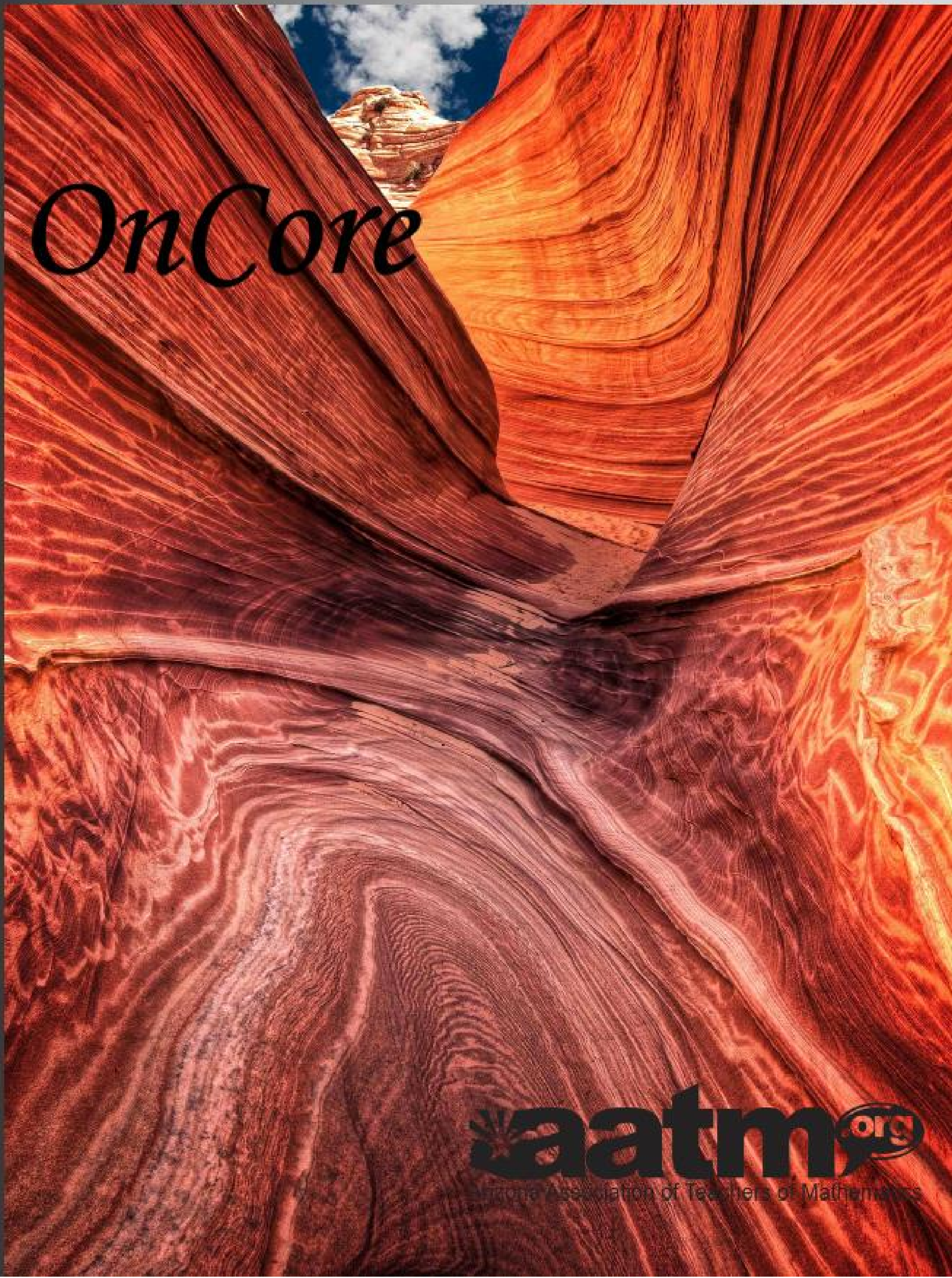
Arizona Association of Teachers of Mathematics

Fall 2018

OnCore



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Arizona Association of Teachers of Mathematics

Dear AATM Colleagues, October 2018

This fall 2018 issue of *OnCore* contains a variety of activities and explorations for students, K-12. *Using Literature to Develop Young Students' Measurement and Graphing Talents* focuses on having students explore ways to measure lengths, represent those measurements in different types of graphs, and relate data collected to the contexts. Motivation for the activities is *How Big is a Foot?* the story about a king who desires to have a bed made for his queen and there are disputes about the appropriate measurement units to employ. *Place It, Build It, Compare It, Calculate It* provides grades 5-10 students with opportunities to construct multi-digit numbers to match specifications. By doing so, students gain greater insight into the relationship between a digit's placement in a multi-digit number and its value. Individual, partner and group activities involve use of the same digits to construct and compare different numbers, and compute with those numbers to produce the greatest (least) product or quotient. *Leap Frog is More Than a Game* uses the familiar Leap Frog game as a venue for students, grades 3-12, to explore number patterns resulting from number of moves, and functions in terms of the relationship between the number of players and the number of moves. Throughout play of the game, students represent data in t-charts and graphs, compare their data, and formulate conclusions. *Make It Simpler* gives new meaning to this problem solving strategy, usually implemented by using simpler numbers, to the use of different contexts. To illustrate these "simplify" methods, a problem is presented and accompanied with others in which the context is changed and graphical representations are presented. *Archimedes, Euler, and Nursing Cats*: *Rethinking the Arithmetic Mean* begins by exploring the concept of mean as "fair sharing," and then considers other examples in the contexts of, for example, rates, the physics of balance, and geometric constructions. Several grand examples of ways in which mathematical concept development is enhanced through the use of applications are provided. This is particularly useful for students in high school. Our final article, *Inside Interactive Engaged Problem Solving in Precalculus: Content and Classroom Moments,* describes a 1-credit course to accompany traditional 3-credit Precalculus courses developed for community college students in primarily Hispanic Serving Institutions. How the course may be adapted and used to support the transition from high school to college mathematics is described. Three Precalculus problems are presented along with strategies to enable peer exploration and discussion.

We hope that you find these articles helpful.
Please consider contributing an article to our spring 2019 *OnCore* magazine.

Very best wishes for a happy fall,

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Using Literature to Develop Young Students' Measurement and Graphing Talents

Mi Yeon Lee and Dionne Francis

Abstract

The Common Core State Standards for Measurement and Data for Grades 1 and 2 (1.MD.2; 1.MD.4; 2.MD.9; 2.MD.10; NGA & CCSSO, 2010) focus on the generation of data by measuring lengths of objects and representing those measurements using different types of graphs. In this article, based on children's literature, we describe activities that require students to measure, collect data, represent those data in graphs, and describe how they produced the graphs. Additional strategies are proposed to foster students' discussions about graphs.

Using Children's Literature to Generate Two Measurement Data Point Representations

Children's books often provide contexts that support students' understanding of measurement relationships, the nature of the data collected, and the relationship of data to the contexts (Clarke, 2001; Cross et al., 2012). This is the story of our work with grade 1 and 2 students.

On Day 1, we began our activity by focusing on the popular children's book, *How Big is a Foot*? (Myller, 1991). In the book, the king asks an apprentice to make a bed for his queen. When determining the size of the bed, a misunderstanding occurs because the king used his foot size to determine the length of the bed and the apprentice used his foot size. We paused at the page that ends with, "Why was the bed too small for the queen?" Some students were genuinely puzzled while others stated that the bed was too small because different foot sizes were used to measure the bed.

To test that conjecture, we first modeled the situation. We had students create their own footprints by tracing their feet on pieces of paper, cutting out the tracings, and then measuring the length of their footprints using paper clips as the measurement unit. Some students measured the perimeter of their footprints or a curvy line along a side of the footprints (See top row of Figure 1). Other students placed paper clips in the middle of their footprints (See bottom row of Figure 1). These measurement strategies are very common in the early grades as students' understanding of the attribute of length (i.e., lengths span fixed distances) and of unit iteration (i.e., tiling the length without gaps or overlaps) are emerging (Sarama & Clements, 2009).

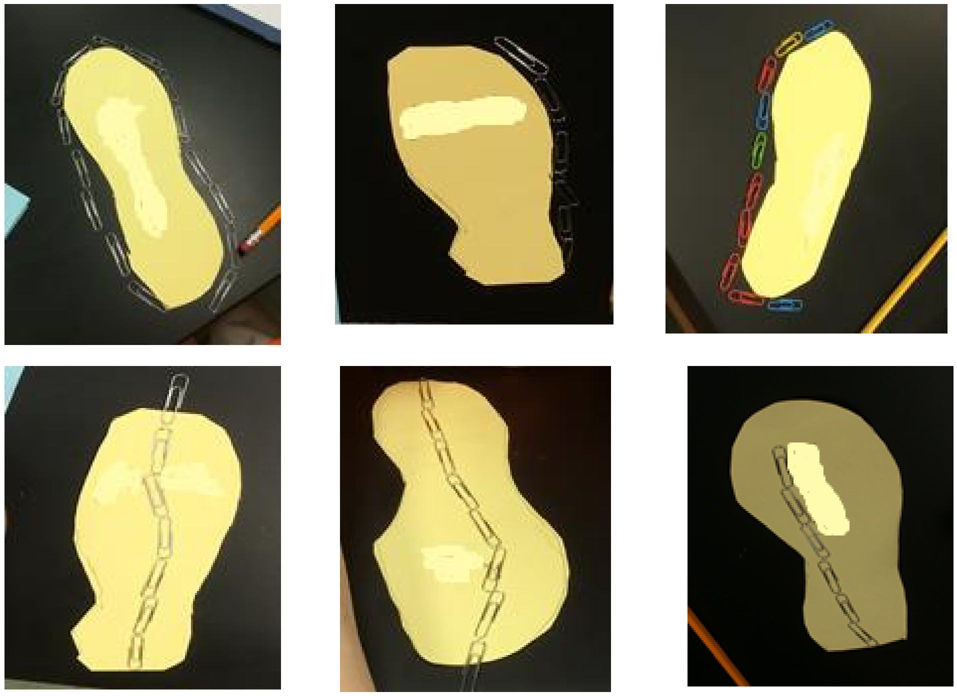

Figure 1. Students' Methods of Measuring Length

On Day 2, we established consensus on which dimension of the footprint (length, width, or perimeter) should be used, and how paper clips should be positioned to measure the footprints. Five key principles about measuring length emerged, including *no overlaps*, *no gaps*, *straight line*, *end-to-end*, and *measuring tip to tip* (Sarama & Clements, 2009). Through exploration, students determined that without adhering to these principles, it's possible to get different measurements for the same object. As one student explained, "If there is overlap, we use more clips than we need. So that's not the real length of the footprint." Students then re-measured their footprints by lining up paper clips vertically, from a bottom line to a top line (See Figure 2), and recorded these measurements in their journals.

Figure 2. Measuring a Footprint with Paper Clips

We then distributed same-size paper feet and sticky notes on which students wrote their names and measurements (See Figure 3). Paper feet had to be of the same size so as not to distort the picture graph.

Figure 3. A Paper Foot and Sticky Note

Creating a Picture Graph

On Day 3, by using the paper foot and sticky note, including their foot sizes, students constructed a picture graph and a bar graph. First, students were asked to state their foot sizes in paper clip measurements in order to identify the values to be entered on the horizontal axis. Because continuous data (like length) cannot be directly used to create picture graphs or bar graphs, we used each paper clip as a discrete unit and represented foot sizes in terms of number of paper clips, such as a 6, 7, 8, or 9-paper clips. Next, students placed their paper feet (with their names in print) above the numbers corresponding to their foot sizes (See Figure 4).

