Weighted Polar Finite Time Control Barrier Functions With Applications To Multi-Robot Systems

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Abstract— **In this paper, we introduce a class of functions inspired by the weighted Lp norm which is used for the control of unicycle robots in planar space. In particular, we prove that these functions are valid finite time control barrier functions. Finite time control barrier functions (FCBFs) provide a formal guarantee for finite time convergence to desired sets in the state space. Traditionally, these barrier functions consider only the position of the robot and not the heading, which makes it difficult to apply this framework in cases where the heading is important in addition to the position. In this paper, a new barrier function defined with the full state of the robot is proposed to achieve finite time convergence to the desired set in the state space and the desired heading angle with controllable error bounds. We propose a quadratic program (QP) based controller, which guarantees finite time convergence to a desired region in the state space. We show that there exists singular sets in the state space where the QP is infeasible. By virtue of the structure of the proposed barrier function, feasibility of the QP is guaranteed. A multi-robot case study is presented, along with simulation and experimental results.**

I. INTRODUCTION

We consider the problem of a unicycle robot converging to a region of interest with a desired heading angle within a finite time using control barrier functions. This problem has applications in multi-agent systems such as in caging [1], and in defender-intruder type games [2]. In all these applications, it is preferable to have a control policy which guarantees finite time convergence to desired regions with a reasonable bound on the heading angles of the robots.

Control barrier functions discussed in [3], [4], guarantee finite time convergence to desired sets in the state space. In these papers, the barrier functions are only a function of the position of the virtual control point of the robot. This is convenient when the finite time reachability requirement is restricted to the position of the center of mass (CoM). However, if one requires stricter convergence guarantees such as convergence to a terminal level set with a desired angle, then using the barrier functions suggested in [3] and [4] will not work. In such a situation, it becomes imperative to consider a finite time control barrier function (FCBF) defined in the entire state of the system. In this paper, a new FCBF is introduced for differential drive mobile robots (unicycle dynamics) inspired by the generalization of the weighted L_p norm [5], [6], [7], [8] in order to incorporate the entire state of the system.

One of the main difficulties associated with a barrier functions based quadratic program (QP) [9] is proving feasibility at all points in the state space. In the case of single integrator dynamics as considered in [3], [10], [11], feasibility is assured due to the single integrator dynamics. However, when one considers nonholonomic systems such as the unicycle dynamics, then proving feasibility is not straightforward. If the framework suggested in [3], [4], [10], [11] is used, then we will encounter points of singularity in the state space where the QP is infeasible. With the proposed barrier function in this paper, we eliminate the singular sets in the state space and thus guarantee feasibility of the QP in the compact domain.

The contributions of this paper are three fold. First, we introduce a new class of finite time control barrier functions for unicycle robots in the planar space. Our proposed work incorporates the full state of the system. This allows us to accomplish stricter convergence specifications such as converging to the desired level set with orientation requirements. Second, since these barrier functions cannot be used directly with the dynamics in a QP framework due to the existence of the singular sets where the QP is infeasible, we leverage the structure of the proposed FCBF which guarantees feasibility of the QP. Last, we detail a control architecture that allows for finite time convergence to a level set at the desired terminal angle.

The paper is organized as follows. Section II discusses the problem statement addressed in this paper. Section III provides background on control barrier functions, weighted polar L_p functions and the near identity diffeomorphism. Section IV discusses weighted polar L_p barrier functions and contains the main results of the paper. Section V discusses the proposed QP based controller which returns a feasible solution. Section VI discusses a multi-robot case study along with simulation and experimental results. Section VII provides concluding remarks.

II. PROBLEM STATEMENT

In this paper, we consider a differential drive mobile robot with the unicycle dynamics

$$\dot{x} = v \cdot \cos(\phi) \tag{1}$$
$$\dot{y} = v \cdot \sin(\phi) \tag{2}$$
$$\dot{\phi} = \omega \tag{3}$$

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where $x, y \in \mathbb{R}$ are the x and y coordinates of the robot, $\phi \in (-\pi, \pi]$ is the heading, $v \in \mathbb{R}$ is the linear velocity, and $\omega \in \mathbb{R}$ is the angular velocity. Denote the domain of the state space as $\mathcal{X} \subset \mathbb{R}^3$. Let $X = \begin{bmatrix} x & y & \phi \end{bmatrix}^T \in \mathcal{X} \subset \mathbb{R}^3$ be the entire state of the robot. Then the dynamics can be rewritten as

$$\dot{X} = \begin{bmatrix} cos(\phi) & 0 \\ sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}.$$

Let $g(X) = \begin{bmatrix} cos(\phi) & 0 \\ sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(\phi)\bar{L} \\ e_2^T \end{bmatrix}$ where $R(\phi)$ is the standard rotation matrix, $\bar{L} \in \mathbb{R}^{2\times 2}$ is a diagonal matrix with $(1, 0)$ as its diagonal elements, and $e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ is the standard basis vector in the y direction. Define $\bar{x} = \begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2$. We define the control that is applied to the robot as $u = \begin{bmatrix} v & \omega \end{bmatrix}^T \in \mathbb{R}^2$. We assume that the linear velocity is bounded as $|v| \le v_{max}$ where $v_{max} \in \mathbb{R}_{>0}$.

