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Application of the Riemann–Hilbert approach to the multicomponent AKNS integrable hierarchies

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A class of Riemann–Hilbert problems on the real axis is formulated for solving the multicomponent AKNS integrable hierarchies associated with a kind of bock matrix spectral problems. Through special Riemann–Hilbert problems where a jump matrix is taken to be the identity matrix, soliton solutions to all integrable equations in each hierarchy are explicitly computed. A class of specific reductions of the presented integrable hierarchies is also made, together with its reduced Lax pairs and soliton solutions.

1. Introduction

The inverse scattering transformation and the Riemann–Hilbert approach are among powerful approaches for studying integrable equations in soliton theory. The first approach gives a generalized Fourier method, which attempts Cauchy problems of integrable equations [1], and the second approach provides an equivalent but more direct technique to solve integrable equations, which particularly generates soliton solutions [2].

The Riemann–Hilbert approach starts from a kind of matrix spectral problems possessing bounded eigenfunctions analytically expendable to the upper or lower half-plane. The normalization conditions at infinity on the real axis in determining the scattering coefficients is used in solving the associated Riemann–Hilbert problems [2]. Once taking a jump matrix to be the identity matrix, reduced Riemann–Hilbert problems engender soliton solutions, whose special limits can yield lump solutions, periodic solutions and complexiton solutions. Quite a few integrable equations, including the multiple wave interaction equations [2],

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the general coupled nonlinear Schrödinger equations [3], the Harry Dym equation [4], the generalized Sasa–Satsuma equation [5], the Dullin–Gottwald–Holm equation [6] and a coupled mKdV system [7], have been attempted by solving the associated Riemann–Hilbert problems. Based on Riemann–Hilbert problems, a dressing method has also been developed to get soliton solutions through gauge transformations [8–10], and it has been generalized for Lax operators in the orthogonal and symplectic Lie algebras [11] and further developed in numerous publications (see, for example, [12,13]). Moreover, the long time asymptotics and the small dispersion limit of integrable equations can be explored through analyzing the asymptotics of related Riemann–Hilbert problems [14,15]. The technique developed in [14] is nowadays called the nonlinear steepest descent method for integrable equations.

The standard procedure for presenting Riemann–Hilbert problems on the real axis is as follows. We begin with a hierarchy of matrix spectral problems of the following form:

$$-i\phi_x = U\phi,\ -i\phi_{t_r} = V^{[r]}\phi,\ U = A(\lambda) + P(u,\lambda),\ V^{[r]} = B^{[r]}(\lambda) + Q^{[r]}(u,\lambda),\ r \geq 0,$$

where i is the unit imaginary number, λ is a spectral parameter, u is a potential, ϕ is an $m \times m$ matrix eigenfunction, $A, B^{[r]}$ are constant commuting $m \times m$ matrices, and $P, Q^{[r]}$ are trace-less $m \times m$ matrices. The compatibility condition of each pair in the hierarchy of matrix spectral problems is the zero curvature equation

$$U_{t_r} - V_x^{[r]} + i[U, V^{[r]}] = 0,$$

where $[\cdot,\cdot]$ is the matrix commutator. To establish a class of Riemann–Hilbert problems for this zero curvature equation, we adopt the following pair of equivalent matrix spectral problems

$$\psi_x = i[A(\lambda),\psi] + \check{P}(u,\lambda)\psi, \psi_{t_r} = i[B^{[r]}(\lambda),\psi] + \check{Q}^{[r]}(u,\lambda)\psi,$$

where ψ is an $m \times m$ matrix eigenfunction, $\check{P} = iP$ and $\check{Q}^{[r]} = iQ^{[r]}$. The commutativity of A and $B^{[r]}$ guarantees the required equivalence. The relation between the two matrix eigenfunctions ϕ and ψ is determined by

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{iA(\lambda)x + iB^{[r]}(\lambda)t_r}.$$

For the second class of matrix spectral problems above, we can find two bounded analytical matrix eigenfunctions with the canonical asymptotic conditions

$$\psi^{\pm}(x,t_r,\lambda) \to I_m,\ \text{when } x, t_r \to \pm\infty.$$

We assume throughout the paper that I_m stands for the identity matrix of size m, and that $\mathbb{C}^+$ denotes the upper half-plane: $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, and $\mathbb{C}^-$, the lower half-plane: $\mathbb{C}^- = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}$; and $\mathbb{C}_0^{\pm}$ are the closures of $\mathbb{C}^{\pm}$, respectively. Based on those two matrix eigenfunctions $\psi^{\pm}(x,t_r,\lambda)$, we can now determine two analytical matrix functions $P^{\pm}(x,t_r,\lambda)$, which are analytical in $\lambda \in \mathbb{C}^{\pm}$ and continuous in $\lambda \in \mathbb{C}_0^{\pm}$, respectively, to formulate a class of Riemann–Hilbert problems on the real axis:

$$G^+(x,t_r,\lambda) = G^-(x,t_r,\lambda)G(x,t_r,\lambda),\ \lambda \in \mathbb{R},$$

where

$$G^+(x,t_r,\lambda) = \lim_{\mu\in\mathbb{C}_0^+,\,\mu\to\lambda} P^+(x,t_r,\mu),\ (G^-)^{-1}(x,t_r,\lambda) = \lim_{\mu\in\mathbb{C}_0^-,\,\mu\to\lambda} P^-(x,t_r,\mu),\ \lambda \in \mathbb{R},$$

and

$$G(x,t_r,\lambda) = P^-(x,t_r,\lambda)P^+(x,t_r,\lambda),\ \lambda \in \mathbb{R}.$$

Upon taking the jump matrix G to be the identity matrix I_m, the resulting Riemann–Hilbert problems can be normally solved to generate soliton solutions, by observing asymptotic behaviors of the matrix functions $P^{\pm}$ at infinity of λ, which also provide the canonical normalization conditions for the presented Riemann–Hilbert problems. In practice, we only need to take the Riemann–Hilbert problems with the space variable into consideration, and determine the time dependence of soliton solutions through exploring the time dependence of vectors in the kernels of $P^{\pm}(x,\lambda)$ generated from the Riemann–Hilbert problems with the variable x.

In this paper, we aim to present an application of the Riemann–Hilbert approach to the multicomponent Ablowitz–Kaup–Newell–Segur (AKNS) integrable hierarchies. A practical procedure for obtaining soliton solutions consists of two steps: Step 1 is to formulate a kind of Riemann–Hilbert problems with the space variable from a spatial matrix spectral problem generating an integrable hierarchy, and Step 2 is to compute soliton solutions to each system in the integrable hierarchy explicitly by solving special associated Riemann–Hilbert problems. This is a similar study on integrable hierarchies to the algebro-geometric solutions to integrable hierarchies [16,17], but higher-order matrix problems bring huge difficulty in presenting exact solutions explicitly and we only did algebro-geometric solutions associated with lower-order matrix spectral problems. Instead of algebraic curves, we would like to use the Riemann–Hilbert technique to present soliton solutions in this paper.

The rest of the paper is organized as follows. In Section 2, within the zero-curvature formulation, we rederive the multicomponent AKNS integrable hierarchies and represent their bi-Hamiltonian structures for ease of reference, based on a kind of new arbitrary order matrix spectral problems suited for the Riemann–Hilbert theory. In Section 3, for all multicomponent systems in each resulting AKNS integrable hierarchy, we analyze analytical properties of matrix eigenfunctions for equivalent spatial matrix spectral problems, and build a class of Riemann–Hilbert problems associated with the newly introduced spatial matrix spectral problems. In Section 4, we compute soliton solutions to the multicomponent AKNS systems from special associated Riemann–Hilbert problems on the real axis, in which a jump matrix is taken to be the identity matrix. In Section 5, we consider a class of specific reductions, and generate soliton solutions to the reduced multicomponent AKNS integrable hierarchies by the reduced Riemann–Hilbert problems with the identity jump matrix. In the last section, we give concluding remarks, along with some further questions.

