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Graphs with large rank numbers and rank numbers of subdivided stars[☆]

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Abstract

A k-ranking of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that if $f(u) = f(v)$ then every uv path contains a vertex w such that $f(w) > f(u)$. The rank number of G, denoted $\chi_r(G)$, is the minimum k such that a k-ranking exists for G. It is known that given a graph G and a positive integer t the question of whether $\chi_r(G) \leq t$ is NP-complete. In this paper we characterize graphs with large rank numbers. In addition, we characterize subdivided stars based on their rank numbers.

Keywords: Vertex ranking; Rank number; Stars; Subdivided stars

1. Introduction

Let G be an undirected graph with no loops and no multiple edges. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a (vertex) k-ranking of G if for $u, v \in V(G)$, $f(u) = f(v)$ implies that every uv path contains a vertex w such that $f(w) > f(u)$. By definition, every ranking is a proper coloring. The rank number of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking. For a graph G, by optimal k-ranking we mean a k-ranking such that $k = \chi_r(G)$. When the value of k is not important, we will call a k-ranking simply a ranking. It is known that $\chi_r(K_n) = n$ and $\chi_r(K_{m,n}) = \min(m, n) + 1$.

Vertex rankings of graphs are applicable to a plethora of other fields including designs of very large scale integration (VLSI) layouts, Cholesky factorizations of matrices in parallel, wi-fi analytics, and scheduling problems of assembly steps in manufacturing systems. The optimal tree node ranking problem is identical to the problem of generating a minimum height node separator tree for a tree. Node separator trees are extensively used in VLSI

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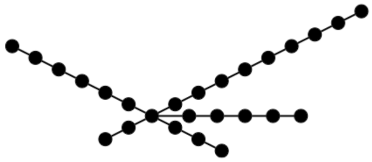

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 1. $S_{9,6,5,3,2}$ with corresponding matrix C.

layout [1]. Ranking of graphs is used in communication networks in which information flow between the nodes has to be monitored. An application of graph ranking to scheduling of assembly steps in manufacturing system is discussed in [2].

The concept of ranking was introduced by Iyer et al., in [3], for trees. It is shown by Bodlaender et al., in [4], that for a graph G and a positive integer t the question of whether $\chi_r(G) \leq t$ is NP-complete. The mathematical studies of vertex rankings were initiated by Ghoshal and Laskar in [5]. Since then, the rank number of numerous families of graphs have been established, for example see [6–14]. A generalization of k-ranking using the l_p norm is discussed in [15].

Let G be a graph of order n. We know that $1 \leq \chi_r(G) \leq n$. In this paper, we study the rankings of subdivided stars and identify graphs with large rank numbers. The rank number of a subdivided star is either $\chi_r(P)$ or $\chi_r(P)+1$, where P is the longest path in the subdivided star. We characterize subdivided stars based on their rank numbers, and hence establish the rank number of a subdivided star. We also identify graphs with rank number $n-1$ or $n-2$.

Let G be a graph and let H be a subgraph of G. Throughout this paper, the graph $G - H$ represents the graph obtained by deleting $E(H)$ from G. If $S \subseteq V(G)$, then $G - S$ denotes the graph obtained by deleting the vertices in S from G, that is, the subgraph induced by $V(G) - S$. For any labeling f of a graph G, let $S_i(f) = \{v \in V(G) \mid f(v) = i\}$. Throughout this paper, we assume that for any optimal k-ranking f, $|S_i(f)| \geq |S_j(f)|$ for $1 \leq i < j \leq k$ because of Lemma 1.1.

Lemma 1.1 (*[5]*). *There is an optimal ranking f of G such that $|S_1(f)| \geq |S_2(f)| \geq \cdots |S_k(f)|$.*

Lemma 1.2 (*[5]*). *Let H be a subgraph of G. Then $\chi_r(H) \leq \chi_r(G)$.*

Lemma 1.3 (*[14]*). *Let H_1 and H_2 be two vertex disjoint graphs such that $\chi_r(H_1) = \chi_r(H_2) = k$. Let G be a connected supergraph of $H_1 \cup H_2$. Then $\chi_r(G) \geq k+1$.*

Lemma 1.4 (*[5]*). *Let G be a graph of order n and let I be an independent set of G. Then $\chi_r(G) \leq n - |I| + 1$.*

Theorem 1.5 (*[4]*). $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$, *where P_n is a path on n vertices.*

2. Subdivided stars

Let r be a positive integer, and let $n_1, n_2, \ldots, n_r$ be positive integers. Let the edges of a complete bipartite graph $K_{1,r}$ be $e_1, e_2, \ldots, e_r$. For $1 \leq i \leq r$, subdivide edge e_i $n_i - 1$ times to obtain a subdivided star denoted by $S_{n_1,n_2,\ldots,n_r}$. We consider the subdivided edges as paths, $P_{n_1}, P_{n_2}, \ldots, P_{n_r}$, and refer to them as branches.