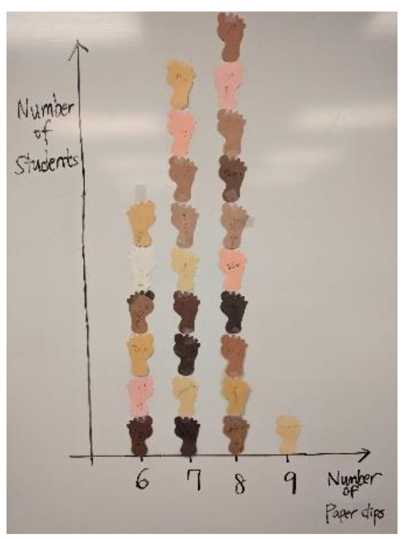

Figure 4. Footprint Picture Graph

As noted, we had prepared copies of the same-size paper foot for graphing purposes, and explained to the students why standardizing the footprint was necessary (considering why they all need to use the same size footprint can lead to a fruitful discussion about misleading graphs). After students had placed their "named" footprints, we asked them what the horizontal axis and the vertical axis represent on the graph. Students could label the horizontal axis (Number of Paper Clips) but struggled to label the vertical axis, as there was no information on the axis. We guided the students to count the number of paper feet in each category of foot size, and showed them how to represent that number of students on the vertical axis. After the picture graph (Figure 4) was created, we engaged students in discussion about how data on the graph could be used to identify the foot size that was the shortest, longest, most common, and least common.

Creating a Bar Graph with Sticky Notes and Using Data to Make Decisions

On Day 4, we used the sticky notes to create a bar graph. We created the axes and asked students to identify the appropriate places to attach their sticky notes on the graph (See Figure 5).

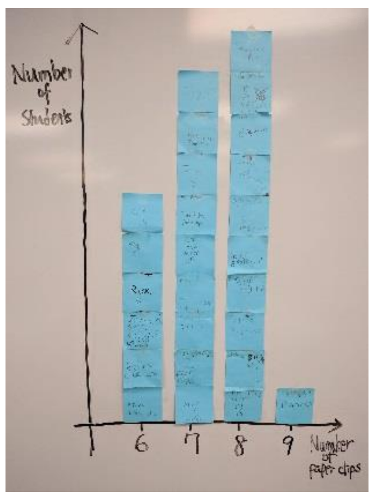

Figure 5. Bar Graph with Sticky Notes

After completing the bar graph, we placed the picture graph and the bar graph side-by-side. We asked students to analyze the graphs by relating them to the story: "Which foot size should be the king's foot?" Trying to answer this question using the data from the graphs launched a very interesting discussion about how to summarize data and the criteria students should use to identify the king's foot. Some students claimed that either the longest or the smallest foot size should be the king's foot. Other students maintained that the most common or least common foot size should be a king's foot. Only one student thought that the "middle" foot size should be the king's foot. Based on the picture of the king as a tall man, most students thought that the maximum foot size would be most suitable. This discussion led students to consider why they would use one foot size over another. Such a discussion is important because it begins to lay the conceptual foundation for the understanding of measures of center, a topic addressed in upper elementary and middle school (NGA & CCSSO, 2010).

Testing Students' Conjectures

On Day 5, after finalizing the foot size for the king, students worked in groups of six to create different-size beds for the queen, by using multiple footprints and tape (See Figure 6). Each of the four groups of students created a bed using the king's dimensions – one used the king's foot, one used the smallest foot size, one used the least common foot size, and the last used the most common foot size. Then students compared the beds and talked about why three of the beds did not fit the queen. Students were encouraged to justify their conclusions, communicate their ideas to others, and respond to arguments of others based on the evidence.

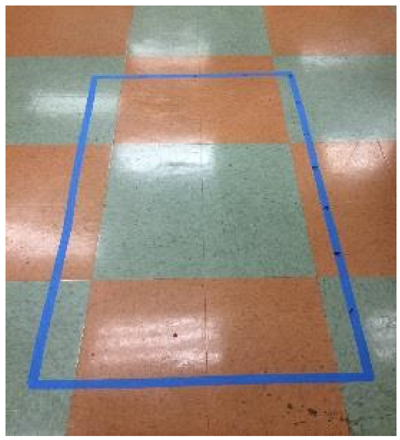

Figure 6. Example of Final Bed

Conclusion

To build statistical literacy, students should be able to do more than create and read graphs. They also should be able to interpret information in the graph, and draw on existing knowledge to make inferences (Curcio, 2010). Accordingly, we conclude our paper by suggesting three strategies to foster discussion that builds statistical literacy: 1) Begin discussion with open-ended questions. 2) Encourage students to compare differently configured graphs. 3) Allow students to pose their own questions.

By asking students what they notice about graphs they have constructed, teachers provide opportunities for students to analyze the data from their own perspectives. According to Whitin and Whitin (2003), teachers tend to begin discussion about graphs with closed questions, such as "how many students have footprints that are six paper clips long?" Questions like this draw students' attention to specific numerical information represented in the graph, thereby restricting their view of the data as determined by the teacher.

Second, teachers should encourage students to compare different graphs that represent the same data. Both *picture graphs* and *bar graphs* are used when data are discrete (e.g., number of countable objects) or categorical (e.g., gender, favorite color), and data on the graphs are organized according to discrete whole numbers or categories.

By comparing two graphs, students can observe that a picture graph, with its symbols, can usually be converted into a bar graph, which is a more universal graphic representation. The experience of converting a picture graph to a bar graph, along with comparing the two graphs, can enable students to see the process of transforming a semi-concrete representation into a more abstract representation (Curcio, 2010). This may support the development of students' abilities to create bar graphs as well as enhance their understanding of how information is preserved in different representations.

Finally, we recommend that teachers give students opportunities to develop their own questions that can be answered by interpreting the graphs. According to Bamberger, Oberdorf, and Schultz-Ferrell (2010), "When students learn to formulate questions a graph is supposed to answer, they organize and display the data in a more meaningful way" (p. 146).

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Place It!
Build It! Compare It! Calculate It!

Sarah Schaefer

Abstract

Do students understand the relationship between each digit and its value as they compose and decompose numbers? Students' understanding of place value begins during the building of multi-digit numbers. Providing students with activities in which they explore and generalize how a digit's position is related to its value, and comparing numbers, will build number sense.

The position of a digit and understanding its magnitude and effect on an operation, such as addition, subtraction, multiplication or division, is foundational in developing number sense (NGA & CCSSO, 2010). The following activities may be adapted for use with students in Grades 1-5.

Build It!

Individual Activity

Students are given 10 cards, one each with the digits 0-9. Cards are turned over. To play, students turn over one card at a time and build a multi-digit number. Depending on the grade level, numbers may have from two to five-digits. For example, consider a four-digit number.

I. The student turns over four cards and arranges them to form a four-digit number. Once a digit is placed, it cannot be moved. After all digits are placed, the student is directed to record the number in three ways: standard form, word form, and expanded form.

II.

3	4	7	2

Standard form: 3 4 7 2
Word form: Three-thousand, four-hundred, seventy-two.
Expanded form: 3000 + 400 + 70 + 2

III. Students are directed to use the same digits to make three more four-digit numbers (4732, 3724, 7432) and then students arrange those four numbers from least to greatest (3472, 3742, 4732, 7432).

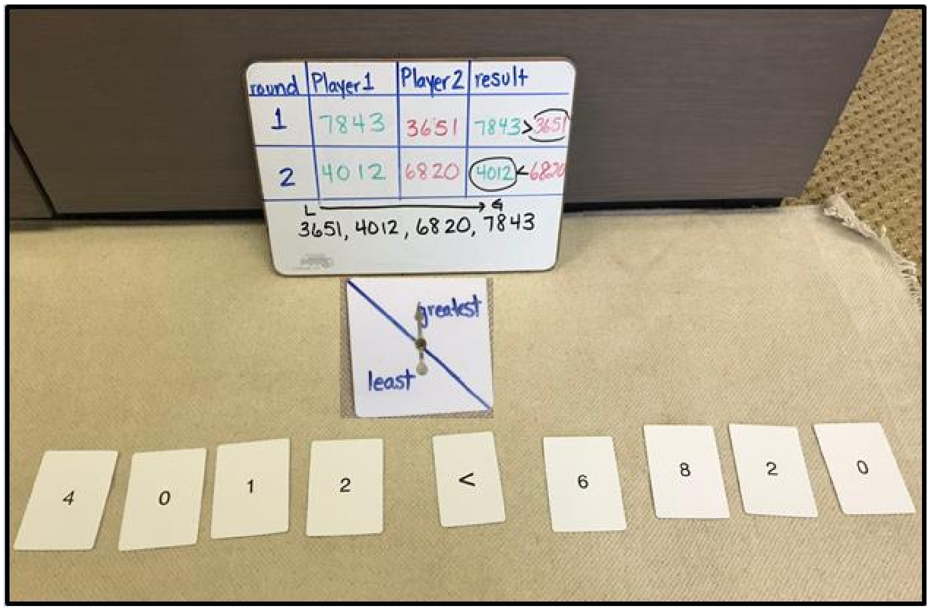
Questions to ask:

"What is the greatest number that can be made with the cards?" (7432)
"What is the least number?" (2347)

Compare It!

Partner Activity:

Students take turns choosing a digit card and placing the digit in one of four places (thousands, hundreds, tens, ones) until no places remain empty. Once the four-digit numbers are formed, students compare numbers. Using a spinner marked *greatest* and *least*, students spin the spinner to determine whether the greatest or least number will get a point. For example, if the spinner lands on *least*, the player with the least number wins a point. Students record each round on a white board. The method of recording can vary based on the grade level. Students should use symbols (> , <) to communicate their results.

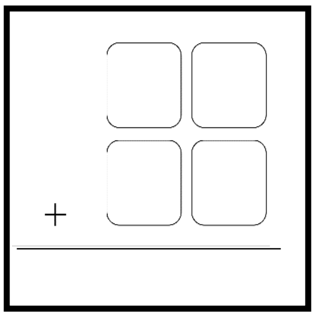


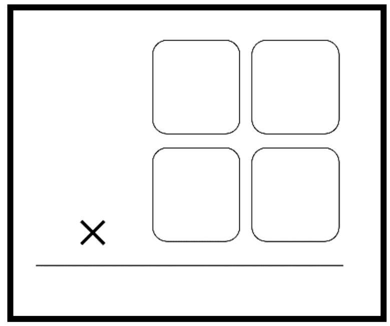
After two rounds, students are directed to record the four numbers in order, least to greatest.

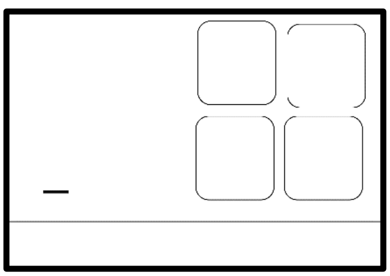
Calculate It!

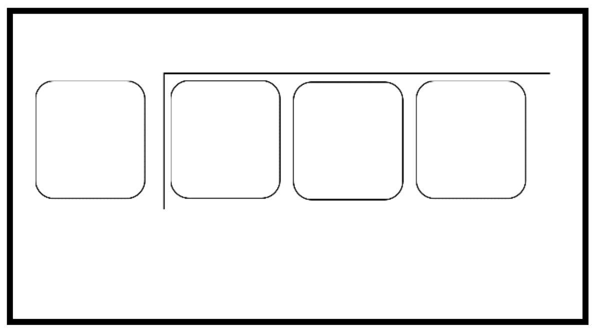
Individual Activity:

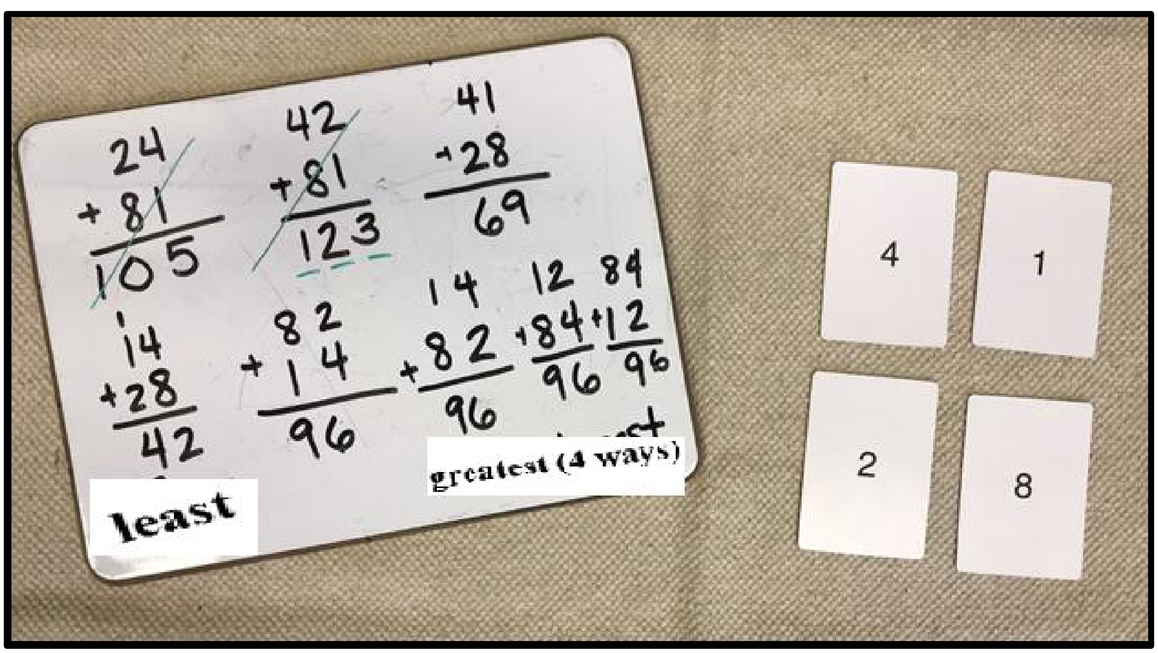
Students are given a template similar to the ones shown below. The format can be structured to include adding two, two-digit numbers that result in a two-digit sum where no digit can be repeated. Once a student picks the digits, you may ask how many different problems can be formed using the same digits and the same template? What is the greatest product? Greatest quotient? Least difference? Students record their work on white boards.