Let $\mathcal{G} = \{X \in \mathcal{X} | h(X) \ge 0\}$ be a region of interest which is defined as the super zero level set of a function $h : \mathcal{X} \to \mathbb{R}$. We require that the robot converge to $\mathcal{G}$ in a finite time $0 < T < \infty$. We now formally define the problem statement that is addressed in this paper.

Problem Statement. *Given a differential drive mobile robot with unicycle dynamics as in* (1)-(3)*, synthesize a finite time control barrier function such that a QP based feedback controller with the finite time barrier function as the constraint allows the robot to converge to $\mathcal{G}$ within a finite time $T \in (0, \infty)$.*

III. MATHEMATICAL BACKGROUND

In this section we provide mathematical background on finite time control barrier functions and weighted polar L_p functions. We use these tools to address the problem statement described in the previous section.

A. Finite Time Convergence Control Barrier Functions

Consider a control affine dynamical system

$$\dot{x} = f(x) + g(x)u, \tag{4}$$

where f and g are locally Lipschitz continuous, $x \in \mathcal{X} \subseteq \mathbb{R}^n$, and $u \in \mathbb{R}^m$. Note that (1)-(3) is in control affine form.

Next, we define finite time convergence control barrier functions [3] which guarantee finite time convergence to desired sets in the state space.

Definition 1 (Finite Time Convergence Control Barrier function)**.** *A continuously differentiable function $h : \mathcal{X} \to \mathbb{R}$ is a* finite time convergence control barrier function *if there exists real parameters $\rho \in [0, 1)$ and $\gamma > 0$ such that for all $x \in \mathcal{X}$,*

$$\sup_{u \in \mathbb{R}^m} \left\{ L_f h(x) + L_g h(x)u + \gamma \cdot sign(h(x)) \cdot |h(x)|^\rho \right\} \ge 0. \tag{5}$$

where $L_f h(x) := \frac{\partial h(x)}{\partial x} f(x)$ and $L_g h(x) := \frac{\partial h(x)}{\partial x} g(x)$ are the Lie derivatives of h in the direction of f and g respectively.

If h is a finite time convergence control barrier function, then there exists a control input $u \in \mathbb{R}^m$ that drives the state of the system x to the target set $\{x \in \mathcal{X} | h(x) \ge 0\}$ in finite time, as formalized next.

Proposition 1 (Proposition III.1, [3])**.** *Let $\mathcal{G} \subseteq \mathcal{X}$ be a target set defined as $\mathcal{G} = \{x \in \mathcal{X} | h(x) \ge 0\}$ where $h : \mathcal{X} \to \mathbb{R}$. If h is a finite time convergence control barrier function for* (4)*, then, for any initial condition $x_0 \in \mathcal{X}$ and any Lipschitz continuous feedback control $u : \mathcal{X} \to \mathbb{R}^m$ satisfying $u(x) \in \mathcal{U}$ where the set $\mathcal{U}$ is defined by*

$$\left\{ u \in \mathbb{R}^m \middle| L_f h(x) + L_g h(x)u + \gamma \cdot sign(h(x)) \cdot |h(x)|^\rho \ge 0 \right\} \tag{6}$$

for all $x \in \mathcal{X}$, the system will be driven to the set $\mathcal{G}$ in a finite time $T \in (0, \infty)$ such that $x(T) \in \mathcal{G}$. Moreover, $\mathcal{G}$ is forward invariant so that the system remains in $\mathcal{G}$ for all $t \ge T$. ■

B. Weighted Polar L_p Functions

In this section, the weighted polar L_p function from an earlier version of [5] is introduced. The idea is to appropriately use a coordinate transformation between the Cartesian system and the polar system to retrieve the surface equation represented by the weighted L_p norm in each domain.

Consider the coordinate transformation $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Gamma(\bar{x}) = \begin{bmatrix} \kappa x \\ \kappa y + 1 \end{bmatrix}.$$

The polar coordinates of $\bar{x}$ transformed by the mapping Γ are given by

$$\mathcal{R}_\Gamma(\bar{x}) = \sqrt{(\kappa x)^2 + (\kappa y + 1)^2} \tag{7}$$

$$\theta_\Gamma(\bar{x}) = \arctan\left(\frac{\kappa y + 1}{\kappa x}\right). \tag{8}$$

Definition 2 (Weighted Polar $2D$ L_p function (Definition 3 in earlier version [5]))**.** *Let $\kappa \neq 0$ and $\sigma = (\sigma_1, \sigma_2)$ be an element-wise positive vector. The σ, κ weighted polar $2D$ L_p function $\Omega : \mathbb{R}^2 \to \mathbb{R}$ is the positive definite function*

$$\Omega(\bar{x}) = \left(\left(\frac{|R_\Gamma(\bar{x}) - 1|}{\sigma_2} \right)^p + \left(\frac{|\theta_\Gamma(\bar{x}) - \theta_0|}{\sigma_1} \right)^p \right)^{\frac{1}{p}}$$

where $\theta_0 = sign(\kappa) \cdot \frac{\pi}{2}$, and p is even.

A full derivation and analysis of the weighted polar L_p function can be found in the earlier version of [5]. In this paper, the terminal level set is defined as the level set of a modified weighted polar L_p function, which has a similar bent rectangular contour as its level set.

