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On Local Automorphisms of $\mathfrak{sl}_2$

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Abstract. We establish that the set of local automorphisms $\mathrm{LAut}(\mathfrak{sl}_2)$ is the group $\mathrm{Aut}^{\pm}(\mathfrak{sl}_2)$ of all automorphisms and anti-automorphisms. An example of a linear map that agrees with automorphisms on each basis element of $\mathfrak{sl}_n$, but is not a local automorphism is constructed.

Keywords: Lie algebra, local automorphism, automorphism, group of automorphisms, group of local automorphisms.

MSC 2010: 08A35, 17A36.

1 Introduction.

The idea of a local automorphism dates to 1988 [14], where for a given set of mappings $\mathcal{S}$ from a set X into a set Y, Larson suggests to call a mapping θ to be interpolating $\mathcal{S}$ if for each $x \in X$ there is an element $S_x \in \mathcal{S}$, depending on x, with $\theta(x) = S_x(x)$. Set $X = Y = \mathcal{A}$ to be an algebra and $\mathcal{S} = \mathrm{Aut}(\mathcal{A})$, then a linear endomorphism Δ of $\mathcal{A}$ is called *a local automorphism* if it interpolates the set of automorphisms of $\mathcal{A}$. Local automorphisms have been introduced and studied for the algebra $B(X)$ of all bounded linear operators on infinite-dimensional Banach space X by Larson and Sourour [10]. It has been proven for the associative algebra of square matrices $\mathcal{M}_n(\mathbb{C})$ that any local automorphism is either an automorphism or an anti-automorphism [10].

Investigation of local automorphisms continues for some subalgebras of $\mathcal{M}_n(\mathbb{C})$. In [3] a description of the local automorphisms of a finite-dimensional CSL algebra is given. A CSL algebra, or a digraph algebra, is an algebra which is spanned by a set of matrices which contains all diagonal matrix units $\{E_{ii}\}_{1\le i\le n}$ and is closed under multiplication. A local automorphism of a finite-dimensional CSL algebra is either an automorphism or an automorphism composed with a map that is the transpose map on a specific direct summand of the algebra and the identity map on the complement to the summand. There is a discussion of local automorphisms of the algebra of niltriangular matrices $N(n, \mathcal{K})$ over an associative commutative ring $\mathcal{K}$ with identity in [5]. The authors of [5] provide a full description for $n = 3$ and construct some non-trivial examples of local automorphisms for $n > 3$.

In our work we consider the simple Lie algebra $\mathfrak{sl}_n$ of traceless $n \times n$ matrices over a field of characteristic zero. With the formal investigations of local automorphisms of the associative matrix algebras in mind, our motivation is to investigate the local automorphisms of $\mathfrak{sl}_n$.

For a finite-dimensional algebra $\mathfrak{g}$, the set of local automorphisms constitutes a group which we denote by $\mathrm{LAut}(\mathfrak{g})$. Clearly, $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of $\mathrm{LAut}(\mathfrak{g})$.

We establish that any anti-automorphism is a local automorphism of $\mathfrak{sl}_n$. In fact, it is the composition of the matrix transposition and an automorphism of the algebra. We obtain a full description of a local automorphism of $\mathfrak{sl}_2$ – it is either an automorphism or an anti-automorphism. Furthermore, $\mathrm{Aut}(\mathfrak{sl}_2)$ is a normal subgroup of $\mathrm{LAut}(\mathfrak{sl}_2)$ of index 2. We use direct computations and our approach differs from the corresponding methods of [1] and [2] that address the problem in full generality. We also construct an example of a linear map that agrees with automorphisms on each basis element of $\mathfrak{sl}_n$ and is not a local automorphism.