2. Multi-component AKNS hierarchies revisited

2.1. Zero curvature formulation

The zero curvature formulation to build integrable hierarchies is outlined as follows [18–20]. Let u be a vector potential and λ, a spectral parameter. Choose a square matrix spectral matrix $U=U(u,\lambda)$ from a given matrix loop algebra, whose underlying Lie algebra could be either semisimple [18,19] or non-semisimple [20]. Assume that there is a formal series solution

$$W=W(u,\lambda)=\sum_{m=0}^{\infty}W_m\lambda^{-m}=\sum_{m=0}^{\infty}W_m(u)\lambda^{-m} \tag{2.1}$$

to the corresponding stationary zero curvature equation

$$W_x=i[U,W]. \tag{2.2}$$

Often W is uniquely determined as long as the initial matrix W_0 is fixed. Using this solution W, we define a series of Lax matrices

$$V^{[r]}=V^{[r]}(u,\lambda)=(\lambda^r W)_+ + \Delta_r,\ r\geq 0, \tag{2.3}$$

where the subscript $+$ stands for the operation of taking a polynomial part in λ, and Δ_r, $r\geq 0$, are some well-selected modification terms. The appropriateness of selecting Δ_r is required to generate an integrable hierarchy

$$u_{t_r}=K_r(u)=K_r(x,t,u,u_x,\ldots),\ r\geq 0, \tag{2.4}$$

from a series of zero curvature equations

$$U_{t_r} - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0, \tag{2.5}$$

successfully. The two matrices U and $V^{[r]}$ are called a Lax pair [21] of the rth evolution equation in the hierarchy (2.4). Clearly, the zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_{t_r} = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi,\ r \geq 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

In order to explore the Liouville integrability of the hierarchy (2.4), we normally furnish a bi-Hamiltonian structure [22]:

$$u_{t_r} = K_r = J_1 \frac{\delta \tilde{H}_{r+1}}{\delta u} = J_2 \frac{\delta \tilde{H}_r}{\delta u},\ r \geq 1, \tag{2.7}$$

where J_1 and J_2 form a Hamiltonian pair and $\frac{\delta}{\delta u}$ denotes the variational derivative (see, e.g., [23]). Such Hamiltonian structures can be usually furnished under the help of the trace identity [18]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^\gamma \mathrm{tr}(W\frac{\partial U}{\partial u})\Big],\ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|,$$

or more generally, the variational identity [20]:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial \lambda}\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^\gamma \langle W, \frac{\partial U}{\partial u}\rangle\Big],\ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|,$$

where $\langle \cdot,\cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [24]. The bi-Hamiltonian structure ensures that there exist infinitely many commuting Lie symmetries $\{K_r\}_{r=0}^\infty$ and conserved quantities $\{\tilde{H}_r\}_{r=0}^\infty$:

$$[K_{r_1}, K_{r_2}] = K'_{r_1}[K_{r_2}] - K'_{r_2}[K_{r_1}] = 0, \tag{2.8}$$

$$\{\tilde{\mathcal{H}}_{r_1}, \tilde{\mathcal{H}}_{r_2}\}_J = \int \Big(\frac{\delta \tilde{\mathcal{H}}_{r_1}}{\delta u}\Big)^T J \frac{\delta \tilde{\mathcal{H}}_{r_2}}{\delta u}\, dx = 0, \tag{2.9}$$

where $r_1, r_2 \geq 0$, $J = J_1$ or J_2, and K' denotes the Gateaux derivative of K with respect to u: $K'(u)[S] = \frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0} K(u+\varepsilon S, u_x + \varepsilon S_x, \ldots)$.

It is known that for an evolution equation with a vector potential u, $\tilde{H} = \int H\, dx$ is a conserved functional iff $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry [25,26], and thus, a Hamiltonian structure links conserved functionals to adjoint symmetries and further symmetries. The existence of an adjoint symmetry is necessary for a totally nondegenerate system of differential equations to admit a conservation law, and a pair of a symmetry and an adjoint symmetry leads to a conservation law for whatever systems of differential equations [26,27]. We point out that when the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations engender typical integrable hierarchies [28,10]; when semisimple, the associated zero curvature equations generate a collection of different integrable hierarchies; and when non-semisimple, we get hierarchies of integrable couplings [29], which require extra care in presenting Hamiltonian structures.

2.2. AKNS hierarchies with multiple potentials

We would like to show the process of generating multicomponent AKNS integrable hierarchies below to see how one presents the Lax pairs and bi-Hamiltonian structures of integrable hierarchies for later use in establishing Riemann–Hilbert problems.

Let n be an arbitrary natural number. We consider the following matrix spectral problem of $n+1$ order:

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ U = (U_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.10}$$

where α_1 and α_2 are real constants, λ is a spectral parameter and u is a $2n$-dimensional potential

$$u = (p, q^T)^T,\ p = (p_1, p_2, \ldots, p_n),\ q = (q_1, q_2, \ldots, q_n)^T. \tag{2.11}$$

A special case of $p_j = q_j = 0,\ 2 \le j \le n$, transforms (2.10) into the standard AKNS matrix spectral problem [30], and therefore (2.10) is called a multicomponent AKNS matrix spectral problem and its associated hierarchy, a multicomponent AKNS integrable hierarchy. A difference from the matrix spectral problem discussed in [31] is an introduction of the unit imaginary number i in (2.10). Because of the existence of a multiple eigenvalue of $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, the matrix spectral problem (2.10) is degenerate.

To rederive the associated multicomponent AKNS integrable hierarchies, we first solve the stationary zero curvature equation (2.2) corresponding to (2.10), as suggested in the general zero curvature formulation. We seek a solution W of the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.12}$$

where a is a scalar, b^T and c are n-dimensional columns, and d is an $n \times n$ matrix. It is direct to see that the stationary zero curvature equation (2.2) becomes

$$\left[\begin{array}{l} a_x = i(pc - bq), \\ b_x = i(\alpha\lambda b + pd - ap), \\ c_x = i(-\alpha\lambda c + qa - dq), \\ d_x = i(qb - cp), \end{array}\right. \tag{2.13}$$

where $\alpha = \alpha_1 - \alpha_2$. We expand W as a formal series:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m},\ W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix},\ m \ge 0, \tag{2.14}$$

where $b^{[m]}, c^{[m]}$ and $d^{[m]}$ are expressed as

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}, \ldots, b_n^{[m]}),\ c^{[m]} = (c_1^{[m]}, c_2^{[m]}, \ldots, c_n^{[m]})^T,\ d^{[m]} = (d_{jl}^{[m]})_{n\times n},\ m \ge 0. \tag{2.15}$$

Then, the system (2.13) equivalently generates the following recursion relations:

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a_x^{[0]} = 0,\ d_x^{[0]} = 0, \tag{2.16a}$$

$$b^{[m+1]} = \frac{1}{\alpha}(-ib_x^{[m]} - pd^{[m]} + a^{[m]}p),\ m \ge 0, \tag{2.16b}$$

$$c^{[m+1]} = \frac{1}{\alpha}(ic_x^{[m]} + qa^{[m]} - d^{[m]}q),\ m \ge 0, \tag{2.16c}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q),\ d_x^{[m]} = i(qb^{[m]} - c^{[m]}p),\ m \ge 1. \tag{2.16d}$$