C. Near Identity Diffeomorphism

The unicycle kinematics model is nonlinear and nonholonomic. Hence, the approach adopted in this paper is to control a virtual point which is not at the center of mass but a point apart from it. The new coordinate which is centered at the virtual point can be obtained via the near identity diffeomorphism discussed in [12]. To that end, let $r = \begin{bmatrix} r_x & r_y \end{bmatrix}^T \in \mathbb{R}^2$ be the new virtual point which is controlled. That is,

$$r = \bar{x} + l \cdot R(\phi) \cdot e_1$$

where $l \in \mathbb{R}_{>0}$, and $e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is the standard basis element along the x direction. Hence we have

$$r = \bar{x} + l \cdot \begin{bmatrix} cos(\phi) \\ sin(\phi) \end{bmatrix}$$

The dynamics of r are then given by

$$\begin{aligned} \dot{r} &= \dot{\bar{x}} + l \cdot \begin{bmatrix} -sin(\phi) \\ cos(\phi) \end{bmatrix} \cdot \dot{\phi} \\ &= \begin{bmatrix} vcos(\phi) - lsin(\phi)\omega \\ vsin(\phi) + lcos(\phi)\omega \end{bmatrix}. \end{aligned}$$

Denote the total state of the robot as $\tilde{r} = \begin{bmatrix} r & \phi \end{bmatrix}^T \in \mathcal{X}$. Define $L \in \mathbb{R}^{2\times 2}$ to be a diagonal matrix with $(1,l)$ as its diagonal elements. Hence we have

$$\dot{\tilde{r}} = \begin{bmatrix} R(\phi)L \\ e_2^T \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{9}$$

Let $\tilde{g}(r) = \begin{bmatrix} R(\phi)L & e_2^T \end{bmatrix}^T$. Moving forward, we will consider this virtual point as the state of the robot and use these dynamics for the analysis of the Lie derivatives and the gradients.

IV. WEIGHTED POLAR L_p BARRIER FUNCTIONS

Consider the original state of the robot $X = \begin{bmatrix} x & y & \phi \end{bmatrix}^T$. We consider the following candidate weighted L_p function $\Upsilon : \mathcal{X} \to \mathbb{R}$:

$$\Upsilon(X) = |\kappa| - \left(\left(\frac{R_\Gamma(\bar{x}) - 1}{\sigma_2} \right)^p + \left(\frac{\theta_\Gamma(\bar{x}) - \theta_0}{\sigma_1} \right)^p + \mu \cdot \left(\frac{|\kappa| \cdot (\phi - \phi_{term})}{\sigma_3} \right)^p \right)^{\frac{1}{p}} \tag{10}$$

where ϕ and ϕ_{term} are wrapped to $(-\pi, \pi]$, $(\phi - \phi_{term})$ is bounded by 2π, p is even and $\mu \in \mathbb{R}_{>0}$ is a design parameter. We will see later that μ influences the error bound on the angle of convergence to the terminal level set.

We define the set of points at which the Lie derivative of (10) in the direction of the actuation vector field g vanishes. More formally, we have the following definition.

Definition 3 (Singular set)**.** *A singular set of the system as in* (1)-(3) *is defined as* $\mathcal{S} = \{X \in \mathcal{X} | L_g\Upsilon(X) = 0\}$*, where* $L_g\Upsilon(X) = \frac{\partial \Upsilon(X)}{\partial X} g(X)$ *is the Lie derivative of* (10) *along the direction of the actuation vector field* g*.*

Since Υ is smooth in the domain $\mathcal{X}$ and the dimension of $\mathcal{S}$ is strictly less than 3, the singular set is measure zero. As μ is also a control parameter, the singular set can be eliminated by choosing a specific value of μ. Before considering a proper choice of μ, the following lemma shows that the singular set $\mathcal{S}$ is always nonempty for any $\mu \in \mathbb{R}_{>0}$ if the original state equation in (1)-(3) is used instead of (9).

Lemma 1. *Consider the weighted polar L_p function as in* (10)*. Let* $\mathcal{S} = \{X \in \mathcal{X} | L_g\Upsilon(X) = 0\}$ *where* $L_g\Upsilon(X) = \frac{\partial \Upsilon(X)}{\partial X} g(X)$ *is the Lie derivative of* Υ *along* g *as in* (1)-(3)*. Then the set* $\mathcal{S}$ *is non empty for any* $\mu \in \mathbb{R}_{>0}$*.*

Proof. Define $\alpha(X) = \left(\left(\frac{R_\Gamma(\bar{x}) - 1}{\sigma_2} \right)^p + \left(\frac{\theta_\Gamma(\bar{x}) - \theta_0}{\sigma_1} \right)^p + \mu \cdot \left(\frac{|\kappa| \cdot (\phi - \phi_{term})}{\sigma_3} \right)^p \right) \in \mathbb{R}_{>0}$. The Lie derivative of the weighted L_p function along g is given by

$$\begin{aligned} L_g\Upsilon(X) &= \frac{\partial \Upsilon(X)}{\partial X} g(X) \\ &= \alpha(X)^{\frac{1-p}{p}} \begin{bmatrix} -\Omega(\bar{x})^{p-1} \nabla\Omega(\bar{x}) & -\frac{\mu|\kappa|^p(\phi-\phi_{term})^{p-1}}{\sigma_3^p} \end{bmatrix} \begin{bmatrix} R(\phi)\bar{L} \\ e_2^T \end{bmatrix}. \end{aligned}$$