2 Preliminaries

Definition 2.1. A vector space $\mathfrak{g}$ over $\mathbb{F}$ is called a Lie algebra if its multiplication (called Lie bracket and denoted by $(x, y) \mapsto [x, y]$) satisfies the identities:

(1) $[x, x] = 0$;

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Every associative algebra $(\mathcal{A}, \ \cdot\ )$ turns into a Lie algebra $(\mathfrak{A}, [-, -])$, where $\mathfrak{A} = \mathcal{A}$ and the Lie bracket is defined by the commutator $[a, b] = a \cdot b - b \cdot a$. This way the associative algebra $\mathcal{M}_n(\mathbb{F})$ of $n \times n$ square matrices over $\mathbb{F}$ turns into the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$. Consider the subset of $\mathcal{M}_n(\mathbb{F})$ of traceless matrices. It is well-known that the product of traceless matrices is not necessarily traceless, hence it is not a subalgebra of $\mathcal{M}_n(\mathbb{F})$. This set is closed under the commutator and is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ denoted by $\mathfrak{sl}_n(\mathbb{F})$, which is the main focus of our work.

In our work we consider a field $\mathbb{F}$ of characteristic zero and all definitions and results are restricted to this field. Hence, we omit $\mathbb{F}$ from now on.

A linear bijective map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is called an automorphism (resp. an anti-automorphism), if it satisfies $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (resp. $\varphi([x, y]) = [\varphi(y), \varphi(x)]$) for all $x, y \in \mathfrak{g}$. Analogous notions are defined in any algebra.

The set of all automorphisms of $\mathfrak{g}$ constitutes a group with respect to composition, which is denoted by $\mathrm{Aut}(\mathfrak{g})$. The set of all anti-automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}^-(\mathfrak{g})$. Due to skew-symmetry of the Lie bracket it is immediate that $\varphi \in \mathrm{Aut}(\mathfrak{g})$ if and only if $-\varphi \in \mathrm{Aut}^-(\mathfrak{g})$. Note that the set $\mathrm{Aut}^\pm(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}) \cup \mathrm{Aut}^-(\mathfrak{g})$ is a group and it is called the *signed automorphisms* group. Moreover, $\mathrm{Aut}(\mathfrak{g})$ is of index two in $\mathrm{Aut}^\pm(\mathfrak{g})$ and therefore is its normal subgroup [6].

A notion of a linear map which agrees with automorphisms at each point is given in the next definition.

Definition 2.2. A linear map $\Delta : \mathfrak{g} \to \mathfrak{g}$ is a local automorphism if for any $X \in \mathfrak{g}$ there exists an automorphism φ_X of $\mathfrak{g}$ such that $\Delta(X) = \varphi_X(X)$.

It follows from Definition 2.2 that a local automorphism is an injective map. The set of all local automorphisms is denoted by $\mathrm{LAut}(\mathfrak{g})$ and obviously there is an inclusion $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{LAut}(\mathfrak{g})$. Moreover, it is straightforward that the composition of two local automorphisms is a local automorphism and $\mathrm{LAut}(\mathfrak{g})$ is a monoid. There is a statement [5, Lemma 4] that $\mathrm{LAut}(\mathfrak{g})$ is a group under the usual composition, which is not true since local automorphisms are generally speaking not surjective (see Theorem 3.11 of [4]). However, the proof given in [5] works if $\mathfrak{g}$ is finite-dimensional. Indeed, then every local automorphism is surjective and given $\Delta \in \mathrm{LAut}(\mathfrak{g})$ we have $Y = \Delta(X) = \varphi_X(X)$ which implies $\Delta^{-1}(Y) = \varphi_X^{-1}(Y)$. Since $\varphi_X^{-1} \in \mathrm{Aut}(\mathfrak{g})$, the inverse of Δ also acts point-wise as an automorphism. Thus, there is the following

Proposition 2.3. *The set* $\mathrm{LAut}(\mathfrak{g})$ *constitutes a group with respect to composition if* $\mathfrak{g}$ *is finite-dimensional.*

For the associative algebra of all square matrices over the complex field, all local automorphisms are described by the following theorem.