We can use a binary matrix to represent a subdivided star. For a subdivided star, $S_{n_1,n_2,\ldots,n_r}$, where $n_1 \geq n_2, \geq \cdots, \geq n_r$, construct an $r \times \chi_r(P_{n_1})$ matrix C where row i is the binary representation of n_i. An example of a subdivided star and its associated matrix C is given in Fig. 1. Using this binary representation of subdivided stars, we characterize subdivided stars based on their rank numbers.

Lemma 2.1. $\chi_r(P_{n_1}) \leq \chi_r(S_{n_1,n_2,\ldots,n_r}) \leq \chi_r(P_{n_1}) + 1$, *where $n_1 \geq n_2 \geq \cdots \geq n_r$.*

Proof. Since P_{n_1} is a subgraph of $S_{n_1,n_2,\ldots,n_r}$, $\chi_r(S_{n_1,n_2,\ldots,n_r}) \geq \chi_r(P_{n_1})$.

Label the center vertex $\chi_r(P_{n_1}) + 1$, and branch P_{n_i} using labels $1, 2, \ldots, \chi_r(P_{n_i})$. Since $n_1 \geq n_2 \geq \cdots \geq n_r$, this will produce a ranking using $\chi_r(P_{n_1}) + 1$ labels. Thus $\chi_r(S_{n_1,n_2,\ldots,n_r}) \leq \chi_r(P_{n_1}) + 1$. □

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Theorem 2.2. *For any subdivided star $S_{n_1,n_2,\ldots,n_r}$, with $n_1 \geq n_2 \geq \cdots \geq n_r$,*

$$\chi_r(S_{n_1,n_2,\ldots,n_r}) = \begin{cases} \chi_r(P_{n_1}), & \text{if } \exists\, i \text{ such that } C_i = [0,0,\ldots,0]^T \\ & \text{and } \forall j < i, C_j \text{ contains at most one } 1 \\ \chi_r(P_{n_1}) + 1, & \text{otherwise.} \end{cases}$$

Proof. Note that, since $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$ and P_{n_1} is a longest branch, the column C_1 will contain at least one 1. Assume that for some i, $C_i = [0,0,\ldots,0]^T$ and $\forall\ j < i$, C_j contains exactly one 1. If $\chi_r(P_{n_1}) = \chi_r(P_{n_2})$, then there would be two 1's in the first column of C, which contradicts the assumption. Thus branch P_{n_1} is the only branch with rank number $\chi_r(P_{n_1})$.

Let $S^1 = S_{n_1,n_2,\ldots,n_r}$ and $C^1 = C$. Label the outermost $2^{\chi_r(P_{n_1})-1} - 1$ vertices in P_{n_1} with $\chi_r(P_{n_1}) - 1$ labels. Label the next outermost vertex, w_1, with label $\chi_r(P_{n_1})$. We have now labeled $2^{\chi_r(P_{n_1})-1}$ vertices on the branch P_{n_1} so the number of vertices left unlabeled has the same binary representation as $|V(P_{n_1})|$ but without the leading 1. Remove the $2^{\chi_r(P_{n_1})-1}$ labeled vertices on P_{n_1} from S^1. Now we have a new subdivided star, $S^2 = S_{m_1,m_2,\ldots,m_k}$ for $0 \leq k \leq r$, where $m_1 \geq m_2 \geq \cdots \geq m_k$. Note that any path between vertices in S^2 and the deleted vertices from S^1 must contain the vertex labeled $\chi_r(P_{n_1})$. Also, $\chi_r(P_{m_1}) = \chi_r(P_{n_1}) - 1$, since C_1 has exactly one 1. The matrix C^2 can be considered a $r \times \chi_r(P_{n_1})$ matrix with the first column consisting of all zeros and the last $\chi_r(P_{n_1}) - 1$ columns the same as that of C. Repeat this process until we get the C^i matrix. The matrix C^i will have zeros on the first i columns and the last $\chi_r(P_{n_1}) - i$ columns the same as that of C. Note that column C_i of C has all zeros.

