad \forall p \in N \\ & \text{s.t.} \frac{\chi - x_k}{v_f} \in [(p-1)T, pT] \\ M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall k \in K, \quad \forall p \in N \\ M_{M_k}(\frac{\xi - x_{k-1}}{w},\xi) \geq \gamma_p(\frac{\xi - x_{k-1}}{w},\xi) & \forall k \in K, \quad \forall p \in N \\ & \text{s.t.} \frac{\xi - x_{k-1}}{w} \in [(p-1)T, pT] \end{cases} \quad (10)$$

$$\begin{cases} M_{\gamma_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in N^2 \\ M_{\gamma_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in N^2 \\ M_{\gamma_n}(nT + \frac{\chi - \xi}{v_f},\chi) \geq \beta_p(nT + \frac{\chi - \xi}{v_f},\chi) & \forall (n,p) \in N^2 \\ & \text{s.t. } nT + \frac{\chi - \xi}{v_f} \in [(p-1)T, pT] \end{cases} \quad (11)$$

$$\begin{cases} M_{\beta_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in N^2 \\ M_{\beta_n}(nT + \frac{\xi - \chi}{w},\xi) \geq \gamma_p(nT + \frac{\xi - \chi}{w},\xi) & \forall (n,p) \in N^2 \\ & \text{s.t. } nT + \frac{\xi - \chi}{w} \in [(p-1)T, pT] \\ M_{\beta_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in N^2 \end{cases} \quad (12)$$

Above all, specific traffic control problems can be modeled as an LP formulation with linear constraints (10)-(12). In such a formulation, the boundary conditions which are upstream and downstream flows are the decision variable, i.e. the objective function can be realized through controlling the inflow and outflow on a traffic link; the objective function can be any linear function of the decision variables. Although there are no strategies yet used to control the boundary flow on a highway link, it is reasonable to assume that all highway links can be controlled in the future.

In this methodology, the initial conditions are known and fixed in the LP model. In reality, however, there is uncertainty in the initial conditions due to measurement error. To deal with this situation, a stochastic programming model was derived for the robust traffic control control when the initial conditions are random variables with normal distributions in the next two sections.

It should be noticed that the fundamental diagram is empirical, and it is more reasonable to define those variables as random variables as well. But in this paper, we only introduced uncertainty into initial conditions because of the complexity with uncertainty in parameters. Although the Moskowitz solutions are piecewise linear function in the initial and boundary conditions, the fundamental diagram parameters are involved in the domain of the function. Also the solutions are bilinear function of some parameters. All of these facts make it difficult to solve the traffic control problem when the uncertainty is inserted to fundamental diagram parameters.

3. RELAXATION OF ROBUST CONTROL WITH UNCERTAINTY IN INITIAL CONDITIONS

In this section, only the initial conditions are regarded with uncertainty and chance constraints are used to deal with the uncertainty. Moreover, we relaxed the problem to make it tractable.

3.1 The stochastic programming formula

In the rest of this paper, the objective functions are only functions of boundary conditions and there are no uncertainties in the objective functions. A general inequality form of an LP problem is

$$\begin{aligned} &Minimize \quad f(x) \\ &s.t. \quad Ax \geq b \end{aligned} \quad (13)$$

When there is uncertainty in the constraints, we can convert this LP problem to a stochastic programming problem with chance constraints,

$$\begin{aligned} &Minimize \quad f(x) \\ &s.t. \quad Pr\{Ax \geq b\} \geq 1 - \alpha \end{aligned} \quad (14)$$

where $f(x)$ is a linear function of x, x is a decision variable vector which is boundary conditions vector in this paper, A is the coefficient matrix, b is the right-hand side vector, and $1-\alpha$ is the confidence level of the chance constraint. Assume $\rho(k)$ is subjected to a normal distribution with mean and standard deviation of (ρ_k, σ_k). In our stochastic traffic control problem, the constraints (10)-(12) needs to be expressed as the chance constraints form. For example, the constraint of

$$M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi), \quad \forall k \in K, \quad \forall p \in N \quad (15)$$

should be converted to,

$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha, \quad \forall k \in K, \quad \forall p \in N \quad (16)$$

To solve the stochastic programming problem with these chance constraints, we converted the chance constraints into deterministic linear constraints. Then the stochastic programming became an LP problem and it could be solved easily.