Theorem 2.4. *([10]) A linear map* $\alpha : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ *is a local automorphism iff* α *is an automorphism or an anti-automorphism, i.e., either* α *is of the form* $X \mapsto AXA^{-1}$ *or* $X \mapsto AX^TA^{-1}$ *for a fixed* $A \in \mathcal{M}_n(\mathbb{C})$*.*

In this work we study local automorphisms of $\mathfrak{sl}_n$. Let us denote the unit matrix with zero entries everywhere but the intersection of the i-th row and j-th column by E_{ij}. Recall that the simple Lie algebra of $n \times n$ matrices of trace zero is generated as

$$\mathfrak{sl}_n = \mathrm{Span}\langle E_{ij}, h_1, \ldots, h_{n-1} \mid 1 \leq i \neq j \leq n \rangle,$$

where $h_i = E_{ii} - E_{i+1,i+1}$.

Before presenting the main results, we introduce the following theorem which is used to prove our results.

Theorem 2.5. *[7] Over an algebraically closed field of characteristic 0, the group of automorphisms of the Lie algebra* $\mathfrak{sl}_2$ *is the set of mappings* $X \mapsto A^{-1}XA$ *and the group of automorphisms of the Lie algebra* $\mathfrak{sl}_n (n \geq 3)$ *is the set of mappings of the form* $X \mapsto A^{-1}XA$ *or* $X \mapsto -A^{-1}X^TA$*.*

Remark 2.6. In the case $n = 2$ since there is only one type of automorphisms which is the conjugation by a matrix, a local automorphism Δ of $\mathfrak{sl}_2$ sends a matrix X to a similar matrix $\Delta(X)$. Since the characteristic polynomials of similar matrices are the same, we have $p_X(\lambda) = p_{\Delta(X)}(\lambda)$. The analogous statement for $n \geq 3$ holds only for the local automorphisms that act at each point as the automorphism $X \mapsto A^{-1}XA$.

3 Main Results

Proposition 3.1. *Every anti-automorphism of $\mathfrak{sl}_n$ is a local automorphism.*

Proof. Consider the transpose map $\Delta(X) = X^T$. It is well-known fact that a square matrix is similar (conjugate) to its transpose [8, Theorem 66]. In this case, by Theorem 2.5 we obtain that Δ is a local automorphism of $\mathfrak{sl}_n$. Furthermore, Δ is an anti-automorphism, and each anti-automorphism is of the form $\Delta \circ \varphi$, where φ is an automorphism. Then $\Delta \circ \varphi$ is a local automorphism as the composition of two local automorphisms. □

Corollary 3.2. *The signed automorphism group* $\mathrm{Aut}^{\pm}(\mathfrak{sl}_n)$ *is a subgroup of* $\mathrm{LAut}(\mathfrak{sl}_n)$.

In general, we could not obtain a full description of local automorphisms of $\mathfrak{sl}_n$. However, in the case $n = 2$ we achieve this goal.

Let us use the matrix representation of the elements of the Lie algebra $\mathfrak{sl}_2$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiplication in this algebra is as follows:

$$[e, f] = h, \quad [h, e] = 2e, \quad [f, h] = 2f.$$

First, let us establish the following result.

Proposition 3.3. *A local automorphism Δ of $\mathfrak{sl}_2$ that fixes h is either an automorphism with matrix* $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *or anti-automorphism with matrix* $\begin{pmatrix} 0 & \mu & 0 \\ \mu^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *in the ordered basis (e, f, h) with $\lambda, \mu \in \mathbb{F}^*$.*

Proof. Let $\Delta : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ be a local automorphism that fixes h. Then by the description of the automorphisms of $\mathfrak{sl}_2$ by Theorem 2.5 we obtain that $\Delta(e) = T_e^{-1} e T_e$ and $\Delta(f) = T_f^{-1} f T_f$ for some invertible matrices T_e and T_f. Simple manipulations show that

$$\Delta(e) = \frac{1}{|T_e|}(\alpha^2 e - \beta^2 f + \alpha\beta h) \text{ and } \Delta(f) = \frac{1}{|T_f|}(-\gamma^2 e + \delta^2 f - \gamma\delta h),$$

where $(\beta\ \alpha)$ is the second row of T_e, $(\delta\ \gamma)$ is the first row of T_f, and $|A|$ is the determinant of A.