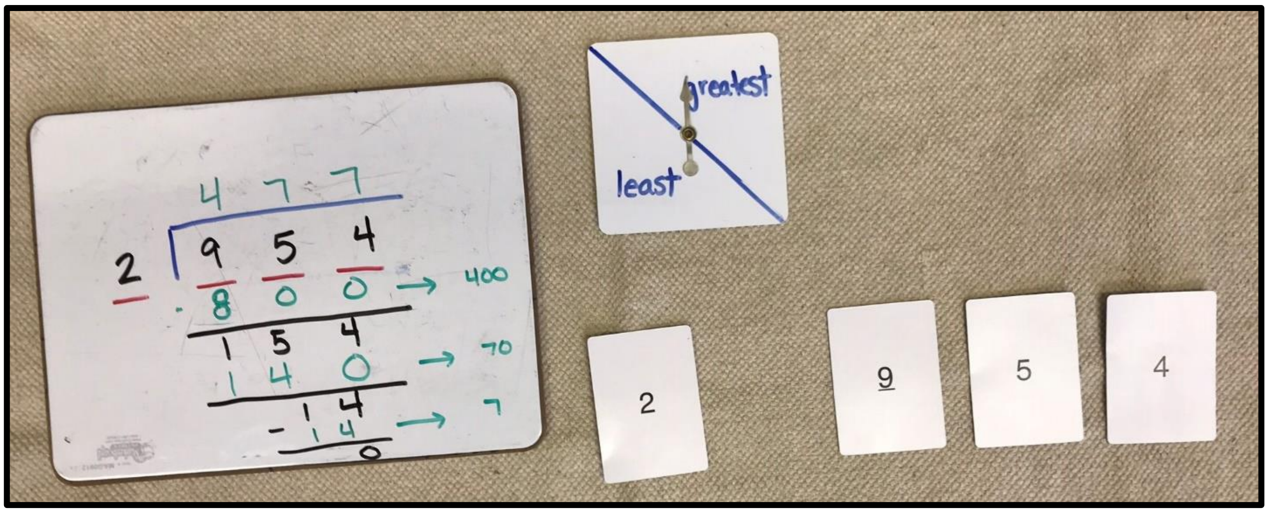






Partner Activity:

Using a greatest/least spinner, students spin the spinner to establish a goal for the round. If the spinner lands on greatest, the player with the greatest result wins a point. Students take turns choosing a card and placing the digit. Once the problem is formed, students compute and compare results. The student with the greatest result is awarded a point. If a student is trying to obtain the greatest product, determining placement of lesser and greater numbers is obviously an important consideration. After the round is played, ask: "What is the greatest possible result that can be obtained with your digits? Least?"



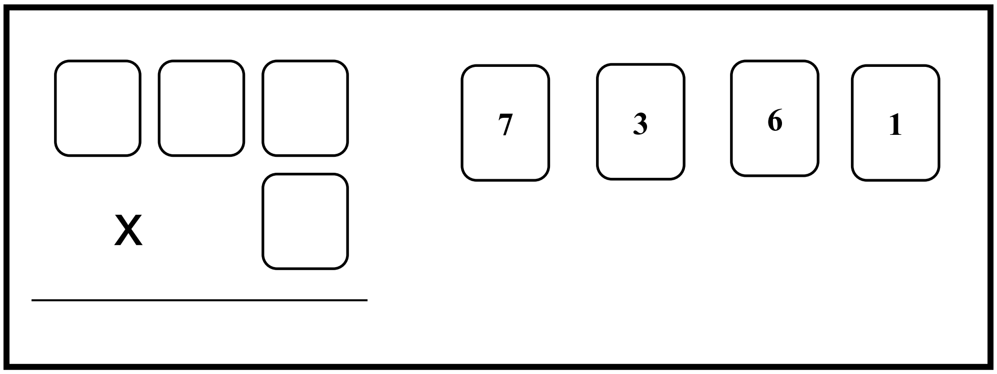
The following **Place It!** activities provide students with opportunities to create their own problems. Some students may work through fewer problems while others create generalizations based on digit placements.

The number of cards in a deck, the digits that are used, the operation, and the number of digits on the template may be altered. Placing requirements include: digits cannot be repeated or the result must be within a predetermined range (e.g. sum must be between 200 and 300). Those requirements can add different layers of complexity.

Once students have had ample opportunities to explore these activities, they should be expected to reflect on the activities by recording their thoughts in journals. Ask students to comment on relationships between placement of numbers and outcomes obtained.

Journal Prompt

Spiky picks the cards shown below and is asked to build a problem that will produce the greatest possible product. Which number should he place first? Why? What is the greatest possible product that can be obtained?

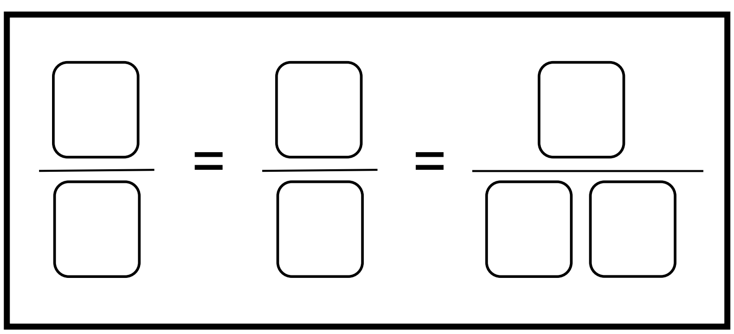


Journal reflections will provide the teacher with insight into a student's thinking. Is the placement of numbers random or did the student systematically place the numbers? Does the student understand that the location of a digit will affect the result? Does the student understand the digit placement required to produce the greatest or least result?

Understanding place value is central in the development of number sense and the growth in reasoning skills, particularly with regard to multi-digit computations. Activities such as these can be used throughout the school year to deepen understanding while providing problem-solving venues for all students. You can even extend the idea to fractions. Try It!

Fraction Challenge

Is it possible to use numbers to create a true statement, where no number is repeated?



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Leap Frog is More than a Game!

Janet M. Herrelko

Abstract

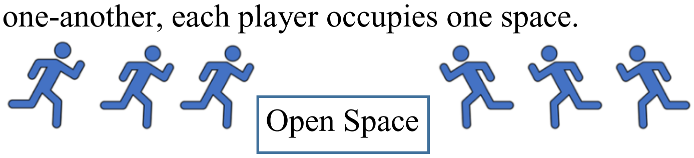
The game of Leap Frog is used as a method for engaging students in the solution of problems by physically acting it out, recording data, drawing conclusions from the data to produce a mathematical generalization. The game can be played with students in grades 3 to 12. For each group of grades, actions are described to push students to higher levels of thinking and problem solving.

Since the turn of the century, experiential learning, including games, has been promoted by James, Lewin, Dewey, Piaget, Vygotsky and other educational theorists (Kolb, 2015). The Leap Frog game is designed to provide students, grades 3 through 12, with exciting mathematics, through explorations involving number patterns, t-charts, graphs (linear, quadratic), and equations. In Professional Development programs, groups of teachers have enjoyed solving the problems as well as students who play Leap Frog.

The Mathematical Leap Frog Game

Instructions

1. Ask for two volunteers, A and B.
    - a) A will record the moves done by the active player.
    - b) B will move the open space (use a piece of colored paper as the open space) to track where it is located after each move.
2. Place other members of the class in two same-size teams.
    - a) Provide some form of identification for each team, as for example, different caps, Groucho Marx glasses, or Pippi Long Stocking headband braids, one color for each group.
    - b) Select a team leader who will be first in line.
    - c) Have students line up in single file to the left (one team) and to the right (other team) of the space tracker volunteer who stands in the middle at the start of the game.
    - d) Teams face one-another, each player occupies one space.



Rules of the Game

1. You may only move forward.
2. There are two types of forward moves you can make:
   a. The JUMP (J). On a turn, you may jump over only one space into an empty space (like in Leap Frog and Checkers games).
   b. The SLIDE (S). You move forward one space to the empty space.
3. As team members move (one by one) from their side to the opposite side, the numbers of moves and types of moves are recorded. Teams are trying for the fewest number of moves.
4. The team with the fewest number of moves, to get all team members from one side to the other, is the winner.
5. Whole class develops a report and presents data recorded: number of moves, types of moves, and number of team members.

Reflection

1. Examine each team's report of number of moves and types of moves.
2. Are there any patterns in the number of moves made by team players?

One caution is the size of teams. If you can keep each team to no more than five students in the earlier grades (3 - 8), all students can participate in the solution of the problem. At the high school level, a maximum of 10 students per team works well.

Grades 3 Through 5

Students do the leap frog moves of jumping and sliding. The teacher or classroom aide records the number of moves (slides and jumps), and directs the jumps. For students who are tall, the "jump" can be made by walking around a player.

The teacher can help students identify patterns between number of players and number of moves by beginning the game with teams of one player on each side. Record the number of moves it takes to get the teams to the opposite sides. Then, increase the number of players by one and play the game with two players on each team. Record number of moves. Repeat until you have five players on each team. Note: Once you have teams of three players or more, the pattern relating number of moves to number of players, becomes more obvious, See Table 1.

Table 1. Recording Chart – Answer Key

Game	Number of Players per Team (*n*)	Number of Moves (M)	Types of Moves
1	1	3	SJS
2	2	8	SJSJJSJS
3	3	15	SJSJJSJJJSJJSJS
4	4	24	SJSJJSJJJSJJJJSJJJSJJSJS
5	5	35	SJSJJSJJJSJJJJSJJJJJSJJJJSJJJSJJSJS

The teacher should lead students in discussion of the patterns in each column: 1) The number of players increases by 1, and 2) the number of moves changes at a different rate.

Grades 6 Through 9

Follow the same procedure described for Grades 3-5, and then:

Step 1: Start by having students create the data that produce the chart.

Step 2: Using the chart numbers, graph the Team Members list, and create a second graph using the Number of Moves. Discuss the shapes of the resulting graphs.

Step 3: Examine the recorded data to determine the number of moves needed for n players on each team. This will result in the algebraic representation for the generalization of: $M = n^2 + 2n$. This quadratic function will be described in the Extension below.

Grades 10 Through 12

Follow Steps 1-3 shown for Grades 6-9 and include Step 4.

Step 4: Extension

a) Number of Players Column: When you subtract the number of players from Game 1 to Game 2, the difference is 1. Repeat that procedure with number of players from Games 2 to 3, Games 3 to 4, Games 4 to 5.
b) Number of Moves Column: Subtract number of Moves 2 from Moves 1, and list the difference. Continue subtracting until you finish with Moves 5 (see Table 2).
c) Construct graphs: Have students examine their graphs and see what connections can be made with the final uniform differences of 1 (producing a linear graph) and the differences of 2 (producing a quadratic graph).
d) Challenge students: Students defend why these subtraction results produce a function with that difference as the exponent, and present their rationales to the class. After each presentation, the class should vote as to whether the presented rationale was convincing or not.
e) Evaluation rubric: A rubric could be created to be used by classmates to enable evaluation of the presentations.