Here, we would show the derivation of deterministic version for the chance constraint $Pr\{M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)\} \geq 1-\alpha$ in detail. Then the integrated deterministic constraints were obtained using the same method.

From (Claudel and Bayen (2011)), the Moskowitz solution at upstream from the initial condition can be explicitly expressed as:

$$M_{M_k}(t,\xi) = \begin{cases} +\infty, \\ \text{if } t \leq \frac{\xi-(k-1)X}{w} \\ -\sum_{i=1}^{k-1}\rho(i)X + \rho_c(tv_f + (k-1)X - \xi), \\ \text{if } t \geq \frac{\xi-(k-1)X}{w}, \quad \text{and } \rho(k) \leq \rho_c \\ -\sum_{i=1}^{k-1}\rho(i)X + \rho(k)(tw + (k-1)X - \xi) - \rho_m tw, \\ \text{if } \frac{\xi-(k-1)X}{w} \leq t \leq \frac{\xi - kX}{w}, \quad \text{and } \rho(k) \geq \rho_c \\ -\sum_{i=1}^{k}\rho(i)X + \rho_c(tw + kX - \xi) - \rho_m tw, \\ \text{if } t \geq \frac{\xi - kX}{w}, \quad \text{and } \rho(k) \geq \rho_c \end{cases} \quad (17)$$

From (17) it is known that $M_{M_k}(t,\xi)$ is a nonincreasing function of $\rho(k)$, as shown in Figure 1. Then the corresponding chance constraint was simply divided into two situations:

(i). $\rho_c \leq \rho_k + z_{1-\alpha}\sigma_k$ as shown left in Figure 1. We should convert the chance constraint to $f_2(\rho_k + z_{1-\alpha}\sigma_k) \geq \gamma_p(pT,\xi) \quad \forall k \in K, \quad \forall p \in N$;

(ii). $\rho_c \geq \rho_k + z_{1-\alpha}\sigma_k$ as shown right in Figure 1. We should convert the chance constraint to $f_1(\rho_k + z_{1-\alpha}\sigma_k) \geq \gamma_p(pT,\xi) \quad \forall k \in K, \quad \forall p \in N$.
where $z_{1-\alpha}$ is defined as z score such that $P(\rho(k) \leq \rho_k + z_{1-\alpha}\sigma_k) = 1-\alpha$.

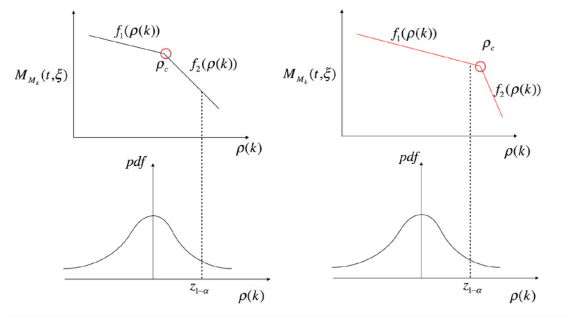

Fig. 1. Solution at upstream from initial condition and associated initial density distribution.

Substituting the expressions of $M_{M_k}(t,\xi)$ and $\gamma_p(pT,\xi)$ into the inequalities above leads to the following linear deterministic constraint:

$$\begin{cases} -\sum_{i=1}^{k-1}\rho(i)x + \rho_c(pTv_f + (k-1)x - \xi) \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } t \geq \dfrac{\xi-(k-1)X}{w}, \qquad \text{and } \rho_k + z_{1-\alpha}\sigma_k \leq \rho_c \\ -\sum_{i=1}^{k-1}\rho(i)X + (\rho(k) + z_{1-\alpha}\sigma_k)(tw + (k-1)X - \xi) - \rho_m tw \\ \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } \dfrac{\xi-(k-1)X}{w} \leq t \leq \dfrac{\xi-kX}{w}, \qquad \text{and } \rho_k + z_{1-\alpha}\sigma_k \geq \rho_c \\ -\sum_{i=1}^{k-1}\rho(i)X - (\rho(k)+z_{1-\alpha}\sigma_k)X + \rho_c(tw + kX - \xi) - \rho_m tw \\ \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } t \geq \dfrac{\xi-kX}{w}, \qquad \text{and } \rho_k + z_{1-\alpha}\sigma_k \geq \rho_c \end{cases} \tag{18}$$