Now define $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\Psi(\phi) = \frac{\mu|\kappa|^p(\phi-\phi_{term})^{p-1}}{\sigma_3^p}$. Then we have

$$L_g\Upsilon(X) = \alpha(X)^{\frac{1-p}{p}} \begin{bmatrix} -\Omega(\bar{x})^{p-1} \nabla\Omega(\bar{x}) & -\Psi(\phi) \end{bmatrix} \begin{bmatrix} R(\phi)\bar{L} \\ e_2^T \end{bmatrix}$$

The Lie derivative, $L_g\Upsilon$, at X is given by

$$L_g\Upsilon(X) = \alpha(X)^{\frac{1-p}{p}} \cdot (-\Omega(\bar{x})^{p-1} \nabla\Omega(\bar{x}) R(\phi)\bar{L} - \Psi(\phi)e_2^T).$$

$L_g\Upsilon(X) = 0_{1\times 2}$ implies that $\nabla\Omega(\bar{x})R(\phi)e_1 = 0$ and $\phi = \phi_{term}$ holds since $\bar{L}$ is singular. This condition implies that $L_g\Upsilon(X) = 0_{1\times 2}$ if the heading is equal to the desired heading ϕ_{term}, and at that point, the gradient of Ω is orthogonal to the heading. Since there always exists a point $X \in \mathcal{X}$ such that the gradient $\nabla\Omega(\bar{x})$ is orthogonal to the heading in $\mathcal{X}$, if the heading of X is equal to ϕ_{term} then $X \in \mathcal{S}$. Hence, there does not exist a $\mu \in \mathbb{R}_{>0}$ such that $\mathcal{S}$ is empty. This concludes the proof. ■

A. Condition For Vanishing Lie Derivative

The following Lemma characterizes the condition at which the Lie derivative is zero in the state space.

Lemma 2. *Consider the unicycle robot with the near identity diffeomorphism kinematics as in* (9)*. The singular set* $\mathcal{S}$ *in Definition 3 is nonempty only if*

$$\nabla\Omega^T(r) + \frac{\Psi(\phi)R(\phi)e_2}{l\Omega(r)^{p-1}} = 0 \tag{11}$$

holds, where $\Psi(\phi) = \frac{\mu|\kappa|^p(\phi-\phi_{term})^{p-1}}{\sigma_3^p}$ *for all* $\tilde{r} = \begin{bmatrix} r & \phi \end{bmatrix}^T \in \mathcal{X}$.

Proof. The Lie derivative of the weighted polar L_p function along $\tilde{g}$ for all $\tilde{r} \in \mathcal{X}$ is given by

$$\begin{aligned} L_{\tilde{g}}\Upsilon(\tilde{r}) &= \frac{\partial \Upsilon(\tilde{r})}{\partial \tilde{r}} \tilde{g} \\ &= \alpha(\tilde{r})^{\frac{1-p}{p}} \begin{bmatrix} -\Omega(r)^{p-1} \nabla\Omega(r) & -\frac{\mu|\kappa|^p(\phi-\phi_{term})^{p-1}}{\sigma_3^p} \end{bmatrix} \begin{bmatrix} R(\phi)L \\ e_2^T \end{bmatrix} \end{aligned}$$

Hence we have

$$L_{\tilde{g}}\Upsilon(\tilde{r}) = \alpha(r)^{\frac{1-p}{p}} (-\Omega(r)^{p-1} \nabla\Omega(r) R(\phi)L - \Psi(\phi)e_2^T) \tag{12}$$

Suppose $L_{\tilde{g}}\Upsilon(\tilde{r}) = 0$. Then we have

$$0 = -\Omega(r)^{p-1} \nabla\Omega(r) R(\phi)L - \Psi(\phi)e_2^T$$

Since $R(\phi)L$ is nonsingular,

$$\nabla\Omega(r)^T = -\frac{\Psi(\phi)R(\phi)e_2}{l\Omega(r)^{p-1}}$$

holds, and the lemma follows. ■

Observe that the contrapositive of Lemma 2 indicates that if $\langle\nabla\Omega(r), R(\phi)e_1\rangle \neq 0$ or $||\nabla\Omega^T(r)|| \neq \frac{|\Psi(\phi)|}{l\Omega(r)^{p-1}}$, then $||L_{\tilde{g}}\Upsilon(\tilde{r})||$ is not equal to zero at $\tilde{r}$. This means that if we pick the value of the control parameter μ such that $||\nabla\Omega^T(r)|| \neq \frac{|\Psi(\phi)|}{l\Omega(r)^{p-1}}$, then we have $L_{\tilde{g}}\Upsilon(\tilde{r}) \neq 0$ for all $\tilde{r} \in \mathcal{X}$.

B. Weighted polar L_p Barrier Functions

Consider the weighted polar L_p function as in (10). We prove that it is a valid finite time convergence control barrier function in order to provide a formal guarantee for convergence to the terminal level set at the desired angle. The following theorem provides a characterization of a new compact domain local to the initial condition $\tilde{r}_0$ of the robot.

