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Solvable Leibniz algebras with quasi-filiform split Lie nilradical

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Maqolada nilradikalli tabiiy usulda gradiurovkalangan kvazi-filiform Li algebrasi o'rganilgan va bunday algebralarni tasniflash metodi keltirilgan.

В настоящей статье мы исследовали алгебр Лейбница чьи нильрадикали является естественно градуированной квази-филиформной алгеброй Ли и представили метод построения и классификации таких алгебр.

Introduction. In recent years, a great deal of study has been conducted on Leibniz algebras. These algebras, introduced by Loday in [11], were conceived as a natural generalization of Lie algebras. Various classical results from the study of Lie algebras have been extended to the study of Leibniz algebras.

Significantly, Levi's Theorem for Leibniz algebras, proven in [3], states that every Leibniz algebra can be represented as a semidirect sum of a semisimple Lie subalgebra and the maximal solvable ideal. Semisimple Lie algebras are direct sums of simple Lie algebras, which are completely classified. Hence, the problem of classification of Leibniz algebras reduces to the classification of those that are solvable. By a result of [6], a solvable Leibniz algebra is a direct sum of its nilradical and a space complimentary to the nilradical. Thus, the classification of Leibniz algebras actually reduces to the study of nilpotent algebras and solvable algebras with these as their nilradicals.

A convenient class of nilpotent Leibniz algebras to consider in this manner are those subject to some restriction on their nilindex. Moreover, it is often useful to restrict focus to naturally graded algebras. In the study of Lie algebras, the naturally graded algebras of maximum nilindex were originally classified in [13]. In the Leibniz case, the maximum nilindex is greater by one and such algebras, as well as the non-Lie naturally graded filiform Leibniz algebras, were classified in [2]. Based upon such results, it is possible to classify solvable Leibniz algebras with nilradicals matching these descriptions, a task which was carried out in [6] and [7] for the null-filiform and naturally graded filiform cases respectively. The classification of some filiform Leibniz algebras was obtained in [9] and [10], allowing for the recent classification in [5] of solvable Leibniz algebras with filiform nilradicals and the surprising result that, when the nilradical is Lie, any such algebra is Lie.

A natural next step is to consider algebras of nilindex one less than filiform, called quasi-filiform. All naturally graded quasi-filiform Lie algebras were classified in [8], with a slight o mission amended in [14], whereas the non-Lie Leibniz algebras of this type were classified in [4]. In this paper, we examine solvable Leibniz algebras with naturally graded quasi-filiform split Lie nilradicals and classify those with the property that the space complementary to the nilradical is of maximum dimension, excluding a particular low-dimensional special case.

Throughout this paper, products omitted from tables of multiplication are assumed to be zero. All algebras are assumed to be finite-dimensional and over the field of complex numbers.

Preliminaries. We now present some preliminary definitions and relevant results.

Definition 1. Let L be a vector space over a field F, endowed with a bilinear bracket operation $[-,-]$. Then we call L a Leibniz algebra if all $x, y, z \in L$ satisfy the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

Note that, in the case where $[x,x] = 0$ for all $x \in L$, L is, in fact, a Lie algebra. The following definitions provide some useful concepts for the study of Leibniz algebras.

Definition 2. Let L be a Leibniz algebra over a field F. Then we call the sets $\mathrm{Ann}_\ell(L) = \{x \in L \mid [x,y] = 0\ \forall y \in L\}$ and $\mathrm{Ann}_r(L) = \{x \in L \mid [y,x] = 0\ \forall y \in L\}$ the left and right annihilators of L respectively. We define the center of L to be the intersection of the left and right annihilators and denote it by $\mathrm{Center}(L)$.

We can conclude from the Leibniz identity that, if L is a Leibniz algebra, $[x,x], ([x,y]+[y,x]) \in \mathrm{Ann}_r(L)$ for any $x, y \in L$.

