AAS 17-474

A PURSUIT-EVASION GAME IN THE ORBITAL PLANE

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We consider a pursuit-evasion game in the Hill's frame, in which a deputy spacecraft aims to capture a chief spacecraft as fast as possible while the chief tries to delay or avoid capture. Capture is defined as positional coincidence of the two spacecraft within a user-specified tolerance, and both spacecraft operate under input constraints. This zero-sum game of pursuit and evasion with free final time is treated as a game with a terminal cost and fixed final time. For the case of circular orbits, a semi analytical solution for the time of capture is presented along with a closed form expression for the optimal control inputs. Furthermore, for a circular orbit, we can precisely characterize the sets of initial conditions that ensure capture of the chief by the deputy.

INTRODUCTION

The study and design of the relative motion between two vehicles or objects orbiting a central body is a basis for various applications such as regulation of satellite orbits, space debris avoidance and manual operations in space. In this work, we consider the problem of achieving positional coincidence between two orbiting spacecraft, where one spacecraft referred to as the deputy tries to reach a pre-defined positional proximity to (or capture) another spacecraft referred to as the chief spacecraft, which maneuvers to avoid positional proximity. We treat this problem as a two-player zero-sum differential game, where the chief spacecraft and the deputy spacecraft respond to each other's actions in an antagonistic manner.

In the general rendezvous problem, the chief spacecraft (target) is passive. For rendezvous in circular orbits of the chief and deputy, we have closed-form solutions for the rendezvous problem in the vicinity of the chief where the linearized Clohessy-Wiltshire equations are valid. Our problem is different from the general rendezvous problem in that the chief is capable of actuating to avoid its capture. The chief and deputy spacecraft are identical in dynamics with similar actuation capabilities. The problem of position coincidence between the chief and deputy spacecraft is to be achieved in optimal time with antagonistic action on the part of the chief spacecraft (in other words, the "optimal value of the game" is the time of capture). The achievement of time-optimal positional proximity (capture) for spacecraft is useful as part of operations such as rendezvous, or even at times to generate impact trajectories, while the dual problem of evasion is applicable for a spacecraft to avoid an antagonistic vehicle in orbit. The game theoretic modeling assumes the most antagonistic maneuvers on the part of the chief spacecraft, which is conservative for most proximity operations and targets that maneuver under faulty control systems. In addition, the chief does not necessarily

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have to be a maneuvering spacecraft. The chief spacecraft can be substituted by an orbiting body whose orbit is subject to some uncertainty (disturbance) that is bounded. In addition, we can design a maneuvering trajectory in a field of multiple non-cooperative targets for a spacecraft, such that the risk of collision is minimized.

The standard relative motion equations described in the orbital plane have coupling between them.[1] This increases the difficulty in analyzing and solving the planar pursuit-evasion problem. To address this issue, we make use of the relative motion equations in Levi-Civita coordinates,[2,3] which are obtained by simple transformations from the inertial coordinates. This coordinate system has been used by Kollin and Akella[2] to solve the minimum-time orbital rendezvous problem for an unforced target. The dynamics expressed in the new coordinate system become decoupled harmonic oscillators, which simplifies the analysis. It is also possible to precisely characterize a controllable set, which is the set of initial conditions for the deputy to be able to capture the chief, given the parameters of the problem.

The problem of spacecraft relative motion has been extensively studied in the literature.[4,1,5] The rendezvous problem in the Hill's frame between two spacecraft has been addressed previously using a game theoretic formulation, often a linear quadratic game framework.[6,7] However, in these works, the terminal time is fixed (in pursuit-evasion games, terminal time is typically a free parameter) and the control inputs are not constrained. Stupik, Pontani and Conway have solved a position coincidence problem in three dimensions with input constraints using particle swarm optimization and kriging.[4] Menon and Calise presented optimal feedback guidance laws for a pursuit-evasion game among spacecraft using a quadratic performance index.[5] The reformulation of a two-player differential game with free terminal time as a game with a terminal cost was first introduced by Gutman et al.[8]

Main contributions: We present a solution to the problem of positional coincidence between two maneuvering spacecraft in the range of proximity operations, with free terminal time and subject to input constraints. The solution is presented as a pair of optimal strategies for both spacecraft, which are obtained from modeling the problem as a two-player zero-sum pursuit-evasion game. When the initial orbits are circular, we provide a simple condition to determine if capture is possible, based on the orbit parameters and the initial positions. We also characterize the level sets of the minimum time of capture, which we obtain in closed form for a circular orbit. The main assumption we make in this work is that the chief and the deputy are initially located at different points on circular orbits that are close to each other. This is valid since most missions operate a separate control strategy until the vicinity of the chief is reached, after which the proximity operation in the terminal phase commences.

The paper is organized as follows. First, we present the formulation of the orbital pursuit-evasion game. Then, we discuss the solution approach to obtain the optimal controls for the two spacecraft involved in the game. We discuss a case where both spacecraft are initially on the same circular orbit and characterize the geometry of the level sets. Then, we extend this analysis to cases where the two spacecraft are initially on different circular orbits. Finally, we present numerical simulations that illustrate the solution method and also provide a visualization of the level sets of the time of capture. In the conclusion, we summarize the central idea of the paper and briefly discuss future work.