At this point none of the branches of the subdivided star S^i have a rank number greater than $\chi_r(P_{n_1}) - i$, (because the first i columns of C^i are zeros), where P_{n_1} is the longest branch of the original subdivided star S. Label the branches of S^i using $\chi_r(P_{n_1}) - i$ labels or less and label the middle vertex $\chi_r(P_{n_1}) - i + 1$.

An example of this procedure is shown in Figs. 2–6. Note that in each of these figures the vertices with labels are the vertices that were deleted from the previous subdivided star.

Now any path between vertices in S^a and the vertices deleted from S^{a+1} must contain the largest labeled vertex, w_a, that was deleted from the largest branch of S^a. Also note that S^{a+1} will not have a vertex labeled the same as the label of w_a because there is exactly one 1 in each of the first $i - 1$ columns of C. Therefore, this labeling will create a ranking using only $\chi_r(P_{n_1})$ labels. This implies $\chi_r(S_{n_1,n_2,\ldots,n_r}) \leq \chi_r(P_{n_1})$, and hence, by Lemma 2.1, $\chi_r(S_{n_1,n_2,\ldots,n_r}) = \chi_r(P_{n_1})$.

Let $\chi_r(S_{n_1,n_2,\ldots,n_r}) = \chi_r(P_{n_1})$ and let f be an optimal labeling of S. Assume there is no i such that $C_i = [0,0,\ldots,0]^T$ and $\forall\ j < i$, C_j contains only one 1. Let column C_{g+1} contain more than one 1 where $0 \leq g \leq \chi_r(P_{n_1}) - 1$. Let $C_1, C_2, C_3, \ldots, C_g$ contain exactly one 1.

If $g = 0$, then $\chi_r(P_{n_1}) = \chi_r(P_{n_2})$. Then by Lemma 1.3, $\chi_r(S_{n_1,n_2,\ldots,n_r}) = \chi_r(P_{n_1}) + 1$, a contradiction.

Suppose $0 < g \leq \chi_r(P_{n_1}) - 1$. Then, since f must use label $\chi_r(P_{n_1})$ on the branch P_{n_1}, and the largest label can only be used once, $f(m) < \chi_r(P_{n_1})$, where m is the middle vertex. When optimally ranking $S_{n_1,n_2,\ldots,n_r}$, the vertex with the highest label of any branch can be placed as close to the middle vertex as possible while still maintaining optimality. Therefore, without loss of generality, assume that f has such property.

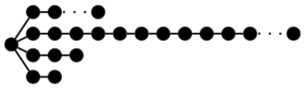
Now, follow the process defined earlier to create the subdivided star S^{g+1} and matrix C^{g+1} whose first g columns contain zeros and the remaining $\chi_r(P_{n_1}) - g$ columns are the same as that of C. Note that in this process, the rank number of the largest branch in S^{a+1} is one less than the rank number of the longest branch in S^a where $1 \leq a \leq g$ (because the first g columns of C have exactly one 1). Thus the longest branch of S^g will have rank number $\chi_r(P_{n_1}) - g + 1$, and thus $f(m) < \chi_r(P_{n_1}) - g + 1$. However, the column C_{g+1} has at least two 1′s, which means that there are at least two branches of S^{g+1} with rank number $\chi_r(P_{n_1}) - g$. Then $f(m) \geq \chi_r(P_{n_1}) - g + 1$, a contradiction.

If every column of C has exactly one 1, then using the above process we get $f(m) < 1$ which is a contradiction.

Thus there is an i such that $C_i = [0,0,\ldots,0]^T$ and $\forall\ j < i$, C_j contains only one 1. □

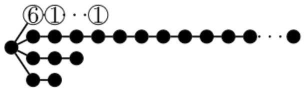
3. Graphs with large rank numbers

We will first characterize graphs with rank number $n - 1$, where n is the order of the graph. In this section, we use the notation $H \subseteq G$ to mean that H is a subgraph of G.



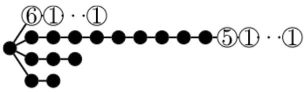
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 2. $S = S^1 = S_{32,25,3,2}$ with corresponding matrix $C = C^1$.



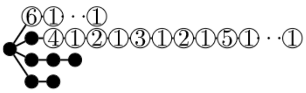
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 3. $S^2 = S_{25,3,2}$ with corresponding matrix C^2.



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 4. $S^3 = S_{9,3,2}$ with corresponding matrix C^3.



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 5. $S^4 = S_{3,2,1}$ with corresponding matrix C^4.