Definition 3. Let L be a Leibniz algebra over a field F. We define lower central series L^k for $k \in \mathbb{N}$ by $L^1 = L$ and $L^{k+1} = [L^k, L]$ and its derived series $L^{[k]}$ for $k \in \mathbb{N}$ by $L^{[1]} = L$ and $L^{[k+1]} = [L^{[k]}, L^{[k]}]$. We say that L is nilpotent if its lower central series is eventually zero and solvable if its derived series is eventually zero; it is clear from the definitions that nilpotency implies solvability. Moreover, if L is nilpotent, we define its index of nilpotency to be the smallest natural number k such that $L^k = \{0\}$, and, if it is solvable, we define its index of solvability to be the smallest natural number m such that $L^{[m]} = \{0\}$. Note that, if $\dim(L) = n$ and L is nilpotent, index of nilpotency of L is at most $n+1$.

The following theorem further relates the notions of nilpotency and solvability [1].

Theorem 1. *Let L be a Leibniz algebra over a field F. Then L is solvable if and only if L^2 is nilpotent.*

Definition 4. We call the maximal nilpotent ideal of a Leibniz algebra its nilradical.

We now introduce some notions which we will use to examine this relationship.

Definition 5. Let L be a Leibniz algebra over a field F and $d : L \to L$ a linear transformation such that, for any $x, y \in L$,

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

Then we say that d is a derivation of L; the set of derivations of L is denoted by $\mathrm{Der}(L)$.

It follows by the Leibniz identity that, for any $x \in L$, the operator $\mathcal{R}_x : L \to L$ of right multiplication by x is a derivation of L. Such derivations are called inner, while all others are called outer.

Definition 6. [12]. Let L be a Leibniz algebra over a field F and $d_1, \dots, d_k$ derivations of L such that $\alpha_1 d_1 + \ldots + \alpha_k d_k$ is not nilpotent for any nonzero $(\alpha_1, \dots, \alpha_k) \in F^k$. Then we say that $d_1, \dots, d_k$ are nil-independent.

The following theorem [6] gives us a bound on the dimension of the space complementary to an algebra's nilradical and will play an important role in our classification.

Theorem 2. *Let L be a Leibniz algebra over a field F, N its nilradical, and k the maximum number of nil-independent derivations of N. Then $\dim(L) - \dim(N) \leq k$.*

We now define terminology for n-dimensional algebras of two specific nilindices, the latter being the type of algebra with which we will concern ourselves.

Definition 7. Let L be an n-dimensional Leibniz algebra over a field F such that, $\dim(L^i) = n - i$ for each $2 \leq i \leq n$. Then we say that L is filiform.

Definition 8. Let L be an n-dimensional nilpotent Leibniz algebra over a field F with index of nilpotency $n-1$. Then we say that L is quasi-filiform.

Finally, we present the classification of naturally graded quasi-filiform split Lie algebras, which follows directly from the classification of all naturally graded quasi-filiform Lie algebras originally published in [8] and corrected in [14].

Theorem 3. *Let $\mathfrak{g}$ be an n-dimensional naturally graded quasi-filiform split Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ is isomorphic to one of the following pairwise non-isomorphic algebras with basis $\{e_0, \dots, e_{n-1}\}$:*

$$\begin{array}{ll} L_{n-1} \oplus \mathbb{C}: & n \geq 4, \\ \quad [e_0, e_i] = -[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-3, \\ Q_{n-1} \oplus \mathbb{C}: & n \geq 7, n- \text{odd}, \\ \quad [e_0, e_i] = -[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-4, \\ \quad [e_i, e_{n-2-i}] = (-1)^{i-1} e_{n-2}, & 1 \leq i \leq n-3. \end{array}$$

Classification of Leibniz Algebras with Naturally Graded Quasi-Filiform Split Lie Nilradical, and Maximal Dimension of the Complementary Space. We now seek to classify the Leibniz algebras with the algebras of Theorem 3 as their nilradicals. In particular, we will examine those which are of maximal dimension for their nilradical; that is, the cases where the inequality of Theorem 2 is an equality.