For simplicity, only $\rho(k)$ for the constraints involving M_{M_k} was considered as a random variable, all of other $\rho(i)'s, i \in \{1,2,...,k-1\}$ were still regarded as fixed values with their corresponding means. In section 4, the complexity of regarding all of the $\rho(i)'s, i \in \{1,2,...,k-1\}$ as random variables will be explained and the accuracy of this relaxation will be demonstrated by Monte Carlo simulation.
The other constraints in (10)-(12) were found in this same manner. For the sake of simplicity and the space limit, those constraints would not be displayed here.

3.2 Case study

We implemented our framework onto a single highway link located between the PeMS vehicle detection stations 400536 and 400284 on Highway I-880 N around Hayward, CA, USA. We divided this spatial domain of 3.858 km into 7 even segments and created a temporal domain of 7 min with 28 even segments. In addition, the parameters in the fundamental diagram were defined as follows: the critical density $\rho_c = 0.03/m$; the free flow speed $v_f = 30m/s$; the jam density $\rho_m = 0.24/m$. We assumed the initial densities have normal distributions. The mean of the initial densities on the seven segments were defined as six piece-wise affine constants in the range $[0.01, 0.07]$. We implemented the IBMIlogCplex solver in Matlab to solve the LPs. The International System of Units was adopted and the units were omitted in the following analysis for simplicity.
Four scenarios with different standard deviation (0.01, 0.02, 0.03 and 0.04) in the initial condition segments were investigated. For each single scenario, the standard deviation in all initial condition segments were the same and denoted by σ. For those scenarios, $1-\alpha$, the confidence level, was set to 97.5%. The following objective function was chosen to be optimized,

$$Maximize \quad \sum_{i=1}^{n_{max}} q_{out}(i) - \sum_{i=2}^{n_{max}} |q_{out}(i) - q_{out}(i-1)| \tag{19}$$

In this objective function, we wanted to maximize and smooth the outflows at the same time. However, the second term in the objective function was not a linear function of decision variables. To solve this problem, we added another variable vector $q_d(i), i \in \{2,3,...n_{max}\}$ into this problem,

$$\begin{aligned} Max \quad & \sum_{i=1}^{n_{max}} q_{out}(i) - \sum_{i=2}^{n_{max}} q_d(i) \\ s.t. \quad & q_d(i) \geq q_{out}(i) - q_{out}(i-1), \quad \forall i \in \{2,3,...,n_{max}\} \\ & q_d(i) \geq q_{out}(i-1) - q_{out}(i), \quad \forall i \in \{2,3,...,n_{max}\} \end{aligned} \tag{20}$$

The optimal density field solutions are shown in Figure 2. In these cases, the confidence level was fixed, so the confidence interval for the initial condition was wider when the standard deviation was larger. Intuitively, the chance constraints forced the solution to satisfy (10)-(12) for all of the values of the initial conditions in the confidence interval. Therefore, the wider the confidence level was, the more restricted the feasible region was. As this is a maximization problem, the optimal value should be lower for the case with larger standard deviation (i.e. with smaller feasible region). Therefore, with increasing standard deviation, the bandwidth of the shock wave (the yellow band) was wider because less vehicles could proceed, as shown in Figure 2. Also, the shockwaves in each scenario are consecutive due to the second term in (19).