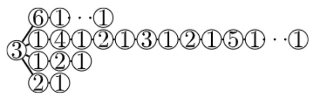


Fig. 6. $\chi_r(S_{32,25,3,2}) = \chi_r(P_{32}) = 6$.

Lemma 3.1. *Let G be a graph of order n, and let G_1 be a subgraph of G such that $|V(G_1)| = n_1$. If $\chi_r(G_1) \le r$ then $\chi_r(G) \le r + (n - n_1)$.*

Proof. Let f be a ranking of G_1 such that $|f| = r$. Now extend f to a ranking of G by assigning distinct labels from the set $\{r+1, r+2, \ldots, r+(n-n_1)\}$ to vertices in $V(G) - V(G_1)$. $\square$

Theorem 3.2. $\chi_r(K_n - G) = n - 1$ *if and only if* $K_n - G \ne K_n$, $K_3 \not\subseteq G$, *and* $C_4 \not\subseteq G$.

Proof. Let $K_n - G \ne K_n$, $K_3 \not\subseteq G$, and $C_4 \not\subseteq G$. Since $K_n - G \ne K_n$, $\chi_r(K_n - G) \le n - 1$.

Consider a ranking f of $K_n - G$. Suppose there exist vertices $a_1, a_2, a_3 \in V(K_n - G)$ such that $f(a_1) = f(a_2) = f(a_3) = a$. Then $\{a_1, a_2, a_3\}$ is an independent set in $K_n - G$, and hence form a K_3 in G, which is a contradiction. Therefore, no label can appear three times or more under f.

Now, suppose there are vertices a_1, a_2, b_1, b_2 such that $f(a_1) = f(a_2) = a$ and $f(b_1) = f(b_2) = b$, and without loss of generality, assume that $a > b$. Then $a_1a_2 \in E(G)$ and $b_1b_2 \in E(G)$ since f is a ranking of $K_n - G$. Moreover, since $K_3 \not\subseteq G$ and $C_4 \not\subseteq G$, there exists at most one edge in G that has endpoints a_1 or a_2 and b_1 or b_2. This implies that in $K_n - G$ either $a_1b_1a_2$ or $a_1b_2a_2$ is a path that violates the requirements of a ranking, a

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contradiction. Therefore two labels cannot appear more than once under f. Since no label can be used three times and two labels cannot be used more than once, $\chi_r(K_n - G) \geq n - 1$. Therefore, $\chi_r(K_n - G) = n - 1$.

Now, suppose $\chi_r(K_n - G) = n - 1$. Then $K_n - G \neq K_n$ as $\chi_r(K_n) = n$. If v_1, v_2, and v_3 form K_3 in G, then $\{v_1, v_2, v_3\}$ form an independent set in $K_n - G$ and thus, by Lemma 1.4, $\chi_r(K_n - G) \leq n - 2$, a contradiction. Lastly, assume that $C_4 \subseteq G$. We know that $\chi_r(\overline{C_4}) = 2$, and thus, by Lemma 3.1, $\chi_r(K_n - G) \leq n - 2$, a contradiction. □

From Theorem 3.2 we get the following corollaries.

Corollary 3.3. *$\chi_r(G) = n - 1$ if and only if $G \neq K_n$, $K_3 \not\subseteq \overline{G}$, and $C_4 \not\subseteq \overline{G}$.*

Corollary 3.4. *For a tree T on $n > 1$ vertices, $\chi_r(\overline{T}) = n - 1$.*

We now look at the characteristics of graphs with rank number $n - 2$, where n is the order of the graph. Let $B = W_5 - uv$, where W_5 is a wheel on 5 vertices with center vertex u and $v \neq u$ is a vertex in W_5. Note that $\overline{B}$ has two components, a P_3 and a P_2 and hence we have the following observation.

Observation 3.5. $\chi_r(\overline{B}) = 2$.

Since the graph $\overline{K_{3,3}}$ has two K_3 as components, we have the following observation.

Observation 3.6. $\chi_r(\overline{K_{3,3}}) = 3$.