FORMULATION OF THE ORBITAL PURSUIT-EVASION GAME

Consider two spacecraft (vehicles) orbiting around a central body on the same circular orbit, initially separated by a phase angle. Both spacecraft, one being the chief spacecraft and the other the deputy, possess independent controls to maneuver in the plane of the orbit. The goal of the deputy spacecraft is to attain a certain positional proximity with the chief spacecraft (in other words, to'capture' the chief) whereas the chief spacecraft maneuvers to prevent the deputy from reaching its desired proximity. The chief and deputy spacecraft are also alternatively referred to in the literature as the target and chaser spacecraft respectively. For an unperturbed orbit around the central body, the planar equations of motion of a spacecraft modeled in Levi Civita coordinates along with a time-transformation take the form of de-coupled linear harmonic oscillators.[3] However, in our case, the spacecraft are capable of independent control, and so they are modeled as forced harmonic oscillators. In particular, the equations of motion for a general spacecraft are:

$$\ddot{\boldsymbol{r}}_S = -\omega^2 \boldsymbol{r}_S + \frac{T}{m}\boldsymbol{u}_S,$$

where $\boldsymbol{r}_S \in \mathbb{R}^2$ is the spacecraft position vector at time t, $T > 0$ is the thrusting capacity of the spacecraft, and $m > 0$ is its mass. The angular velocity of the vehicle in the chosen orbit is given by ω. Note that in the circular orbit, the vehicle's radius of orbit, that is, $r := \|\boldsymbol{x}_S\|$ and speed v are related by the linear relation $v = r\omega$. The input vector is denoted by $\boldsymbol{u}_S$ and is constrained to have unit magnitude. Then, the relative motion of the deputy spacecraft with respect to the chief at time t is described by the following equation:

$$\ddot{\boldsymbol{r}}_D - \ddot{\boldsymbol{r}}_C = -\omega^2(\boldsymbol{r}_D - \boldsymbol{r}_C) + \frac{T_D}{m_D}\boldsymbol{u}_D - \frac{T_C}{m_C}\boldsymbol{u}_C, \tag{1}$$

where the position vectors, input vectors, mass and thrust limits for the chief and deputy are denoted with the subscripts C and D respectively. Note that the set of admissible inputs consists of all piece-wise continuous functions taking values on or within the unit circle at each time t. It is reasonable to assume that in the interest of time, the vehicles use the limit of actuation available to them at all times. Hence, the vector inputs are constrained to have unit magnitude, and we can express them in terms of trigonometric functions of angles. In particular, using the angle ϕ for the chief and angle ψ for the deputy, we have $\boldsymbol{u}_D = [\cos\phi \;\; \sin\phi]^{\mathrm{T}}$ and $\boldsymbol{u}_C = [\cos\psi \;\; \sin\psi]^{\mathrm{T}}$. Now, let the relative position vector be denoted by $\boldsymbol{r} := \boldsymbol{r}_D - \boldsymbol{r}_C$ and the relative velocity vector by $\boldsymbol{v} := \dot{\boldsymbol{r}}_D - \dot{\boldsymbol{r}}_C$. In addition, let $\boldsymbol{x} := [\boldsymbol{r} \; \boldsymbol{v}]^{\mathrm{T}}$ denote the relative state vector of the deputy with respect to the chief in the local orbital frame. The equation for relative motion between the two spacecraft can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}_D + \boldsymbol{C}\boldsymbol{u}_C, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2}$$

where $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\omega^2\boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix}$, and $\boldsymbol{B} = \frac{T_D}{m_D}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \end{bmatrix}$ and $\boldsymbol{C} = -\frac{T_C}{m_C}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \end{bmatrix}$.

We are interested in the deputy spacecraft achieving positional coincidence or proximity (as the case requires) to the chief spacecraft in the presence of non-cooperative maneuvering by the chief, in optimal time. This constitutes an orbital pursuit-evasion game. In this context, the chief spacecraft is labeled as the evader and the deputy is labelled as the pursuer. The orbital pursuit-evasion problem is described as follows:

Problem 1. Given the initial positions for the chief and deputy at time $t = 0$, the period ω of the circular orbit, and the system in (2), find the optimal final time t_f of the game and the admissible control inputs $\boldsymbol{u}_C^*(t)$ and $\boldsymbol{u}_D^*(t)$ for each instant of time t, such that when applied to the system in (2), the control $\boldsymbol{u}_D^*(t)$ can guide the deputy spacecraft to achieve at time t_f the desired positional proximity with (that is, capture of) the chief spacecraft which maneuvers according to the input $\boldsymbol{u}_C^*(t)$. That is,

$$
\begin{aligned}
&\min_{\boldsymbol{u}_D(\cdot)} \max_{\boldsymbol{u}_C(\cdot)} \;\; t_f, \\
&\text{subject to} \\
&\|\boldsymbol{r}(t_f)\| = l, \\
&\boldsymbol{x}(0) = \boldsymbol{x}_0, \\
&\text{eqn (2)},
\end{aligned}
$$

where the user-specified tolerance for capture by positional coincidence is given by $l \geq 0$.

SOLUTION TO THE ORBITAL PURSUIT-EVASION GAME

Following the approach proposed by Gutman, Esh and Gefen,[8] we now introduce a game with only a terminal cost that is equivalent to the pursuit-evasion game defined in Problem 1. In particular, we define a state transformation $\boldsymbol{y}(t) := \boldsymbol{D}\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t)\boldsymbol{x}(t)$, where $\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t)$ is the state transition matrix associated with the matrix $\boldsymbol{A}$, and $\boldsymbol{D} = \begin{bmatrix}\boldsymbol{I}_2 & \boldsymbol{0}\end{bmatrix}$ serves to extract the position vector from the state vector. Then, using equation (2) and the fact that $\dot{\boldsymbol{\Phi}}_{\boldsymbol{A}}(t_f, t) = -\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t)\boldsymbol{A}(t)$, we have the time derivative of the transformed state:

$$
\dot{\boldsymbol{y}}(t) = \boldsymbol{\mathcal{B}}(t_f, t)\boldsymbol{u}_D(t) + \boldsymbol{\mathcal{C}}(t_f, t)\boldsymbol{u}_C(t). \tag{3}
$$

where $\boldsymbol{\mathcal{B}}(t_f, t) := \boldsymbol{D}\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t)\boldsymbol{B}(t)$ and $\boldsymbol{\mathcal{C}}(t_f, t) := \boldsymbol{D}\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t)\boldsymbol{C}(t)$. Specifically, for the problem formulated in this paper, with the initial time at $t = 0$,

$$
\begin{aligned}
\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t) &= \begin{bmatrix} \cos(\omega(t_f - t))\boldsymbol{I}_2 & \sin(\omega(t_f - t))/\omega \boldsymbol{I}_2 \\ -\omega \sin(\omega(t_f - t))\boldsymbol{I}_2 & \cos(\omega(t_f - t))\boldsymbol{I}_2 \end{bmatrix}, \\
\boldsymbol{\mathcal{B}}(t_f, t) &= \frac{T_D}{\omega m_D} \sin(\omega(t_f - t))\boldsymbol{I}_2, \\
\boldsymbol{\mathcal{C}}(t_f, t) &= -\frac{T_C}{\omega m_C} \sin(\omega(t_f - t))\boldsymbol{I}_2.
\end{aligned}
$$