Next we take the initial values:

$$a^{[0]} = \beta_1,\ d^{[0]} = \beta_2 I_n, \tag{2.17}$$

where β_1, β_2 are arbitrary real constants, and set constants of integration in (2.16d) to zero, that is, demand

$$W_m|_{u=0} = 0,\ m \ge 1. \tag{2.18}$$

This way, with $a^{[0]}$ and $d^{[0]}$ being given by (2.17), all matrices $W_m,\ m \geq 1$, are uniquely determined. For example, a direct calculation, based on (2.16), presents that

$$b_j^{[1]} = \frac{\beta}{\alpha} p_j,\ c_j^{[1]} = \frac{\beta}{\alpha} q_j,\ a^{[1]} = 0,\ d_{jl}^{[1]} = 0; \tag{2.19a}$$

$$b_j^{[2]} = -\frac{\beta}{\alpha^2} i p_{j,x},\ c_j^{[2]} = \frac{\beta}{\alpha^2} i q_{j,x},\ a^{[2]} = -\frac{\beta}{\alpha^2} pq,\ d_{jl}^{[2]} = \frac{\beta}{\alpha^2} p_l q_j; \tag{2.19b}$$

$$b_j^{[3]} = -\frac{\beta}{\alpha^3}[p_{j,xx} + 2pqp_j],\ c_j^{[3]} = -\frac{\beta}{\alpha^3}[q_{j,xx} + 2pqq_j], \tag{2.19c}$$

$$a^{[3]} = -\frac{\beta}{\alpha^3} i(pq_x - p_x q),\ d_{jl}^{[3]} = -\frac{\beta}{\alpha^3} i(p_{l,x} q_j - p_l q_{j,x}); \tag{2.19d}$$

$$b_j^{[4]} = \frac{\beta}{\alpha^4} i[p_{j,xxx} + 3pqp_{j,x} + 3p_x q p_j], \tag{2.19e}$$

$$c_j^{[4]} = -\frac{\beta}{\alpha^4} i[q_{j,xxx} + 3pqq_{j,x} + 3pq_x q_j], \tag{2.19f}$$

$$a^{[4]} = \frac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_x q_x + p_{xx} q], \tag{2.19g}$$

$$d_{jl}^{[4]} = -\frac{\beta}{\alpha^4}[3p_l pqq_j + p_{l,xx} q_j - p_{l,x} q_{j,x} + p_l q_{j,xx}]; \tag{2.19h}$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq j, l \leq n$. Based on (2.16d), we can get, from (2.16b) and (2.16c), a recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m \geq 1, \tag{2.20}$$

where Ψ is a $2n \times 2n$ matrix integro-differential operator

$$\Psi = \frac{i}{\alpha} \begin{bmatrix} (\partial + \sum_{j=1}^{n} q_j \partial^{-1} p_j) I_n + q\partial^{-1} p & -q\partial^{-1} q^T - (q\partial^{-1} q^T)^T \\ p^T \partial^{-1} p + (p^T \partial^{-1} p)^T & -(\partial + \sum_{j=1}^{n} p_j \partial^{-1} q_j) I_n - p^T \partial^{-1} q^T \end{bmatrix}. \tag{2.21}$$

To derive the multicomponent AKNS integrable hierarchies, we take the following Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{jl}^{[r]})_{(n+1)\times(n+1)} = (\lambda^r W)_+ = \sum_{m=0}^{r} W_m \lambda^{r-m},\ r \geq 0, \tag{2.22}$$

where the modification terms are all set to zero and each W_m is defined in (2.14). The compatibility conditions of (2.6), i.e., the zero curvature equations (2.5), engender the so-called multicomponent AKNS integrable hierarchies:

$$u_{t_r} = \begin{bmatrix} p^T \\ q \end{bmatrix}_{t_r} = K_r = i \begin{bmatrix} \alpha b^{[r+1]T} \\ -\alpha c^{[r+1]} \end{bmatrix},\ r \geq 0. \tag{2.23}$$

The first two nonlinear integrable systems in the above hierarchies (2.23) are as follows:

$$p_{j,t_2} = -\frac{\beta}{\alpha^2} i[p_{j,xx} + 2(\sum_{l=1}^{n} p_l q_l) p_j],\ 1 \leq j \leq n, \tag{2.24a}$$

$$q_{j,t_2} = \frac{\beta}{\alpha^2} i[q_{j,xx} + 2(\sum_{l=1}^{n} p_l q_l) q_j],\ 1 \leq j \leq n, \tag{2.24b}$$

and

$$p_{j,t_3} = -\frac{\beta}{\alpha^3}[p_{j,xxx} + 3(\sum_{l=1}^{n} p_l q_l) p_{j,x} + 3(\sum_{l=1}^{n} p_{l,x} q_l) p_j],\ 1 \leq j \leq n, \tag{2.25a}$$

$$q_{j,t_3} = -\frac{\beta}{\alpha^3}[q_{j,xxx} + 3(\sum_{l=1}^{n} p_l q_l) q_{j,x} + 3(\sum_{l=1}^{n} p_l q_{l,x}) q_j],\ 1 \leq j \leq n, \tag{2.25b}$$

where $n \geq 1$. These are the multicomponent versions of the AKNS systems of coupled nonlinear Schrödinger (NLS) equations and coupled modified Korteweg–de Vries (mKdV) equations, respectively. When $n = 2$, under a special kind of symmetric reductions, the multicomponent AKNS systems (2.24) can be reduced to the Manakov system [32], for which a decomposition into finite-dimensional integrable Hamiltonian systems was made in [33], whileas the multicomponent AKNS systems (2.25) contain various systems of mKdV equations, for which there exist different kinds of integrable decompositions originated from symmetry constraints (see, e.g., [34,35]). A relation between the multicomponent NLS systems (2.24) and symmetric spaces has been noticed for the first time in [36], and further, more general multicomponent NLS systems have been studied, on the basis of symmetric spaces (see, for example, [37,38]).

The multicomponent AKNS integrable hierarchies (2.23) possess bi-Hamiltonian structures [25,31], which can be presented through applying the trace identity [18], or more generally, the variational identity [20]. Actually, we have

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = \alpha_1 a + \alpha_2 \mathrm{tr}(d) = \sum_{m=0}^{\infty}(\alpha_1 a^{[m]} + \alpha_2 \sum_{j=1}^{n} d_{jj}^{[m]})\lambda^{-m},$$

and

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial u}) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{m\geq 0} G_{m-1}\lambda^{-m}.$$

Plugging these into the trace identity and checking the case of $m = 2$ tells $\gamma = 0$ in the trace identity, and thus, we have

$$\frac{\delta \tilde{H}_m}{\delta u} = iG_{m-1},\ \tilde{H}_m = -\frac{i}{m}\int(\alpha_1 a^{[m+1]} + \alpha_2 \sum_{j=1}^{n} d_{jj}^{[m+1]})\,dx,\ G_{m-1} = \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix},\ m \geq 1. \tag{2.26}$$

This tells the following bi-Hamiltonian structures for the multicomponent AKNS systems in (2.23):

$$u_{t_r} = K_r = J_1 G_r = J_1 \frac{\delta \tilde{H}_{r+1}}{\delta u} = J_2 \frac{\delta \tilde{H}_r}{\delta u},\ r \geq 1, \tag{2.27}$$

where each Hamiltonian pair $(J_1, J_2 = J_1 \Psi)$ is given by

$$J_1 = \begin{bmatrix} 0 & \alpha I_n \\ -\alpha I_n & 0 \end{bmatrix}, \tag{2.28a}$$

$$J_2 = i\begin{bmatrix} p^T\partial^{-1}p + (p^T\partial^{-1}p)^T & -(\partial + \sum_{j=1}^{n} p_j\partial^{-1}q_j)I_n - p^T\partial^{-1}q^T \\ -(\partial + \sum_{j=1}^{n} p_j\partial^{-1}q_j)I_n - q\partial^{-1}p & q\partial^{-1}q^T + (q\partial^{-1}q^T)^T \end{bmatrix}. \tag{2.28b}$$

Thus, each of the operators $\Phi = \Psi^\dagger = J_2 J_1^{-1}$ presents a recursion operator [39] for every hierarchy with a fixed integer $n \geq 1$ in (2.23). Adjoint symmetry constraints (or equivalently symmetry constraints) decompose each multicomponent AKNS system into two commuting finite-dimensional Liouville integrable Hamiltonian systems [31].