Theorem 1. *Let $\tilde{r}_0 = \begin{bmatrix} r_0 & \phi_0 \end{bmatrix}^T \in \mathcal{X}$ be the initial configuration of the robot. Define $\partial\mathcal{G} = \{\tilde{r} \in \mathcal{X} | \Upsilon(\tilde{r}) = 0\}$. Then, there exists $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, |\kappa|)$, $\mu \in (0, \hat{\mu}_{max}]$, $\mathcal{D} := \mathcal{M} \backslash \mathcal{N}$ where $\tilde{r}_0 \in \mathcal{D}$ and*

$$\begin{aligned} \mathcal{M} &:= \{\begin{bmatrix} r & \phi \end{bmatrix}^T \in \mathcal{X} | |\kappa| - \Omega(r) \leq \varepsilon_2\} \\ \mathcal{N} &:= \{\begin{bmatrix} r & \phi \end{bmatrix}^T \in \mathcal{X} | |\kappa| - \Omega(r) < -\varepsilon_1\}, \end{aligned}$$

such that $\partial\mathcal{G} \subset \mathcal{D}$.

Proof. We choose $\varepsilon_1 > 0$ to be large enough such that $\Upsilon(\tilde{r}_0) \geq -\varepsilon_1$. This is possible since $\phi - \phi_{term} \in (-2\pi, 2\pi]$ is bounded. In fact, this ε_1 can be chosen such that $\sup_{\tilde{r} \in \mathcal{N}^c} \Upsilon(\tilde{r}) \leq -\varepsilon_1$. Observe that this implies that $\Omega(r_0) \geq -\varepsilon_1$, and so $\tilde{r}_0 \in \mathcal{N}^c$ where $\mathcal{N}^c$ is the complement of $\mathcal{N}$.

Now, pick $\varepsilon_2 \in (0, |\kappa|)$, then

$$\begin{aligned} \Upsilon(\tilde{r}) &= |\kappa| - \left((|\kappa| - \varepsilon_2)^p + \mu \cdot \frac{|\kappa|^p \cdot (\phi - \phi_{term})^p}{\sigma_3^p}\right)^{\frac{1}{p}} \\ &\geq |\kappa| - \left((|\kappa| - \varepsilon_2)^p + \mu \cdot \frac{|\kappa|^p \cdot (2\pi)^p}{\sigma_3^p}\right)^{\frac{1}{p}} \qquad (13) \end{aligned}$$

holds for any $\tilde{r} \in \partial M$. Now pick $\mu \leq \frac{\bar{\delta} \cdot \sigma_3^p}{(2\pi)^p}$ for some $\bar{\delta} \in (0,1)$ defined as $\bar{\delta} = 1 - \frac{(|\kappa| - \varepsilon_2)^p}{|\kappa|^p}$. By substituting μ to (13), $\Upsilon(\tilde{r}) \geq 0$ holds for all $\tilde{r} \in \partial\mathcal{M}$.

Define $\hat{\mu}_{max} = \frac{\bar{\delta} \cdot \sigma_3^p}{(2\pi)^p}$. Then from the choice of $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, |\kappa|)$ and $\mu \in (0, \hat{\mu}_{max})$, the supremum of Υ on N^c is strictly negative, and the infimum of Υ on $\partial\mathcal{M}$ is non-negative. It is also true that $\inf_{\tilde{r} \in \mathcal{M}^c} \Upsilon(\tilde{r}) > 0$, and therefore, $\partial G \cap \mathcal{D}^c$ is empty. Hence $\partial G \subset \mathcal{D}$. ■

Now, having clearly defined the domain, we prove that there is no singular set (Definition 3) within the domain $\mathcal{D}$ for a fixed μ which invokes Theorem 1.

Lemma 3. *There exists a uniform upper bound $\mu_{max} > 0$, such that if $\mu \in (0, \mu_{max}]$, then $||L_{\tilde{g}}\Upsilon(\tilde{r})|| > 0$ for all $\tilde{r} \in \mathcal{D}$.*

Proof. Observe that $\mathcal{D}$ is compact in $\mathcal{X}$ since $\mathcal{N}$ is open, and Υ is continuously differentiable in $\mathcal{D}$. Therefore, there exists a minimum,

$$K := \min_{\tilde{r} \in \mathcal{D}} ||\nabla\Omega(r)|| \quad \text{and} \quad \Omega_{min} := \min_{\tilde{r} \in \mathcal{D}} \Omega(r)$$

Note that $\Omega(r) \neq 0$ for all $\tilde{r} \in \mathcal{D}$ since Ω is positive definite and $\begin{bmatrix} 0 & 0 \end{bmatrix}^T \notin \mathcal{D}$. Now, pick $\mu^* > 0$ such that

$$\frac{\mu^* |\kappa|^p (2\pi)^{p-1}}{l\sigma_3^p \Omega_{min}^{p-1}} \leq \frac{K}{\beta}, \text{ where } \beta > 1$$

Now suppose that if $\tilde{r} \in \mathcal{D}$, then by its definition, $\Psi(\phi) = 0$ if and only if $\phi = \phi_{term}$ since $\mu > 0$. Observe that if $\Psi(\phi) = 0$, then $||L_{\tilde{g}}\Upsilon(\tilde{r})|| \neq 0$ since $||\nabla\Omega(r)|| \neq 0$. And so it is enough to prove that $||L_{\tilde{g}}\Upsilon(\tilde{r})|| \neq 0$ for $\phi \neq \phi_{term}$.