Then the transformed state $\boldsymbol{y}(t)$ is given by

$$
\boldsymbol{y}(t) = \cos(\omega(t_f - t))\boldsymbol{r}(t) + \frac{\sin(\omega(t_f - t))}{\omega}\boldsymbol{v}(t). \tag{4}
$$

At any time t, let the terminal cost for the game be denoted by $\mathcal{J}(\boldsymbol{y}, t_f, t) := \|\boldsymbol{y}(t_f)\|$. From the definition of the state transformation, it is clear that $\|\boldsymbol{y}(t_f)\| = \|\boldsymbol{r}(t_f)\|$ since $\boldsymbol{\Phi}_{\boldsymbol{A}}(t_f, t_f) = \boldsymbol{I}_2$. If we consider a function defined on $\boldsymbol{y}(t)$ as $w(t) := \|\boldsymbol{y}(t)\|$, then the time-derivative of $w(t)$ is

$$
\dot{w}(t) = \frac{\boldsymbol{y}^{\mathrm{T}}(t)}{\|\boldsymbol{y}(t)\|}\boldsymbol{y}(t).
$$

Consequently, the expression for the terminal cost can be written as follows:

$$\mathcal{J}(\boldsymbol{y}, t_f, t) = \|\boldsymbol{y}(t_f)\| = \|\boldsymbol{y}(t)\| + \int_t^{t_f} \frac{\boldsymbol{y}^{\mathrm{T}}(\eta)}{\|\boldsymbol{y}(\eta)\|}\dot{\boldsymbol{y}}(\eta)\mathrm{d}\eta.$$

The free-final time PEG now can be re-formulated as a terminal cost game:
Find the control inputs $\boldsymbol{u}_C^*$ and $\boldsymbol{u}_D^*$ for each time instant t, to achieve the min-max value (saddle point) $\mathcal{J}^*(\boldsymbol{y}, t_f, t)$ of the terminal cost given by:

$$\begin{aligned}
\mathcal{J}^*(\boldsymbol{y}, t_f, t) &= \min_{\boldsymbol{u}_D(\cdot)} \max_{\boldsymbol{u}_C(\cdot)} \; \mathcal{J}(\boldsymbol{y}, t_f, t) \\
&= \|\boldsymbol{y}^*(t_f)\| \\
&= \min_{\boldsymbol{u}_D(\cdot)} \max_{\boldsymbol{u}_C(\cdot)} \; \|\boldsymbol{y}(t)\| + \int_t^{t_f} \frac{\boldsymbol{y}^{\mathrm{T}}(\eta)}{\|\boldsymbol{y}(\eta)\|}\dot{\boldsymbol{y}}(\eta)\mathrm{d}\eta \\
&= \|\boldsymbol{y}(t)\| + \int_t^{t_f} \min_{\boldsymbol{u}_D(\eta)} \max_{\boldsymbol{u}_C(\eta)} \; \frac{\boldsymbol{y}^{\mathrm{T}}(\eta)}{\|\boldsymbol{y}(\eta)\|}\dot{\boldsymbol{y}}(\eta)\mathrm{d}\eta.
\end{aligned} \tag{5}$$

The optimal terminal state attained by applying the optimal controls is denoted by $\boldsymbol{y}^*(t)$. Note that we also want to find the optimal final time $t_f > 0$ at which $\mathcal{J}^*(\boldsymbol{y}, t_f, t) = \|\boldsymbol{y}^*(t_f)\| = l$. From inspecting the min-max term inside the integral in equation (5), and recalling the expression for the time derivative of the transformed state $\dot{\boldsymbol{y}}(t)$ in (3), we obtain the saddle point control pair as follows:

$$\begin{aligned}
\boldsymbol{u}_D^*(t) &= -\mathrm{sgn}(\boldsymbol{\mathcal{B}}(t_f, t))\frac{\boldsymbol{y}^*(t)}{\|\boldsymbol{y}^*(t)\|} = -\frac{\boldsymbol{y}^*(t)}{\|\boldsymbol{y}^*(t)\|}, \\
\boldsymbol{u}_C^*(t) &= \mathrm{sgn}(\boldsymbol{\mathcal{C}}(t_f, t))\frac{\boldsymbol{y}^*(t)}{\|\boldsymbol{y}^*(t)\|} = -\frac{\boldsymbol{y}^*(t)}{\|\boldsymbol{y}^*(t)\|}.
\end{aligned} \tag{6}$$

It is interesting to note that for optimal play, the controls for the pursuer and the evader spacecraft are the same. Also, the above structure for the optimal control pair is only valid if the value of $\|\boldsymbol{y}^*(t)\|$ never goes to zero at any time $t \in [0, t_f]$. Hence, when $\|\boldsymbol{y}^*(t)\|$ does become zero, the optimal controls must also be set to zero. If we assume $m_C = m_D = m$, then for all $t \in [0, t_f]$,

$$l = \|\boldsymbol{y}^*(t_f)\| = \|\boldsymbol{y}(t)\| - \frac{(T_D - T_C)(1 - \cos(\omega(t_f - t)))}{m\omega^2}.$$

For perfect positional coincidence, $l = 0$, and

$$0 = \|\boldsymbol{y}(t)\| - \frac{(T_D - T_C)(1 - \cos(\omega(t_f - t)))}{m\omega^2}. \tag{7}$$