We will make use of the following lemma:

Lemma 1. *Let R be a Leibniz algebra with a Lie subalgebra N, and $R^2 \subseteq N$. Then for every $u \in N^2$ and every $p \in R$, $[p, u] = -[u, p]$. That is, N^2 anticommutes with R.*

Proof. Every element of N^2 is a finite sum of products of elements in N. By linearity, it suffices to consider a single such product, $u = [v, w]$ for some $v, w \in N$. Since $R^2 \subseteq N$, $[a, b] + [b, a] \in \text{Ann}_r(R) \cap N \subseteq \text{Center}(N)$ for all $a, b \in R$. Then for arbitrary $p \in R$ and some $\alpha, \beta \in \text{Center}(N)$,

$$[p, u] = [p, [v, w]] = [[p, v], w] - [[p, w], v] = [-[v, p] + \alpha, w] - [-[w, p] + \beta, v] =$$

$$= [-[v, p], w] - [-[w, p], v] = -[[v, p], w] - [v, [w, p]] = -[u, p], \text{ since } N \text{ is Lie.}$$

□

We are now prepared to begin our classification.

Case: $N = L_{n-1} \oplus \mathbb{C}$. The first quasi-filiform naturally graded Lie algebra is $L_{n-1} \oplus \mathbb{C}$, for $n \geq 5$ with basis $\{e_0, \dots, e_{n-1}\}$ and multiplication table [1]

$$[e_0, e_i] = -[e_i, e_0] = e_{i+1}, \quad 1 \le i \le n-3.$$

We wish to examine solvable Leibniz algebras R with this nilradical. Observe that $N^2 = \mathrm{span}\{e_1, \ldots, e_{n-2}\}$ and $\mathrm{Center}(N) = \mathrm{span}\{e_{n-2}, e_{n-1}\}$.

Derivations on $L_{n-1} \oplus \mathbb{C}$. Consider arbitrary $f \in \mathrm{Der}(L_{n-1} \oplus \mathbb{C})$, represented as a matrix. Since f is a derivation, $f([a,b]) = [f(a), b] + [a, f(b)]$ for any elements a, b. By examining these constraints obtained from products of basis elements of the forms

$$[e_1, e_2], \ [e_1, e_{n-1}], \ [e_0, e_i], \quad 1 \le i \le n-1,$$

we obtain that f must be of the form

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & a_0 + b_1 & 0 & \cdots & 0 \\ a_3 & b_3 & b_2 & 2a_0 + b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{n-2} & b_{n-2} & b_{n-3} & \cdots & (n-3)a_0 + b_1 & d \\ a_{n-1} & b_{n-1} & 0 & \cdots & 0 & c \end{pmatrix}$$

Moreover, it can be verified directly that every matrix of this form is a derivation, so this is a complete description of $\mathrm{Der}(L_{n-1} \oplus \mathbb{C})$.

Some elements of $\mathrm{Der}(L_{n-1} \oplus \mathbb{C})$ are inner. Specifically, the matrix with $b_2 = -1$ and other coefficients zero corresponds to right-multiplication by e_0, while the matrix with $a_{i+1} = 1$ and other coefficients zero corresponds to right multiplication by e_i for $1 \le i \le n-3$.

Nil-Independent Derivations. Consider the matrix form given above. Since three parameters a_1, b_2, c_3 appear on the diagonal, it is clear that we can obtain three nil-independent derivations, such as those corresponding to $(a_0, b_1, c) = (1,0,0)$, $(0,1,0)$, and $(0,0,1)$; call these $\mathcal{R}_x$, $\mathcal{R}_y$, and $\mathcal{R}_z$ respectively. We claim that this is maximal; there cannot be a fourth nil-independent derivation.

For consider any set of four derivations. Then, since the subspace of $\mathbb{C}^n$ corresponding to the diagonal entries of derivations of n is 3-dimensional, we can take a linear combination to obtain a derivation of the above form with

[1] $L_{n-1} \oplus \mathbb{C}$ can be defined for $n = 4$ as well, however, it is a special case which we do not examine here.

$a_0 = b_1 = c = 0$. By taking the change of basis $e'_{n-2} = e_{n-1}$, $e'_{n-1} = e_{n-2}$, this becomes strictly lower triangular, and thus nilpotent. Since the change of basis preserves nilpotency, A is nilpotent as well, so these four derivations are not nil-independent. Therefore the maximum number of nil-independent derivations on $L_{n-1} \oplus \mathbb{C}$ is 3. Then $\dim(R) - n \leq 3$.