Table 2. Number of Players and Number of Moves

Number of Players per Team and Differences			Number of Moves and Differences		
Game 1	**1**		**3**		
		1		**5**	
Game 2	**2**		**8**		**2**
		1		**7**	
Game 3	**3**		**15**		**2**
		1		**9**	
Game 4	**4**		**24**		**2**
		1		**11**	
Game 5	**5**		**35**		

Note: Activities like Leap Frog engage students in the Mathematics Practices of the Common Core State Standards (NGSA & CCSSO, 2010) and the NCTM Teaching Practices (2014). Those are listed in Table 3.

Table 3. Mathematical Practices for Students and Teachers

CCSSM - Mathematical Practices	NCTM Mathematical Teaching Practices
1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.	1. Focus learning with mathematics goals 2. Implement tasks that promote reasoning and problem solving. 3. Use and connect math representations. 4. Facilitate meaning-full discourse. 5. Pose purposeful questions. 6. Build procedural fluency from conceptual understanding. 7. Support productive struggle. 8. Elicit and use evidence of student thinking.

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Make It Simpler

William R. Speer

Abstract

This article describes the problem-solving strategy of "Make It Simpler" in a context that goes beyond using more convenient numbers to exploring metaphor and context change. Two challenging problems that do not appear to have clear pathways to solution are used to illustrate how "making it simpler" through metaphor and context shifts can lead to increased discourse and deeper comprehension.

Among the strategies identified as useful tools to use in solving problems are, for example, drawing a picture, making a table, looking for a pattern, or acting it out. A strategy often referred to as Make it Simpler, is frequently illustrated by inserting "easier" numbers in a problem while at the same time eliminating extraneous information. Although this method works in certain situations, there is a better way to make the problem simpler.

For example, consider the following Milk-Syrup Problem.

> There are two containers, one with milk, and the other with chocolate syrup. A certain amount of syrup is poured into the milk and then the mixture is fully stirred. Next, the same amount of the mixture of the "chocolate milk" is poured back into the syrup. The question: Is there more syrup in the milk container, more milk in the syrup container, or the same amount of milk and syrup in each container?

Often, the response is that there is now more syrup in the milk container. This answer is typically justified by an explanation that "pure syrup" was poured in, but that mixed syrup and milk was poured back. A second common answer is that there is no way to tell since we don't know the size of the containers – which, it turns out, is not a relevant factor

Yellow Beads and Blue Beads

Changing the context, while maintaining relationships in the original problem, may help alleviate certain confounding variables. In this case, emphasize the fact that syrup is not milk and milk is not syrup. Instead of containers of milk and syrup, use two jars, one containing 1000 yellow beads and the other 300 blue beads. Here, it is clear that yellow beads are not blue and blue beads are not yellow. If one carries out the action described in the Milk-Syrup problem, it will become clear that placing X (some number) blue beads in the yellow beads jar and mixed, followed by taking a random X beads back to the blue jar will result in the same number of blue in the yellow jar as there are yellow in the blue jar. A number of trials, changing the value of X each time, will provide convincing evidence that since X beads "went over" and X beads "came back," then for every blue bead that did not "come back," a yellow bead must have taken its place.

Boys and Girls Bus Problem

Taking this additional step, consider two buses full of children, one with all boys and the other with all girls. In this case, we will allow for different size buses (as we could have for the syrup/milk and the blue/yellow versions), but now we limit the capacity of the containers – the buses will be full so no extra seats available. The reason for this adjustment will become obvious in the problem set up.

> Two buses on a field trip stop for a break. Bus #1 is filled with girls and Bus #2 is filled with boys. X number of boys get off the Boys' bus and board the Girls' bus. When the driver is ready to leave, he notices that not everyone on the bus has a seat, so he tells X extra passengers to go to the other bus. X students get off – some boys and some girls, and board the other bus. Are there now more girls on the Boys' bus, or more boys on the Girls' bus, or are there the same number of boys and girls on each bus?

This bus version of the problem may be modeled by using red/black playing cards, two different colored chips, or heads/tails of coins. Begin by making two piles, one red and the other black (or two separate groups of colored chips or two piles of coins, one pile Heads up and the other Tails up). Move X (some number) from the first pile to the second pile. Mix them up. Move X (same number) back from the second pile to the first. You will now see, for example, 5 red cards were moved over and 5 any color cards were moved back. If it happened that only 2 red cards were moved back, then 3 black cards must have come with them to make a total of 5 (while 3 red cards and 2 black cards left behind). Regardless of what the actual number are that are moved, it will always result in the same number of "odd" colored cards in each black and red pile.

This version of the Make it Simpler strategy accomplishes so much more than simply changing numbers. It changes the focus from making the arithmetic easier, to focus on the discourse of reasoning rational thought, as well as to finding connections between models and contexts.

Map Temperature Problem

Another version of Make it Simpler can be found by building your way up to the level of sophistication required for solution. Frequently, this application involves stating a problem AND the answer to the problem, but then asking, "How do we know?"

Consider the following:

> Given a map of any region with a circle drawn on the map (of any radius). There will always be two points on the opposite ends of a diameter that will have the same temperature. How do we know?

Take a moment and think about the problem and its implications. The problem does not limit the size of the circle. It also does not ask you to actually find the two points – it simply assures you that at any given point in time, there will be two such points. The real problem is not in finding the two points, but instead, finding a way to explain how we know there are ALWAYS such points.

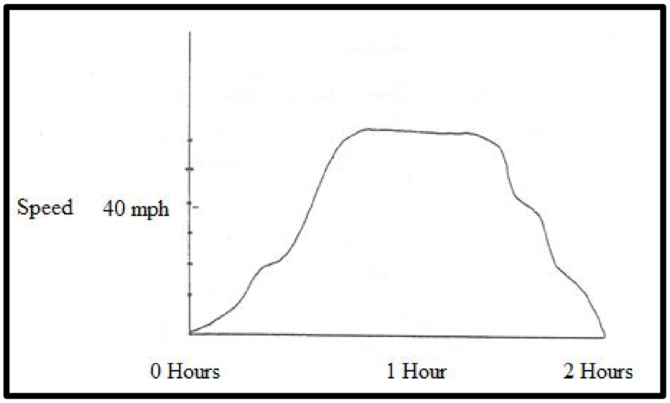
How can we go about tackling this problem? To set a plan of attack on this map problem, let's consider a simpler, perhaps seemingly unrelated, problem – specifically, a rate/time/distance query that will lead us toward a way of thinking about continuity, the cornerstone of one solution to the posed problem.

The Car Travel Problem

> A family began a car trip at 2:00 PM. After driving for 2 hours, the car traveled a total of 80 miles. Was there ever a time when the car's speedometer showed "exactly" 40 miles per hour?

This car trip problem has a vague connection to the temperature problem in that both seem to be asking for an existence "proof" of a proposed conclusion. If one can find a path to solving one problem, it may help in solving the other. What other potential analogies exist between these two problems. If one considers the starting point as one endpoint (A) of a line segment and the point 2 hours later as the second endpoint (B), then to get from A to B you must have traveled 80 miles in 2 hours thereby averaging 40 miles per hour. Of course, you started at zero miles an hour so you didn't travel 40 mph for the entire trip – there must have been at least one period on the trip where you travelled faster than 40mph to make up for the time that you spent driving under 40 mph. Consequently, there are at least two points between A and B where your speedometer reads exactly 40 mph. (Bonus: MIGHT there be three such points, or do the number of points that indicate you travelled 40 mph have to be an even number?)

A graphical representation of the car travel problem might help us make the next connection in our journey to solve the temperature problem. See Graph 1.



Graph 1: Car Travel Problem

We don't have the specific details of the various speeds at any given point in time except that, at the start the car is "at rest" and begins increasing speed to a point beyond 40 mph at some point. Also, we know that the car's speed will have to revert back to 0 mph at the end of the trip. Consequently, the graph shows a possible record of speed and distance, travelling from point A to point B. If the travelling line stayed below the horizontal 40 mph for this trip, it would not be possible to average a speed of 40 mph. Therefore, there MUST be at least two points during this trip that the car speedometer read "40 miles per hour."

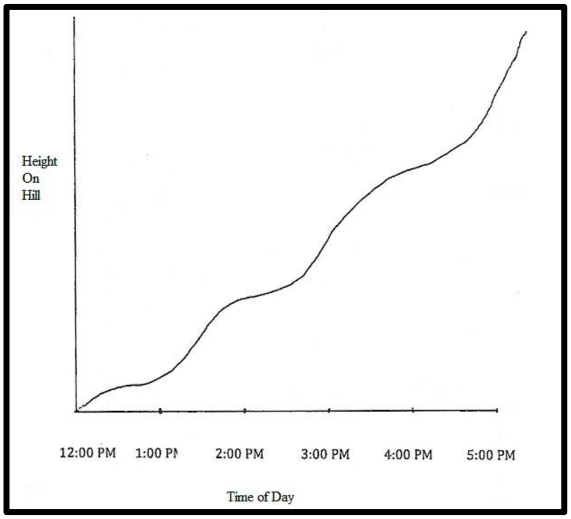
Now, consider another variation on this theme that involves a trip with a starting point, a stopping point, and a measurable attribute, such as time or speed.

Hiking Problem

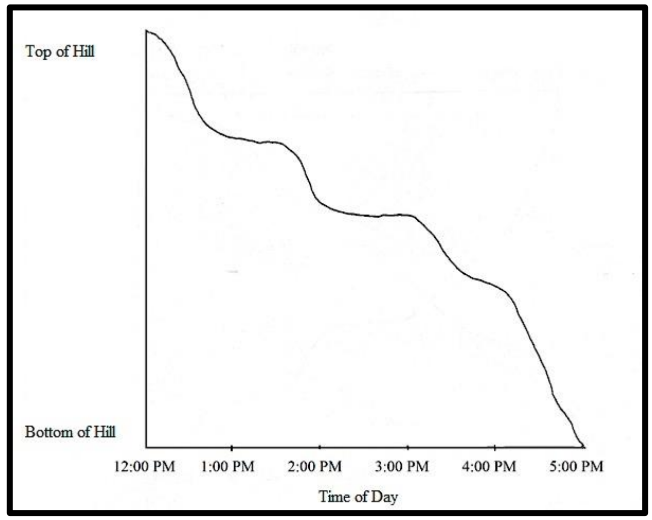
> At noon, a hiker began to climb a hill following a single path. She arrived at the top of the hill at 5pm. After spending the night at the top, at noon the next day, the hiker began the trip down following the same path, and arrived at the bottom at 5pm. After a moment's reflection, the hiker realized that there was at least one place on the way down where she was at exactly the same place at the same time the day before. How does she know?

In this problem, a coordinate graph with the horizontal axis denoting time and the vertical axis showing the height of the hill, can be used to chart possible trips the hiker could take, starting at the bottom of the hill on Day 1, and starting at the top of the hill on Day 2. As with the Car Travel Problem, we do not know the actual events of the hike other than the beginning time, the ending time, and the trail followed. We DO NOT have any information on rest stops, running, snack breaks, etc.

A graphical representation of the Hiking Problem (similar to the one used to interpret the Car Travel Problem) might again help us make the next connection to solve the Temperature Problem. Graph 2 shows a potential pathway for the hike up (the curve starting at the bottom of the hill at noon).