Applying eqn (4) to eqn (7),

$$1 - \cos(\omega(t_f - t)) = \frac{m\omega^2}{(T_D - T_C)}\|\cos(\omega(t_f - t))\boldsymbol{r}(t) + \frac{\sin(\omega(t_f - t))}{\omega}\boldsymbol{v}(t)\|$$

or

$$\begin{aligned}
(1 - \cos(\omega(t_f - t)))\frac{(T_D - T_C)}{m\omega^2} = \big(&\cos^2(\omega(t_f - t))\|\boldsymbol{r}(t)\|^2 + \frac{\sin^2(\omega(t_f - t))}{\omega^2}\|\boldsymbol{v}(t)\|^2 \\
&+ 2\frac{\cos(\omega(t_f - t))\sin(\omega(t_f - t))}{\omega}(\boldsymbol{r}(t)^{\mathrm{T}}\boldsymbol{v}(t))\big)^{\frac{1}{2}}.
\end{aligned} \tag{8}$$

To find the optimal time of capture from the initial conditions, we set $t = 0$ in equation (8).

$$\begin{aligned}(1-\cos(\omega t_f))\frac{(T_D - T_C)}{m\omega^2} &= \big(\cos^2(\omega t_f)\|\boldsymbol{r}(0)\|^2 + \frac{\sin^2(\omega t_f)}{\omega^2}\|\boldsymbol{v}(0)\|^2\big)2R^2(1-\cos\theta)\\ &\quad + 2\frac{\cos(\omega t_f)\sin(\omega t_f)}{\omega}(\boldsymbol{r}(0)^{\mathrm{T}}\boldsymbol{v}(0))\big)^{\frac{1}{2}}.\end{aligned}$$

For a circular orbit at $t = 0$, the position and velocity vectors for each spacecraft are perpendicular to each other. Let us recall here that $\boldsymbol{r} = \boldsymbol{r}_D - \boldsymbol{r}_C$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}_D - \dot{\boldsymbol{r}}_C = \boldsymbol{v}_D - \boldsymbol{v}_C$ for all time t. If we consider the initial phase separation between the chief and deputy to be denoted by the angle θ, we see that the angle between $\boldsymbol{r}_C(0)$ and $\boldsymbol{r}_D(0)$ is equal to θ and so is the angle between the vectors $\boldsymbol{v}_C(0)$ and $\boldsymbol{v}_D(0)$. Let R denote the radius of the circular orbit such that $\|\boldsymbol{r}_C(0)\| = \|\boldsymbol{r}_D(0)\| = R$, and V denote the speed on the orbit such that $\|\boldsymbol{v}_C(0)\| = \|\boldsymbol{v}_D(0)\| = V$. Note that $V = \omega R$. From the law of cosines, we determine that $\|\boldsymbol{r}(0)\|^2 = R^2 + R^2 - 2R^2\cos\theta$ and $\|\boldsymbol{v}(0)\|^2 = V^2 + V^2 - 2V^2\cos\theta$. Substituting these values in eqn (8),

$$\begin{aligned}(1-\cos(\omega t_f))\frac{(T_D - T_C)}{m\omega^2} &= \big(2\cos^2(\omega t_f)R^2(1-\cos\theta) + 2\frac{\sin^2(\omega t_f)}{\omega^2}V^2(1-\cos\theta)\\ &\quad + 2\frac{\cos(\omega t_f)\sin(\omega t_f)}{\omega}(\boldsymbol{r}(0)^{\mathrm{T}}\boldsymbol{v}(0))\big)^{\frac{1}{2}}.\end{aligned}$$

Making use of the expressions for $\boldsymbol{r}(0)$ and $\boldsymbol{v}(0)$,

$$\begin{aligned}\boldsymbol{r}(0)^{\mathrm{T}}\boldsymbol{v}(0) &= \boldsymbol{r}_D(0)^{\mathrm{T}}\boldsymbol{v}_D(0) + \boldsymbol{r}_C(0)^{\mathrm{T}}\boldsymbol{v}_C(0) - \boldsymbol{r}_D(0)^{\mathrm{T}}\boldsymbol{v}_C(0) + \boldsymbol{r}_C(0)^{\mathrm{T}}\boldsymbol{v}_D(0)\\ &= -\boldsymbol{r}_D(0)^{\mathrm{T}}\boldsymbol{v}_C(0) + \boldsymbol{r}_C(0)^{\mathrm{T}}\boldsymbol{v}_D(0)\\ &= -RV(\cos(\pi/2+\theta) + \cos(\pi/2-\theta))\\ &= 0.\end{aligned}$$

Then,

$$\begin{aligned}(1-\cos(\omega t_f))\frac{(T_D - T_C)}{m\omega^2} &= \big(2\cos^2(\omega t_f)R^2(1-\cos\theta) + 2\frac{\sin^2(\omega t_f)}{\omega^2}V^2(1-\cos\theta)\big)^{\frac{1}{2}}\\ &= \big(2R^2(\cos^2(\omega t_f) + \sin^2(\omega t_f))(1-\cos\theta)\big)^{\frac{1}{2}}\\ &= \big(2R^2(1-\cos\theta)\big)^{\frac{1}{2}}.\end{aligned}\tag{9}$$

We will only consider the positive root of the term inside the paranthesis since the norm function is non-negative. Alternatively,

$$(1-\cos(\omega t_f)) = \frac{\sqrt{2}mR\omega^2(1-\cos\theta)^{\frac{1}{2}}}{(T_D - T_C)}.\tag{10}$$

The smallest real positive value of t_f which satisfies equation (10) corresponds to the first time that the deputy spacecraft captures the chief spacecraft. If no such value exists for t_f, then capture of the chief by the deputy is not possible provided that the chief spacecraft maneuvers optimally. In equation (10), the expression on the left hand side always takes values between 0 and 2. Thus, we obtain a condition that must be satisfied for capture to be possible:

$$0 \leq \frac{mR\omega^2(1-\cos\theta)^{\frac{1}{2}}}{(T_D - T_C)} \leq \sqrt{2}.\tag{11}$$

Alternatively, this condition bounds the initial phase separation between the two spacecraft that can result in capture for a given circular orbit and values for m, T_C and T_D. We also recover the expected condition that the actuation capability of the deputy (pursuer) must be greater than that of the chief (evader) for capture to be guaranteed.

