On the Model Order in Parameter Estimation using Virtual Compensators

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Abstract. Processing signals from open-loop system realizations can replace real-time operation using actuators in the design of closed-loop eigenstructures. One merit of the signal processing-based implementation is that it, in principle, allows virtual compensators of user-defined model order since the closed-loop systems are not to be realized during physical testing. The present paper explores the implication of the virtual compensator order in terms of the Fisher information on unknown parameters to be estimated in a model updating context. A numerical example with a structural system of engineering interest is presented that demonstrates the basic points outlined in the paper.

Keywords: Model updating, Output feedback, Eigenstructure assignment, Virtual implementation, Complex gains

1 Introduction

Calibrated numerical models are used extensively within structural and mechanical engineering for design, analysis, health monitoring, and control [1]. The calibrated models are often obtained through conventional updating schemes, in which the discrepancy between poles from a model, $\mathscr{M}$, and target poles estimated from the physical system, $\mathscr{P}$, is minimized [2]. An issue in this context—which will typically prevail when using poles as targets in structural and mechanical engineering applications—is that the system of equations to be solved in the minimization problem is ill-conditioned [3, 4]. Proposals to increase the target space and, in this way, resolve the condition issue by testing the system under known perturbations have been made [5, 6], but practicality is often limited. Another way to address the target space issue is to interrogate the structure in closed loop with different gains and, as such, increase the number of poles that can be identified. However, the closed-loop interrogation scheme has not yet had an important impact in applications, which, presumably, is due to the practical overhead associated with real-time operation.

Recently, it has been shown that the noted overhead can be eliminated by use of a virtual approach, in which multiple closed-loop systems can be computed based on a single open-loop realization [7–10]. The virtual implementation also removes stability and actuator design constraints [11], enables the use of complex gains because the control forces do not have to be physically delivered [10], and allows for a user-defined model order of the closed-loop system. In the present paper, focus will be on the last item, namely, to examine the feasibility of increasing the closed-loop system model order as an alternative approach to increase the target space. The theoretical outset in the examination is the Fisher information on the parameters to be updated; in particular, how the information increases as function of the model order and how this is reflected in the accuracy of the model updating.

The paper is organized as follows: section 2 outlines the fundamental theory of dynamic output feedback, followed by a description of the virtual implementation in section 3. Section 4 describes the concept of Fisher information, and subsequently, in section 5, a model updating scheme based on minimizing the discrepancy between closed-loop target poles and model-based poles is outlined. In section 6, a numerical example is presented to demonstrate the points made, and in section 7 some concluding remarks are given to close the paper.

2 Output feedback

Let a structural domain, $\mathscr{P}$, be described as a linear, time-invariant system in discrete time as

$$x(k+1) = A_d x(k) + B_d u(k) \tag{1a}$$

$$y(k) = Cx(k). \tag{1b}$$

Here, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^r$ the control input, $y(k) \in \mathbb{R}^m$ the output, and $A_d \in \mathbb{R}^{n\times n}, B_d \in \mathbb{R}^{n\times r}$, and $C \in \mathbb{R}^{m\times n}$ the system matrices. It should be noted that when measurements are displacements, velocities, non-collocated accelerations or collocated accelerations where the direct transmission term has been subtracted, Eq. (1b) holds directly. It is assumed throughout this paper that one of the mentioned conditions is met.

In case of dynamic output feedback, u is the output of a discrete-time, finite-dimensional linear time-invariant system driven by y, thus

$$x_f(k+1) = A_f x_f(k) + B_f y(k) \tag{2a}$$

$$u(k) = C_f x_f(k) + D_f y(k) + v(k) \tag{2b}$$

for some excitation v and coefficient matrices $A_f \in \mathbb{C}^{q\times q}, B_f \in \mathbb{C}^{q\times m}, C_f \in \mathbb{C}^{r\times q}$, and $D_f \in \mathbb{C}^{r\times m}$. Augmenting Eq. (2a) with Eq. (1a), using Eqs. (1b) and (2b), yields

$$\begin{Bmatrix} x(k+1) \\ x_f(k+1) \end{Bmatrix} = \begin{bmatrix} A_d + B_d D_f C & B_d C_f \\ B_f C & A_f \end{bmatrix} \begin{bmatrix} x(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} v(k), \tag{3}$$

which is referred to as the compensator. Defining $\tilde{x}(k) = \{x(k)^T \quad x_f(k)^T\}^T$, Eq. (3) can, as shown in [7, 11], be rewritten as