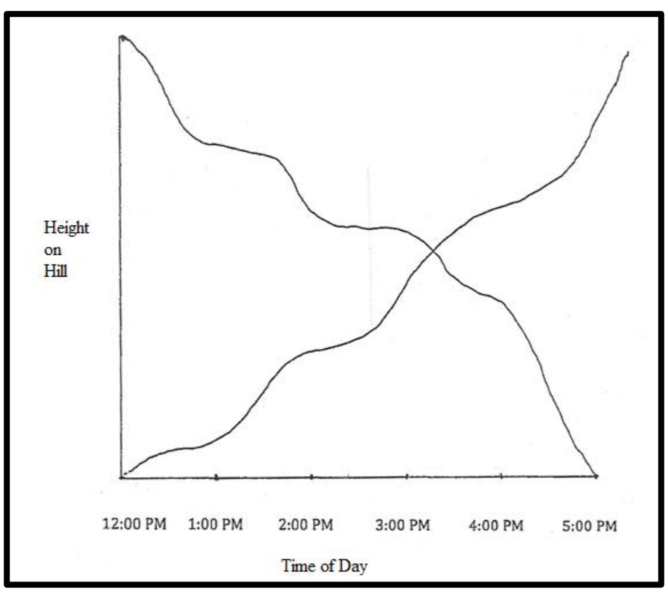


Graph 2: Hiking Problem – Path Up



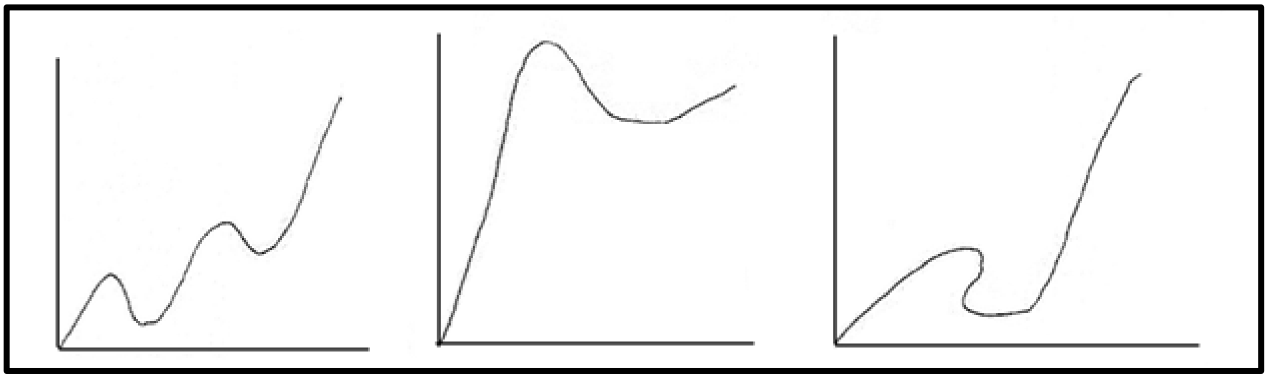
Graph 3: Hiking Problem – Path Down

Although Graph 3 shows a potential pathway for the hike down (the curve starting at the top of the hill at 12 pm), we can see that each trip (hiking up and hiking down) that took 5 hours to complete and, when you overlap the two days, the graphs must intersect (at least once) during those 5 hours. The intersection point represents the hiker at the same time of day at the same point on the trail.



Graph 4: Hiking Up and Down Crossing

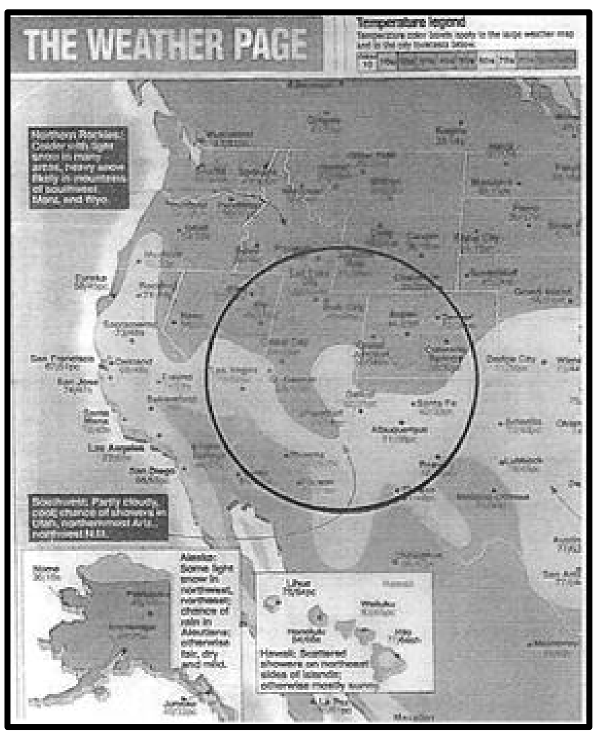
It is interesting to draw several graphs representing different scenarios to see if it is possible to have this phenomenon occur more than once. BTW: As an aside, it is also interesting to consider the feasibility or possibility of hiking up graphs as shown in Graph 5:



Graph 5: Other Hiking Up Maps

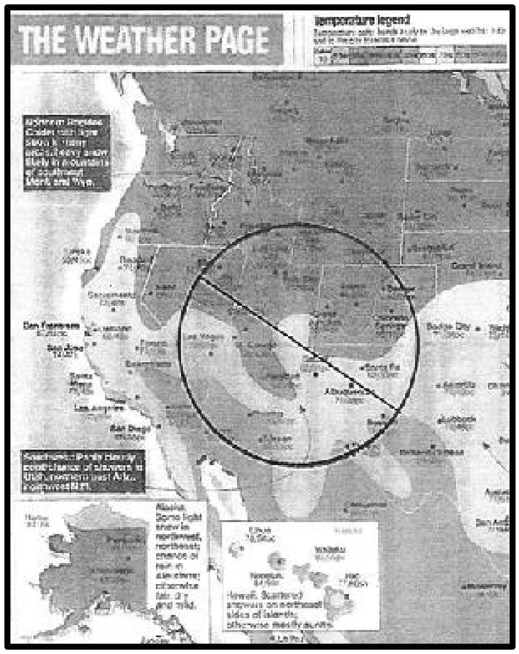
Another Look: Map Temperature Problem

Given a map of any region with a circle drawn on the map (of any radius), there will always be two points on the opposite ends of a diameter that will have the same temperature. How do we know? See Graph 6. Perhaps we can identify two endpoints, a path to follow, an attribute to measure, and draw a graph to relate these.



Graph 6: Weather Map

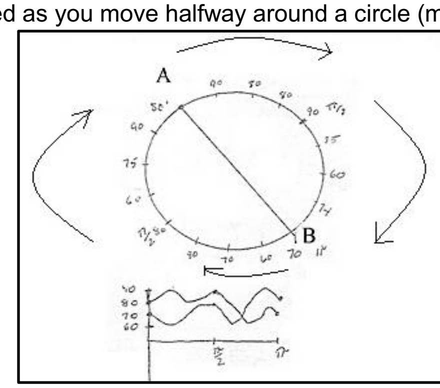
A diameter drawn on the circle can be used to identify our two endpoints (See Graph 7). Each endpoint has a temperature. This temperature will form the attribute whose change will be measured in this problem. If these temperatures are the same, then our task is over – that is, we will have found two points, the ends of a diameter, with the same temperature. Of course, it is quite likely that the two points we identify will have different temperatures, and like the Hiking Problem, these will be our two distinct starting points.



Graph 7: Weather Map with Diameter

Consider an instantaneous trip along one of the semi-circumferences of the circle, from A to B, and record the temperatures as you go on Graph 8. Next, take an instantaneous trip along the other semi-circumference of the circle from B to A and record these temperatures as you go on the graph.

Think of A and B as endpoints of a random diameter. If A and B are the same temperature, the problem is solved. If A and B have different temperatures, then consider the graph of temperature change and distance traveled as you move halfway around a circle (moving from point A to point B).

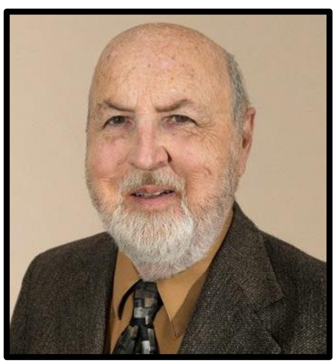


Graph 8: Recording Temperatures

The solution lies in the fact that temperature is a continuous function (especially when you allow for temperature instrumentation error). That is to say, in a linear sense, if it is 80 degrees HERE and 70 Degrees THERE, then somewhere between HERE and THERE it must be a place where it is 74 degrees. Yes, it COULD also jump above 80 and below 70, but it MUST hit every degree between 80 and 70 in somewhere between. The existence of a diameter with endpoints having the same temperature is "proven" by the intersection of the two temperature graphs. The continuity of temperature ensures that such an intersection will take place at least once. It is not possible to go from X degrees to X + 5 degrees without going through X + 1, X + 2, X + 3, and X + 4 degrees – no matter how quickly it seems the temperature is changing. Combining the Intermediate Value Theorem on continuous functions with one-to-one correspondence of the ruler postulate yields the fact that at least one pair of endpoints to a diameter must have the same temperature.

Conclusion

Problem solving strategies can be useful tools when applying them to attack problems, but they can also be rich tools to explore on their own. The process of reading, analyzing, formulating and verifying solutions takes on a less mechanical air and leads to an experience where the depth of learning is rewarded through transfer.



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Archimedes, Euler, and Nursing Cats: Rethinking the Arithmetic Mean

James A. Middleton

Abstract

The concept of the arithmetic mean has humble beginnings in children's understanding of fractions as fair-sharing. It becomes more sophisticated as we think of it as point of balance of a set of measurements. Later, it gains broader meaning as a moment, the point of balance across multiple dimensions including weighted values. These ideas of the mean are presented as a trajectory of learning, where earlier concepts are necessary understandings for the development of later concepts. Examples, activities, and useful models and tools are provided, beginning in the early elementary grades, and leading to high school and beyond.

The Mean as Compensation: Mother Cats

The other day, I found myself a bit bored and began to look at YouTube videos. The one that caught my attention was of a mother cat, buried in a pile of 12 kittens, each scrambling to find a "faucet" from which their breakfast would flow, that is, if they could only fight off their siblings. My wife had just rescued a mother cat who subsequently gave birth to two kittens. This seemed hardly fair. The video of kittens (12) was subject to "survival of the fittest," while my wife's rescue cat and her 2 kittens had a surplus of milk.

"The Humane Society and other animal welfare organizations should have a policy," I proposed, "where cats with few kittens share the burden of those with too many! After all, if the 14 kittens were distributed evenly across the two mothers, each would have seven to feed, all would be nourished, and the violence among litter mates would be reduced." My wife expressed her enthusiasm by gleefully handing me a Bernie Sanders flyer and pouring herself a stiff drink!

I then made this proposal to statistics students in my class. "Suppose my policy was adopted by the Humane Society. How could the Society divvy up the kittens to maximize benefit?"

"Just use the mean!" A voice shouted from the black void that constituted the back of the classroom. "It shares all kittens fairly across all mother cats." This was greeted with enthusiasm by the other students.

It turned out that my students had hit upon a key conceptual understanding of the arithmetic mean: The arithmetic average of a set of numbers fairly shares the total amount in the set across all of the objects being measured.

Early Elementary Grades: Fair-Sharing

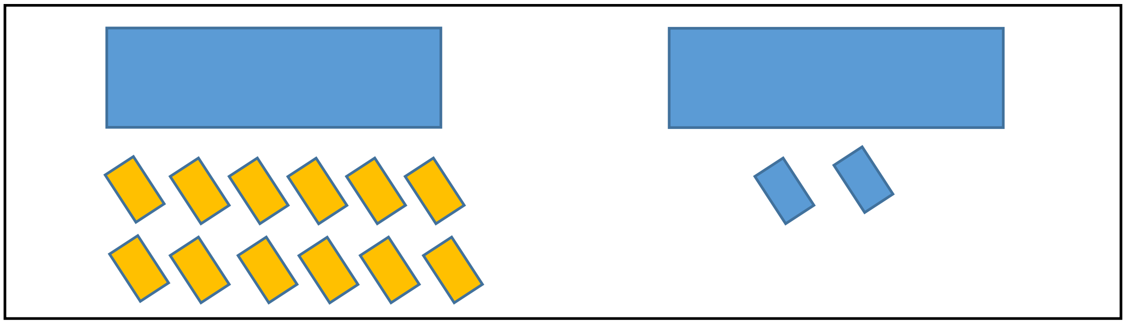

Figure 1. Two Cats, Each with Different Numbers of Kittens

For the two cats shown in Figure 1, we can reallocate the kittens from one to the other, creating a fair share, as shown in Figure 2.