3. Associated Riemann–Hilbert problems

Let us fix the integer $n \geq 1$ and begin with the matrix spectral problems of the rth multicomponent AKNS system (2.23):

$$-\,i\phi_x = U\phi = U(u,\lambda)\phi, \tag{3.1}$$

$$-\,i\phi_{t_r} = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \tag{3.2}$$

where the Lax pair reads

$$U = \lambda \Lambda + P,\ V^{[r]} = \lambda^r \Omega + Q^{[r]}, \tag{3.3}$$

with $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1, \beta_2 I_n)$, and

$$P = \left[\begin{array}{cc} 0 & p \\ q & 0 \end{array}\right],\ Q^{[r]} = \sum_{m=1}^{r} W_m \lambda^{r-m} = \sum_{m=1}^{r} \left[\begin{array}{cc} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{array}\right] \lambda^{r-m}. \tag{3.4}$$

Here u, p, q are defined by (2.11), and $a^{[m]}, b^{[m]}, c^{[m]}, d^{[m]}$, $1 \le m \le r$, are determined in (2.19).

In what follows, we discuss the scattering and inverse scattering for the multicomponent AKNS system (2.23) using the Riemann–Hilbert approach [2] (see also [9,10,38]). The resulting results will lay the groundwork for soliton solutions in the next section. Assume that all the potentials rapidly vanish when $x \to \pm\infty$ or $t_r \to \pm\infty$ and satisfy the integrable conditions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^{m_1} |t_r|^{m_2} \sum_{j=1}^{n} (|p_j| + |q_j|)\, dx dt_r < \infty,\ m_1, m_2 = 0, 1. \tag{3.5}$$

In order to facilitate the expression, we also assume that

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \beta = \beta_1 - \beta_2 < 0. \tag{3.6}$$

For the matrix spectral problems (3.1) and (3.2), we note, under (3.5), that when $x, t_r \to \pm\infty$, we have the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda \Lambda x + i\lambda^r \Omega t_r}$. Therefore, upon making the variable transformation

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{i\lambda \Lambda x + i\lambda^r \Omega t_r}, \tag{3.7}$$

we can have the canonical normalization $\psi \to I_{n+1}$, when $x, t_r \to \pm\infty$. Similarly as suggested in the introduction, set $\check{P} = iP$ and $\check{Q}^{[r]} = iQ^{[r]}$, and the equivalent pair of matrix spectral problems to (3.1) and (3.2) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.8}$$

$$\psi_{t_r} = i\lambda^r[\Omega, \psi] + \check{Q}^{[r]}\psi, \tag{3.9}$$

Applying a generalized Liouville's formula [40] leads to

$$\det \psi = 1, \tag{3.10}$$

due to $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$.

Let us now present a class of associated Riemann–Hilbert problems with the variable x. In the scattering problem, we first take two matrix solutions $\psi^{\pm}(x, \lambda)$ of (3.8) with the asymptotic conditions

$$\psi^{\pm} \to I_{n+1},\ \text{when } x \to \pm\infty, \tag{3.11}$$

respectively. The above superscripts refer to which end of the x-axis the boundary conditions are required for. By (3.10), we find that $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since

$$\phi^{\pm} = \psi^{\pm} E,\ E = \mathrm{e}^{i\lambda \Lambda x}, \tag{3.12}$$

are both matrix solutions of (3.1), they are linearly dependent, and hence, we have

$$\psi^{-} E = \psi^{+} E S(\lambda),\ \lambda \in \mathbb{R}, \tag{3.13}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is the scattering matrix. Note that $\det S(\lambda) = 1$ because of $\det \psi^{\pm} = 1$.

Then applying the method of variation of parameters, we can transform the matrix spectral problem (3.8) into the following Volterra integral equations for $\psi^{\pm}$ [2]:

$$\psi^{-}(\lambda,x)=I_{n+1}+\int_{-\infty}^{x}\mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^{-}(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.14}$$

$$\psi^{+}(\lambda,x)=I_{n+1}-\int_{x}^{\infty}\mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^{+}(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.15}$$

where the boundary condition (3.11) has been used. Thus, $\psi^{\pm}$ allows analytical continuations off the real axis $\lambda\in\mathbb{R}$ provided that the integrals on their right hand sides converge. Based on the diagonal form of Λ and the first assumption $\alpha<0$ in (3.6), we can readily see that the integral equation for the first column of ψ^{-} involves only the exponential factor $\mathrm{e}^{-i\alpha\lambda(x-y)}$, which decays because of $y<x$ in the integral, when λ is in the closed upper half-plane $\mathbb{C}_0^{+}$, and the integral equation for the last n columns of ψ^{+} involves only the exponential factor $\mathrm{e}^{i\alpha\lambda(x-y)}$, which also decays because of $y>x$ in the integral, when λ is in the closed upper half-plane $\mathbb{C}_0^{+}$. Therefore, those $n+1$ columns can be analytically continued to the closed upper half-plane $\mathbb{C}_0^{+}$. Similarly, we can see that the last n columns of ψ^{-} and the first column of ψ^{+} can be analytically continued to the closed lower half-plane $\mathbb{C}_0^{-}$.

Below we would like to determine two matrix eigenfunctions $P^{\pm}(x,\lambda)$, which are analytically continued to the upper and lower half-planes, respectively. First, let us express

$$\psi^{\pm}=(\psi_1^{\pm},\psi_2^{\pm},\ldots,\psi_{n+1}^{\pm}), \tag{3.16}$$

namely, $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1\le j\le n+1)$, and then the matrix solution

$$P^{+}=P^{+}(x,\lambda)=(\psi_1^{-},\psi_2^{+},\ldots,\psi_{n+1}^{+})=\psi^{-}H_1+\psi^{+}H_2 \tag{3.17}$$

is analytic in $\lambda\in\mathbb{C}^{+}$ and continuous in $\lambda\in\mathbb{C}_0^{+}$, and the matrix solution

$$(\psi_1^{+},\psi_2^{-},\ldots,\psi_{n+1}^{-})=\psi^{+}H_1+\psi^{-}H_2 \tag{3.18}$$

is analytic in $\lambda\in\mathbb{C}^{-}$ and continuous in $\lambda\in\mathbb{C}_0^{-}$, where H_1 and H_2 are defined by

$$H_1=\mathrm{diag}(1,\underbrace{0,\ldots,0}_{n}),\ \ H_2=\mathrm{diag}(0,\underbrace{1,\ldots,1}_{n}). \tag{3.19}$$

Moreover, from the Volterra integral equations (3.14) and (3.15), we can find that

$$P^{+}(x,\lambda)\to I_{n+1},\ \text{ when } \lambda\in\mathbb{C}_0^{+}\to\infty, \tag{3.20}$$

and

$$(\psi_1^{+},\psi_2^{-},\ldots,\psi_{n+1}^{-})\to I_{n+1},\ \text{ when } \lambda\in\mathbb{C}_0^{-}\to\infty. \tag{3.21}$$