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{u}(k) \tag{4}$$

with the output

$$\tilde{u}(k) = \tilde{G}\tilde{C}\tilde{x}(k) + \begin{Bmatrix} 0 \\ v(k) \end{Bmatrix} \tag{5}$$

and

$$\tilde{A} = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & B_d \\ I & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}. \tag{6}$$

As shown in Eq. (4), using the feedback law in Eq. (5), the compensator can be viewed as a static output feedback system of order $n+q$ and with the system matrices $\tilde{A} \in \mathbb{R}^{(n+q)\times(n+q)}$, $\tilde{B} \in \mathbb{R}^{(n+q)\times(r+q)}$, $\tilde{C} \in \mathbb{R}^{(m+q)\times(n+q)}$ and $\tilde{G} \in \mathbb{C}^{(r+q)\times(m+q)}$.

3 Virtual implementation

As seen in section 2, implementation of the compensator to close the loop changes the model order of the system from n to $n+q$, and the inputs and outputs increase by q. The virtual implementation of the compensator model is achieved using the relation between the open- and closed-loop transfer matrices, $H(z)$ and $\mathscr{H}(z)$, which in a system controlled with positive static output feedback with the gain G is defined by [12]

$$\mathscr{H}(z) = (I - H(z)G)^{-1}H(z), \tag{7}$$

from which it follows that the closed-loop system can be identified from an open-loop realization. The open-loop transfer matrix can, in order to comply with the compensator model, be defined as [8]

$$H_C(z) = \begin{bmatrix} \frac{1}{z}I & 0 \\ 0 & H(z) \end{bmatrix}, \tag{8}$$

so by substituting Eq. (8) into Eq. (7), we get the compensator transfer matrix

$$\tilde{H}(z) = \begin{bmatrix} I - \frac{1}{z}A_f & -\frac{1}{z}B_f \\ -H(z)C_f & I - H(z)D_f \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{z}I & 0 \\ 0 & H(z) \end{bmatrix}. \tag{9}$$

Worth of explicit note is that when the identification of the open-loop system is conducted in the frequency domain, the implementation of the virtual compensator follows directly. If, however, the system is identified in the time-domain, one must transform to z-domain in order to calculate Eq. (9). An approach for this is provided in [8], and it is based on mapping observer Markov parameters to $H(z)$. Furthermore, it should be noted that the inverse z-transformation filters unstable poles, hence removing the system stability constraint required in physical testing.

4 Fisher Information on the Parameters

The amount of information that the closed-loop system carries on the system parameters to be updated, gathered in $\theta \in \mathbb{R}^s$, can be assessed, qualitatively, by use of the Fisher information matrix, $\mathcal{I} \in \mathbb{C}^{s \times s}$.

Let Y be an observable variable, carrying information on θ, then the Fisher information matrix can be expressed in terms of the likelihood function, f, with the entries

$$\mathcal{I}_{j,k} = -E\left(\frac{\partial^2 \ln f(Y;\theta)}{\partial\theta_j \partial\theta_k}\right) \quad \forall j,k \in [1,s], \tag{10}$$

where E is the expectation operator. Assuming $Y \sim \mathcal{N}(\mu(\theta), \Sigma)$, the Fisher information matrix can be expressed as [13]

$$\mathcal{I} = \mathcal{J}^H \Sigma^{-1} \mathcal{J}, \tag{11}$$

where superscript H indicates the conjugate transpose, $\Sigma \in \mathbb{C}^{s \times s}$ is the covariance matrix of Y, and $\mathcal{J} \in \mathbb{C}^{q+m \times s}$ is the Jacobian matrix containing the first-order derivatives of Y with respect to θ. If Y contains $q+m$ poles of the closed-loop system, which will be the case in this study, the Jacobian is given as