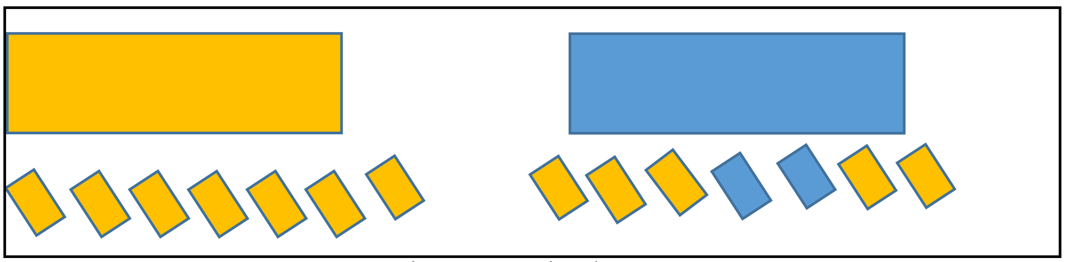

Figure 2. Fair Share

This is true for any number of cats, with any number of kittens. Numerically, it is a simple fraction: $\frac{total\ number\ of\ kittens}{total\ number\ of\ cats}$. We call this fraction the *mean* of the set of numbers.

Conceptually, its cognitive origin is the notion of a *fair share*, or what is called in the literature on students' thinking, *partitive division* (Kouba, 1989). Anytime you take a set of objects (kittens) and "deal" them out fairly across a second set of objects (cats), the total number the second set receives is the arithmetic mean. We often term this mean value as a *rate*, the number of kittens *per cat*.

This notion of the mean is easily accessible to students in first or second grade (given small enough quantities). They intuitively understand fair share. As long as the numbers are whole numbers, evenly divisible, students can dole out equal numbers of cookies per student, or pennies per pocket.

Middle Elementary Grades: Dealing with Remainders

A leap in understanding comes when the number of objects to be shared is not evenly divisible by the grouping factor. What do you do with a remainder? In work on children's mathematical thinking, we found that by posing the same problem and only changing numbers to include a remainder for the quotient, we could make a nice transition from a situation students understand (fair share with evenly divisible whole numbers) to the inclusion of a fraction. Patricia Moyer introduced the children's book, *A Remainder of One* (Pinczes 2000), to her students to help them make this transition.

Figure 3. A Hershey™ Bar as a Transitional Model for Fractions.

In research performed by one of my former doctoral students, we found that it is important to use an object that can be easily divided into equal pieces (first graders gasp in horror when I suggest sharing a leftover kitten equally) (Toluk & Middleton, 2004). My colleague, Leen Streefland (1991) made good use of chocolate bars (Hershey™ bars are best), because a whole bar can be divided into three strips horizontally, 4 strips vertically, or 12 pieces individually. Consider the problem: "After a birthday party, Sandra has 3 candy bars. Enrique has 4, and Monica has 5. If they share their candy bars equally, how many will each receive?" That problem can be extended to: "After a birthday party, Sandra has 3 candy bars, Enrique has 4, and Monica has 6. If they share their candy bars equally, how many would each receive?" The first problem is fairly easy for students who have solved similar problems in the past. The second creates a dilemma: *What do we do with the leftover candy bar?* Because a Hershey™ bar can be separated into 3 strips or 12 individual pieces, it doesn't take long for children to break the extra bar into thirds or 4 pieces of the 12. The different answers make for a lively discussion about equivalent fractions. Egg cartons are another good model, easily divisible by 1, 2, 3, 4, and 6.

Upper Elementary and Middle School: Rates

In middle school, we used brownies, submarine sandwiches, or other continuous models to help students see that fair sharing of a lesser number by a greater number yields a quotient less than 1, a fraction. By extension any number *a*, divided by another number *b*, yields a quotient with a value of $\frac{a}{b}$, whether or not the fraction is less than one. In grades 5 and 6, the quantity of sandwiches can be considered as fractional pieces in tune with the emphasis on ratio and proportional reasoning. That concept and reasoning method are core to middle school mathematics.

What is important in all of these early forays, is that rates ($\frac{13}{3}$ or $4\frac{1}{3}$ *candy bars per student*) represent the mean value of the numerator per unit of the denominator.

In your statistics unit, you will find the formula for the mean to be introduced early, but with little connection to these fundamental conceptions of partitive division. $\frac{\sum x_i}{n}$ is deceptively easy to calculate without developing adequate understanding of the meaning of the *mean.* In the middle school years, the mean should begin to take on a more geometrical conception: That of a *balancing point.*

The Mean as Balancing Point: Archimedes

Early Middle School: Levers

One of the important ideas in all of physics is that of a lever, a simple machine. In the middle school science curriculum, it is common for students to explore Archimedes' remarkable finding that there is a proportional relationship between the length of a lever and the mass that it can move. See Figure 4.

$$W_1 d_1 = W_2 d_2$$

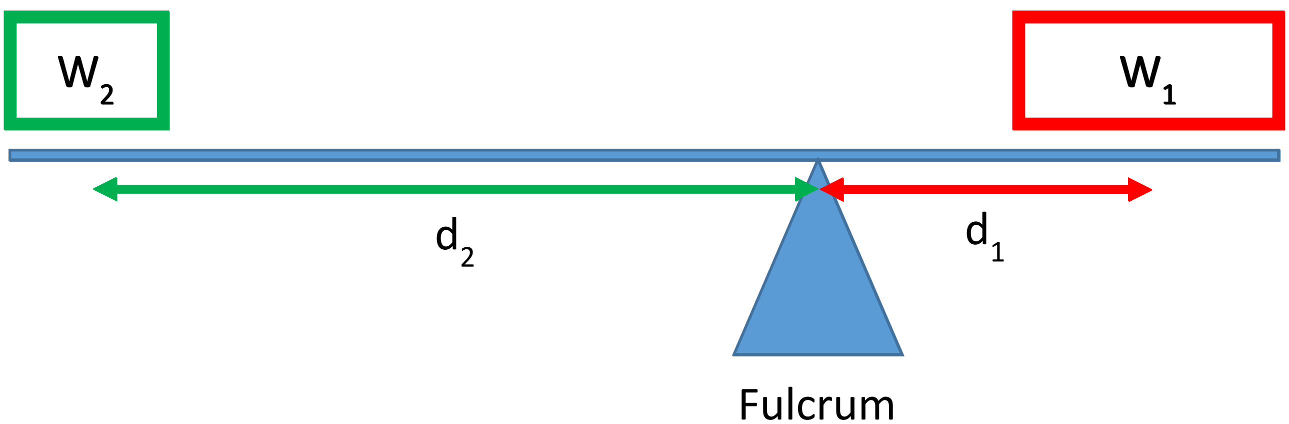

Figure 4. Archimedes' Law of the Lever

In any Type I lever, where the fulcrum lies between the masses, and the lever is balanced, the heavier weight has to compensate for its mass by taking up a shorter length of the lever. The lighter weight gets proportionally more lever. If there are multiple objects placed at different points on the lever, for the lever to balance, the sum of weights "times" distances on one side of the fulcrum has to equal the sum of weights "times" distances on the other.

$$\sum W_1 d_1 = \sum W_2 d_2$$

Math teachers can use this to their advantage when developing the concept of the mean. If we take the distances of unit masses on each side of the fulcrum, the sum of those distances have to be equal to each other. To illustrate this, we decided that we would work with grade 5 teachers to explore this concept using unit weights (pennies) and paint stirring sticks donated from the local home improvement store. See Figure 5.

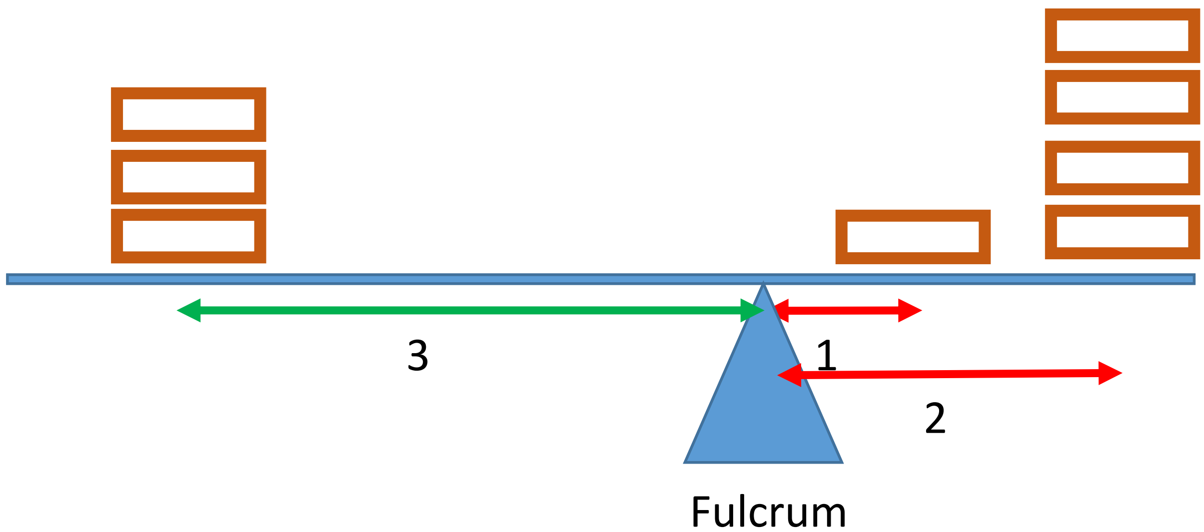

Figure 5. Using a Lever to Illustrate the Arithmetic Mean

In Figure 5, the lever is balanced because the three pennies on the left are each located 3 inches from the *center of mass*, while the 5 pennies on the right are distributed: one at 1 unit and four at 2 units from the center of mass. So, the total weight "times" distance on the left (3×3) is equal to the total weight times distance on the right (1×1 + 4×2). The fulcrum will always balance when this relationship holds.

The key to making this a good introduction to the mean is to place the fulcrum at zero, provide students with a pile of pennies stacked to one side, and then ask them to predict where a second stack of pennies has to be placed for the lever to balance, *before* actually placing them. This is best done in groups of three; one student can't hold the fulcrum steady and place all the pennies in their proper places at the same time. Keep track of the number of pennies and their distance to the left and to the right of the fulcrum. I suggest beginning with only two piles, one on each side, and then moving to more complex situations. A table like the one shown in Figure 5 comes in handy. The first problem contains the quantities from Figure 5.

1. Use the provided lever, fulcrum, and weights (pennies) to predict the distance that balances each system.

System	Left # of pennies	Left Distance	Left W x D	Right # of pennies	Right Distance	Right W x D
A	3	3	9	1	1	1
				4	2	8
Total			**9**			**9**
B	5	2	10	3	4	12
	1	3	3	?	?	
Total			**13**	**?**	**?**	**?**
C	2	2	4	6	2	12
	?	?	?	7	1	7
Total			**?**			**19**

2. Now that you have balanced a few levers, write a rule that would work for ANY number of pennies. Your answer should be in the form of instructions to a person who has never tried this problem before.

Figure 6. Balancing Stacks of Pennies

After this activity, you can discuss responses to Question 2, consolidating their thinking into a rule. The discussion should include the integers, keeping track of the pennies on the left (negative numbers) and on the right (positive numbers). Because the magnitude of the total W × D quantities are *always* equal, the balance point will *always* be zero, the mean location of all pennies.

Following this discussion, a typical activity using histograms may be introduced. A histogram is just a depiction of a lever, with the bars representing the number of measurements across intervals of the variable being measured (just like the number of pennies (weight) is stacked at each distance). The histogram in Figure 7 gives a discrete example, where we are counting whole hotdogs and whole people.