Since $\phi \in (-\pi, \pi]$, then for all $\tilde{r} \in \mathcal{D}$, the following holds,

$$\frac{(\phi - \phi_{term})^{p-1}}{\sigma_3^p \Omega(r)^{p-1}} \leq \frac{(2\pi)^{p-1}}{\sigma_3^p \Omega_{min}^{p-1}}$$

since $p > 1$. This implies that

$$\left|\left|\frac{\Psi(\phi)R(\phi)e_2}{l\Omega(r)^{p-1}}\right|\right| \leq \frac{\mu|\kappa|^p \cdot (2\pi)^{p-1}}{l\sigma_3^p \Omega_{min}^{p-1}}$$

since $R(\phi)$ is a unitary matrix. Now, pick

$$\mu < \min\{\mu^*, \hat{\mu}_{max}\}, \qquad (14)$$

where $\hat{\mu}_{max}$ is as in Theorem 1. Then for all $\tilde{r} \in \mathcal{D}$ we have

$$\left|\left|\frac{\Psi(\phi)R(\phi)e_2}{l\Omega(r)^{p-1}}\right|\right| \leq \frac{K}{\beta} < ||\nabla\Omega(r)||$$

since $\beta > 1$. By invoking the contrapositive of Lemma 2, $||L_{\tilde{g}}\Upsilon(\tilde{r})|| \neq 0$ holds for all $\tilde{r} \in \mathcal{D}$. Hence, the upper bound $\min\{\mu^*, \hat{\mu}_{max}\}$ suffices the requirement. ■

Next, we show that the weighted polar L_p function is indeed a valid finite time control barrier function in $\mathcal{D}$.

Theorem 2. *In the domain $\mathcal{D}$ with the desired level set $\mathcal{G} = \{\tilde{r} \in \mathcal{X} | \Upsilon(\tilde{r}) \geq 0\}$, the weighted polar L_p function Υ renders* (5) *true for all $\tilde{r} \in \mathcal{D}$ and there exists a finite time $0 < T < \infty$ such that $\Upsilon(\tilde{r}(T)) \in \mathcal{G}$.*

Proof. Let $\mathcal{D}$ be the domain as defined in Theorem 1. Pick μ as proposed in Lemma 3. Then, from Lemma 3, we have $L_{\tilde{g}}\Upsilon(\tilde{r}) \neq 0$ for all $\tilde{r} \in \mathcal{D}$. Now for all $\tilde{r} \in \mathcal{D}$, pick

$$u = \frac{L_{\tilde{g}}\Upsilon(\tilde{r}) \cdot (-\gamma \cdot sign(\Upsilon(\tilde{r})) \cdot |\Upsilon(\tilde{r})|^\rho)}{||L_{\tilde{g}}\Upsilon(\tilde{r})||_2^2} \in \mathbb{R}^2$$

This gives us $L_{\tilde{g}}\Upsilon(\tilde{r})u + \gamma \cdot sign(\Upsilon(\tilde{r})) \cdot |\Upsilon(\tilde{r})|^\rho = 0$. Hence, for all $\tilde{r} \in \mathcal{D}$, there exists a $u \in \mathbb{R}^2$ such that (5) is satisfied. Thus in the domain $\mathcal{D}$, Υ is a valid finite time control barrier function. Note that since we wrap ϕ, ϕ_{term} in (10), $\tilde{r}$ remains in $\mathcal{D}$. Since $\partial\mathcal{G} \subset \mathcal{D}$, invoking Proposition 1, and the fact that Υ is positive definite, there exists $T \in (0, \infty)$ such that $\tilde{r}(t) \in \mathcal{D}$ for all $t \in [0, T)$ and $\Upsilon(\tilde{r}(T)) \in \mathcal{G}$. ■

C. Bound on Angle of Convergence

Proposition 2. *Given a domain $\mathcal{D}$ as defined in Theorem 1 and a desired level set $\mathcal{G} = \{\tilde{r} \in \mathcal{X} | \Upsilon(\tilde{r}) \geq 0\} \subset \mathcal{D}$ with Υ as the associated weighted polar L_p barrier function, any continuous controller $u : \mathcal{D} \to \mathbb{R}^2$ such that $u(\tilde{r}) \in \mathcal{U}$ as in* (6) *drives the robot to $\mathcal{G}$ in a finite time $0 < T < \infty$. Moreover, the angle of convergence of the robot to the desired level set is bounded by*

$$\phi_{term} - \frac{\sigma_3}{\mu^{1/p}} \leq \phi(T) \leq \phi_{term} + \frac{\sigma_3}{\mu^{1/p}}$$

where $\phi_{term} \in (-\pi, \pi]$ is the desired heading angle.