In view of eqn (9), eqn (10) can be written in terms of the norm of the initial relative position vector $\|\boldsymbol{r}(0)\|$ as follows:

$$(1-\cos(\omega t_f)) = \frac{m\omega^2\|\boldsymbol{r}(0)\|}{(T_D - T_C)}. \tag{12}$$

Using this expression, we can visualize the controllable sets, that is, the initial conditions for which the pursuer can reach the origin (capture the evader) in finite time. These controllable sets also represent the reachable sets if we consider the inertial frame, for the deputy to reach the chief. The visualization of these sets is obtained by expanding in the position space the contour that corresponds to each finite real solution for the time of capture. The projection of the contour (isochrone) for each value of t_f on the position plane is circular in this case and centered around the position of the chief spacecraft, as shown in eqn (13).

$$\|\boldsymbol{r}(0)\| = \frac{(1-\cos(\omega t_f))(T_D - T_C)}{m\omega^2}. \tag{13}$$

The upper bound on the relative distance between the two spacecraft for capture to be possible in finite time is given by:

$$\|\boldsymbol{r}(0)\|_{\max} = \frac{2(T_D - T_C)}{m\omega^2}. \tag{14}$$

While the capture condition indicates the possibility of capture given a set of initial conditions, it is to be noted that the relative equations of motion in their simplified form of the forced harmonic oscillator are most valid close to the original circular orbit, and as we move further away from the orbit, in practice, there will be increase in position error from following the control inputs calculated using this model. For cases where only positional proximity is desired, that is, $l > 0$,

$$\begin{aligned} l + (1-\cos(\omega t_f))\frac{(T_D - T_C)}{m\omega^2} &= \Big(2\cos^2(\omega t_f)R^2(1-\cos\theta) + 2\frac{\sin^2(\omega t_f)}{\omega^2}V^2(1-\cos\theta) \\ &\quad + 2\frac{\cos(\omega t_f)\sin(\omega t_f)}{\omega}(\boldsymbol{r}(0)^{\mathrm{T}}\boldsymbol{v}(0))\Big)^{\frac{1}{2}}, \end{aligned}$$

and we have in place of eqn (10),

$$l + (1-\cos(\omega t_f))\frac{(T_D - T_C)}{m\omega^2} = \big(2R^2(1-\cos\theta)\big)^{\frac{1}{2}}.$$

The smallest positive real solution of this equation will be the time of desired positional proximity (capture).

SPACECRAFT INITIALLY ON DIFFERENT CIRCULAR ORBITS

Next, let us consider the case where the chief and deputy are initially on different circular orbits, that is, $\omega_C \neq \omega_D$. Then, the relative motion described in eqn (1) is now described as

$$\begin{aligned} \ddot{\boldsymbol{r}}_D - \ddot{\boldsymbol{r}}_C &= -\omega_D^2(\boldsymbol{r}_D) + \omega_C^2(\boldsymbol{r}_C) + \frac{T_D}{m_D}\boldsymbol{u}_D - \frac{T_C}{m_C}\boldsymbol{u}_C, \\ &= -\omega_D^2(\boldsymbol{r}_D - \boldsymbol{r}_C) + (\omega_C^2 - \omega_D^2)(\boldsymbol{r}_C) + \frac{T_D}{m_D}\boldsymbol{u}_D - \frac{T_C}{m_C}\boldsymbol{u}_C. \end{aligned} \tag{15}$$

The above equation cannot directly be expressed in state space form using the relative states alone. By choosing the new state vector to be an extension of the previous state, that is, $\tilde{\boldsymbol{x}} := [\boldsymbol{r}\ \boldsymbol{v}\ \boldsymbol{r}_C\ \boldsymbol{v}_C]^{\mathrm{T}}$, we get a state space model of the system that is of the form

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{B}}\boldsymbol{u}_D + \tilde{\boldsymbol{C}}\boldsymbol{u}_C, \quad \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0, \tag{16}$$

where the matrices are as follows:

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 & \boldsymbol{0} & \boldsymbol{0} \\ -\omega_D^2\boldsymbol{I}_2 & \boldsymbol{0} & (\omega_C^2-\omega_D^2)\boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} & -\omega_C^2\boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix}, \tilde{\boldsymbol{B}} = \frac{T_D}{m_D}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \text{ and } \tilde{\boldsymbol{C}} = -\frac{T_C}{m_C}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \\ \boldsymbol{0} \\ -\boldsymbol{I}_2 \end{bmatrix}.$$

Also, the matrix $\tilde{\boldsymbol{D}} := \begin{bmatrix}\boldsymbol{I}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$. The expression in eqn (3) holds now with the replacement of $\boldsymbol{A}$ by $\tilde{\boldsymbol{A}}$, $\boldsymbol{B}$ by $\tilde{\boldsymbol{B}}$, and $\boldsymbol{C}$ by $\tilde{\boldsymbol{C}}$. With the initial time fixed at $t = 0$, we compute the state transition matrix $\boldsymbol{\Phi}_{\tilde{\boldsymbol{A}}}(t_f, t)$ using an inverse laplace transform and subsequently the matrices $\boldsymbol{\mathcal{B}}(t_f, t)$ and $\boldsymbol{\mathcal{C}}(t_f, t)$. For compactness of expression in the matrices, we denote $(t_f - t)$ by the variable τ.