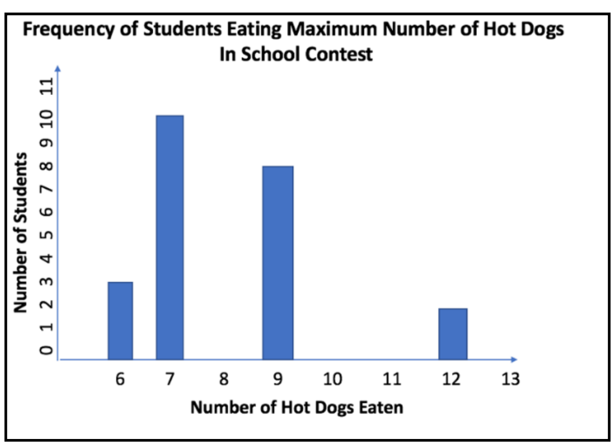

Figure 7. Histogram of Hot Dog Eaters

Given a histogram, ask: "Where does the system balance?" Just as if the horizontal scale were a lever, a student can do two lovely conceptual things: 1) Estimate by intuitively sliding a finger along the axis to a point where it looks like the "weights" (hot dog eaters) are balanced, or 2) calculate the mean by adding the weights on each side of some point to determine the balance. By inspecting the graph, students can intuit that the balance will be somewhere between 7 and 9 hot dogs. A quick calculation (6x3) + (7x10) = 88, and (9x8) + (12 x 2) = 96 confirms this relationship.

We can use the arithmetic mean, $\frac{\sum x_i}{n}$ to find the exact balance point.

$$\frac{(6x3)+(7x10)+(9x8)+(12x2)}{(3+10+8+2)} = \frac{184}{23} = 8.$$

Thinking back to what we learned with our levers, we know that the sum of the hot dog eaters x their distance away from this central value on the left, must equal the sum of hot dog eaters x their distance on the right.

$$3(8-6) + 10(8-7) = 8(9-8) + 2(12-8)$$
$$3(2) + 10(1) = 8(1) + 2(4)$$
$$6 + 10 = 8 + 8$$
$$16 = 16$$

When we engaged fifth graders in this set of activities, they easily intuited the location of the fulcrum, and became fairly adept at predicting where to set piles of pennies to balance the load. For a culminating activity, we brought 2" x 10" boards and made a large lever about 10 feet long. The school principal was an ex-football player, so students weighed him and predicted how many fifth graders they would need, and where to place them on the fulcrum, to achieve balance for different locations of the principal (Guerra et al, 2010).

A bonus of using this approach to teach the concept of mean, is that students are engaged in proportional reasoning in a non-trivial context, and we are able to provide early instruction in the use of integers, and the concept of negative numbers being merely those quantities on the left side of the fulcrum, when it is at zero. Negative numbers have real meaning to students when they experience them hands-on, minds-on.

The Mean as a Moment: Euler

As students move into high school Algebra and Geometry courses, they will develop two key conceptions that will help them flesh out their understanding of the mean as a balance point: The concept of *function*, and the *concept of center of mass*. It is an important property for any Real function f(*x*), that we can always find some point that balances all its values over the independent variable. In the lever example, since all W x D on the left side of the fulcrum equal the W x D on the right side, when added (accounting for negative values), the sum is zero.

The lever example is a good starting model for this, because we have to account for the height of the stacks of pennies (i.e., their frequencies), not just their distances from the fulcrum. In the field of statistics, what is usually called *univariate* statistics is not really univariate. Whenever we keep track of the frequencies of a measurement occurring in a data set, we are really making a function f(x) where the frequencies are dependent upon values of the variable we are measuring. Just like any Real function, we can graph it as:

$$y = f(x)$$

Weighted Averages

I am currently working with my college students on a research study to identify types of tasks and teacher support that help students become engaged in mathematics. We ask high school students questions like, "Rate on a scale of 1 to 5 how successful you felt in the activity you just completed." We had 894 students rate an activity in which they had been engaged. The results are shown in Table 1.

Table 1. High School Freshman Degree of Success

Rating	**Frequency**
1	46
2	63
3	220
4	339
5	226
Total	894

Just like the lever activity, we can try to balance the set of data by sliding our finger along the horizontal axis and intuit that the balance point is somewhere between 3 and 4. But what is that exact balance point? In the graph of the data shown in Figure 8, this balance point becomes more obvious.

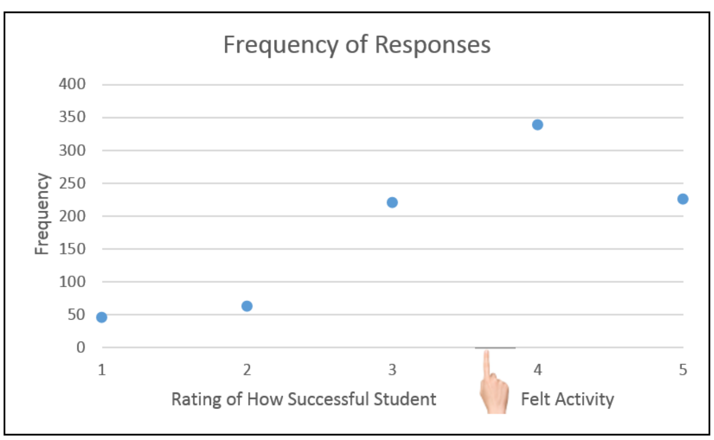

Figure 8. Frequencies of Student Responses

In high school we will extend the lever activity to now use what mathematicians call *moment*. The *first moment* of a real function (for example, f (rating)) is called its *center of mass*, because it balances the function, i.e., the point where the "mass" or the total quantity of the dependent variable is balanced on x-axis. Mathematically, this is expressed as:

$$M = \sum_{i=1}^{n} x_i \frac{f(x_i)}{n}$$

The first moment of a discrete distribution of data is the sum of the individual weights ($\frac{f(x_i)}{n}$) times their respective distances (x_i). For our example of student responses to the perceived success question, it looks like this:

$$M = \sum_{i=1}^{n} x_i \frac{f(x_i)}{n}$$

$$= 1\left(\frac{46}{894}\right) + 2\left(\frac{63}{894}\right) + 3\left(\frac{220}{894}\right) + 4\left(\frac{339}{894}\right) + 5\left(\frac{226}{894}\right)$$

$$= 3.71$$

Looking back at Figure 7, 3.71 *does* look like it balances the overall data. But to show that it does, we can use the property of *moments* to help us. If this sum is correct, the distances of all points in the function about the moment will be zero.

For our sample data:

$$0 = [46(1 - 3.71)] + [63(2 - 3.71)] + [220(3 - 3.71)] + [339(4 - 3.71)] + [226(5 - 3.71)]$$

$$0 = 46(-2.71) + 63(-1.71) + 220(-0.71) + 339(0.29) + 226(1.29)$$

$$0 = -124.66 - 107.73 - 156.20 + 98.31 + 291.54$$

$$0 = 0 \text{ (within floating point error)}$$

Now, why is this rather complicated understanding of the arithmetic mean important for high school students? First, as we have established earlier, it builds on the essential idea of fair share and center of mass. That is, the mean is the point in a distribution of data where all of the weighting (frequencies) is shared equally among all values of the independent variable, no matter how far apart they are. Moreover, *because* each data point shares the weighting, this central value is the balancing point for the distribution (there is equal mass above the mean and below the mean).

Connections and Extensions in High School

But, there is more. High school students are increasingly being asked to learn about the mean and variance as parameters of the normal distribution. This happens both in mathematics classes (typically in the Algebra sequence, but sometimes in Geometry, and of course in AP statistics), and in AP Biology and Physics (momentum is not surprisingly derived (literally) from the concept of moment). The mean, as we have just shown, is the first moment of a distribution of data. The variance is the *second* moment. It is the point where the sum of the *squared* distances from the mean equals zero. Its square root, the standard deviation, is the standard measure by which we compare the width of two distributions of data. Understanding the standard deviation as the average *Euclidean distance* every data point has from the center of mass of the distribution, provides a more intuitive meaning for the measure. Typically, these concepts, like the mean, are taught using just the formula for calculation.

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n},\ \sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n},\ \text{and}\ \ \sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$$

Thinking of it as a *moment* does more than just eliminate the problem of summing to zero. It ties the standard deviation to the Pythagorean Theorem in Geometry, to quadratics in Algebra, and to the points of inflection on the symmetric, bell-shaped distribution that we call Normal in Statistics and Calculus. As students progress towards Statistics and Calculus courses, these concepts will reappear in slightly different form. Summation will be replaced by integration across a continuous variable, and frequencies will be supplanted by probabilities. However, their underlying meanings are tied to these simpler ideas of fair share (compensation), balance, and moment.

A nice introduction to these ideas stems from Geometry, where we commonly introduce the *centroid of a triangle* as its center of mass. This is typically introduced with the median of each angle, and the notion of the center of mass is fairly intuitive. If half the area of a triangle is on one side of a median, then the other half of the area has to be on the other side. Each of the media of a triangle shares this property, and for the triangle to balance across all dimensions, the three media intersect at a single point. But, you really only need two dimensions to define the center of mass. The third is redundant.

Another way to find the centroid of a triangle is by using the *moments* of each dimension (i.e., the mean of the coordinate in each dimension) to find the center of mass. See Figure 9.

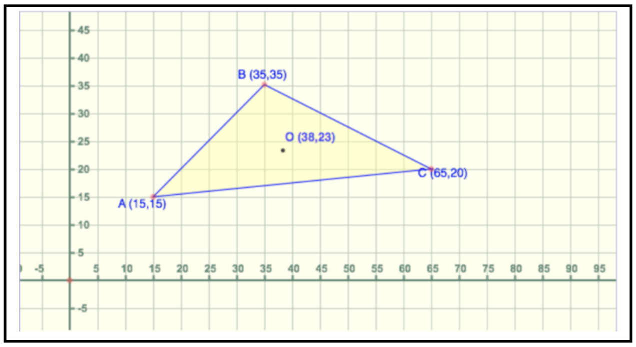

Figure 9. Centroid of a Triangle as a Center of Mass (from www.mathopenref.com/coordcentroid.html)

We also know that a triangle can be defined as three vertices in space: $(x_A, y_A), (x_B, y_B), (x_C, y_C)$, where x is arranged on one coordinate axis (the independent variable), and y is the height of that point on the other axis (the dependent variable). The notation I am using is meant to illustrate that the vertices of a triangle can be conceived of as a function: f(*x*) = *y*.

x	***y***
15	15
35	35
65	20

If this is so, then there *has* to exist some center of mass for that function in the horizontal direction, and if we invert the function, there has to be a center of mass in the vertical direction. The mean of the x-values is the center of mass for x. The mean of the y-values is the center of mass for y:

$$\mu_x = \frac{15 + 35 + 65}{3}$$

$$= 38.33$$

$$\mu_y = \frac{15 + 35 + 20}{3}$$

$$= 23.33$$

Thus, the centroid of this triangle must lie on the line: x = 38.33 because the mass on one side of the line equals the mass on the other. Ditto for our y values: The center of mass must lie on this line because the figure is balanced across it. The intersection of these two lines, that demarcate the center of mass of each dimension, give us the center of mass of the triangle. *Math Open Reference* has a nice Applet (see Figure 8) that allows students to drag vertices of a triangle to illustrate how the centroid is always the mean of the coordinates (μ_x, μ_y).

This concept can be extended to weighted quantities by considering the vertices to be three objects with mass:

x	*y*	**mass**
15	15	2 units
35	35	3 units
65	20	4 units

The coordinates of the center of mass will be:

$$M_x = \frac{15(2) + 35(3) + 65(4)}{2 + 3 + 7} = 32.92$$

and

$$M_y = \frac{15(2) + 35(3) + 20(4)}{2 + 3 + 7} = 17.92$$

It is the same formula we used before for calculating the mean of a set of data:

$$M = \sum_{i=1}^{n} x_i \frac{f(x_i)}{n}$$

Summary: Thinking of a Learning Trajectory

So, we have seen how this intuitive concept of *compensation*, *balance,* and *center of mass* has humble, but powerful beginnings in the elementary activity of *fair sharing*. It extends to embody fractions and proportional reasoning, the integers and rational numbers in the middle grades, and extends to the reals, where they are applied to summary statistics, and continuous functions in high school. I purposefully put all of these conceptions into the same manuscript because it is important for each of us, regardless of the level we teach, to see what students bring to us, and where we are guiding them.

Each of us, if armed with an understanding of the more basic conceptions of the mean, can use those same models for struggling students to help them understand what, to most students, are pretty heady concepts: Center of Mass and Moment. Moreover, for students who seem to do well with Statistics, the extensions I have provided here will enable them to develop a deeper understanding of the mean that goes beyond "add 'em up and divide by *n*" by connecting their statistical understanding to Algebra, Geometry, and beyond.