Using the linearity of Δ we have

$$\Delta(e+h) = \frac{1}{|T_e|}(\alpha^2 e - \beta^2 f + \alpha\beta h) + h = \begin{pmatrix} \frac{\alpha\beta}{|T_e|}+1 & \frac{\alpha^2}{|T_e|} \\ \frac{-\beta^2}{|T_e|} & -\frac{\alpha\beta}{|T_e|}-1 \end{pmatrix}.$$

The characteristic polynomial of the last matrix is $-\dfrac{2\alpha\beta}{|T_e|} + x^2 - 1$, and by Remark 2.6 is equal to the characteristic polynomial, $p_{e+h}(x) = x^2 - 1$, of $e+h$. This yields $\alpha\beta = 0$. Similarly, $x^2 - 1 = p_{f+h}(x) = p_{\Delta(f+h)}(x) = \dfrac{2\gamma\delta}{|T_f|} + x^2 - 1$ implies $\delta\gamma = 0$. Moreover, $x^2 - 1 = p_{e+f}(x) = p_{\Delta(e+f)}(x) = x^2 - \dfrac{1}{|T_e||T_f|}(\alpha\delta - \gamma\beta)^2$ implies $(\alpha\delta - \gamma\beta)^2 = |T_e||T_f|$. Since T_e and T_f are invertible, α and β cannot be zero at the same time. Similarly for α and γ, δ and γ, δ and β. This gives us the following cases:

Case 1. $\alpha \neq 0$, $\beta = 0$, $\delta \neq 0$, $\gamma = 0$. This case leads to $|T_e||T_f| = (\alpha\delta)^2$.

Case 2. $\alpha = 0$, $\beta \neq 0$, $\delta = 0$, $\gamma \neq 0$. In this case $|T_e||T_f| = (\gamma\beta)^2$.
The only possible result is two local automorphisms:

$$\Delta_1(e) = \lambda e,\ \Delta_1(f) = \tfrac{1}{\lambda} f,\ \Delta_1(h) = h$$

and

$$\Delta_2(e) = \mu f,\ \Delta_2(f) = \tfrac{1}{\mu} e,\ \Delta_2(h) = h,$$

where $\lambda = \dfrac{\alpha^2}{|T_e|}$ and $\mu = \dfrac{-\beta^2}{|T_e|}$. One can check that Δ_1 is an automorphism and Δ_2 is an anti-automorphism of $\mathfrak{sl}_2$. $\square$

We now establish the full description of the local automorphisms of $\mathfrak{sl}_2$.

Theorem 3.4. *The group* $\mathrm{LAut}(\mathfrak{sl}_2)$ *coincides with* $\mathrm{Aut}^{\pm}(\mathfrak{sl}_2)$.

Proof. Let Δ' be an arbitrary local automorphism of $\mathfrak{sl}_2$. For every $x \in \mathfrak{sl}_2$ we have $\Delta'(x) = \varphi_x(x)$ for some $\varphi_x \in \mathrm{Aut}(\mathfrak{sl}_2)$. Then $\varphi_h^{-1} \circ \Delta'$ is a local automorphism as the composition of two local automorphisms and $(\varphi_h^{-1} \circ \Delta')(h) = \varphi_h^{-1}(\varphi_h(h)) = h$. By Proposition 3.3 $\varphi_h^{-1} \circ \Delta'$ is either equal to the automorphism Δ_1 or to the anti-automorphism Δ_2 defined in the proof of Proposition 3.3. Hence, Δ' is either an automorphism or an anti-automorphism, and $\mathrm{LAut}(\mathfrak{sl})_2 \subseteq \mathrm{Aut}^{\pm}(\mathfrak{sl}_2)$. However, Corollary 3.2 claims the converse inclusion. Thus, $\mathrm{LAut}(\mathfrak{sl}_2) = \mathrm{Aut}^{\pm}(\mathfrak{sl}_2)$. $\square$

Corollary 3.5. *Every local automorphism of* $\mathfrak{sl}_2$ *is either an automorphism* $X \mapsto A^{-1}XA$ *or an anti-automorphism* $X \mapsto A^{-1}X^TA$ *for any invertible* A.

