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Stationary Harmonic Measure and DLA in the Upper Half Plane

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Abstract
In this paper, we introduce the stationary harmonic measure in the upper half plane. By bounding this measure, we are able to define both the discrete and continuous time diffusion limited aggregation (DLA) in the upper half plane with absorbing boundary conditions. We prove that for the continuous model the growth rate is bounded from above by $o(t^{2+\epsilon})$. Moreover we prove that all the moments are finite for the size of the aggregation. When time is discrete, we also prove a better upper bound of $o(n^{2/3+\epsilon})$, on the maximum height of the aggregate at time n. An important tool developed in this paper, is an interface growth process, bounding any process growing according to the stationary harmonic measure. Together with [12] one obtains non zero growth rate for any such process.

1 Introduction

In this paper, we consider the stationary harmonic measure in the upper half plane and the corresponding diffusion limited aggregation (DLA). The diffusion limited aggregation (DLA) in $\mathbb{Z}^2$ was introduced in 1983 by Witten and Sander [16] as a simple model to study the geometry and dynamics of physical systems governed by diffusive laws. The DLA is defined recursively as a process on subsets $A_n \in \mathbb{Z}^2$. Starting from $A_0 = \{(0, 0)\}$, at each time a new point a_{n+1} sampled from the harmonic probability measure on the outer vertex boundary of A_n (denoted $\partial^{out} A_n$) is added to A_n. Intuitively, a_{n+1} is the first place that a random walk starting from infinity visits $\partial^{out} A_n$.

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Although DLA is straightforwardly defined and easily simulated on a computer, very little about it is known rigorously. One of the notable exceptions is shown by Kesten where a polynomial upper bound, which equal to $n^{2/3}$ when $d = 2$ and $n^{2/d}$ when $d \geq 3$, of the growth rate on DLA arms is given, see Corollary in Reference [7] or Theorem in Reference [8]. In a later work [9] Kesten improved the upper bound for DLA when $d = 3$ to $\sqrt{n \log(n)}$, and when $d > 3$ to $n^{2/(d+1)}$. No non-trivial lower bounds have been proved till present day. It is in fact still open to rule out that the DLA converges to a ball, although numerical simulations clearly exclude this.

Recently, this topic is re-visited by Benjamini and Yadin [2] where they "'clean up' Kesten's argument, and make it more robust". They proved upper bounds on the growth rates of DLA's on "transitive graphs of polynomial growth, graphs of exponential growth, non-amenable graphs, super-critical percolation on Z^d and high dimensional pre-Sierpinski carpets".

In this paper, we further extend the reach of Kesten's idea to non-transitive graphs. We define the (horizontally) translation invariant stationary harmonic measure on the upper half plane with absorbing boundary condition and show the growth of such stationary harmonic measure in a connected subset intersecting x-axis is sub-linear with respect to the height. With the bounds found for our stationary harmonic measure, we will be able to define a continuous time DLA on the upper half plane and give upper bound on its growth rates.

One of the goals of this paper is to construct the theoretical theory for defining an infinite stationary version of DLA in the upper half plane. The motivation for defining the stationary DLA (SDLA) is twofold. First DLA growing from an absorbing boundary models many natural phenomenon which are not modeled well by the classical DLA (e.g. [11], [15, Chapter 1: Fractals in nature]). Indeed in an experimental setting it is actually very hard to avoid boundary effects. Second SDLA exhibits new physical and mathematical phenomenon. SDLA enjoys a translation invariance symmetry. This effects the geometry of the DLA arms as they tend to be thinner and "upward" growing. Mathematically one gains the power of ergodic theory for geometric control, making the SDLA more amenable to analyze than the classical DLA. As an example one can use mass transport principle to control the expected width of SDLA arms. Moreover it is conjectured that at time infinity and arms of the SDLA are finite, which again is an effect of the infinite boundary. Another motivation is that locally, SDLA behaves like DLA growing from a long line [14], thus geometric results on the more symmetric SDLA can be transferred to the classical DLA.

We first define several sets and stopping times for our problem. Let $\mathcal{H} = \{(x, y) \in \mathbb{Z}^2, y \geq 0\}$ be the upper half plane (including x-axis), and $S_n, n \geq 0$ be a 2-dimensional simple random walk. For any $x \in \mathbb{Z}^2$, we will write

$$x = (x_1, x_2)$$

with x_i denote the ith coordinate of x, and $\|x\| = \|x\|_1 = |x_1| + |x_2|$. Then let $L_n, D_n \subset \mathbb{Z}^2$ be defined as follows: for each nonnegative integer n, define

$$L_n = \{(x, n),\ x \in \mathbb{Z}\},$$
$$V_n = \{(0, k),\ 0 \leq k \leq n\},$$

and

$$U_n = L_0 \cup V_n.$$

I.e., L_n is the horizontal line of height n while U_n is x-axis plus the vertical line segment between $(0, 0)$ and $(0, n)$. And we let $i \cdot n = (0, n)$ be the "end point" of V_n. Moreover, we use $\mathcal{P}_n \subset \mathcal{H}$ for an arbitrary finite path in the upper half plane connecting $i \cdot n$ and the x-axis. One can immediately see that V_n is one such path.

And for each subset $A \subset \mathbb{Z}^2$ we define stopping times

$$\bar{\tau}_A = \min\{n \geq 0,\ S_n \in A\}$$

and

$$\tau_A = \min\{n \geq 1,\ S_n \in A\}.$$

For any subsets $A_1 \subset A_2$ and B and any $y \in \mathbb{Z}^2$, by definition one can easily check that

$$\begin{aligned} P_y\left(\tau_{A_1} < \tau_B\right) &\leq P_y\left(\tau_{A_2} < \tau_B\right), \\ P_y\left(\bar{\tau}_{A_1} < \bar{\tau}_B\right) &\leq P_y\left(\bar{\tau}_{A_2} < \bar{\tau}_B\right), \end{aligned} \tag{1}$$

and that

$$\begin{aligned} P_y\left(\tau_B < \tau_{A_2}\right) &\leq P_y\left(\tau_B < \tau_{A_1}\right), \\ P_y\left(\bar{\tau}_B < \bar{\tau}_{A_2}\right) &\leq P_y\left(\bar{\tau}_B < \bar{\tau}_{A_1}\right), \end{aligned} \tag{2}$$

where $P_y(\cdot) = P(\cdot|S_0 = y)$. Now we define the stationary harmonic measure on $\mathcal{H}$ which will serve as the Poisson intensity in our continuous time DLA model. For any connected $B \subset \mathcal{H}$, any edge $\vec{e} = x \to y$ with $x \in B$, $y \in \mathcal{H} \setminus B$ and any N, we define

$$H_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} P_z\left(S_{\bar{\tau}_{B \cup L_0}} = x, S_{\bar{\tau}_{B \cup L_0}-1} = y\right) \tag{3}$$