We focus on the case where this difference is exactly 3; that is, $R = \langle e_0, \cdots, e_{n-1}, x, y, z\rangle$, where $\mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_z$ are obtained by right multiplication by x, y, z respectively. Then, by taking linear combinations and relabeling of x, y, z, we may assume without loss of generality that the products $[N, \{x, y, z\}]$ are given by

$$
\begin{aligned}
&[e_0, x] = e_0 + \sum_{j=1}^{n-1} a_j e_j, && [e_0, y] = \sum_{j=1}^{n-1} a'_j e_j,\\
&[e_1, x] = \sum_{j=2}^{n-1} b_j e_j, && [e_1, y] = e_1 + \sum_{j=2}^{n-1} b'_j e_j,\\
&[e_i, x] = (i-2)e_i + \sum_{j=i+1}^{n-3} b_{j-i+1} e_j, && [e_i, y] = e_i + \sum_{j=i+1}^{n-3} b'_{j-i+1} e_j,\\
&[e_{n-1}, x] = d e_{n-2}, && [e_{n-1}, y] = d' e_{n-2},\\
&[e_0, z] = \sum_{j=1}^{n-1} a''_j e_j, && [e_i, z] = \sum_{j=i+1}^{n-3} b''_{j-i+1} e_j,\\
&[e_1, z] = \sum_{j=2}^{n-1} b''_j e_j, && [e_{n-1}, z] = d'' e_{n-2} + e_{n-1},
\end{aligned}
$$

where $2 \leq i \leq n-2$.

Elimination of Parameters from $[N, \{x, y, z\}]$. We take a change of basis on x, y, z to remove the component of the derivation which is inner. As we noted earlier, b_2 and a_i for $2 \leq i \leq n-2$ are inner derivations corresponding to right-multiplication by elements of $L_{n-1} \oplus \mathbb{C}$. By taking a change of basis, subtracting multiples of these elements from x, y, and z, we can set the corresponding parameters to zero: namely a_i, a'_i, a''_i for $2 \leq i \leq n-2$, and b_2, b'_2, b''_2.

By taking a change of basis $e'_1 = e_1 + \sum_{j=3}^{n-2} A_j e_j$, $e'_{i+1} = [e_1, e'_i]$ for $2 \leq i \leq n-3$, we may choose the coefficients A_j to obtain $[e'_1, x] = b_{n-1}e_{n-1}$ and $[e'_i, x] = (i-1)e'_i$ for $2 \leq i \leq n-2$, eliminating the parameters b_i for $3 \leq i \leq n-2$. By redefining the coefficients a'_i, a''_i, b'_i, b''_i we have no change to other products.

Note that, since R is Leibniz and $R^2 \subseteq N$, for any elements a, b we have

$$[a, b] + [b, a] \in \mathrm{Ann}_r(R) \cap N \subseteq \mathrm{Center}(N) = \mathrm{span}\{e_{n-2}, e_{n-1}\}.$$

That is, $[b,a] = -[a,b] + \alpha e_{n-2} + \beta e_{n-1}$ for some coefficients α, β. Lemma 1 shows that these coefficients are zero for products where at least one of a and b is in N^2; since $N^2 = \mathrm{span}\{e_2, \cdots, e_{n-2}\}$, this means exactly that x, y, z anticommute with e_i for $2 \leq i \leq n-2$.

We now consider a number of applications of the Leibniz identity to specific basis elements to further pare down our parameters. By examining the Leibniz identity on

$$\begin{array}{llll} [e_0,[x,y]], & [e_1,[x,y]], & [e_0,[x,z]], & [e_1,[x,z]], \\ [e_{n-1},[x,z]], & [e_0,[y,z]], & [e_1,[y,z]], & \end{array}$$

we eliminate or relate parameters, obtaining the following products for $[N, \{x,y,z\}]$:

$$\begin{array}{ll} [e_0,x] = e_0 + a_1 e_1 + a_{n-1} e_{n-1}, & [e_0,y] = -a_1 e_1 - a_1 b'_{n-1} e_{n-1}, \\ [e_1,x] = 0, & [e_1,y] = e_1 - b'_{n-1} d' e_{n-2} + b'_{n-1} e_{n-1}, \\ [e_i,x] = (i-1)e_i, & [e_i,y] = e_i, \\ [e_{n-1},x] = (n-3)d' e_{n-2}, & [e_{n-1},y] = d' e_{n-2}, \\ [e_0,z] = (a_1 b'_{n-1} - a_{n-1}) e_{n-1}, & [e_1,z] = b'_{n-1} d' e_{n-2} - b'_{n-1} e_{n-1}, \\ [e_{n-1},z] = -d' e_{n-2} + e_{n-1}, & \end{array}$$