In general, we may not only want to maximize the outflow, but also to minimize the congestion. There are several ways to realize this objective, such as adding another constraints to represent the worst level of service, change the objective function, and so on. Here, we formulated this problem as follow:

$$\begin{aligned} min \quad & -\lambda \sum_{i=1}^{n_{max}} q_{out}(i) + (1-\lambda)Q \\ s.t. \quad & Q \geq \sum_{j=1}^{i}(q_{in}(j) - q_{out}(j)), \quad \forall i \in N \\ & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{21}$$

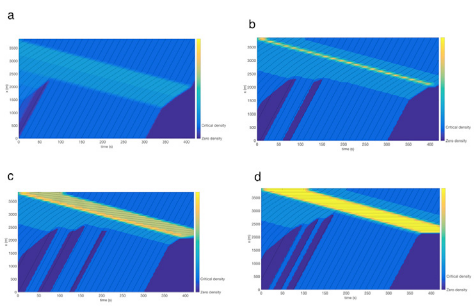

Fig. 2. Solution to robust control problem (19): a) $\sigma = 0.01$; b) $\sigma = 0.02$; c) $\sigma = 0.03$; d) $\sigma = 0.04$.

where $Q + \sum_{k=1}^{k_{max}} \rho(k)$ is the maximum number of vehicles stuck in the link during the simulation, λ and $1-\lambda$ are the weights of total outflow and Q, respectively. $Ax \geq b$ represent the chance constraints we derived in the previous section. The sum of weighted negative total outflow and Q is the new objective function.

The standard deviation of the initial conditions was 0.02 and the confidence level of the chance constraints was 97.5%. To make the result more intuitive, we defined the level of service as $LoS = -Q$. Optimal solutions for different weightings (Figure 3) show there is a tradeoff between outflow and the level of service. With an increase in λ, more vehicles can go through the highway link with a poorer level of service. When $\lambda \geq 0.4$, the optimal values do not change much.

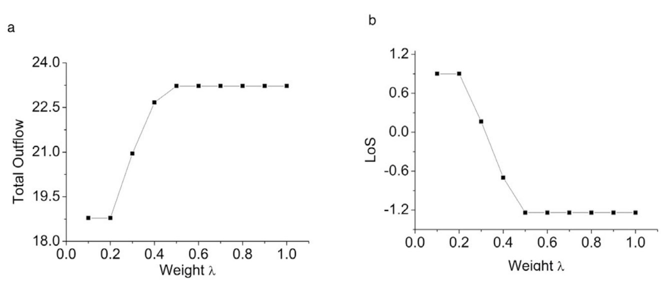

Fig. 3. Optimal values for: a) total outflow; b) level of service

4. MONTE CARLO SIMULATION WITHOUT RELAXATION

In the previous section, some constraints were relaxed, because only $\rho(k)$ for the constraints involving M_{M_k} was considered as a random variable while all of other $\rho(i)'s, i \in \{1, 2, ..., k-1\}$ were regarded as fixed with their corresponding means.

For simplicity, the Moskowitz solutions from the initial conditions can be expressed as the following form:

$$M_{M_k}(t,x) = \begin{cases} f_1(\rho(i)), & \text{if } \rho(k) \geq \rho_c, i = 1, 2, ..., K_{max} \\ f_2(\rho(i)), & \text{if } \rho(k) < \rho_c, i = 1, 2, ..., K_{max} \end{cases} \quad (22)$$

where f_1 and f_2 indicate two linear functions and K_{max} is the number of the initial condition segments. Therefore, the typical chance constraint involving M_{M_k} can be expressed as,

$$\begin{aligned} Pr(M_{M_k}(t,x) \geq g(q)) = Pr(f_1(\rho(i)) \geq g(q), \rho(k) \geq \rho_c) \\ + Pr(f_2(\rho(i)) \geq g(q), \rho(k) < \rho_c) \end{aligned} \quad (23)$$

where g is a linear function of boundary conditions.