By definition, a necessary condition for $H_{B,N}(\vec{e}) > 0$ (although at this point we have not yet ruled out the possibility it equals to infinity) is $y \in \partial^{out} B$ and $|x - y| = 1$. And for all $x \in B$, we can also define

$$H_{B,N}(x) = \sum_{y:\ \vec{e}=(x,y)} H_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} P_z\left(S_{\bar{\tau}_{B \cup L_0}} = x\right). \tag{4}$$

And for each point $y \in \partial^{out} B$, we can also define

$$\begin{aligned} \hat{H}_{B,N}(y) &= \sum_{\vec{e}=(x,y),\ x \in B} H_{B,N}(\vec{e}) \\ &= \sum_{z \in L_N \setminus B} P_z\left(\tau_B \leq \tau_{L_0}, S_{\bar{\tau}_{B \cup L_0}-1} = y\right). \end{aligned} \tag{5}$$

By coupling and strong Markov property, we show that $N \to H_{A,N}(e)$ is bounded and monotone in N. Thus

Proposition 1 *For any B and $\vec{e}$ as above, there is a finite $H_B(\vec{e})$ such that*

$$\lim_{N \to \infty} H_{B,N}(\vec{e}) = H_B(\vec{e}). \tag{6}$$

And we call $H_B(\vec{e})$ the stationary harmonic measure of $\vec{e}$ with respect to B. Thus we immediately have the limits $H_B(x) = \lim_{N \to \infty} H_{B,N}(x)$ and $\hat{H}_B(y) = \lim_{N \to \infty} \hat{H}_{B,N}(y)$ also exists and we call them the stationary harmonic measure of x and y with respect to B. Although now we have the limit $H_B(x)$ exists, it can be zero everywhere for certain B. We do not need to worry about this when B is finite. For each finite B, we let

$$H_B = \sum_{x \in B} H_B(x) = \sum_{y \in \partial^{out} B} \hat{H}_B(y)$$

be the harmonic measure of B. Then we have H_B is non-decreasing as B gets larger:

Proposition 2 *For any finite subsets $B_1 \subset B_2 \subset \mathcal{H}$,*

$$H_{B_2} \geq H_{B_1}.$$

Remark 1 However, for infinite subset of $\mathcal{H}$, it is possible to have the harmonic measure equal to 0 everywhere. In fact, we prove that as long as B has a linear spatial growth horizontally, $H_B(\cdot)$ is uniformly 0. On the other hand, we have also proved that for any B with certain sub-linear spatial growth, it can have non-zero stationary harmonic measure. These results are presented in a separate paper [12].

After presenting the basic properties of our stationary harmonic measure and similar to the regular harmonic measure, the following upper bounds holds for $H_{B,N}(x)$:

Theorem 1 *There is some constant $C < \infty$ such that for each connected $B \subset \mathcal{H}$ with $L_0 \subset B$ and each $x = (x_1, x_2) \in B \setminus L_0$, and any N sufficiently larger than x_2*

$$H_{B,N}(x) \leq C x_2^{1/2}. \tag{7}$$

Remark 2 In this paper, we use C and c as constants in $(0, \infty)$ independent of the change of variables like N or n. But their exact values can be different from place to place.

At the same time, we can also have the following result showing that for a point of height n, say $i \cdot n$ without loss of generality, the harmonic measure is maximized (up to multiplying a constant) by U_n. I.e.

Theorem 2 *There is some constant $c > 0$ such that for all $N > n$,*

$$H_{U_n,N}(i \cdot n) \geq c n^{1/2}. \tag{8}$$

With Proposition 2, Theorems 1 and 2 and the bounds estimates in their proofs, we can further show that

Theorem 3 *There are constants $0 < c, C < \infty$ such that for any finite and connected B in $\mathcal{H}$,*

$$H_B \leq C \left(\min_{x \in B} \{x_2\} + |B| \right), \tag{9}$$

while

$$H_B \geq c \max_{x \in B} \{x_2\} \log^{-1} \left(\max_{x \in B} \{x_2\} \right), \tag{10}$$

$$H_B \geq 1 \tag{11}$$

when $\max_{x \in B}\{x_2\} = 0$.

And again, we also have the total harmonic measure is maximized (up to multiplying a constant) by the vertical line segment V_n over all connected finite subsets with the same cardinality and intersecting L_0.

Theorem 4 *There is a constant $c > 0$ such that for any n,*

$$H_{V_n} \geq cn. \tag{12}$$

With the stationary harmonic measure bounded in Theorem 1, we are now able to define our DLA in the upper half plane as a continuous time stochastic process $A_t, t \geq 0$ taking values on finite subsets of $\mathcal{H}$. First we have $A_0 = \{0\}$. For each $t \geq 0$. A_t grows at a Poisson rate of H_{A_t} and add a new point on $\partial^{out} A_t$ according to the probability distribution

$$\tilde{p}(A_t, y) = \frac{\hat{H}_{A_t}(y)}{H_{A_t}}, \; y \in \mathcal{H}.$$

Similarly, we can also define the discrete DLA model $\{A_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ which is the embedded Markov chain of A_t. I.e., at each n, $A_{n+1} = A_n \cup \{y\}$ where y is sampled according to $\tilde{p}(A_n, y)$.

First, by introducing a pure growth interacting particle system that dominates the continuous time process, we show that A_t is well defined and estimate an upper bound on the growth rate of its arms. For any finite A define

$$\|A\| = \max\{\|x\|, x \in A\}.$$

Theorem 5 *A_t is well defined on $t \in [0, \infty)$. And for any $\epsilon > 0$, we have with probability one*

$$\limsup_{t \to \infty} t^{-2-\epsilon} \|A_t\| = 0. \tag{13}$$

Furthermore, we show that for any time t, $\|A_t\|$ has a finite mth order moment for all $m \geq 1$.