Theorem 3.7. *$\chi_r(K_n - G) = n - 2$ if and only if $K_3 \subseteq G$ or $C_4 \subseteq G$, $K_4 \not\subseteq G$, $B \not\subseteq G$ and $K_{3,3} \not\subseteq G$.*

Proof. Suppose that $\chi_r(K_n - G) = n - 2$. If $K_3 \not\subseteq G$ and $C_4 \not\subseteq G$, then either $\chi_r(K_n - G) = n$ or, by Theorem 3.2, $\chi_r(K_n - G) = n - 1$. If $K_4 \subseteq G$, then $K_n - G$ contains an independent set of size 4, and thus, by Lemma 1.4, $\chi_r(K_n - G) \leq n - 3$. If either $B \subseteq G$ or $K_{3,3} \subseteq G$, then by Observations 3.5 and 3.6 and by Lemma 3.1, we get $\chi_r(K_n - G) \leq n - 3$. Thus if $\chi_r(K_n - G) = n - 2$ then $K_3 \subseteq G$ or $C_4 \subseteq G$, $K_4 \not\subseteq G$, $B \not\subseteq G$ and $K_{3,3} \not\subseteq G$.

Let $K_3 \subseteq G$ or $C_4 \subseteq G$, $K_4 \not\subseteq G$, $B \not\subseteq G$ and $K_{3,3} \not\subseteq G$. Since $K_3 \subseteq G$, or $C_4 \subseteq G$, as discussed in the proof of Theorem 3.2, we can find a ranking f of $K_n - G$ with $|f| \leq n - 2$. Thus $\chi_r(K_n - G) \leq n - 2$.

Suppose that $\chi_r(K_n - G) \leq n - 3$. Let f be an optimal ranking of $K_n - G$. Then, since $|f| \leq n - 3$, one of the following three cases must occur.

Case 1: There exist vertices $v_1, v_2, v_3, v_4 \in V(K_n - G)$ such that $f(v_1) = f(v_2) = f(v_3) = f(v_4) = 1$. This means that these vertices form a K_4 in G, a contradiction.

Case 2: There exist vertices $x_1, x_2, x_3, y_1, y_2 \in V(K_n - G)$ such that $f(x_1) = f(x_2) = f(x_3) = 1$ and $f(y_1) = f(y_2) = 2$. Then the vertices x_1, x_2, x_3 must form a K_3 in G and the vertices y_1, y_2 must be adjacent in G. Also, each of x_1, x_2, x_3 must be adjacent to either y_1 or y_2 in G. (If x_i were adjacent to neither y_1 nor y_2 for some $1 \leq i \leq 3$ then the path $y_1 x_i y_2$ would exist in $K_n - G$, making f not a ranking.) If each of the x_i's is adjacent to the same y_j for some $1 \leq j \leq 2$ then x_1, x_2, x_3, and y_j form a K_4 in G. If, without loss of generality, $x_1 y_1 \notin E(G)$, $x_1 y_2 \in E(G)$, $x_2 y_1 \in E(G)$, and $x_3 y_1 \in E(G)$, then the subgraph of G induced by x_1, x_2, x_3, y_1, and y_2 contains a B with x_2 as the center vertex. However, we assumed that $K_4 \not\subseteq G$ and $B \not\subseteq G$.

Case 3: There exist vertices $x_1, x_2, y_1, y_2, z_1, z_2 \in V(K_n - G)$ such that $f(x_1) = f(x_2) = 1$, $f(y_1) = f(y_2) = 2$, and $f(z_1) = f(z_2) = 3$. Then $\{x_1x_2, y_1y_2, z_1z_2\} \in E(G)$. Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, and $V_3 = \{z_1, z_2\}$. Since f is a ranking, paths such as $y_1 x_1 y_2$ or $z_1 x_1 z_2$, or $z_1 y_1 z_2$ do not exist in $K_n - G$, and thus in G there must be at least two edges between every V_i and V_j, for all $1 \leq i < j \leq 3$. Moreover, the subgraph of $K_n - G$ induced by $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ must be disconnected, since the highest label 3 is used twice in the subgraph. This means $\overline{J}$ is disconnected, where J is the subgraph of G induced by $\{x_1, x_2, y_1, y_2, z_1, z_2\}$. Thus J must contain one of the four graphs on six vertices given in Fig. 7.

Note that the first three graphs contain B as a subgraph, and the last graph is a $K_{3,3}$. This is a contradiction, as we assumed that $B \not\subseteq G$ and $K_{3,3} \not\subseteq G$.