If all of the initial conditions are independently normal distributed, $(f(\rho(i)), \rho(k))$ is subject to a bivariate normal distribution $n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,

$$\boldsymbol{\mu} = [\mu_{f(\rho(i))}, \mu_{\rho(k)}], \quad \boldsymbol{\Sigma} = \begin{bmatrix} Var(f(\rho(i)) & Cov(f(\rho(i), \rho(k))) \\ Cov(f(\rho(i), \rho(k))) & Var(\rho(k)) \end{bmatrix} \quad (24)$$

Although the pdf of a bivariate normal distribution can be obtained, there is no closed-form to calculate the corresponding cumulative distribution function (cdf). In fact, there is no closed-form for the cdf of an univariate normal distribution either. The integral of a normal pdf is an error function, which is nonlinear. The reason why we can convert the chance constraint into linear form is that the normal table is available. Unfortunately, for a bivariate normal distribution, there is no such table that can be used to find the critical value for a corresponding confidence level.

To test our relaxed model, Monte Carlo simulation were used to convert the chance constraints into a linear form. The algorithm for constraints $Pr(M_{M_k}(t,x) \geq \gamma(t,x)) >= 1-\alpha$ is as follows.

Step 1. Generate N random numbers from the normal distribution for each initial condition segment. In this paper, $N = 1000$. $\rho(k_i)$ is the ith number for the kth segment.

Step 2. Calculate $M_{M_{k(i)}}(t,x), i = 1, 2, ...N$ using the ith number from each segment from Step 1.

Step 3. Sort $M_{M_k(i)}(t,x)$ into ascending order. Find the corresponding critical value. For example, if the confidence level is 97.5%, then the critical value should be $M_{M_k(25)}(t,x)$ in the ordered sequence.

Step 4. Replace the constraints involving M_{M_k} and γ of $Pr(M_{M_k}(t,x) \geq \gamma(t,x)) >= 1-\alpha$ with $M_{M_k(N\alpha)}(t,x) \geq \gamma(t,x)$.

Other constraints can be obtained by the same manner. The comparison of optimal total outflows from (19) between this Monte Carlo methodology and the relaxation formulation is shown in Figure 4.

In this example, the mean value vector of initial densities was $\rho = (0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07)$ and the standard deviations were 0.02. As we mentioned before, optimal value decreases with the variability increase. The largest difference between these two methodologies is less than 2 percent. Because this is a maximization problem, the optimal value from the relaxation formulation is larger than that from the original problem. The results shown in Figure 4 coincide with this proposition. Above all, our relaxed stochastic program fitted well with the Monte Carlo simulation.

5. DISCUSSION

We used a set of individual chance constraints in this framework, we are also interested in joint chance constraints and that will be a promising topic in the future work. In addition, finding a stochastic programming formulation to deal with the uncertainty in fundamental diagram parameters will be an interesting research direction.

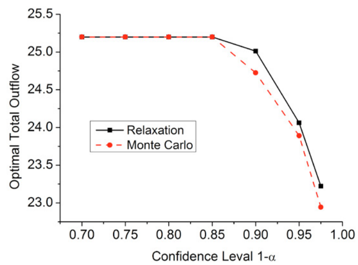

Fig. 4. Comparison between two methodologies: $\sigma(i) = 0.02, \quad \forall i \in K$.

What is more,applying this robust control framework on the highway network to check its efficiency is necessary.

REFERENCES

Aubin, J.P., Bayen, A.M., and Saint-Pierre, P. (2008). Dirichlet problems for some hamilton–jacobi equations with inequality constraints. *SIAM Journal on Control and Optimization*, 47(5), 2348–2380.

Barron, E. and Jensen, R. (1990). Semicontinuous viscosity solutions for hamilton–jacobi equations with convex hamiltonians. *Communications in Partial Differential Equations*, 15(12), 293–309.

Bekiaris-Liberis, N. and Bayen, A.M. (2015). Nonlinear local stabilization of a viscous hamilton-jacobi pde. *IEEE Transactions on Automatic Control*, 60(6), 1698–1703.

Canepa, E.S. and Claudel, C.G. (2012). Exact solutions to traffic density estimation problems involving the lighthill-whitham-richards traffic flow model using mixed integer programming. In *Intelligent Transportation Systems (ITSC), 2012 15th International IEEE Conference on*, 832–839. IEEE.