Theorem 6 *For any integer $m \geq 1$ and any $t \geq 0$*

$$E\left[\|A_t\|^m\right] < \infty. \tag{14}$$

Remark 3 In our construction we are able to define the dominating interacting particle system starting from any initial configuration in $\{0, 1\}^{\mathcal{H}}$, whose growth rate is given by the upper bound of the stationary harmonic measure found in Theorem 1 . This, together with Reference [12], may allows us to define a horizontally translation invariant infinite DLA on $\mathcal{H}$ and estimate its (non-zero) growth rate. We call this the stationary DLA model, and it will be presented in Reference [13]. We refer the reader to look at recent results on other stationary aggregation processes [1,3].

For the discrete time process let $h_n = \max_{x \in A_n}\{x_2\}$. By Theorem 1 and (10), we see that the probability that a new point y is added to the aggregation A_n is no larger than $\log(h_n)/\sqrt{h_n}$. Then the Borel-Cantelli argument in Step (ii) of Reference [8] easily gives us a stronger upper bound on h_n:

Theorem 7 *For any $\epsilon > 0$, we have with probability one*

$$\limsup_{n \to \infty} n^{-\epsilon - 2/3} h_n = 0.$$

The structure of this paper is as follows: In Sect. 2 we prove the more basic properties of the stationary harmonic measure, i.e., Propositions 1 and 2. Theorem 2 is proved in Sect. 3 and Theorem 2 in Sect. 4. In Sect. 5 we use the bounds found the the previous two sections and show Theorems 3 and 4 inductively. In Sect. 6, we use an interacting particle system argument to define the dominating process and prove Theorems 5 and 6. After that, Theorem 7 follows immediately. Our main contribution in this paper is the proof of Theorems 5 and 6, where a interface particle system with unbounded transition rate is introduced and studied to stochastically bound the growth of the continuous time DLA model.

2 Properties of Stationary Harmonic Measure

2.1 Proof of Proposition 1

To show Proposition 1, we first need to verify that the infinite summation defined in (4) converges. Note that for $x_2 > 0$ and any $N > x_2$ and any $z \in L_N \setminus B$,

$$P_z\left(S_{\bar{\tau}_B} = x\right) = \sum_{k=1}^{\infty} P_z\left(\bar{\tau}_{B\cup L_0} = k, S_k = x\right).$$

And by time reversal, see Reference [7] for example, and symmetry of simple random walk, we have

$$\begin{aligned} P_z\left(\bar{\tau}_B = k, S_k = x\right) &= P_z\left(S_k = x, S_1, S_2, \cdots, S_{k-1} \notin B \cup L_0\right) \\ &= P_x\left(S_k = z, S_1, S_2, \cdots, S_{k-1} \notin B \cup L_0\right) \\ &= P_x\left(S_k = z, \tau_{B\cup L_0} > k\right). \end{aligned}$$

Thus

$$\begin{aligned} P_z\left(S_{\bar{\tau}_B} = x\right) &= \sum_{k=1}^{\infty} P_x\left(S_k = z, \tau_{B\cup L_0} > k\right) \\ &= E_x\Big[\text{number of visits to } z \text{in time interval } [0, \tau_{B\cup L_0})\Big]. \end{aligned}$$

Then taking the summation over all $z \in L_N \setminus B$, we have

$$H_{B,N}(x) = E_x\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{B\cup L_0})\Big]. \tag{15}$$

Then noting that $\tau_{L_0} \geq \tau_{B\cup L_0}$, we have

$$H_{B,N}(x) \leq E_x\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{L_0}]\Big].$$

Moreover, for $N > x_2$, note that if we trace the jumps on the second coordinate of S_n, it gives an (embedded) 1-dimensional simple random walk. We can use the strong Markov property of random walk on stopping time $\bar{\tau}_{L_N} \wedge \tau_{L_0}$

$$\begin{aligned} &E_x\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{L_0}]\Big] \\ &\quad = \sum_{w \in L_N} P_x(\tau_{L_N} < \tau_{L_0}, S_{\tau_{L_N}} = w) E_w\Big[\text{number of visits to } L_N \text{in time interval} [0, \tau_{L_0}]\Big]. \end{aligned}$$

Note that for each $w \in L_N$,

$$E_w\Big[\text{number of visits to } L_N \text{in time interval} [0, \tau_{L_0}]\Big] = \frac{4}{P_{w-(0,1)}\left(\tau_{L_0} < \tau_{L_N}\right)} = 4N \tag{16}$$

is actually independent of the choice of w, and that for all $N > x_2$

$$P_x(\tau_{L_N} < \tau_{L_0}) = \frac{x_2}{N}.$$

We have

$$E_x\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{L_0}]\Big] = 4N \cdot P_x(\tau_{L_N} < \tau_{L_0}) = 4x_2.$$

Thus we have shown that

$$H_{B,N}(x \to y) \le H_{B,N}(x) \le 4x_2 < \infty. \tag{17}$$

Similarly, we can also show that for $x_2 = 0$,

$$H_{B,N}(x \to y) \le H_{B,N}(x) \le 1 < \infty. \tag{18}$$

With $H_{B,N}(x \to y)$ uniformly bounded for all N, we next show that $H_{B,N}(x \to y)$ is monotonically decreasing with respect to N i.e., for any $N > M > x_2 + 1$ we want to show that

$$H_{B,N}(x \to y) \le H_{B,M}(x \to y). \tag{19}$$

Recalling that

$$H_{B,N}(x \to y) = \sum_{z \in L_N \setminus B} P_z \left(S_{\bar{\tau}_{B \cup L_0}} = x,\ S_{\bar{\tau}_{B \cup L_0} - 1} = y \right),$$

for each N we can define $S_n^{(0,N)}$ be a simple random walk in some probability space $P(\cdot)$ starting at $(0, N)$, and $S_n^{(k,N)} = S_n^{(0,N)} + (k, 0)$ for all $k \in \mathbb{Z}$. Noting that $S_n^{(k,N)}$ is a simple random walk starting at (k, N), we have

$$H_{B,N}(x) = \sum_{k:\ (k,N) \in L_N \setminus B} P \left(S^{(k,N)}_{\bar{\tau}_{B \cup L_0}} = x,\ S^{(k,N)}_{\bar{\tau}_{B \cup L_0} - 1} = y \right). \tag{20}$$