$$\mathcal{J} = \begin{bmatrix} \frac{\partial\lambda_1}{\partial\theta_1} & \frac{\partial\lambda_1}{\partial\theta_2} & \cdots & \frac{\partial\lambda_1}{\partial\theta_s} \\ \frac{\partial\lambda_2}{\partial\theta_1} & \frac{\partial\lambda_2}{\partial\theta_2} & \cdots & \frac{\partial\lambda_2}{\partial\theta_s} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\lambda_{q+m}}{\partial\theta_1} & \frac{\partial\lambda_{q+m}}{\partial\theta_2} & \cdots & \frac{\partial\lambda_{q+m}}{\partial\theta_s} \end{bmatrix} \in \mathbb{C}^{q+m \times s} \tag{12}$$

when λ_i denotes the ith pole.

One gathers that the Fisher information can be used to assess the qualitative implication of the model order, q, as it is contented that an increase in Fisher information on the parameters will improve the estimation of these in model updating schemes. Another way to appreciate this is from the Cramér-Rao lower bound, $\mathcal{C} \in \mathbb{C}^{s \times s}$, which is defined as

$$\mathcal{C} = \mathcal{I}^{-1} \tag{13}$$

and composes a measure of the minimum covariance that any unbiased estimator of θ can achieve. In other words, $\mathcal{C}$ provides a lower bound on the variance by which we can estimate each component in θ, and one anticipates that this variance will decrease asymptotically as q increases.

5 Model Updating using Virtual Compensator

The estimation of the parameters in $\theta \in \mathbb{R}^s$ is formulated as the constrained optimization problem

$$\begin{aligned} \underset{\theta \in \mathbb{R}^s}{\text{argmin}} \quad & ||\Lambda_{\mathscr{M}}(\theta) - \Lambda_{\mathscr{P}}|| \\ \text{subject to} \quad & \forall i \in [1,\text{s}] : \alpha_i \leq \theta_i \leq \beta_i, \end{aligned} \tag{14}$$

where $\Lambda_{\mathscr{P}} \in \mathbb{C}^{q+m}$ is a subset of poles estimated from the physical system, $\Lambda_{\mathscr{M}}(\theta) \in \mathbb{C}^{q+m}$ the corresponding model-predicted poles, and $\alpha_i, \beta_i \in \mathbb{R}$ lower and upper bounds on θ_i. The idea is to ensure that $q + m \geq s$, such that the system to be solved in the optimization scheme is (over)determined, which can be achieved by stacking identified poles from multiple closed-loop systems into a target vector [14], or by increasing the compensator order. In this paper, we opt for the latter and use the least square method described in [11] to design a single gain configuration to place $q + m \geq s$ poles, hence yielding the target vector

$$\Lambda_{\mathscr{P}} = \{\lambda_{\mathscr{P}_1}, \ldots, \lambda_{\mathscr{P}_{q+m}}\}^T \in \mathbb{C}^{q+m}. \tag{15}$$

Since the least square procedure uses arbitrary complex constants to collapse the eigenvector basis, different gain configurations yielding the same pole subset placement are achievable. One approach to investigate the error in the realization of the closed-loop poles, using the virtual dynamic approach with the gain $\tilde{G}$, is through the poles' sensitivity with respect to some parameter, g, of the gain, that is,

$$\frac{\partial \lambda_j}{\partial g} = \psi_j^T \frac{\partial \mathscr{A}_{cs}}{\partial g} \phi_j = \psi_j^T \tilde{B} \frac{\partial \tilde{G}}{\partial g} \tilde{C} \phi_j \tag{16}$$

where ψ_j and ϕ_j are the jth left and right eigenvectors of the compensator state matrix, $\mathscr{A}_{cs}$, presented in Eq. (3). In order to omit gains that cause undue error, we use the heuristic gain selection procedure proposed in [10], which circumvents calculation of the gain derivative, to select from a pool of candidates. In particular, we use the initial model of the closed-loop system to define the metric

$$\gamma_i = \|\Psi_{\mathscr{M}_i} \tilde{B}\| \|\tilde{C} \Phi_{\mathscr{M}_i}\|, \tag{17}$$

where $\Psi_{\mathscr{M}_i}$ and $\Phi_{\mathscr{M}_i}$ are the left and right eigenvectors of the closed-loop system model with gain $\tilde{G}_i$.