Canepa, E.S. and Claudel, C.G. (2013). Spoofing cyber attack detection in probe-based traffic monitoring systems using mixed integer linear programming. In *Computing, Networking and Communications (ICNC), 2013 International Conference on*, 327–333. IEEE.

Carlson, R.C., Papamichail, I., and Papageorgiou, M. (2011). Local feedback-based mainstream traffic flow control on motorways using variable speed limits. *IEEE Transactions on Intelligent Transportation Systems*, 12(4), 1261–1276.

Claudel, C.G. and Bayen, A.M. (2010a). Lax–hopf based incorporation of internal boundary conditions into hamilton–jacobi equation. part i: Theory. *IEEE Transactions on Automatic Control*, 55(5), 1142–1157.

Claudel, C.G. and Bayen, A.M. (2010b). Lax–hopf based incorporation of internal boundary conditions into hamilton-jacobi equation. part ii: Computational methods. *IEEE Transactions on Automatic Control*, 55(5), 1158–1174.

Claudel, C.G. and Bayen, A.M. (2011). Convex formulations of data assimilation problems for a class of hamilton–jacobi equations. *SIAM Journal on Control and Optimization*, 49(2), 383–402.

Courant, R., Friedrichs, K., and Lewy, H. (1928). Über die partiellen differenzengleichungen der mathematischen physik. *Mathematische annalen*, 100(1), 32–74.

De Wit, C.C. (2011). Best-effort highway traffic congestion control via variable speed limits. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, 5959–5964. IEEE.

Frankowska, H. (1993). Lower semicontinuous solutions of hamilton–jacobi–bellman equations. *SIAM Journal on Control and Optimization*, 31(1), 257–272.

Kurzhanskiy, A.A. and Varaiya, P. (2012). Guaranteed prediction and estimation of the state of a road network. *Transportation research part C: emerging technologies*, 21(1), 163–180.

Li, Y., Canepa, E., and Claudel, C. (2014). Optimal control of scalar conservation laws using linear/quadratic programming: Application to transportation networks. *IEEE Transactions on Control of Network Systems*, 1(1), 28–39.

Lighthill, M.J. and Whitham, G.B. (1955). On kinematic waves. ii. a theory of traffic flow on long crowded roads. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 229, 317–345. The Royal Society.

Lu, X.Y., Varaiya, P., and Horowitz, R. (2009). An equivalent second order model with application to traffic control. *IFAC Proceedings Volumes*, 42(15), 375–382.

Moskowitz, K. (1965). Discussion of 'freeway level of service as influenced by volume and capacity characteristics' by dr drew and cj keese. *Highway Research Record*, 99, 43–44.

Papageorgiou, M., Hadj-Salem, H., and Blosseville, J.M. (1991). Alinea: A local feedback control law for on-ramp metering. *Transportation Research Record*, 1320(1), 58–67.

Pasquale, C., Papamichail, I., Roncoli, C., Sacone, S., Siri, S., and Papageorgiou, M. (2015). Two-class freeway traffic regulation to reduce congestion and emissions via nonlinear optimal control. *Transportation Research Part C: Emerging Technologies*, 55, 85–99.

Pisarski, D. and Canudas-de Wit, C. (2016). Nash game-based distributed control design for balancing traffic density over freeway networks. *IEEE Transactions on Control of Network Systems*, 3(2), 149–161.

Richards, P.I. (1956). Shock waves on the highway. *Operations research*, 4(1), 42–51.

Tampère, C.M. and Immers, L. (2007). An extended kalman filter application for traffic state estimation using ctm with implicit mode switching and dynamic parameters. In *Intelligent Transportation Systems Conference, 2007. ITSC 2007. IEEE*, 209–216. IEEE.

Tan, M.C., Feng, L.B., and Xu, J.M. (2007). Traffic flow prediction based on hybrid arima and ann model. *Zhongguo Gonglu Xuebao(China Journal of Highway and Transport)*, 20(4), 118–121.

Zhang, L., Ma, J., and Zhu, C. (2012). Theory modeling and application of an adaptive kalman filter for short-term traffic flow prediction. *Journal of Information and Computational Science*, 9(16), 5101–5109.