Proof. Since Υ is a finite time control barrier function, from Proposition 2, the robot converges to $\mathcal{G}$ in a finite time $0 < T < \infty$. That is, we have $\Upsilon(\tilde{r}) \geq 0$ for all $t \geq T$. Since the robot is also within the $2D$ level set $\Omega(r)$ we have $\Omega(r) = m \cdot |\kappa|$ where $0 < m \leq 1$. That is, we have

$$\begin{aligned}
&0 \leq |\kappa| - \left(m^p \cdot |\kappa|^p + \frac{\mu \cdot |\kappa|^p}{\sigma_3^p} \cdot (\phi(T) - \phi_{term})^p \right)^{\frac{1}{p}} \\
\iff \quad & m^p + \frac{\mu}{\sigma_3^p} \cdot (\phi(T) - \phi_{term})^p \leq 1 \\
\iff \quad & \frac{\mu}{\sigma_3^p} \cdot (\phi(T) - \phi_{term})^p \leq (1 - m^p)
\end{aligned}$$

Let $\xi = 1 - m^p$. Note that $\xi < 1$. Hence we have

$$(\phi(T) - \phi_{term})^p \leq \frac{\xi \cdot \sigma_3^p}{\mu} \iff |\phi(T) - \phi_{term}| \leq \frac{\xi' \cdot \sigma_3}{\mu^{\frac{1}{p}}}$$

where $\xi' = \xi^{1/p}$. Since $\frac{\xi' \cdot \sigma_3}{\mu^{\frac{1}{p}}} \leq \frac{\sigma_3}{\mu^{\frac{1}{p}}}$ we have the inequality

$$\phi_{term} - \frac{\sigma_3}{\mu^{1/p}} \leq \phi(T) \leq \phi_{term} + \frac{\sigma_3}{\mu^{1/p}}$$

and hence the theorem holds. ∎

V. QUADRATIC PROGRAM BASED CONTROLLER SYNTHESIS

We adopt a QP based controller which incorporates the weighted polar L_p finite time barrier function as a constraint, which returns the minimum energy control law point wise in the state space:

$$\begin{aligned}
\min_{u \in \mathbb{R}^2} \quad & ||u||_2^2 \\
\text{s.t} \quad & \frac{\partial \Upsilon(\tilde{r})}{\partial \tilde{r}} \dot{\tilde{r}} \geq -\gamma \cdot sign(\Upsilon(\tilde{r})) \cdot |\Upsilon(\tilde{r})|^{\rho}
\end{aligned} \tag{15}$$

where $\gamma > 0$, $\rho \in [0, 1)$, and $\Upsilon : \mathcal{D} \to \mathbb{R}$ is the weighted polar L_p barrier function of the form

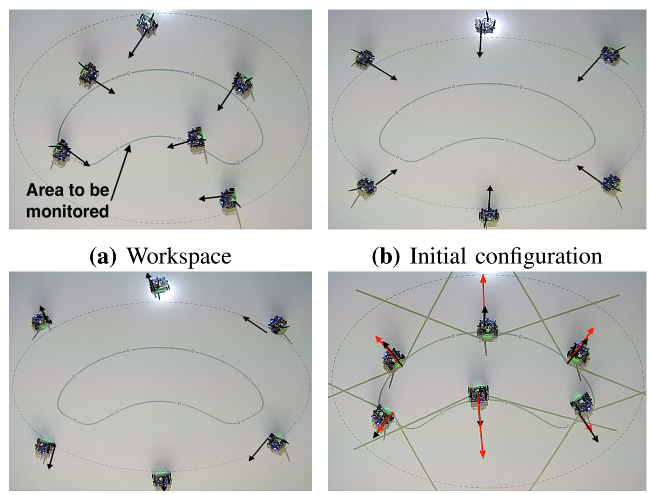
$$\Upsilon(\tilde{r}) = |\kappa| - \left(\mu_1 \cdot \left(\frac{R_\Gamma(r) - 1}{\sigma_2} \right)^p + \mu_1 \cdot \left(\frac{\theta_\Gamma(r) - \theta_0}{\sigma_1} \right)^p + \mu_2 \cdot \left(\frac{|\kappa| \cdot (\phi - \phi_{term})}{\sigma_3} \right)^p \right)^{\frac{1}{p}} \tag{16}$$

where $\mu_1 \in \mathbb{R}_{\geq 0}, \mu_2 \in \mathbb{R}_{>0}$ are design parameters.

Algorithm 1 QP Based Controller

Input : $L_{\tilde{g}} \Upsilon(\tilde{r})$

if DeadLock = 1 **then**
  Pick $\mu_1 = 1$, μ_2 such that (14) holds
  while $\Upsilon(\tilde{r}) < 0$ **do**
    Solve the QP as in (15)
  end while
  Pick $\mu_1 = 0$, $\mu_2 > 0$ to be large
  while $\Upsilon(\tilde{r}) < 0$ **do**
    Solve the QP as in (15)
  end while
else
  Pick $\mu_1 = 1$, $\mu_2 > 0$ to be large
  Solve the QP as in (15)
  if $||L_g \Upsilon(\tilde{r})|| < \delta$ **then**
    DeadLock ← 1
  else
    DeadLock ← 0
  end if
end if



(a) Workspace **(b)** Initial configuration

(c) Intermediate configuration **(d)** Final configuration

Fig. 1: In (b), the robots initialize around the target level set described by (10) with the parameters in the case study. Using the QP based controller in Algorithm 1, the robots converge to the terminal level set in the direction of the gradient of the level set as seen in (d).

We propose Algorithm 1 which guarantees feasibility of the QP as well as small value of error on the final heading angle.

Stage 1: In this stage, we choose $\mu_1 = 1$ and μ_2 as in (14). Then for all $\tilde{r} \in \mathcal{D}$, the weighted polar L_p barrier function is given by (16) The QP as in (15) is solved in order to converge to the desired level set within a finite time $0 < T_1 < \infty$. This signifies the end of stage 1.