Recalling that $N > M > x_2$, a random walk starting at L_N must first visit L_M before it can ever reach x. Thus for stopping time

$$\bar{\tau}_{L_M} = \inf \left\{ n : \ S_n^{(0,N)} \in L_M \right\}$$

note that by definition we also have

$$\bar{\tau}_{L_M} = \inf \left\{ n : \ S_n^{(k,N)} \in L_M \right\}$$

and

$$S^{(k,N)}_{\bar{\tau}_{L_M}} = (k, 0) + S^{(0,N)}_{\bar{\tau}_{L_M}}$$

for all $k \in \mathbb{Z}$. Thus by strong Markov property, we have for each k such that $(k, N) \in L_N \setminus B$

$$\begin{aligned}
&P \left(S^{(k,N)}_{\bar{\tau}_{B \cup L_0}} = x,\ S^{(k,N)}_{\bar{\tau}_{B \cup L_0} - 1} = y \right) \\
&= \sum_{j \in Z} P \left(S^{(0,N)}_{\bar{\tau}_{L_M}} = (j, M),\ \bar{\tau}_{L_M} \le \bar{\tau}_{B-(k,0)} \right) \\
&\quad P_{(j+k,M)} \left(S_{\bar{\tau}_{B \cup L_0}} = x,\ S_{\bar{\tau}_{B \cup L_0} - 1} = y \right) \\
&\le \sum_{j \in Z} P \left(S^{(0,N)}_{\bar{\tau}_{L_M}} = (j, M) \right) P_{(j+k,M)} \left(S_{\bar{\tau}_{B \cup L_0}} = x,\ S_{\bar{\tau}_{B \cup L_0} - 1} = y \right).
\end{aligned}$$

Taking summation over all k,

$$H_{B,N}(x) \le \sum_{j\in\mathbb{Z}} P\left(S^{(0,N)}_{\bar{\tau}_{L_M}} = (j, M)\right)$$
$$\sum_{k:\ (k,N)\in L_N\setminus B} P_{(j+k,M)}\left(S_{\bar{\tau}_{B\cup L_0}} = x,\ S_{\bar{\tau}_{B\cup L_0}-1} = y\right). \tag{21}$$

Note that for any $(i, M) \in B$, $P_{(i,M)}\left(S_{\bar{\tau}_{B\cup L_0}} = x\right) = 0$. Thus

$$\begin{aligned}&\sum_{k:\ (k,N)\in L_N\setminus B} P_{(j+k,M)}\left(S_{\bar{\tau}_{B\cup L_0}} = x,\ S_{\bar{\tau}_{B\cup L_0}-1} = y\right)\\ &\le \sum_{k:\ (k,M)\in L_M\setminus B} P_{(k,M)}\left(S_{\bar{\tau}_{B\cup L_0}} = x,\ S_{\bar{\tau}_{B\cup L_0}-1} = y\right) = H_{B,M}(x\to y).\end{aligned} \tag{22}$$

Combining (21) and (22) we have (19). The fact that any monotonically decreasing nonnegative sequence is convergent finishes the proof of Proposition 1. □

2.2 Proof of Proposition 2

To show Proposition 2 for finite subsets, recalling the definition and the fact that both B_1 and B_2 are finite, for any sufficiently large N such that $L_N \cap B_2 = \varnothing$, we have

$$H_{B_1} = \sum_{x\in B_1}\sum_{z\in L_N} P_z(S_{\bar{\tau}_{B_1\cup L_0}} = x)$$

and

$$H_{B_2} = \sum_{x\in B_2}\sum_{z\in L_N} P_z(S_{\bar{\tau}_{B_2\cup L_0}} = x).$$

Changing the order of both summations we have

$$\begin{aligned}H_{B_1} &= \sum_{z\in L_N}\sum_{x\in B_1} P_z(S_{\bar{\tau}_{B_1\cup L_0}} = x) = \sum_{z\in L_N} P_z(S_{\bar{\tau}_{B_1\cup L_0}} \in B_1)\\ &= \sum_{z\in L_N} P_z(\tau_{B_1} \le \tau_{L_0})\end{aligned}$$

which is smaller than or equal to

$$\begin{aligned}H_{B_2} &= \sum_{z\in L_N}\sum_{x\in B_2} P_z(S_{\bar{\tau}_{B_2\cup L_0}} = x) = \sum_{z\in L_N} P_z(S_{\bar{\tau}_{B_2\cup L_0}} \in B_2)\\ &= \sum_{z\in L_N} P_z(\tau_{B_2} \le \tau_{L_0})\end{aligned}$$

by (1). □

3 Uniform Upper Bounds on Harmonic Measure

In this section, we improve the linear bound in (17) to Theorem 1. Without loss of generality we can assume $x_2 = n$. According to the definition of $H_{B,N}(x)$ and (2), we first note that for any $B' \subset B$, with $x \in B'$ and $L_0 \subset B'$,

$$H_{B,N}(x) \le H_{B',N}(x).$$

Since B is connected and $L_0 \subset B$, there must be a finite nearest neighbor path

$$\mathcal{P}_n = \{x = P_0, P_1, P_2, \cdots, P_{k_n} \in L_0\}$$

connecting x and L_0, where $|P_i - P_{i+1}| = 1$. And since $d(x, L_0) = n$, $|x - P_{k_n}| \geq n$, define

$$m_n = \inf\{i : |P_i - x| \geq n\} - 1$$

and

$$\mathcal{Q}_n = \{P_0, P_1, P_2, \cdots, P_{m_n}\}.$$

One can immediately see that

$$\mathcal{Q}_n \subset B(x, 2n) := \{y :\in \mathbb{Z}^2 : \|y - x\|_2 \leq 2n\}.$$