The next proposition shows that for a linear map $\Delta : \mathfrak{sl}_n \to \mathfrak{sl}_n$ to be a local automorphism it is not enough to check that Δ acts as an automorphism on each basis element of $\mathfrak{sl}_n$.

Proposition 3.6. *Let $n \geq 3$ and $\alpha \in \mathbb{F}^*$. Define a linear map $\Delta_\alpha : \mathfrak{sl}_n \to \mathfrak{sl}_n$ by setting*

$$\Delta_\alpha(E_{1,n-1}) = E_{n1} + \alpha E_{n,n-1}, \qquad \Delta_\alpha(E_{1n}) = E_{1,n-1}$$
$$\Delta_\alpha(E_{n1}) = E_{n,n-1}, \qquad \Delta_\alpha(E_{n,n-1}) = E_{1n}$$

and fixing all the other matrix units. Then Δ_α agrees with automorphisms on each basis element of $\mathfrak{sl}_n$, but Δ_α is not a local automorphism of $\mathfrak{sl}_n$.

Proof. Consider the matrices

$$\begin{aligned}
T_1 &= I + E_{1n} + E_{n-1,1} + (\alpha-1)E_{n-1,n-1} + E_{n,n-1} - E_{nn},\\
T_2 &= I - E_{n-1,n-1} - E_{nn} + E_{n-1,n} + E_{n,n-1},\\
T_3 &= I - E_{11} + E_{1,n-1} + E_{n-1,1} - E_{n-1,n-1},\\
T_4 &= I - E_{11} + E_{1,n-1} - E_{n-1,n-1} + E_{n-1,n} + E_{n1} - E_{nn}.
\end{aligned}$$

Note that they are invertible and the following equalities hold:

$$\begin{aligned}
E_{1,n-1}T_1 =& E_{1,n-1} + E_{11} + (\alpha-1)E_{1,n-1} = E_{11} + \alpha E_{1,n-1}\\
=&(E_{n1} + E_{11} - E_{n1}) + \alpha(E_{n,n-1} - E_{n,n-1} + E_{1,n-1})\\
=&T_1(E_{n1} + \alpha E_{n,n-1});\\
E_{1n}T_2 =& E_{1n} - E_{1n} + E_{1,n-1} = E_{1,n-1} = T_2E_{1,n-1};\\
E_{n1}T_3 =& E_{n1} - E_{n1} + E_{n,n-1} = E_{n,n-1} = T_3E_{n,n-1};\\
E_{n,n-1}T_4 =& E_{n,n-1} - E_{n,n-1} + E_{nn} = E_{nn} = E_{1n} - E_{1n} + E_{nn} = T_4E_{1n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta_\alpha(E_{1,n-1}) &= E_{n1} + \alpha E_{n,n-1} = T_1^{-1}E_{1,n-1}T_1,\\
\Delta_\alpha(E_{1n}) &= E_{1,n-1} = T_2^{-1}E_{1n}T_2,\\
\Delta_\alpha(E_{n1}) &= E_{n,n-1} = T_3^{-1}E_{n1}T_3,\\
\Delta_\alpha(E_{n,n-1}) &= E_{1n} = T_4^{-1}E_{n,n-1}T_4.
\end{aligned}$$

Since on the other basis elements of $\mathfrak{sl}_n$ the map Δ_α acts as the identity, we obtain that Δ_α is a linear transformation that acts as matrix conjugations on each basis element of $\mathfrak{sl}_n$.

Note that $\Delta_\alpha^2(E_{1,n-1}) = E_{n,n-1} + \alpha E_{1n}$ is a matrix of rank 2. Since $\mathrm{LAut}(\mathfrak{sl}_n)$ is a group, if Δ_α is a local automorphism, then so is Δ_α^2. However, the automorphisms of $\mathfrak{sl}_n$ by Theorem 2.5 preserve rank of matrices. Therefore, we have a contradiction and Δ_α is not a local automorphism. □

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