for $2 \leq i \leq n-2$, as well as

$$\begin{aligned} [x,y] &= d'(a_{n-1} + (n-4)a_1 b'_{n-1}) e_{n-3} + C_{xy} e_{n-2} + D_{xy} e_{n-1}, \\ [x,z] &= d'((n-4)a_{n-1} - (n-2)a_1 b'_{n-1}) e_{n-3} + C_{xz} e_{n-2} + D_{xz} e_{n-1}, \\ [y,z] &= d'(a_{n-1} - a_1 b'_{n-1}) e_{n-3} + C_{yz} e_{n-2} + D_{yz} e_{n-1}, \end{aligned}$$

where $C_{xy}, D_{xy}, C_{xz}, D_{xz}, C_{yz}, D_{yz} \in \mathbb{C}$.

Now apply the Leibniz identity on $[y,[x,z]]$ to obtain that $a_1 b'_{n-1} d' = 0$. The change of basis given by setting $e'_0 = e_0 + a_1 e_1 - a_{n-1} d' e_{n-2} + a_{n-1} e_{n-1}$, $e'_1 = e_1 + b'_{n-1} e_{n-1}$, $e'_{n-1} = e_{n-1} - d' e_{n-2}$, $x' = x - a_{n-1} d' e_{n-3}$, and $z' = z + a_{n-1} d' e_{n-3}$, with other basis elements unchanged, preserves the multiplication on $L_{n-1} \oplus \mathbb{C}$ and yields

$$\begin{array}{lll} [e_0,x] = e_0, & [e_1,y] = e_1, & [e_{n-1},z] = e_{n-1}, \\ [e_i,x] = (i-1)e_i, & [e_i,y] = e_i, & 2 \leq i \leq n-2. \end{array}$$

Other Products, and Anticommutators. We are now in a position to resume in greater detail our examination right annihilator of R. Observe that $[e_0, x] = e_0$, $[e_1, y] = e_1$, and $[e_{n-1}, z] = e_{n-1}$, which are all nonzero, so $x, y, z \notin \mathrm{Ann}_r(R)$. We claim that $\mathrm{Ann}_r(R) \subseteq \mathrm{span}\{e_{n-1}\}$.

Take an arbitrary element $r \in \mathrm{Ann}_r(R)$ and separate it into components $r = \eta+\xi x+\upsilon y+\zeta z$, where $\eta \in N$ and $\xi, \upsilon, \zeta \in \mathbb{C}$. Then consider the following product:

$$0 = [e_0+e_1+e_{n-1}, \eta+\xi x+\upsilon y+\zeta z] = [e_0+e_1+e_{n-1}, \eta]+\xi e_0+\upsilon e_1+\zeta e_{n-1}.$$

Since $[e_0+e_1+e_{n-1}, \eta] \in N^2$, while e_0, e_1, e_{n-1} are not in N^2, we have $\xi = \upsilon = \zeta = 0$ by linear independence. That is, $\mathrm{Ann}_r(R) \subset N$. Since $\mathrm{Ann}_r(R) \cap N \subseteq \mathrm{Center}(N)$, we have $\mathrm{Ann}_r(R) \subseteq \mathrm{Center}(N) = \mathrm{span}\{e_{n-2}, e_{n-1}\}$.

Now consider arbitrary $\alpha e_{n-2} + \beta e_{n-1} \in \mathrm{Ann}_r(R)$ for $\alpha, \beta \in \mathbb{C}$. Then we have $0 = [x, \alpha e_{n-2} + \beta e_{n-1}] = -\alpha(n-3)e_{n-2} + \beta[x, e_{n-1}]$; since $0 + [x, e_{n-1}] = [e_{n-1}, x]+[x, e_{n-1}] \in \mathrm{Ann}_r(R)$, we can write $[x, e_{n-1}] = \gamma e_{n-2}+\delta e_{n-1}$ to obtain $0 = -\alpha(n-3)e_{n-2} + \beta(\gamma e_{n-2} + \delta e_{n-1}) = (\beta\gamma - \alpha(n-3))e_{n-2} + \beta\delta e_{n-1}$. As such, either $\beta = 0$ or $\delta = 0$. If the former is true, we have $\beta\gamma = 0$ immediately; if the latter, we can see that $\gamma e_{n-2} \in \mathrm{Ann}_r(R)$, so $0 = [x, \gamma e_{n-2}] = -\gamma(n-3)e_{n-2}$, implying $\gamma = 0$ and thus $\beta\gamma = 0$. Consequently, we must have $\alpha = 0$, so $\mathrm{Ann}_r(R) \subseteq \mathrm{span}\{e_{n-1}\}$, as desired.