Secondly, we construct the analytic counterpart of P^{+} defined in the lower half-plane $\mathbb{C}^{-}$ from the adjoint matrix spectral problems. The adjoint equations of (3.1) and (3.8) are defined by

$$i\tilde{\phi}_x=\tilde{\phi}U, \tag{3.22}$$

and

$$i\tilde{\psi}_x=\lambda[\tilde{\psi},\Lambda]+\tilde{\psi}P, \tag{3.23}$$

respectively. Since $\phi^{\pm}$ and $\psi^{\pm}$ are solutions to (3.1) and (3.8), the inverse matrices $\tilde{\phi}^{\pm}=(\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm}=(\psi^{\pm})^{-1}$ solve the above two adjoint equations, respectively. Upon expressing $\tilde{\psi}^{\pm}$ as follows:

$$\tilde{\psi}^{\pm}=(\tilde{\psi}^{\pm,1},\tilde{\psi}^{\pm,2},\ldots,\tilde{\psi}^{\pm,n+1})^{T}, \tag{3.24}$$

namely, $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^{\pm}$ $(1 \leq j \leq n+1)$, we can show in an analogous manner that the adjoint matrix solution

$$P^{-} = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \ldots, \tilde{\psi}^{+,n+1})^{T} = H_1\tilde{\psi}^{-} + H_2\tilde{\psi}^{+} = H_1(\psi^{-})^{-1} + H_2(\psi^{+})^{-1} \tag{3.25}$$

is analytic in $\lambda \in \mathbb{C}^{-}$ and continuous in $\lambda \in \mathbb{C}_0^{-}$, and the other matrix solution

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \ldots, \tilde{\psi}^{-,n+1})^{T} = H_1\tilde{\psi}^{+} + H_2\tilde{\psi}^{-} = H_1(\psi^{+})^{-1} + H_2(\psi^{-})^{-1} \tag{3.26}$$

is analytic in $\lambda \in \mathbb{C}^{+}$ and continuous in $\lambda \in \mathbb{C}_0^{+}$. Similarly, we can determine that

$$P^{-}(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \mathbb{C}_0^{-} \to \infty, \tag{3.27}$$

and

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \ldots, \tilde{\psi}^{-,n+1})^{T} \to I_{n+1}, \text{ when } \lambda \in \mathbb{C}_0^{+} \to \infty. \tag{3.28}$$

Now we have constructed the two matrix functions, $P^{+}(x,\lambda)$ and $P^{-}(x,\lambda)$, which, as functions of λ, are analytic in $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$, and continuous in $\mathbb{C}_0^{+}$ and $\mathbb{C}_0^{-}$, respectively. Further defining

$$G^{+}(x,\lambda) = \lim_{\mu \in \mathbb{C}_0^{+},\, \mu \to \lambda} P^{+}(x,\mu), \ (G^{-})^{-1}(x,\lambda) = \lim_{\mu \in \mathbb{C}_0^{-},\, \mu \to \lambda} P^{-}(x,\mu), \ \lambda \in \mathbb{R}, \tag{3.29}$$

we can directly verify that on the real line, the two matrix functions G^{+} and G^{-} are related to each other as follows:

$$G^{+}(x,\lambda) = G^{-}(x,\lambda)G(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.30}$$

where by (3.13), the jump matrix G can be computed as follows:

$$\begin{aligned} G(x,\lambda) &= E(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1} \\ &= E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ s_{n+1,1} & 0 & \cdots & 0 & 1 \end{bmatrix} E^{-1} \end{aligned} \tag{3.31}$$

with $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. The two equations in (3.30) and (3.31) exactly present the class of associated matrix Riemann–Hilbert problems we would like to build for the multicomponent AKNS systems in (2.23). The asymptotic properties

$$P^{\pm}(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \mathbb{C}_0^{\pm} \to \infty, \tag{3.32}$$

generate the canonical normalization conditions

$$G^{\pm}(x,\lambda) \to I_{n+1}, \text{ when } \lambda \in \mathbb{C}_0^{\pm} \to \infty, \tag{3.33}$$

for the above presented Riemann–Hilbert problems.

To complete the direct scattering transform, let us evaluate the derivative of (3.13) with time t_r and use the vanishing conditions of the potentials at infinity of t_r. Clearly, we can prove that the scattering matrix S has to satisfy

$$S_{t_r} = i\lambda^{r}[\Omega, S], \tag{3.34}$$

which tells that the time evolution of the time-dependent scattering coefficients are given by

$$s_{1,j} = s_{1,j}(\lambda,0)\mathrm{e}^{i\beta\lambda^{r}t_r}, \ s_{j,1} = s_{j,1}(\lambda,0)\mathrm{e}^{-i\beta\lambda^{r}t_r}, \ 2 \leq j \leq n+1, \tag{3.35}$$

and all other scattering coefficients are independent of the time variable t_r.

4. Soliton solutions by the Riemann–Hilbert problems

The Riemann–Hilbert problems with zeros yield soliton solutions and can be solved by turning into the ones without zeros [2]. The uniqueness of solutions to each associated Riemann–Hilbert problem, defined by (3.30) and (3.31), does not hold unless the zeros of $\det P^{\pm}$ in the upper and lower half-planes are specified and the structures of $\ker P^{\pm}$ at those zeros are determined [41–43].

Thanks to $\det \psi^{\pm} = 1$, it follows from the definitions of $P^{\pm}$ in (3.17) and (3.25), and the scattering relation between ψ^+ and ψ^- in (3.13) that

$$\det P^+(x,\lambda) = s_{11}(\lambda),\ \det P^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{4.1}$$

where, because of $\det S = 1$, we have

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & \cdots & s_{2,n+1} \\ s_{32} & s_{33} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}. \tag{4.2}$$

Let N be another arbitrary natural number and assume that the function s_{11} has N zeros $\{\lambda_k \in \mathbb{C}^+,\ 1 \le k \le N\}$, and the function $\hat{s}_{11}$ has N zeros $\{\hat{\lambda}_k \in \mathbb{C}^-,\ 1 \le k \le N\}$. To compute N-soliton solutions, we also assume that all those zeros, λ_k and $\hat{\lambda}_k,\ 1 \le k \le N$, are simple. Therefore, each of $\ker P^+(x,\lambda_k)$, $1 \le k \le N$, contains only a single basis column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker P^-(x,\hat{\lambda}_k)$, $1 \le k \le N$, a single basis row vector, denoted by $\hat{v}_k$, $1 \le k \le N$. So, we have

$$P^+(x,\lambda_k)v_k = 0,\ \hat{v}_k P^-(x,\hat{\lambda}_k) = 0,\ 1 \le k \le N. \tag{4.3}$$

It is known that the Riemann–Hilbert problems, determined by (3.30) and (3.31), with the canonical normalization conditions in (3.33) and the zero structures in (4.3) can be solved explicitly [2,44]. To compute N-soliton solutions, we take $G = I_{n+1}$ in each above Riemann–Hilbert problem. This can be realized if we take that $s_{j,1} = \hat{s}_{1,j} = 0,\ 2 \le j \le n+1$, which equivalently requires that no reflection exists in the scattering problem. This resulting special Riemann–Hilbert problem has the solutions (see, e.g., [2,44] for details):

$$P^+(x,\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ P^-(x,\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_l}, \tag{4.4}$$

where M is a square matrix with entries being defined by

$$M = (m_{kl})_{N\times N},\ m_{kl} = \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k},\ 1 \le k,l \le N. \tag{4.5}$$