$$\begin{aligned}
\boldsymbol{\Phi}_{\tilde{\boldsymbol{A}}}(t_f,t) &= \begin{bmatrix} \cos\omega_D\tau\boldsymbol{I}_2 & \frac{\sin\omega_D\tau}{\omega_D}\boldsymbol{I}_2 & (\cos\omega_D\tau-\cos\omega_C\tau)\boldsymbol{I}_2 & \frac{\omega_C\sin\omega_D\tau-\omega_D\sin\omega_C\tau}{\omega_C\omega_D}\boldsymbol{I}_2 \\ -\omega_D\sin\omega_D\tau\boldsymbol{I}_2 & \cos\omega_D\tau\boldsymbol{I}_2 & (\omega_C\sin\omega_C\tau-\omega_D\sin\omega_D\tau)\boldsymbol{I}_2 & (\cos\omega_D\tau-\cos\omega_C\tau)\boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} & \cos\omega_C\tau\boldsymbol{I}_2 & \frac{\sin\omega_C\tau}{\omega_C}\boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} & -\omega_C\sin\omega_C\tau\boldsymbol{I}_2 & \cos\omega_C\tau\boldsymbol{I}_2 \end{bmatrix}, \\
\boldsymbol{\mathcal{B}}(t_f,t) &= \frac{T_D}{\omega_D m_D}\sin(\omega_D\tau)\boldsymbol{I}_2, \\
\boldsymbol{\mathcal{C}}(t_f,t) &= -\frac{T_C}{\omega_C\omega_D m_C}(2\omega_C\sin\omega_D\tau - \omega_D\sin\omega_C\tau)\boldsymbol{I}_2.
\end{aligned}$$

By repeating the analysis for the case $\omega_C = \omega_D$ to this case, the transformed state is

$$\begin{aligned}
\boldsymbol{y}(t) &= \frac{1}{\omega_C\omega_D}\big(\omega_C\sin\omega_D\tau\boldsymbol{v}(t) + (\omega_C\sin\omega_D\tau - \omega_D\sin\omega_C\tau)\boldsymbol{v}_C(t) \\
&\quad + \omega_C\omega_D\cos\omega_D\tau\boldsymbol{r}(t) + \omega_C\omega_D(\cos\omega_D\tau - \cos\omega_C\tau)\boldsymbol{r}_C(t)\big) \\
&= \frac{\sin\omega_D\tau\boldsymbol{v}_D(t)}{\omega_D} - \frac{\sin\omega_C\tau\boldsymbol{v}_C(t)}{\omega_C} + \cos\omega_D\tau\boldsymbol{r}_D(t) - \cos\omega_C\tau\boldsymbol{r}_C(t).
\end{aligned}$$

It is still true that $\|\boldsymbol{y}(t_f)\| = \|\boldsymbol{r}(t_f)\|$. The optimal control pair is given by eqn (6). For $m_C = m_D = m$ and $l = 0$,

$$0 = \|\boldsymbol{y}(t)\| - \frac{(T_D - 2T_C)}{m}\frac{(1-\cos(\omega_D(t_f-t)))}{\omega_D^2} - \frac{T_C(1-\cos(\omega_C(t_f-t)))}{m\omega_C^2}. \tag{17}$$

Let R_C and V_C be the radius and speed of the chief's orbit and R_D and V_D be the radius and speed of the deputy's orbit respectively. At time $t = 0$, using properties of a circular orbit as before, and assuming that the chief is ahead (in terms of phase angle) of the deputy in its orbit, we have that

$$
\begin{aligned}
\|\boldsymbol{y}(0)\| &= \Big(\frac{\sin^2\omega_D t_f V_D^2}{\omega_D^2} + \frac{\sin^2\omega_C t_f V_C^2}{\omega_C^2} + \cos^2\omega_D t_f R_D^2 + \cos^2\omega_C t_f R_C^2 \\
&\quad - 2\big(V_C V_D \frac{\sin\omega_D t_f \sin\omega_C t_f}{\omega_C\omega_D}\cos\theta + R_C V_D \frac{\sin\omega_D t_f \cos\omega_C t_f}{\omega_D}\sin\theta \\
&\quad - V_C R_D \frac{\sin\omega_C t_f \cos\omega_D t_f}{\omega_C}\sin\theta + R_C R_D \cos\omega_D t_f \cos\omega_C t_f \cos\theta\big)\Big)^{\frac{1}{2}} \\
&= \big(R_D^2 + R_C^2 - 2\big(R_C R_D \cos((\omega_D-\omega_C)t_f)\cos\theta + R_C R_D \sin((\omega_D-\omega_C)t_f)\sin\theta\big)\big)^{\frac{1}{2}} \\
&= \big(R_C^2 + R_D^2 - 2R_C R_D \cos((\omega_D-\omega_C)t_f-\theta)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Note that the above equation applies when the two spacecraft are moving in the same direction in their orbits initially. If otherwise, the derivation of this equation is slightly different and involves a sum of the angular rates instead of their difference. From eqn (17),

$$
\begin{aligned}
\big(R_C^2 + R_D^2 - 2R_C R_D \cos((\omega_D-\omega_C)t_f-\theta)\big)^{\frac{1}{2}} &= \frac{(T_D-2T_C)}{m}\frac{(1-\cos(\omega_D t_f))}{\omega_D^2} \\
&\quad + \frac{T_C(1-\cos(\omega_C t_f))}{m\omega_C^2}.
\end{aligned} \tag{18}
$$