Taking advantage of the fact that $[a, a], ([a, b] + [b, a]) \in \mathrm{Ann}_r(R)$ for all $a, b \in R$ and recalling our previous anticommutativity result, we examine the Leibniz identity on the products

$$\begin{array}{llll} [x,[e_0,x]], & [x,[e_1,y]], & [x,[x,e_{n-1}]], & [y,[e_0,x]], \\ [y,[e_1,y]], & [y,[y,e_{n-1}]], & [z,[e_0,x]], & [z,[e_1,y]], \\ [z,[z,e_{n-1}]], & [x,[y,z]], & [x,[z,x]], & [y,[x,z]], \\ [y,[y,z]], & [z,[x,z]], & [z,[y,z]], & [z,[z,z]], \end{array}$$

to show that $[\{x, y, z\}, \{x, y, z\}] \subseteq \mathrm{Ann}_r(R)$ and obtain the remaining products. Then, by taking a change of basis on x, y and z, we obtain the classification.

Theorem 4. *Any solvable Leibniz algebra R of dimension $n + 3$, $n \geq 5$, with nilradical $L_{n-1} \oplus \mathbb{C}$, is isomorphic to an algebra with basis $\{e_0, \dots, e_{n-1}, x, y, z\}$ and the following multiplication table, where $\zeta \in$*

$\{0, 1\}$.

$$\begin{aligned}
&[e_0, e_i] = -[e_i, e_0] = e_{i+1}, && 1 \le i \le n-3,\\
&[e_i, x] = -[x, e_i] = |i-1|e_i, && 0 \le i \le n-2,\\
&[e_i, y] = -[y, e_i] = e_i, && 1 \le i \le n-2,\\
&[e_{n-1}, z] = e_{n-1}, &&\\
&[z, e_{n-1}] = (\zeta - 1)e_{n-1}. &&
\end{aligned}$$

If $\zeta = 0$ this algebra is Lie, while if $\zeta = 1$ it is not. Therefore these two cases are non-isomorphic.

Case: $N = Q_{n-1} \oplus \mathbb{C}$. Using the same methods as above, we also obtain a classification for solvable Leibniz algebras of maximum dimension with nilradical $Q_{n-1} \oplus \mathbb{C}$.

Theorem 5. *Any solvable Leibniz algebra of dimension $n+3$, $n \ge 7$ odd, with nilradical $Q_{n-1} \oplus \mathbb{C}$, is isomorphic to an algebra with basis $\{e_0, \ldots, e_{n-1}, x, y, z\}$ and the following multiplication table, where $\zeta \in \{0, 1\}$.*

$$\begin{aligned}
&[e_0, e_i] = -[e_i, e_0] = e_{i+1}, && 1 \le i \le n-4,\\
&[e_i, e_{n-2-i}] = (-1)^{i-1}e_{n-2}, && 1 \le i \le n-3,\\
&[e_i, x] = -[x, e_i] = |i-1|e_i, && 0 \le i \le n-3,\\
&[e_{n-2}, x] = -[x, e_{n-2}] = (n-4)e_{n-2}, &&\\
&[e_i, y] = -[e_i, y] = e_i, && 1 \le i \le n-3,\\
&[e_{n-2}, y] = -[y, e_{n-2}] = 2e_{n-2}, &&\\
&[e_{n-1}, z] = e_{n-1}, && [z, e_{n-1}] = (\zeta - 1)e_{n-1}.
\end{aligned}$$

If $\zeta = 0$ this algebra is Lie, while if $\zeta = 1$ it is not. Therefore these two cases are non-isomorphic.

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