Then for $B_n = L_0 \cup \mathcal{Q}_n$, to prove Theorem 1, it suffices to show that

$$H_{B_n,N}(x) \leq Cn^{1/2}. \tag{23}$$

And since simple random walk is translation invariant, we can without loss of generality assume that $x_1 = 0$. To show (23), by (15), strong Markov property, and (16)

$$\begin{aligned}
&H_{B_n,N}(i \cdot n) \\
&= E_{i\cdot n}\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{B_n}]\Big] \\
&= \sum_{w \in L_N} P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}, S_{\tau_{L_N}} = w) \\
&\quad E_w\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{B_n}]\Big] \\
&\leq \sum_{w \in L_N} P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}, S_{\tau_{L_N}} = w) \\
&\quad E_w\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{L_0}]\Big] \\
&= 4N \cdot P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}).
\end{aligned} \tag{24}$$

So in order to show (23) and thus Theorem 1, it is sufficient to prove that

$$P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}) \leq \frac{Cn^{1/2}}{N}. \tag{25}$$

To show (25), define $\mathcal{S}_n = \partial^{out} B(i \cdot n, 2n) \cap \{(x, y) \in \mathbb{Z}^2, y \geq 1\}$. Note that if a simple random walk starting at $i \cdot n$ wants to reach L_N before returning to B_n, it has to visit some point in $\mathcal{S}_n$ first. Thus once again by strong Markov property,

$$P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}) = \sum_{z \in \mathcal{S}_n} P_{i\cdot n}(\tau_{\mathcal{S}_n} < \tau_{B_n}, S_{\tau_{\mathcal{S}_n}} = z) P_z(\bar{\tau}_{L_N} < \bar{\tau}_{B_n}). \tag{26}$$

Note that for each $z \in \mathcal{S}_n$, by (2) and the fact that $L_0 \subset B_n$,

$$P_z(\bar{\tau}_{L_N} < \bar{\tau}_{B_n}) \leq P_z(\bar{\tau}_{L_N} < \bar{\tau}_{L_0}) \leq \frac{3n}{N}.$$

Plugging this uniform upper bound into (26), we now have

$$P_{i\cdot n}(\tau_{L_N} < \tau_{B_n}) \leq P_{i\cdot n}(\tau_{\mathcal{S}_n} < \tau_{B_n}) \cdot \frac{3n}{N}.$$

Thus for Theorem 1 it is sufficient to show that

$$P_{i\cdot n}(\tau_{\mathcal{S}_n} < \tau_{B_n}) \le Cn^{-1/2}. \tag{27}$$

Noting that $\mathcal{S}_n \subset \partial^{out} B(i \cdot n, 2n)$, and that $\mathcal{Q}_n \subset B_n$, then by (1) and (2),

$$P_{i\cdot n}(\tau_{\mathcal{S}_n} < \tau_{B_n}) \le P_{i\cdot n}(\tau_{\partial^{out} B(i\cdot n,2n)} < \tau_{\mathcal{Q}_n}).$$

Since simple random walk is translation invariant,

$$P_{i\cdot n}(\tau_{\partial^{out} B(i\cdot n,2n)} < \tau_{\mathcal{Q}_n}) = P_0(\tau_{2n} < \tau_{D_n}),$$

where $D_n = \mathcal{Q}_n - i \cdot n$, which is a connected subset of $B(0, n)$ containing 0. Then

$$P_0(\tau_{2n} < \tau_{D_n}) \le Cn^{-1/2}$$

is guaranteed by Theorem 1 of Reference [10] with $k = \kappa = 1$, which concludes the proof of Theorem 1. □

4 Subset Maximizing the Stationary Harmonic Measure

In this section we prove Theorem 2. Then together with the uniform upper bound we had in Theorem 1, one can see that $U_n = V_n \cup L_0$ is the subset maximizing harmonic measure up to multiplying a constant.

Before we start with the details, an outline of the proof of Theorem 2 is presented. See also Fig. 1. The detailed proof will piece together everything we need in the list below, although the order that each lemma is proved may not be precisely consistent with the outline.

- (i) We have found that $H_{U_n,N}(i \cdot n)$ equals to the expected number of visits to L_N before a simple random walk S starting from $i \cdot n$ returns to U_n. If the random walk reaches L_N first before returning to U_n, the expected number of (re-)visits is $4N + o(N)$.
- (ii) For S to reach L_N first before returning to U_n, it has to reach L_{2n} first. Once it reaches L_{2n}, the probability of success from there is at least of order n/N.
- (iii) If S reached the upper outer boundary of the L_1 ball $B_1(i\cdot n, n/3) = \{|x|+|y-n| \le n/3\}$ before returning to V_n, by the invariance principle there is a positive probability for it to continue to L_{2n} before returning to U_n.
- (iv) The probability that S_n exits $B_1(i \cdot n, n/3)$ before returning to V_n is at least $O(n^{-1/2})$.
- (v) Given S_n exits $B_1(i \cdot n, n/3)$ before returning to V_n, it is more likely to exit from the upper half than the lower half.

Without loss of generality, we only need to prove this theorem for n sufficiently large and N sufficiently larger than n.

4.1 Lower Bound on Escaping Probability

We first show that S_n exits $B_1(i \cdot n, n/3)$ before returning to V_n with probability at least $O(n^{-1/2})$, define $V_n' = \{(0, y),\ n - [n/3] \le y \le n\}$ and

$$\mathcal{S}_{1,n} = \partial B_1(i \cdot n, [n/3]).$$

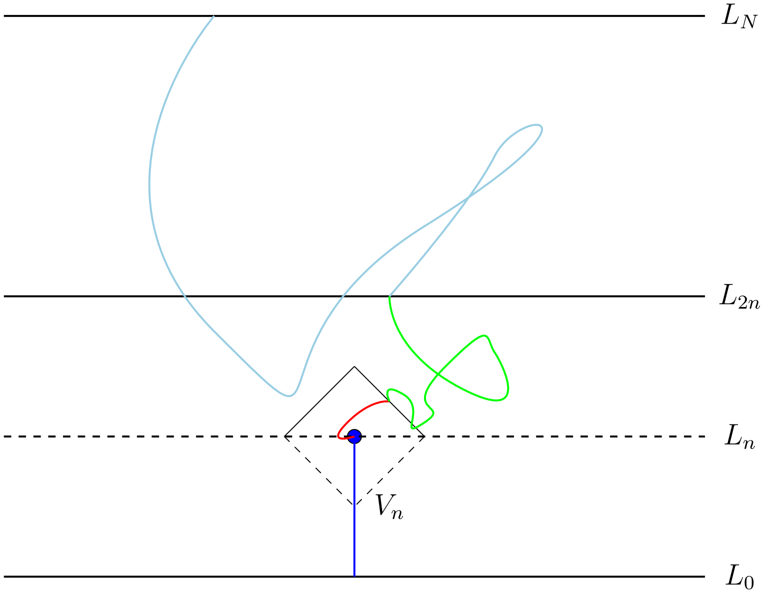

Fig. 1 Outline for the lower bound

Here note that for L_1 ball $B_1(i \cdot n, [n/3])$ we do not need to specify if the boundary is in or out. Then for $C_n' = \partial^{out} B(i \cdot n, n/2)$, note that for a 2-dimensional simple random walk starting at $i \cdot n = (0, n)$ we always have