Because the zeros λ_k and $\hat{\lambda}_k$, $1 \le k \le N$, are constants, i.e., space and time independent, we can easily determine the spatial and temporal evolutions for the vectors, $v_k(x,t_r)$ and $\hat{v}_k(x,t_r)$, $1 \le k \le N$, in the kernels $\ker P^{\pm}$. For example, let us compute the x-derivative of both sides of the first set of equations in (4.3). By using (3.8) first and then again the first set of equations in (4.3), we can derive

$$P^+(x,\lambda_k)\Big(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\Big) = 0,\ 1 \le k \le N. \tag{4.6}$$

It then follows that for each $1 \le k \le N$, the vector $\frac{dv_k}{dx} - i\lambda_k \Lambda v_k$ must be in the kernel of $P^+(x,\lambda_k)$ and so a constant multiple of the vector v_k. Without loss of generality, we consider the simplest case and assume

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k,\ 1 \le k \le N. \tag{4.7}$$

The time dependence of v_k:

$$\frac{dv_k}{dt_r} = i\lambda_k^r \Omega v_k,\ 1 \le k \le N, \tag{4.8}$$

can be worked out similarly by applying the temporal matrix spectral problem (3.9). To sum up, we can have

$$v_k(x,t_r) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^r \Omega t_r} w_k,\ 1 \le k \le N, \tag{4.9}$$

$$\hat{v}_k(x,t_r) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^r \Omega t_r},\ 1 \le k \le N, \tag{4.10}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively.

Finally, we can work out the potential matrix P from $P^+(x,\lambda)$ as follows. Note that P^+ is a solution to the matrix spectral problem (3.8). Therefore, once we expand P^+ at large λ as

$$P^+(x,\lambda) = I_{n+1} + \frac{1}{\lambda} P_1^+(x) + \mathrm{O}(\frac{1}{\lambda^2}),\ \lambda \to \infty, \tag{4.11}$$

plugging this series expansion into (3.8) and balancing O(1) terms generate

$$\check{P} = -i[\Lambda, P_1^+]. \tag{4.12}$$

This equivalently tells that the potential matrix reads:

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & -\alpha(P_1^+)_{12} & -\alpha(P_1^+)_{13} & \cdots & -\alpha(P_1^+)_{1,n+1} \\ \alpha(P_1^+)_{21} & 0 & 0 & \cdots & 0 \\ \alpha(P_1^+)_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(P_1^+)_{n+1,1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{4.13}$$

where $P_1^+ = ((P_1^+)_{jl})_{(n+1)\times(n+1)}$. In other words, the $2n$ potentials p_j and q_j, $1 \le j \le n$, can be evaluated as follows:

$$p_j = -\alpha(P_1^+)_{1,j+1},\ q_j = \alpha(P_1^+)_{j+1,1},\ 1 \le j \le n. \tag{4.14}$$

Now from the λ-dependence of the solutions in (4.4), we have

$$P_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.15}$$

and thus further through the solution expressions in (4.14), obtain the N-soliton solution to the multicomponent AKNS system in (2.23):

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1},\ q_j = -\alpha \sum_{k,l=1}^{N} v_{k,j+1} (M^{-1})_{kl} \hat{v}_{l,1},\ 1 \le j \le n, \tag{4.16}$$

where the matrix M is defined by (4.5), and $v_k = (v_{k,1}, v_{k,2}, \ldots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \ldots, \hat{v}_{k,n+1})$, $1 \le k \le N$, are defined by (4.9) and (4.10), respectively.

5. Specific reductions

Let us take a class of specific reductions for the potential matrix P:

$$P^\dagger = CPC^{-1},\ C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \tag{5.1}$$

where † stands for the Hermitian transpose of a matrix and Σ is a constant invertible $n \times n$ Hermitian symmetric matrix: $\Sigma^\dagger = \Sigma$ and Σ^{-1} exists (see [45] for a general reduction problem). In what follows, we assume that $\bar{z}$ denotes the complex conjugate of a complex quantity z, and $A^\dagger(\bar\lambda) = (A(\lambda))^\dagger$ and $A^{-1}(\bar\lambda) = (A(\bar\lambda))^{-1}$ for a matrix $A(\lambda)$ depending on the spectral parameter λ.

If $\psi(\lambda)$ is a matrix eigenfunction of (3.8), then in addition to a known matrix adjoint eigenfunction $C\psi^{-1}(\bar\lambda)$, we have another matrix adjoint eigenfunction

$$\tilde{\psi}(\bar\lambda) = \psi^\dagger(\bar\lambda)C, \tag{5.2}$$

associated with an eigenvalue $\bar\lambda$, i.e., $\psi^\dagger(\bar\lambda)C$ solves the adjoint equation (3.23) with $\bar\lambda$ replacing λ. Therefore, upon observing the asymptotic properties for $\psi^\pm$ at infinity of λ (see, e.g., (3.32)), the uniqueness of solutions determines

$$(\psi^\pm)^\dagger(\bar\lambda) = C(\psi^\pm)^{-1}(\bar\lambda)C^{-1}. \tag{5.3}$$

Further from the definitions of $P^\pm$ in (3.17) and (3.25), we find that the two matrix solutions $P^\pm$ have the involution property

$$(P^+)^\dagger(\bar\lambda) = CP^-(\bar\lambda)C^{-1}. \tag{5.4}$$

and from the definition of the scattering matrix $S(\lambda)$ in (3.13), we find that the scattering matrix $S(\lambda)$ satisfies the involution relation

$$S^\dagger(\bar\lambda) = CS^{-1}(\bar\lambda)C^{-1}. \tag{5.5}$$

Due to (5.4), we have $s_{11}(\bar\lambda) = \hat{s}_{11}(\lambda)$ by (4.1), and so, the zeros of $\det P^\pm$ satisfy the involution relation:

$$\hat\lambda_k = \bar\lambda_k,\ 1 \le k \le N. \tag{5.6}$$

To obtain involution eigenvectors v_k and $\hat{v}_k$, $1 \le k \le N$, we evaluate the Hermitian transpose of the first set of equations in (4.3):

$$0 = v_k^\dagger (P^+(\lambda_k))^\dagger = v_k^\dagger C P^-(\hat\lambda_k)C^{-1},\ 1 \le k \le N,$$

where (5.4) and (5.6) have been used. This way, we can take

$$\hat{v}_k = v_k^\dagger C,\ 1 \le k \le N, \tag{5.7}$$

as the solutions to the second set of equations in (4.3). It then follows that we have the following involution eigenvectors:

$$v_k(x,t_r) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^r \Omega t_r} w_k,\ \hat{v}_k(x,t_r) = w_k^\dagger \mathrm{e}^{-i\bar\lambda_k \Lambda x - i\bar\lambda_k^r \Omega t_r} C,\ 1 \le k \le N, \tag{5.8}$$

where w_k are arbitrary constant column vectors as before and the matrix C is defined in (5.1).