Stage 2: We next choose $\mu_1 = 0$ and $\mu_2 = \mu_{large} > 0$ (a large value). Then for all $\tilde{r} \in \mathcal{D}$, the weighted polar L_p barrier functions is given by (16). Again, the QP as in (15) is solved and the robot converges to the desired terminal angle in a finite time $0 < T_2 < \infty$ within an error formalized in Proposition 3.

Proposition 3. *Given a desired terminal angle* $\phi_{term} \in (-\pi, \pi]$, $\mu_1 = 0$, $\mu_2 = \mu_{large} > 0$ *with* Υ *as the weighted polar* L_p *barrier function as in* (16)*, any continuous controller* $u : \mathcal{D} \to \mathbb{R}^2$ *such that* $u(\tilde{r}) \in \mathcal{U}$ *as in* (6)*, drives the robot to a final heading angle given by*

$$\phi_{term} - \frac{\sigma_3}{\mu_{large}^{1/p}} \le \phi(T) \le \phi_{term} + \frac{\sigma_3}{\mu_{large}^{1/p}}$$

within a finite time $0 < T < \infty$.

Proof. For all $\tilde{r} \in \mathcal{D}$, observe that if $\mu_1 = 0$, then from (12) we have $L_{\tilde{g}}\Upsilon(\tilde{r}) \neq 0$ for all $\phi \neq \phi_{term}$. Similar to the argument in Theorem 2, Υ is a finite time barrier function. Hence, $\Upsilon(\tilde{r}) \ge 0$ for all $\tilde{r} \in \mathcal{D}$, $t \ge T$ where $0 < T < \infty$ which yields

$$\begin{aligned} & 0 \le |\kappa| - \left(\frac{\mu_{large} \cdot |\kappa|^p}{\sigma_3^p} \cdot (\phi(T) - \phi_{term})^p \right)^{\frac{1}{p}} \\ \iff \quad & \frac{\mu_{large}}{\sigma_3^p} \cdot (\phi(T) - \phi_{term})^p \le 1 \\ \iff \quad & (\phi(T) - \phi_{term})^p \le \frac{\sigma_3^p}{\mu_{large}} \\ \iff \quad & |\phi(T) - \phi_{term}| \le \frac{\sigma_3}{\mu_{large}^{1/p}} \end{aligned}$$

and thus the proposition holds. ■

VI. CASE STUDY

Consider a team of 6 differential drive mobile robots indexed by the set $\mathcal{I} = \{1, 2, \ldots, 6\}$ each with dynamics as in (1)-(3). Since we use the NID, we consider the virtual point $\tilde{r}_i \in \mathcal{X}$ for all $i \in \mathcal{I}$ where $\tilde{r}_i$ is the total state of agent i and $r_i = \begin{bmatrix} r_x^i & r_y^i \end{bmatrix}^T$. Let $\mathcal{G} = \{\tilde{r} \in \mathbb{R}^3 | \Upsilon(\tilde{r}) \ge 0\}$ for all $i \in \mathcal{I}$ be the terminal level set defined as the super level set of the weighted polar L_p barrier function parametrized by $\sigma = (0.7, 0.2, \frac{\pi}{20})$, $\mu = 100$, $\kappa = 1.220$, and R_Γ, θ_Γ are as in (7) and (8), $\phi_{i,term}$ is the gradient direction for all $i \in \mathcal{I}$.

We require the agents to protect an asset which lies within the terminal level set described by (10) as shown in Fig 1a. The agents are equipped with wide field of view (FOV) sensors. The agents must converge to the terminal level set and align themselves in the direction of the gradient vector to the level set so that any incoming threat is detected by at least one agent. At the same time, the agents must also avoid collisions with each other. The collision avoidance barrier certificate is given by

$$h_{safety}(r_i) = ||r_i - r_j||_2^2 - d_{safe}^2 \text{ for all } i \in \mathcal{I}, j \in \mathcal{N}_i$$

where $d_{safe} \in \mathbb{R}_{>0}$ is a safety radius and $\mathcal{N}_i$ is the set of all agents that lie within the neighborhood of agent i.

By solving a decentralized QP as in (15) with the additional collision avoidance barrier functions online and point wise, the agents converge to the terminal level set in the direction of the gradient as shown in Fig. 1. The red arrows indicate the desired heading, the black arrows indicate the final heading of the robots, and the green lines are the FOV for the robots. We chose $\mu = 100$ which is large. Hence, from Proposition 2, the error on the final angle of convergence is also very small. No deadlocks were detected since (11) was always violated. We provide a video of the experiment (`https://youtu.be/oTLoJNgs3bo`) conducted on the Robotarium multi-robot testbed at Georgia Tech [13].

VII. CONCLUDING REMARKS

To conclude, in this paper, we introduced a new finite time control barrier function inspired by the weighted L_p norm. This function encodes the full state of the unicycle, as opposed to existing methods in literature. We also proved that existing methods cannot be used directly since there exist singular sets in the state space. We characterized the condition for the singular sets and determined the expression for the control parameters in order to eliminate these sets. We proposed a QP based controller which returns feasible solutions. We provided a multi-robot case study in addition to the theoretical framework.

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