Note: Without a good understanding of fair sharing, students will not develop an understanding of compensation. Without compensation, balance becomes exceptionally difficult. Without balance, moment and center of mass are inaccessible, except on a merely calculation basis.

With these concepts, students will be introduced to a stronger conception of one of the most important concepts in all of data analysis: the arithmetic mean. With Statistics and Data Science becoming more and more important in today's information-saturated world, such attention will surely benefit students.

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Inside Interactive-Engaged Problem-Solving in Precalculus: Content and Classroom Moments

Guadalupe Lozano

Abstract

The classroom deployment of a co-curricular approach for supporting the learning of Precalculus through mentoring, engagement in interactive problem-solving with peers, and faculty professional development, provide the focus of this article. Our project, funded by the National Science Foundation (#1644899), was conducted at a Southwestern 2-year Hispanic Serving Institution (HSI). In this article, aspects of the project curricula and student-centered classrooms practices that are of value for supporting the high-school to college transition in mathematics, are described. Specific attention is given to the types of problems that support productive peer-to-peer classroom collaborations, as well as the teaching moves and choices that support the development of students' problem-solving abilities.

Introduction

Over the last decade, much progress has been made in bringing student-centered active-learning to university classrooms, particularly in mathematics (e.g., Abell, et al., 2018; Epstein, 2013). Yet, progress has been uneven across institutions, and not always assessed from an equity perspective (Lozano, Franco, and Subbian, 2018). This article shows how instructors launched and supported student-centered, interactive problem-solving lessons around challenging mathematics problems in the Precalculus course curriculum.

The Problem-solving Curriculum

The Co-Precalculus course is designed as a 1-credit companion to the main Precalculus course, and includes: 1) 11 challenging problems sets with 3-4 problems in each, and 2) a forum to generate productive peer interactions aimed at sharpening problem-solving skills. Problem sets were structured to include three types of problems: open-ended word problems, scaffolded word problems, and problems focused primarily on algebraic skills.

Problem Examples

What follows are three examples of open-ended word problems (See Figures 1-3). The Project Speed Problem (Collins, 2017) was developed for this project. The Tethered Cow Problem (Stewart, Redlin, and Watson, 2016) was part of the Co-Precalculus materials at the 4-year partner school. The Which Function? Problem (Lozano, 2007) was part of an open-source bank of Precalculus Assessments (math.lsa.umich.edu/courses/105/Exams/index.html).

In September of 2016, Denise Mueller set a women's Paced Bicycle Land Speed Record at the Bonneville Salt Flats. See TheProjectSpeed.Com website for details about this effort. Denise used the custom bike pictured below. The bike featured 17" diameter high speed motorcycle wheels and tires to lower the center of gravity. The bike utilized a short travel suspension system to dampen road vibration and a steering stabilizer to eliminate any speed wobble due to high speeds. The frame is elongated for stability and to accommodate the double reduction single speed drivetrain.

The double reduction drive train consists of a large front chainring (15 inch diameter) connected by a chain to a 3 inch diameter gear that drives the smaller 14 inch diameter chainring. The 14 inch chainring is connected to the drive axle (rear wheel) with another 3 inch diameter gear.

a) Assume the motorcycle tire adds 2.1 inches to the radius of the motorcycle wheel. If Denise's maximum sustained pedal cadence was 100 revolutions per minute, what is the new women's land speed record in miles per hour?
b) Does your answer seem plausible?
c) Does the answer seem more plausible if you consider that this is a highly trained professional athlete attempting to set a world record?

Figure 1. Project Speed Problem

A cow is tethered by a 100 ft rope to the inside corner of an *L*-shaped building as shown in the figure. Find the area that the cow can graze.

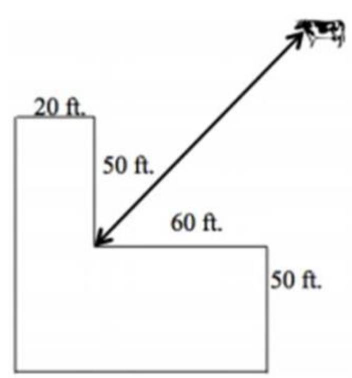

Figure 2. Tethered Cow Problem

Circle the correct answer to each of the following problems. No explanations are necessary.

a) A certain type of sweater at American Eagle Outfitters has been discounted several times since the beginning of the winter. Each time the price of the sweater has dropped to 4/5 of the previous sale price. Let *g* give the sale price of the sweater as a function of the number of discounts given. Then *g* is best described by which of the following types of function?

LINEAR QUADRATIC EXPONENTIAL

b) Adam's weight went up by a pound during the first week of the new year, then down by a pound during the second week. Let *w* give Adam's weight as a function of time, in weeks, since New Year's day. Then *w* is best described by which of the following types of function?

LINEAR QUADRATIC EXPONENTIAL

c) The number of fish in a certain lake has nearly tripled since 2005, as it has grown by 1,000 fish per half year since that year. Let *p* give the fish population as a function of time since the fish were first introduced into the lake. Then *p* is best described by which of the following types of function?

LINEAR QUADRATIC EXPONENTIAL

Figure 3. Which Function? Problem

These problems have common features that make them ideal for rich group negotiations around precalculus content. First, the problems are stated conceptually—the students must decide what to do first, and what mathematics tools might be useful for solving the problems. Second, the problems can be tackled algebraically, graphically, or numerically. These characteristics give flexibility not just to students, but also to instructors in their facilitator roles.

The Pedagogy: Kicking-off Group Work and Tuning into Peer Discussions

Students work in groups of 3-4 on the whiteboard. This set up requires sufficient white board space for each group to sketch and write without erasing ideas worth coming back to, or alternate ways of thinking. For a class of 15-20 students, a team of two people, a facilitator (main instructor) and a co-facilitator (a second instructor, an advanced peer, or a graduate student) ensures timely interaction with all groups.

The nature and liveliness of the peer interactions will vary by class, by group, by problem, and by solution approach. Thus, the work of productively supporting peer discussions is not prescriptive but organic, adaptive. Yet, in all situations, skilled facilitators *tune into the ongoing peer discussion* looking to propel forward each group's work *building on the progress done, rather than changing the approach* chosen by the group.

Tuning into student work so as to support productive thinking without changing it, is as important as it is challenging. But there are some rules of thumb we can follow. Before illustrating what tuning into student work looks like, consider an authentic vignette.

Recommendation: Think about how to solve the Tethered Cow problem prior to reading the vignette. Use your sketch and ideas as a reference, as you read below.

Peer Discussion

Anne: I think we need to use the formula for area of a circle, $A=2\pi r$, where r is the radius.

Santos: Yeah, and the radius is the length of the rope.

Dulip: That sounds right. [Speaking to Rita] Is that what you have?

Rita: Mhh… I think so. But I drew two circles…well, part of them.

Santos: Yeah…actually, I don't think the full circle formula is right. We need some other formula…I remember some formula for sectors of circles… should we look that up?

Anne: Ah, yes…maybe.

Santos: Let's draw the picture. [Drawing on the board] We have a quarter circle here, and a smaller one here, because the rope is longer than the wall.

Rita: Yeah, that's what I have. [Pausing] I think that the cow can also go left, so we have another quarter circle.

Santos: Oh, yes! A little one. But how big is it?

Rita: Well, the wall there is 20 feet long, so that is the radius of the little quarter circle.

Santos: Mhh…yes, I think so, maybe…but the left over rope is longer than 20 feet. Should the radius be longer?

As teachers we may first notice errors, things that happened counter to what we anticipated, or be overwhelmed by the unwieldiness of the discussion. We may notice:

- From the start, the formula is incorrect. The group should calculate area not perimeter, and the formula does not get fixed as the discussion develops.
- There is confusion about what the right area formula is—circular sectors versus full circle.
- There is confusion about length of radii and length of the rope.
- No one seems to have a fully correct solution, or uses the same solution approach.

As facilitators of productive student-centered work, we also notice things we can leverage to *build on the progress done from where it is*, as opposed to changing the flow of discussion. We may observe:

- Two viable solution approaches were introduced—one based on dividing full-circle areas, and the other based on the formula for circular sectors.
- The group is not clearly aware that both approaches are viable in this case (since all circular sectors are just quarter circles!). This is an opportunity to bring back Dulip and Anne into the discussion, who seemingly worked with the full circle area formula, so that the group might discover both approaches work, in this case.
- Once students begin comparing calculations, there will be a natural opportunity to draw attention to the incorrect formula, and fix it.

In our role as facilitator, we notice difficulties, but we avoid drawing attention to our own way of correcting them, or pushing a group towards efficient solution approaches. Instead, we look for productive entry points to: 1) foster a group's line of reasoning, 2) leverage learning opportunities that arise spontaneously (e.g., alternate approaches), 3) bring back silent participants in productive ways, and 4) encourage self-sufficiency in validating a group's work.

Tips for Supporting Productive Peer Interactions

Many books and papers have been written about orchestrating productive peer discussions (e.g., Smith and Stein, 2011), and a few excellent ones about productively supporting them (e.g., Herbst, 2008). Most recently, a multi-institutional mathematics faculty effort led to the publication of the Mathematics Association of America (MAA) *Instructional Practices Guide*, a document combining research, examples, and practical tips for implementing modest-to-bold active-learning practices in our classrooms (Abell, et al., 2018). In this section, simple asset-based teaching moves to support the discussions like the one in the vignette, are described.

Some Asset-based Guidelines

1. When necessary, ask students to write or draw on the board--**never hold a marker yourself.**
2. Engage a group by asking questions only—**avoid making statements.**
3. **Study what a group has written on the board** prior to engaging them. Look for mathematical progress, and assess the need to interact or move on at a given point in time. Look for entry points to engage productively.
4. **Observe the group dynamic** prior to engaging with students. Look for listening and questioning patterns, and assess whether participation in the group appears useful for all members. Look for entry points to engage productively.

The first two guidelines are perhaps the most important for **enabling students to problem-solve resourcefully on their own**. They require intentionality on the part of the facilitators because they call for doing exactly the opposite of what we are expected to do, and used to doing, in traditional classroom settings. The third guideline is most important for **ensuring that facilitators' build productively on the discussion taking place, rather than changing the story line**. This is also counterintuitive from the perspective of the traditional teaching or tutoring of students. In those settings, we tend to present approaches that match our ways of reasoning, are efficient, or make good sense to us.

The work of fostering self-sufficiency in problem solving requires facilitators to *problem-solve around students' problem-solving*, rather than presenting a "clean" mathematics approach to them. As skilled facilitators, we strive to discern what is being done, then support our students to further develop viable approaches on their own.

To illustrate, consider the Tethered Cow Problem. The second asset-based guideline above suggests some possibly useful questions for the group:

- How does your sketch relate to the formula you wrote on the board?
- Do you all agree with the idea of using the circular sector formula to solve this problem?
- Is it possible to use the full circle formula to solve this problem?
- Do you all agree on how many circular sections need to be combined to find the area?
- How do you find the radius for this circular section? How about for this one?
- Do you all agree with the sketch on the board?
- Do you all agree with what Dulip (or other student) just explained?

Depending on a group's response and the mathematical progress shown on the board, each of these questions (especially the yes/no questions) may be followed with a call for the group to discuss further, before the facilitator comes back to question again.

Let's consider the third question in the list. If not everyone agrees with the idea of using circular sectors (some students disagree, or remain quiet), this question creates an opportunity to further the group's thinking on whether or not both approaches (circular sector vs. full circle area, divided by 4) are viable. On the other hand, if everyone agrees, a similar group-call to justify why this is the case could be productive.

Conclusion

This article does not discuss much of the structure of the course itself, or the importance of cross-institutional faculty collaborations in robustly streamlining the transition to the university. Yet, the problems, vignette, and guidelines presented, exemplify the type of asset-based teaching used in the co-precalculus course and in need of more wide-spread implementation across diverse classrooms, institutions, and student populations (e.g. Abell et al., 2018). The choice to offer a close-up look at content and teaching used to productively support student mathematical growth in our course, aims to plainly illustrate practices we seldom experience, but often talk about, making replicability possible.

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