The reduction relation (5.1) implies

$$p = q^\dagger \Sigma, \tag{5.9}$$

which provides many interesting specific reductions depending on the selection of Σ. Based on the recursion relation (2.16), it also follows from (5.1) that

$$W_m^\dagger = CW_mC^{-1},\ m \ge 0, \tag{5.10}$$

where each W_m is defined by (2.14). This further implies that

$$(V^{[r]})^\dagger(\bar\lambda) = CV^{[r]}(\bar\lambda)C^{-1}, \tag{5.11}$$

i.e.,

$$(Q^{[r]})^\dagger(\bar\lambda) = CQ^{[r]}(\bar\lambda)C^{-1}, \tag{5.12}$$

where $V^{[r]}$ and $Q^{[r]}$ are defined in (3.3) (or (2.22)) and (3.4), respectively. Thus, we see that the reductions in (5.1) work for both of the spatial and temporal matrix spectral problems (3.1) and (3.2). Under (5.1), the multicomponent NLS system (2.24) and the multicomponent mKdV system (2.25) are reduced to

$$q_{j,t_2} = \frac{\beta}{\alpha^2} i[q_{j,xx} + 2(\sum_{k,l=1}^{n} \bar{q}_k \sigma_{kl} q_l) q_j],\ 1 \le j \le n. \tag{5.13}$$

and

$$q_{j,t_3} = -\frac{\beta}{\alpha^3}[q_{j,xxx} + 3(\sum_{k,l=1}^{n} \bar{q}_k \sigma_{kl} q_l) q_{j,x} + 3(\sum_{k,l=1}^{n} \bar{q}_k \sigma_{kl} q_{l,x}) q_j],\ 1 \le j \le n, \tag{5.14}$$

where $\Sigma = (\sigma_{jl})_{n\times n}$. More specifically, we have the reduced multicomponent NLS system and mKdV system:

$$q_{j,t_2} = \frac{\beta}{\alpha^2} i[q_{j,xx} + 2\sigma(\sum_{l=1}^{n} |q_l|^2) q_j],\ 1 \le j \le n, \tag{5.15}$$

and

$$q_{j,t_3} = -\frac{\beta}{\alpha^3}[q_{j,xxx} + 3\sigma(\sum_{l=1}^{n} |q_l|^2) q_{j,x} + 3\sigma(\sum_{l=1}^{n} \bar{q}_l q_{l,x}) q_j],\ 1 \le j \le n, \tag{5.16}$$

respectively, if $\Sigma = \sigma I_n,\ \sigma \in \mathbb{R}$, is taken.

In order to compute the N-soliton solutions to the reduced multicomponent AKNS systems, we check the involution property for P_1^+ determined in (4.15). By using (5.6) and (5.7), a direct computation can really show that

$$(P_1^+)^\dagger = -C P_1^+ C^{-1}, \tag{5.17}$$

where P_1^+ is defined by (4.15). Thus, the potential matrix P determined through (4.13) satisfies the reduction relation (5.1). Finally, the reduced N-soliton solution of (2.23) presents the N-soliton solution to the reduced multicomponent AKNS system, including (5.13) and (5.14):

$$q_j = -\alpha \sum_{k,l=1}^{N} v_{k,j+1} (M^{-1})_{kl} \hat{v}_{l,1},\ 1 \le j \le n, \tag{5.18}$$

where the matrix M is defined by (4.5), and $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \dots, \hat{v}_{k,n+1})$, $1 \le k \le N$, are given by (5.8).

6. Concluding remarks

This study aims to formulate Riemann–Hilbert problems associated with matrix spectral problems to compute soliton solutions of integrable hierarchies. One of the important steps is to introduce a kind of equivalent matrix spectral problems so that bounded analytical eigenfunctions in the upper or lower half-plane can be guaranteed to exist. We considered a kind of high-order degenerate AKNS spatial matrix spectral problems and regenerated the corresponding integrable hierarchies possessing bi-Hamiltonian structures. For all multicomponent AKNS systems, we built their Riemann–Hilbert problems and presented an explicit formula for jump matrices in the resulting Riemann–Hilbert problems. Upon taking the identity jump matrix, we computed soliton solutions to all considered multicomponent AKNS systems in each resulting integrable hierarchy. A class of specific reductions was successfully made for each hierarchy and the corresponding N-soliton solutions were generated for all reduced multicomponent AKNS systems.

The Riemann–Hilbert approach is very powerful in constructing soliton solutions, indeed (see also, e.g., [3–7]). The approach has been recently generalized to solve initial–boundary value problems of integrable equations on the half-line and the finite interval [46–49]. Many other approaches to soliton solutions are also available in the field of integrable equations, among which are the Hirota direct method [50], the generalized bilinear technique [51,52], the Wronskian technique [53,54] and the Darboux transformation [55]. It would be important to explore relations among those different approaches.

We also remark that it would be interesting to find other kinds of exact solutions to integrable equations, including positon and complexiton solutions [56,57], lump solutions [58–61], and algebro-geometric solutions [16,17,62,63], based on Riemann–Hilbert problems. Particular lump solutions to the $(2+1)$-dimensional KP and BKP equations have been computed by symbolic computations [64,65]. Can such solutions be generated from Riemann–Hilbert problems or generalized Riemann–Hilbert problems, called $\bar{\partial}$ problems (see, e.g., [1,66])?

There are many other different studies on coupled mKdV equations, which include integrable couplings [67,68], super hierarchies [69,70] and fractional analogous equations [71]. Therefore, another important question for further study is how to formulate Riemann–Hilbert problems for solving those generalized integrable counterparts. It is hoped that our results could be helpful in recognizing exact solutions to generalized integrable equations or hierarchies, from the perspective of the Riemann–Hilbert technique.

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References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[2] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: the Inverse Scattering Method, Consultants Bureau, New York, 1984.
[3] D.S. Wang, D.J. Zhang, J. Yang, Integrable properties of the general coupled nonlinear Schrödinger equations, J. Math. Phys. 51 (2010) 023510.
[4] Y. Xiao, E.G. Fan, A Riemann-Hilbert approach to the Harry-Dym equation on the line, Chinese Ann. Math. Ser. B 37 (2016) 373–384.
[5] X.G. Geng, J.P. Wu, Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation, Wave Motion 60 (2016) 62–72.
[6] D. Shepelsky, L. Zielinski, The inverse scattering transform in the form of a Riemann-Hilbert problem for the Dullin–Gottwald–Holm equation, Opuscula Math. 37 (2017) 167–187.
[7] W.X. Ma, Riemann-Hilbert problems and N-soliton solutions for a coupled mKdV system, J. Geom. Phys. 132 (2018) 45–54.
[8] V.E. Zakharov, A.B. Shabat, Integration of the nonlinear equations of mathematical physics by the method of inverse scattering II, Funct. Anal. Appl. 13 (1979) 166–173.
[9] E.V. Doktorov, S.B. Leble, A Dressing Method in Mathematical Physics, in: Mathematical Physics Studies, vol. 28, Springer, Dordrecht, 2007.
[10] V.S. Gerdjikov, G. Vilasi, A.B. Yanovski, Integrable Hamiltonian Hierarchies: Spectral and Geometric Methods, Springer-Verlag, Berlin, 2008.
[11] V.E. Zakharov, A.V. Mikhailov, On the integrability of classical spinor models in two-dimensional space–time, Comm. Math. Phys. 74 (1980) 21–40.
[12] V.S. Gerdjikov, Algebraic and analytic aspects of soliton type equations, Contemp. Math. 301 (2002) 35–68.
[13] R. Ivanov, On the dressing method for the generalised Zakharov-Shabat system, Nuclear Phys. B 694 (2004) 509–524.