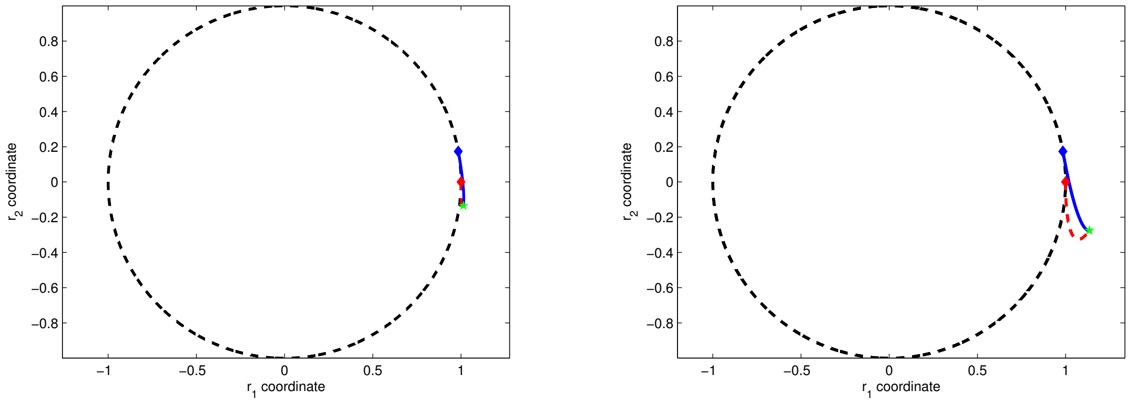
The smallest positive real solution to eqn (18) is the time of capture of the chief by the deputy when the two spacecraft start from different circular orbits. The solution to this transcendental equation can be obtained using a numerical solver. From inspection of eqn (18), we see that all the terms are positive. Hence, the left hand side terms must sum upto a positive real number or zero. If this condition is not satisfied, capture can be avoided under the given initial conditions.

NUMERICAL SIMULATIONS

In this section, we present numerical simulations for a few sample cases. We consider the two spacecraft (chief and deputy) to be of mass $m = 1$ MU (MU: mass unit) each, on a circular orbit of radius $R = 1$ LU (LU: length unit) and $\omega = \frac{\pi}{4}$ rad/TU (TU: time unit). The initial phase separation between the chief and deputy $\theta = 10°$.

The trajectories of the chief and deputy are shown in Figure 1. The initial circular orbit is shown in dotted line. We observe that capture happens sooner with higher T_D, as expected. In addition, the level sets of the time of capture are in Figure 2 with time on the vertical axis against the orbital plane. In Figures 2(b) and 2(d), we see that due to the periodicity of the dynamics, there is an upper bound on the area covered by the projection of the controllable sets on the position plane. This bound is characterized by eqn (14). Note that the level sets as shown in Figure 2 represent the magnitude of the relative position vector, and are centered at the actual position of the chief spacecraft. In Figure 2(e), we see that with more time, the controllable sets do not cover more relative initial conditions in the orbital plane.

Next, we consider a scenario where the chief and deputy are initially in different circular orbits. In particular, we pick the deputy to be on an orbit with $R_D = 1.05$ LU and $\omega_D = \frac{1.1\pi}{4}$ rad/TU, and the chief is on an orbit with $R_C = 1.0$ LU and $\omega_C = \frac{\pi}{4}$ rad/TU and the initial phase separation between the two spacecraft is again $10°$ with the chief ahead of the deputy in its orbit. All thrust values are expressed in MU LU / TU^2.



(a) $T_D = 2, T_C = 0.9$ (b) $T_D = 1.2, T_C = 0.9$

Figure 1. Trajectories of the chief (blue) and deputy (red) ending in capture (positional coincidence). The diamonds mark the initial positions of the two spacecraft, and the green mark represents the final positional coincidence. All values are in LU, TU and MU.

It is evident that with higher actuation limit, the deputy can achieve positional coincidence with the chief in lesser time.

CONCLUSION

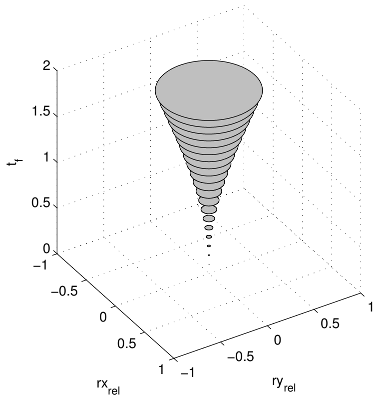
We have solved the pursuit-evasion game in the orbital frame between two maneuvering spacecraft, in the presence of input constraints for both vehicles. The solution proposed here is an open loop solution that depends only on the current relative state of the deputy with respect to the chief. In addition, we have derived a condition for capture that is based on the initial conditions provided for the two spacecraft and other fixed parameters. We have also provided a simple method based on analytical and closed form expressions to extract the level sets of the time-of-capture and consequently obtain the controllable sets for the deputy spacecraft. We will extend this framework to solve the problem of relative velocity minimization which for instance, is relevant for spacecraft in orbit to minimize impact during a meteor shower.[9] We will also analyze the effectiveness of our game theoretic solution for position coincidence problems where the chief (target) is in an orbit that is non-circular. Finally, the optimal value of the time of capture in the game we have addressed in this work can be interpreted as a metric for the risk of collision. In future, we will incorporate these ideas in other problems such as path planning for autonomous spacecraft operating near unpredictably moving objects in orbit.[10]

ACKNOWLEDGMENT

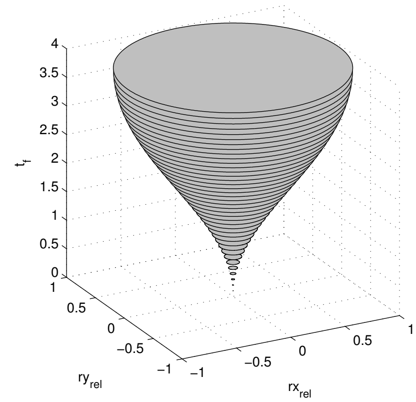
This work has been supported in part by NSF-CMMI grant, Award Number: 1562339. The first author also acknowledges support from the Zonta International Foundation for the year 2016-2017.