6 Numerical Examination

We consider the 10-DOF shear building depicted in Fig 1 in the context of model updating for damage characterization. The example explores what influence the compensator order, q, has on the Fisher information and how this relates to the performance of the model updating when using a single closed-loop system.

The terms *simulation model* and *nominal model* are used to refer to, respectively, the model used for simulating the output for system identification and the numerical model to be updated. In this example, the simulation model is drawn from the manifold containing the virtual nominal model, which implies that in the absence of noise there is a set of parameters for which the model to be updated coincides with the simulation model. This will, of course, not be realizable in practice. The simulation model, which is assigned a 20% stiffness perturbation in the 6th floor, is excited by white Gaussian noise in the first floor, and the resulting displacements are measured in the 1st, 5th, and 9th floor for a

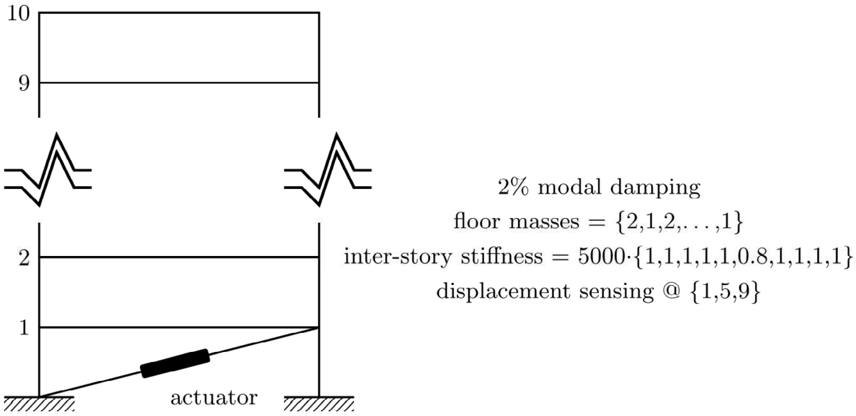

Fig. 1. 10-DOF shear building, where the inter-story stiffness and floor masses are in any consistent set of units, used for numerical illustrations.

duration of 5 minutes with a sampling frequency of 100 Hz. The displacements are contaminated with 2% white Gaussian noise, and the open-loop system identification is performed using the Eigensystem Realization Algorithm [15].

We consider compensator orders of 7, 11, and 15, which result in target vectors with 10, 14, and 18 poles, hence making the optimization problem determined for $q = 7$ and overdetermined for $q = 11$ and $q = 15$. The gain is designed such that a subset of the closed-loop nominal model poles is assigned as $\{\lambda_1, \frac{3}{2}\lambda_1, \frac{4}{2}\lambda_1, \ldots, \frac{q+m+1}{2}\lambda_1\}$, with λ_1 being the first pole of the open-loop nominal model. To compute the Fisher information for each configuration of the compensator order, we estimate the covariance matrix in a Monte Carlo setting with 100 simulations and, subsequently, calculate the Fisher information using Eq. 11. Fig. 2 presents the condition number of each Fisher information matrix,

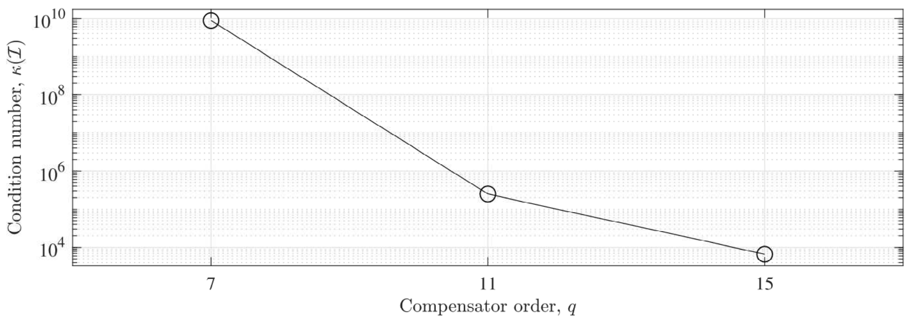

Fig. 2. Condition number of the Fisher information matrix for different configurations of the compensator order, q.