$$\tau_{\mathcal{S}_{1,n}} < \tau_{C_n'}$$

for all sufficiently large n. Thus for the escaping probability we want to bound from below, we have

$$P_{i \cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{U_n}\right) = P_{i \cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V_n'}\right) \geq P_{i \cdot n}\left(\tau_{C_n'} < \tau_{V_n'}\right).$$

By the translation invariance of S,

$$\begin{aligned} P_{i \cdot n}\left(\tau_{C_n'} < \tau_{V_n'}\right) &= P_0\left(\tau_{C_n' - i \cdot n} < \tau_{V_n' - i \cdot n}\right) \\ &= P_0\left(\tau_{\partial^{out} B(0, n/2)} < \tau_{V_n' - i \cdot n}\right) \geq c n^{-1/2} \end{aligned} \tag{28}$$

where the last inequality is guaranteed by Proposition 12 of Reference [10], and the reflection invariance of simple random walk.

4.2 Spatial Distribution at the Escaping Time

Now (28) shows that a 2-dimensional simple random walk starting at $i \cdot n$ will escape $B_1(i \cdot n, [n/3]) := \{x \in \mathbb{Z}^2 : \|x - i \cdot n\|_1 \leq [n/3]\}$ before returning to V_n' and thus U_n with probability at least some constant times $n^{-1/2}$. We next show that, given the random walk successfully escapes, it is more likely to escape from the upper half of $\mathcal{S}_{1,n}$ than the lower half of it. To make it precise, define

$$\mathcal{S}_{1,n}^U = \mathcal{S}_{1,n} \cap \{(x, y), y \geq n\},$$

and

$$\mathcal{S}_{1,n}^{L}=\mathcal{S}_{1,n}\cap\{(x,y),\,y\le n\}.$$

Then for stopping time $\sigma=\tau_{\mathcal{S}_{1,n}}\wedge\tau_{V'_n}$, we want to show

$$P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}}<\tau_{V'_n},\,S_\sigma\in\mathcal{S}_{1,n}^{U}\right)\ge P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}}<\tau_{V'_n},\,S_\sigma\in\mathcal{S}_{1,n}^{L}\right).\tag{29}$$

To show this we can again use translation invariance to move everything centered at 0. For integer $m\ge 1$, let

$$A_m^+=\{(x,y)\in\mathbb{Z}^2,\ x+y=m,\ x\in[0,m]\}\cup\{(x,y)\in\mathbb{Z}^2,\\ -x+y=m,\ x\in[-m,0]\}$$

and

$$A_m^-=\{(x,y)\in\mathbb{Z}^2,\ x+y=-m,\ x\in[-m,0,]\}\cup\{(x,y)\in\mathbb{Z}^2,\\ -x+y=-m,\ x\in[0,m]\}$$

be the upper and lower half of $\partial B_1(0,m)$. Then define $C_m^-=\{(0,-i),i=0,1,\cdots,m\}$, and $C_m^+=\{(0,i),i=0,1,\cdots,m\}$. To show (29), it suffices to prove the following lemma:

Lemma 4.1 *For all integer m, define set*

$$E_m^-=A_m^+\cup A_m^-\cup C_m^-$$

and stopping time

$$\sigma_m^-=\tau_{E_m^-}=\tau_{A_m^+}\wedge\tau_{A_m^-}\wedge\tau_{C_m^-}.$$

We have

$$P_0\left(\tau_{A_m^+}=\sigma_m^-\right)\ge P_0\left(\tau_{A_m^-}=\sigma_m^-\right).\tag{30}$$

Remark 4 Before presenting the detailed argument, we first briefly discuss the idea in the proof of Lemma 4.1: using symmetry together with a decomposition on the **last time** that a simple random walk hits C_m^+, one only need to show that starting from $C_m^+\setminus\{(0,0),(0,m)\}$, a simple random walk is more likely to first hit A_m^+ than A_m^- before returning to $C_m^+\cup C_m^-$, see (39) for precise formula. To prove (39) one can again use symmetry/ reflection principle to form one-to-one mappings between nearest neighbor trajectories.

Proof For

$$E_m=A_m^+\cup A_m^-\cup C_m^-\cup C_m^+,$$

and stopping time

$$\sigma_m=\tau_{E_m}=\tau_{A_m^+}\wedge\tau_{A_m^-}\wedge\tau_{C_m^-}\wedge\tau_{C_m^+},$$

by symmetry we have

$$P_0\left(\tau_{A_m^+}=\sigma_m\right)=P_0\left(\tau_{A_m^-}=\sigma_m\right).$$

At the same time,

$$\begin{aligned} P_0\left(\tau_{A_m^+}=\sigma_m^-\right) &= P_0\left(\tau_{A_m^+}\le\tau_{C_m^+},\tau_{A_m^+}=\sigma_m^-\right)\\ &\quad + P_0\left(\tau_{A_m^+}>\tau_{C_m^+},\tau_{A_m^+}=\sigma_m^-\right)\\ &= P_0\left(\tau_{A_m^+}=\sigma_m\right)+P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\right), \end{aligned} \tag{31}$$

and

$$\begin{aligned} P_0\left(\tau_{A_m^-}=\sigma_m^-\right) &= P_0\left(\tau_{A_m^-}\le\tau_{C_m^+},\tau_{A_m^-}=\sigma_m^-\right)\\ &\quad + P_0\left(\tau_{A_m^-}>\tau_{C_m^+},\tau_{A_m^-}=\sigma_m^-\right)\\ &= P_0\left(\tau_{A_m^-}=\sigma_m\right)+P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^-}=\sigma_m^-\right). \end{aligned} \tag{32}$$