Therefore, $\chi_r(K_n - G) \geq n - 2$, and thus $\chi_r(K_n - G) = n - 2$. □

Corollary 3.8. *$\chi_r(G) = n - 2$ if and only if $K_3 \subseteq \overline{G}$ or $C_4 \subseteq \overline{G}$, $K_4 \not\subseteq \overline{G}$, $B \not\subseteq \overline{G}$ and $K_{3,3} \not\subseteq \overline{G}$.*

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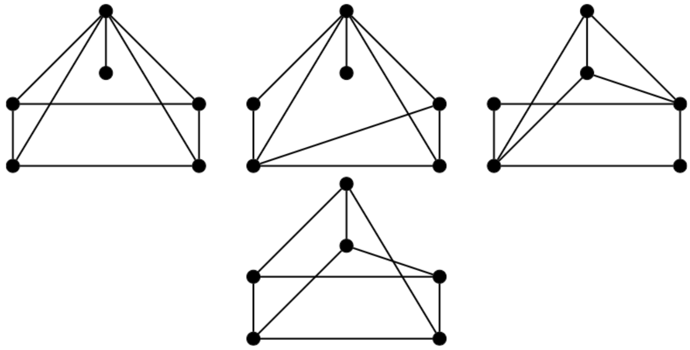

Fig. 7. J must contain one of these four graphs.

4. Conclusion

Finding the rank number of a general graph is known to be extremely difficult. In this paper, we identified graphs with rank numbers $n-1$ and $n-2$, where n is the order of the graph. The idea we used for finding $\chi_r(K_n-G)$ would still work for classifying graphs with smaller rank numbers. However, the number of labeling schemes that needs to be considered grows exponentially as the rank number decreases. Each of the labeling scheme produces multiple forbidden subgraphs, and hence this method, while not incorrect, will not be feasible for classifying graphs with smaller rank numbers. An interesting related question would be to identify the minimum number of edges required in an n vertex graph G such that rank number of G is either $n-1$ or $n-2$. We also characterized subdivided stars based on their rank numbers, thus establishing the rank number of all subdivided stars. Rankings of some other classes of trees have also been studied by others, for example see [7,10,14], however, the rank number of an arbitrary tree has not been established.

References

[1] C. Leiserson, Area efficient graph layouts for VLSI, in: Proc. 21st Ann IEEE Symp. FOCS, 1980, pp. 270–281.

[2] A.V. Iyer, H.D. Ratliff, G. Vijayan, On an edge ranking problem of trees and graphs, Discrete Appl. Math. 30 (1) (1991) 43–52.

[3] A.V. Iyer, H.D. Ratliff, G. Vijayan, Optimal node ranking of trees, Inform. Process. Lett. 28 (5) (1988) 225–229.

[4] H.L. Bodlaender, J.S. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Müller, Z. Tuza, Rankings of graphs, in: Graph-Theoretic Concepts in Computer Science (Herrsching, 1994), in: Lecture Notes in Comput. Sci., vol. 903, Springer, Berlin, 1995, pp. 292–304.

[5] J. Ghoshal, R. Laskar, D. Pillone, Minimal rankings, Networks 28 (1) (1996) 45–53.

[6] H. Alpert, Rank numbers of grid graphs, Discrete Math. 310 (23) (2010) 3324–3333.

[7] B. Blake, E. Field, J. Jacob, Rank numbers of graphs that are combinations of paths and cycles, Involve A J. Math. 6 (3) (2013) 369–381.

[8] E. Bruoth, M. Horňák, On-line ranking number for cycles and paths, Discuss. Math. Graph Theory 19 (2) (1999) 175–197, The Seventh Workshop “3in1” Graphs ’98 (Krynica).

[9] D. Dereniowski, A. Nadolski, Vertex rankings of chordal graphs and weighted trees, Inform. Process. Lett. 98 (3) (2006) 96–100.

[10] S. Hsieh, On vertex ranking of a starlike graph, Inform. Process. Lett. 82 (3) (2002) 131–135.

[11] S. Novotny, J. Ortiz, D.A. Narayan, Minimal k-rankings and the rank number of P_n^2, Inform. Process. Lett. 109 (3) (2009) 193–198.

[12] J. Ortiz, A. Zemke, H. King, D. Narayan, M. Horňák, Minimal k-rankings for prism graphs, Involve 3 (2) (2010) 183–190.

[13] P. Richter, E. Leven, A. Tran, B. Ek, J. Jacob, D.A. Narayan, Rank numbers for bent ladders, Discuss. Math. Graph Theory 34 (2) (2014) 309–329.

[14] E. Sergel, P. Richter, A. Tran, P. Curran, J. Jacob, D.A. Narayan, Rank numbers for some trees and unicyclic graphs, Aequationes Math. 82 (1–2) (2011) 65–79.

[15] B.C. Jacob, J. Jacob, l_p-Optimal rankings and max-optimal rankings are different, Graphs Combin. 33 (6) (2017) 1473–1483.

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