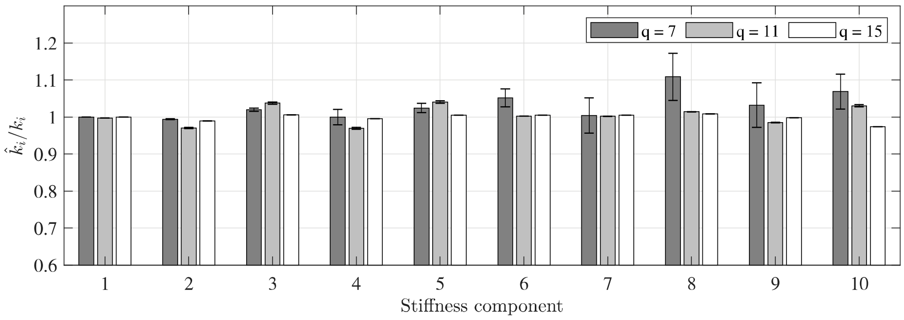

Fig. 3. Model updating results for the shear building for three configurations of q, with $\hat{k}_i$ and k_i being the converged estimate and true value of the stiffness components.

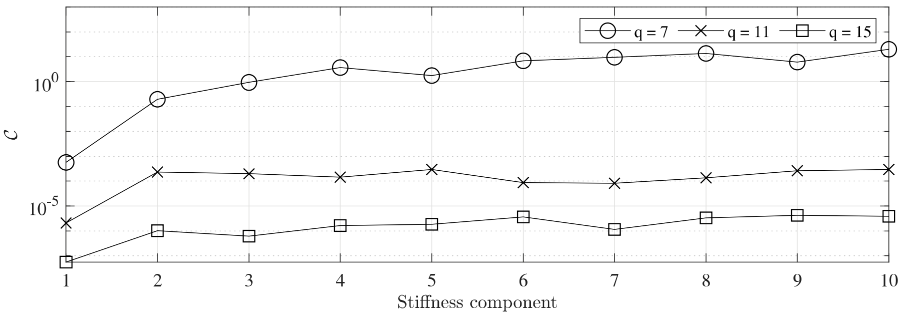

Fig. 4. Cramér-Rao lower bound on the individual parameters to be estimated for three compensator order configurations.

which decreases as q increases. Further numerical examination has confirmed that the decrease is asymptotic.

For the model updating, the target vectors are formed as $\Lambda_{\mathscr{P}} \in \mathbb{C}^{q+m}$ where the corresponding poles from the nominal models, $\Lambda_{\mathscr{M}}(\theta) \in \mathbb{C}^{q+m}$, are taken as the ones yielding the lowest discrepancy to the identified poles. The nominal models are updated for each realization of the target vectors by use of the "fmincon" algorithm in MATLAB® to solve the optimization problem in Eq. (14). It is assumed that no prior knowledge of the perturbation location exists, thus θ contains all the inter-story stiffness values in the shear building model.

The mean and variance of the converged estimates, which are normalized with respect to the true stiffness, are visualized in Fig. 3. The parameters are estimated with a mean absolute percentage error of 3.16%, 2.06%, and 0.37% for a compensator order of 7, 11, and 15, thus the Fisher information increase is reflected in the estimation accuracy. One gathers from Fig. 3 that the variance decreases as q increases, which is in agreement with the tendency of the Cramér-Rao lower bounds depicted in Fig. 4. For each model order, we also find reasonable correlation between which parameters that have the lowest Cramér-

Rao lower bounds and what is observed in terms of relative estimation variance for the parameters in the model updating.

7 Conclusion

The paper addresses model updating by use of a recently proposed virtual approach based on processing open-loop signals to form closed-loop compensators. In particular, we explore the feasibility of resolving the issue of ill-conditioned model updating by (over)determining the system of equations to be solved through an increase of the compensator order.

Model updating of a shear building in a Monte Carlo setting shows the mean of the estimated parameters to approach (asymptotically) the true values when increasing the compensator order. Furthermore, the variance of the estimated parameters and the condition number of the Fisher information matrix decrease asymptotically as the compensator order increases.

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