Thus it is sufficient to show

$$P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\right)\ge P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^-}=\sigma_m^-\right). \tag{33}$$

Under event $\{\tau_{C_m^+}<\sigma_m^-\}$, let random variable N_m^+ be the **last time** S visits C_m^+ in $[0,\sigma_m^- -1]$. Note that N_m^+ is not a stopping time so we cannot use strong Markov property. But we can nonetheless have the decomposition:

$$\begin{aligned} &P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\right)\\ &=\sum_{k=1}^{\infty}\sum_{\substack{x_1,x_2,\cdots,x_{k-1}\notin E_m^-\\ x_k\in x_k\in C_m^+\setminus\{0,(0,m)\}}} P_0\left(S_1=x_1,\cdots,S_k=x_k,N_m^+=k,\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\right). \end{aligned} \tag{34}$$

and

$$\begin{aligned} &P_0\left(\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^-}=\sigma_m^-\right)\\ &=\sum_{k=1}^{\infty}\sum_{\substack{x_1,x_2,\cdots,x_{k-1}\notin E_m^-\\ x_k\in x_k\in C_m^+\setminus\{0,(0,m)\}}} P_0\left(S_1=x_1,\cdots,S_k=x_k,N_m^+=k,\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^-}=\sigma_m^-\right). \end{aligned} \tag{35}$$

Note that for each k, $x_1,x_2,\cdots,x_{k-1}\notin E_m^-$, and $x_k\in x_k\in C_m^+\setminus\{0,(0,m)\}$, we have

$$\begin{aligned} &\{S_1=x_1,\cdots,S_k=x_k,N_m^+=k,\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\}\\ =&\{S_1=x_1,\cdots,S_k=x_k,\ S_{k+1+\cdot}\text{ visit }A_m^+\text{no later than it first visits }A_m^-\cup C_m^+\cup C_m^-\}. \end{aligned}$$

So by Markov property, we have

$$\begin{aligned} &P_0\left(S_1=x_1,\cdots,S_k=x_k,N_m^+=k,\tau_{C_m^+}<\sigma_m^-,\tau_{A_m^+}=\sigma_m^-\right)\\ &=P_0\left(S_1=x_1,\cdots,S_k=x_k\right)P_{x_k}\left(\tau_{A_m^+}=\sigma_m\right). \end{aligned} \tag{36}$$

Plugging back in (34) we have

$$P_0\left(\tau_{C_m^+} < \sigma_m^-, \tau_{A_m^+} = \sigma_m^-\right)$$
$$= \sum_{k=1}^{\infty} \sum_{\substack{x_1, x_2, \cdots, x_{k-1} \notin E_m^- \\ x_k \in x_k \in C_m^+ \setminus \{0, (0,m)\}}} P_0\left(S_1 = x_1, \cdots, S_k = x_k\right) P_{x_k}\left(\tau_{A_m^+} = \sigma_m\right), \tag{37}$$

while the same argument for A_m^- gives us

$$P_0\left(\tau_{C_m^+} < \sigma_m^-, \tau_{A_m^-} = \sigma_m^-\right)$$
$$= \sum_{k=1}^{\infty} \sum_{\substack{x_1, x_2, \cdots, x_{k-1} \notin E_m^- \\ x_k \in x_k \in C_m^+ \setminus \{0, (0,m)\}}} P_0\left(S_1 = x_1, \cdots, S_k = x_k\right) P_{x_k}\left(\tau_{A_m^-} = \sigma_m\right). \tag{38}$$

Comparing (37) and (38) term by term, one can see it suffices to show that for all $z = (0, j) \in x_k \in C_m^+ \setminus \{0, (0, m)\}$,

$$P_z\left(\tau_{A_m^+} = \sigma_m\right) \geq P_z\left(\tau_{A_m^-} = \sigma_m\right). \tag{39}$$

To show (39), one first sees that on $\{\tau_{A_m^+} = \sigma_m\}$ or $\{\tau_{A_m^-} = \sigma_m\}$, a random walk starting at z has to move horizontally at the first step then remain in the right or left half triangle of $B_1(0, m)$ until it exits from A_m^+ or A_m^-. Then for all integer $i \in [0, m]$ we define

$$\begin{aligned} C_{m,i} &= \{(0, y),\ 2i - m \leq y \leq m\}, \\ A_{m,i}^{+,r} &= \{(x, y) \in \mathbb{Z}^2,\ x + y = m,\ x \in [0, m - i]\} \end{aligned}$$

and

$$A_{m,i}^{-,r} = \cup\{(x, y) \in \mathbb{Z}^2,\ -x + y = 2i - m,\ x \in [0, m - i]\}.$$

Now we have by symmetry

$$P_z\left(\tau_{A_m^+} = \sigma_m\right) = \frac{1}{2} P_{(1,j)}\left(\bar{\tau}_{A_{m,0}^{+,r}} \leq \bar{\tau}_{A_{m,0}^{-,r}}, \bar{\tau}_{A_{m,0}^{+,r}} \leq \bar{\tau}_{C_{m,0}}\right). \tag{40}$$

and

$$P_z\left(\tau_{A_m^-} = \sigma_m\right) = \frac{1}{2} P_{(1,j)}\left(\bar{\tau}_{A_{m,0}^{-,r}} \leq \bar{\tau}_{A_{m,0}^{+,r}}, \bar{\tau}_{A_{m,0}^{-,r}} \leq \bar{\tau}_{C_{m,0}}\right). \tag{41}$$

The right hand side of the Eq. (41) equals to 0 when $j = m - 1$. Otherwise, note that if a random walk starting from $(1, j)$ want to visit $A_{m,0}^{-,r}$ before visiting $A_{m,0}^{+,r}$ or $C_{m,0}$, it has to first get through $A_{m,j}^{-,r}$ before visiting $A_{m,0}^{+,r}$ or $C_{m,0}$. Thus

$$P_{(1,j)}\left(\bar{\tau}_{A_{m,0}^{-,r}} \leq \bar{\tau}_{A_{m,0}^{+,r}}, \bar{\tau}_{A_{m,0}^{-,r}} \leq \bar{\tau}_{C_{m,0}}\right) \leq P_{(1,j)}\left(\bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{A_{m,0}^{+,r}}, \bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{C_{m,0}}\right).$$