REFERENCES

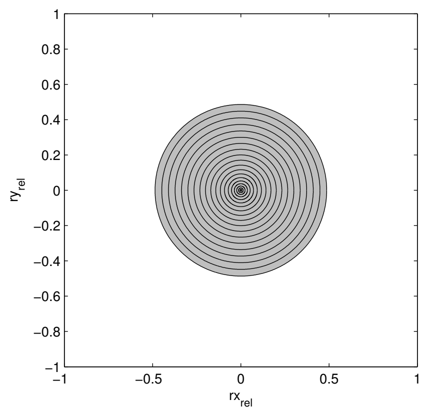
[1] A. Weiss, C. Petersen, M. Baldwin, R. S. Erwin, and I. Kolmanovsky, “Safe Positively Invariant Sets for Spacecraft Obstacle Avoidance,” *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 4, 2014, pp. 720–732.



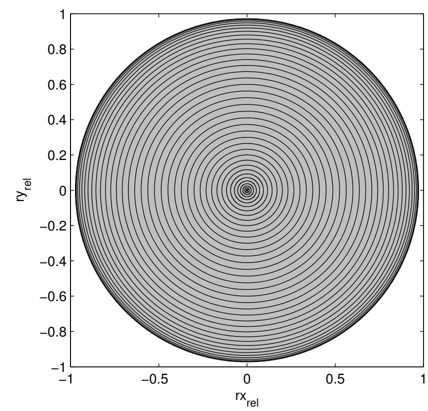
(a) Level sets in the position plane, expanded until $t_f = 2$.



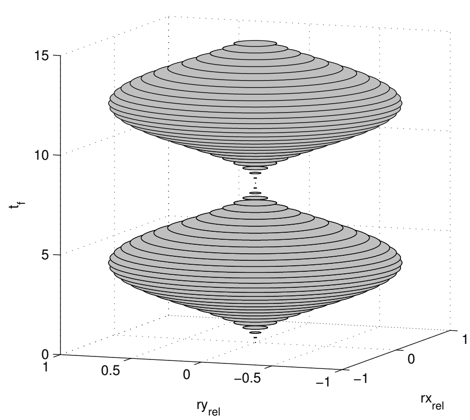
(b) Level sets expanded in the position plane until $t_f = 4$.



(c) Projection of level sets for $t_f = 2$ on the position plane.

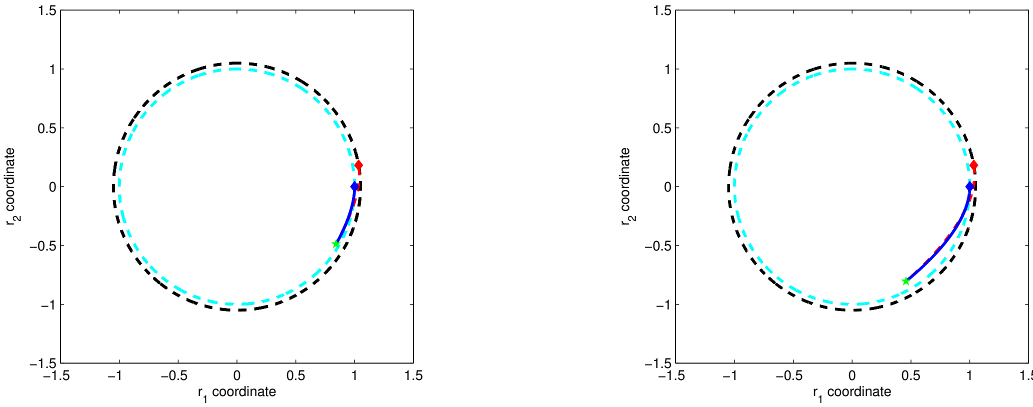


(d) Projection of level sets for $t_f = 4$ on the position plane.



(e) Level sets in the position plane, expanded until $t_f = 15$.

Figure 2. Level sets of the time of capture for $T_D = 1.2$ and $T_C = 0.9$. All values are in LU, TU and MU.



(a) $T_D = 2, T_C = 0.9$ (b) $T_D = 1.2, T_C = 0.9$

Figure 3. Trajectories of the chief (blue) and deputy (red) ending in capture starting from different circular orbits. The original orbits of the spacecraft are shown in cyan (chief) and black (deputy). All values are in LU, TU and MU.

[2] E. Kollin and M. R. Akella, "Autonomous Time-Optimal Spacecraft Rendezvous and Proximity Operations using Stabilized Continuation," *Proceedings of the 26th AAS-AIAA Space Flight Mechanics Meeting*, Feb, 2016.

[3] S. Hernandez and M. R. Akella, "Lyapunov-based guidance for orbit transfers and rendezvous in levicivita coordinates," *Journal of Guidance, Control, and Dynamics*, Vol. 37, No. 4, 2014, pp. 1170–1181.

[4] J. Stupik, M. Pontani, and B. Conway, "Optimal pursuit/evasion spacecraft trajectories in the hill reference frame," *AIAA/AAS Astrodynamics Specialist Conference*, 2012, p. 4882.

[5] P. Menon and A. Calise, "Interception, evasion, rendezvous and velocity-to-be-gained guidance for spacecraft," *Journal of Guidance Control Dynamics*, Vol. 10, 1987, pp. 334–341.

[6] V. Tartaglia and M. Innocenti, "Game Theoretic Strategies for Spacecraft Rendezvous and Motion Synchronization," *AIAA Guidance, Navigation, and Control Conference*, No. January, 2016, pp. 1–13, doi:10.2514/6.2016-0873.

[7] D. Repperger and A. Koivo, "Optimal Terminal Rendezvous as a Stochastic Differential Game Problem," *IEEE Transactions on Aerospace and Electronic Systems*, No. 3, 1972, pp. 319–326.

[8] S. Gutman, M. Esh, and M. Gefen, "Simple linear pursuit-evasion games," *Computers & Mathematics with Applications*, Vol. 13, No. 1-3, 1987, pp. 83–95.

[9] G. E. Peterson, *Dynamics of Meteor Outbursts and Satellite Mitigation Strategies*. AIAA, 1999.

[10] J. Selvakumar and E. Bakolas, "Evasion from a group of pursuers with a prescribed target set for the evader," *American Control Conference (ACC), 2016*, IEEE, 2016, pp. 155–160.