[14] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, Ann. of Math. 137 (1993) 295–368.
[15] P. Deift, S. Venakides, X. Zhou, An extension of the steepest descent method for Riemann-Hilbert problems: the small dispersion limit of the Korteweg–de Vries (KdV) equation, Proc. Natl. Acad. Sci. USA 95 (1998) 450–454.
[16] W.X. Ma, Trigonal curves and algebro-geometric solutions to soliton hierarchies I, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 473 (2017) 20170232.
[17] W.X. Ma, Trigonal curves and algebro-geometric solutions to soliton hierarchies II, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 473 (2017) 20170233.
[18] G.Z. Tu, The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems, J. Math. Phys. 30 (1989) 330–338.
[19] W.X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, Chin. Ann. Math. A 13 (1992) 115–123; Chin. J. Contemp. Math. 13 (1992) 79–89.
[20] W.X. Ma, M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, J. Phys. A: Math. Gen. 39 (2006) 10787–10801.
[21] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21 (1968) 467–490.
[22] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162.
[23] W.X. Ma, B. Fuchssteiner, Integrable theory of the perturbation equations, Chaos Solitons Fractals 7 (1996) 1227–1250.
[24] W.X. Ma, Variational identities and applications to Hamiltonian structures of soliton equations, Nonlinear Anal. TMA 71 (2009) e1716–e1726.
[25] W.X. Ma, R.G. Zhou, Adjoint symmetry constraints leading to binary nonlinearization, J. Nonlinear Math. Phys. 9 (2002) 106–126.
[26] W.X. Ma, Conservation laws of discrete evolution equations by symmetries and adjoint symmetries, Symmetry 7 (2015) 714–725.
[27] W.X. Ma, Conservation laws by symmetries and adjoint symmetries, Discrete Contin. Dyn. Syst. Ser. S 11 (2018) 707–721.
[28] V.G. Drinfel'd, V.V. Sokolov, Equations of Korteweg–de Vries type, and simple Lie algebras, Soviet Math. Dokl. 23 (1982) 457–462.
[29] W.X. Ma, X.X. Xu, Y.F. Zhang, Semi-direct sums of Lie algebras and continuous integrable couplings, Phys. Lett. A 351 (2006) 125–130.
[30] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-sixier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315.
[31] W.X. Ma, R.G. Zhou, Adjoint symmetry constraints of multicomponent AKNS equations, Chin. Ann. Math. B 23 (2002) 373–384.
[32] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys.—JETP 38 (1974) 248–253.
[33] S.T. Chen, R.G. Zhou, An integrable decomposition of the Manakov equation, J. Comput. Appl. Math. 31 (2012) 1–18.
[34] W.X. Ma, Symmetry constraint of MKdV equations by binary nonlinearization, Phys. A 219 (1995) 467–481.
[35] J. Yu, R.G. Zhou, Two kinds of new integrable decompositions of the mKdV equation, Phys. Lett. A 349 (2006) 452–461.
[36] A.P. Fordy, P.P. Kulish, Nonlinear Schrödinger equations and simple Lie algebras, Comm. Math. Phys. 89 (1983) 427–443.
[37] V.S. Gerdjikov, G.G. Grahovski, On the multi-component NLS type systems and their gauge equivalent: Examples and reductions, AIP Conf. Proc. 729 (2004) 162–169.
[38] V.S. Gerdjikov, Basic aspects of soliton theory, in: I.M. Mladenov, A.C. Hirshfeld (Eds.), Geometry, Integrability and Quantization, Softex, Sofia, 2005, pp. 78–125.
[39] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981) 47–66.
[40] W.X. Ma, X.L. Yong, Z.Y. Qin, X. Gu, Y. Zhou, A generalized Liouville's formula, preprint, 2016.
[41] V.S. Shchesnovich, Perturbation theory for nearly integrable multicomponent nonlinear PDEs, J. Math. Phys. 43 (2002) 1460–1486.
[42] V.S. Shchesnovich, J. Yang, General soliton matrices in the Riemann-Hilbert problem for integrable nonlinear equations, J. Math. Phys. 44 (2003) 4604–4639.
[43] J. Yang, Nonlinear Waves in Integrable and Nonintegrable Systems, SIAM, Philadelphia, 2010.
[44] T. Kawata, Riemann spectral method for the nonlinear evolution equation, in: Advances in Nonlinear Waves Vol. I, in: Res. Notes in Math., vol. 95, Pitman, Boston, MA, 1984, pp. 210–225.
[45] A.V. Mikhailov, The reduction problem and the inverse scattering problem, Physica D 3 (1981) 73–117.
[46] A.S. Fokas, J. Lenells, The unified method: I. Nonlinearizable problems on the half-line, J. Phys. A 45 (2012) 195201.
[47] J. Lenells, A.S. Fokas, The unified method: III. Nonlinearizable problems on the interval, J. Phys. A 45 (2012) 195203.
[48] J. Xu, E.G. Fan, Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: without solitons, J. Differential Equations 259 (2015) 1098–1148.
[49] B.B. Hu, T.C. Xia, W.X. Ma, Riemann-Hilbert approach for an initial–boundary value problem of the two-component modified Korteweg–de Vries equation on the half-line, Appl. Math. Comput. 332 (2018) 148–159.
[50] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, New York, 2004.
[51] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011) 140–144.
[52] W.X. Ma, Y. Zhang, Y.N. Tang, J.Y. Tu, Hirota bilinear equations with linear subspaces of solutions, Appl. Math. Comput. 218 (2012) 7174–7183.
[53] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg–de Vries and Kadomtsev–Petviashvili equations: the Wronskian technique, Phys. Lett. A 95 (1983) 1–3.

[54] W.X. Ma, Y. You, Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc. 357 (2005) 1753–1778.

[55] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer, Berlin, 1991.

[56] V.B. Matveev, Generalized Wronskian formula for solutions of the KdV equations: first applications, Phys. Lett. A 166 (1992) 205–208.

[57] W.X. Ma, Complexiton solutions to the Korteweg–de Vries equation, Phys. Lett. A 301 (2002) 35–44.

[58] J. Satsuma, M.J. Ablowitz, Two-dimensional lumps in nonlinear dispersive systems, J. Math. Phys. 20 (1979) 1496–1503.

[59] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differential Equations 264 (2018) 2633–2659.

[60] W.X. Ma, Y. Zhou, R. Dougherty, Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations, Int. J. Modern Phys. B 30 (2016) 1640018.

[61] Y. Zhang, H.H. Dong, X.E. Zhang, H.W. Yang, Rational solutions and lump solutions to the generalized (3 +1)-dimensional shallow water-like equation, Comput. Math. Appl. 73 (2017) 246–252.

[62] E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, Algebro-geometric Approach to Nonlinear Integrable Equations, Springer, Berlin, 1994.

[63] F. Gesztesy, H. Holden, Soliton Equations and Their Algebro-geometric Solutions: (1+1)-Dimensional Continuous Models, Cambridge University Press, Cambridge, 2003.

[64] W.X. Ma, Lump solutions to the Kadomtsev–Petviashvili equation, Phys. Lett. A 379 (2015) 1975–1978.

[65] J.Y. Yang, W.X. Ma, Lump solutions to the BKP equation by symbolic computation, Int. J. Modern Phys. B 30 (2016) 1640028.

[66] M.J. Ablowitz, A.S. Fokas, Complex Variables: Introduction and Applications, second ed., Cambridge University Press, Cambridge, 2003.

[67] X.X. Xu, An integrable coupling hierarchy of the MKdV_{-} integrable systems, its Hamiltonian structure and corresponding nonisospectral integrable hierarchy, Appl. Math. Comput. 216 (2010) 344–353.

[68] X.R. Wang, X.E. Zhang, P.Y. Zhao, Binary nonlinearization for AKNS-KN coupling system, Abstr. Appl. Anal. 2014 (2014) 253102.

[69] W.X. Ma, J.S. He, Z.Y. Qin, A supertrace identity and its applications to superintegrable systems, J. Math. Phys. 49 (2008) 033511.

[70] H.H. Dong, K. Zhao, H.W. Yang, Y.Q. Li, Generalised (2+1)-dimensional super MKdV hierarchy for integrable systems in soliton theory, East Asian J. Appl. Math. 5 (2015) 256–272.

[71] H.H. Dong, B.Y. Guo, B.S. Yin, Generalized fractional supertrace identity for Hamiltonian structure of NLS-MKdV hierarchy with self-consistent sources, Anal. Math. Phys. 6 (2016) 199–209.