Then note that in order to have a random walk starting from $(1, j)$ get to $A_{m,j}^{-,r}$ before visiting $A_{m,0}^{+,r}$ or $C_{m,0}$, it only need to avoid $A_{m,j}^{+,r}$ and $C_{m,j}$. So we have

$$P_{(1,j)}\left(\bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{A_{m,0}^{+,r}}, \bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{C_{m,0}}\right) = P_{(1,j)}\left(\bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{A_{m,j}^{+,r}}, \bar{\tau}_{A_{m,j}^{-,r}} \leq \bar{\tau}_{C_{m,j}}\right).$$

By symmetry one can see

$$P_{(1,j)}\left(\bar{\tau}_{A^{-,r}_{m,j}} \le \bar{\tau}_{A^{+,r}_{m,j}}, \bar{\tau}_{A^{-,r}_{m,j}} \le \bar{\tau}_{C_{m,j}}\right) = P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{A^{-,r}_{m,j}}, \bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{C_{m,j}}\right).$$

Moreover, note that a random walk starting from $(1, j)$ must exist the smaller triangle bounded by $A^{+,r}_{m,j}$, $A^{-,r}_{m,j}$, and $C_{m,j}$ before exiting the larger on bounded by $A^{+,r}_{m,0}$, $A^{-,r}_{m,0}$, and $C_{m,0}$ i.e.,

$$\sigma^r_j = \bar{\tau}_{A^{+,r}_{m,j}} \wedge \bar{\tau}_{A^{-,r}_{m,j}} \wedge \tau_{C_{m,j}} \le \sigma^r = \bar{\tau}_{A^{+,r}_{m,0}} \wedge \bar{\tau}_{A^{-,r}_{m,0}} \wedge \tau_{C_{m,0}}.$$

Thus

$$\begin{aligned} &P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{A^{-,r}_{m,j}}, \bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{C_{m,j}}\right) \\ &\quad = P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \sigma^r_j\right) \\ &\quad \le P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \sigma^r\right) \\ &\quad \le P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{A^{-,r}_{m,0}} \wedge \tau_{C_{m,0}}\right) \\ &\quad = P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{A^{-,r}_{m,0}}, \bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{C_{m,0}}\right) \\ &\quad \le P_{(1,j)}\left(\bar{\tau}_{A^{+,r}_{m,0}} \le \bar{\tau}_{A^{-,r}_{m,0}}, \bar{\tau}_{A^{+,r}_{m,j}} \le \bar{\tau}_{C_{m,0}}\right). \end{aligned} \tag{42}$$

Finally note that the right hand side of the last inequality in (42) is exactly the right hand side of (40). □

With Lemma 4.1, we immediately get (29) from translation invariance.

4.3 Proof of Theorem 2

Now we have all the tools we need to finish the proof of Theorem 2. Recall (24) and apply it to U_n and $i \cdot n$,

$$\begin{aligned} &H_{U_n,N}(i \cdot n) \\ &= E_{i\cdot n}\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{U_n}]\Big] \\ &= \sum_{w\in L_N} P_{i\cdot n}(\tau_{L_N} < \tau_{U_n}, S_{\tau_{L_N}} = w) E_w\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{U_n}]\Big]. \end{aligned}$$

Note that for all $w \in L_N$,

$$P_w(\tau_{L_n} \le \tau_{U_n}) = 1.$$

We have

$$\begin{aligned} &H_{U_n,N}(i \cdot n) \\ &\ge \sum_{w\in L_N} P_{i\cdot n}(\tau_{L_N} < \tau_{U_n}, S_{\tau_{L_N}} = w) E_w\Big[\text{number of visits to } L_N \text{in time interval } [0, \tau_{L_n}]\Big] \\ &= 4P_{i\cdot n}(\tau_{L_N} < \tau_{U_n})(N - n). \end{aligned}$$

Then according again to strong Markov property and the fact that a random walk starting from $i \cdot n$ has to visit L_{2n} before L_N,

$$P_{i\cdot n}(\tau_{L_N} < \tau_{U_n}) = \sum_{w\in L_{2n}} P_{i\cdot n}(\tau_{L_{2n}} < \tau_{U_n}, S_{\tau_{L_{2n}}} = w) P_w(\tau_{L_N} < \tau_{U_n}).$$

Again, note that for all $w \in L_{2n}$

$$P_w(\tau_{L_N} < \tau_{U_n}) \geq P_w(\tau_{L_N} < \tau_{L_n}) = \frac{n}{N-n}.$$

Thus to prove Theorem 2 it is sufficient to show that for N sufficiently larger than n,

$$P_{i\cdot n}(\tau_{L_{2n}} < \tau_{U_n}) \geq cn^{-1/2} \tag{43}$$

To show (43), we have

$$\begin{aligned} P_{i\cdot n}(\tau_{L_{2n}} < \tau_{U_n}) &\geq \sum_{w\in \mathcal{S}^U_{1,n}} P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V_n}, S_{\tau_{\mathcal{S}_{1,n}}} = w \in \mathcal{S}^U_{1,n}\right) P_w(\tau_{L_{2n}} < \tau_{U_n}) \\ &= \sum_{w\in \mathcal{S}^U_{1,n}} P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V'_n}, S_\sigma = w \in \mathcal{S}^U_{1,n}\right) P_w(\tau_{L_{2n}} < \tau_{U_n}). \end{aligned}$$

Note that by invariance principle there is a constant c such that for any sufficiently large n and $w \in \mathcal{S}^U_{1,n}$,

$$P_w(\tau_{L_{2n}} < \tau_{U_n}) \geq c.$$

Thus

$$P_{i\cdot n}(\tau_{L_{2n}} < \tau_{U_n}) \geq c P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V'_n}, S_\sigma = w \in \mathcal{S}^U_{1,n}\right). \tag{44}$$

Then by (28) and (29), we have

$$P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V'_n}, S_\sigma \in \mathcal{S}^U_{1,n}\right) \geq \frac{1}{2} P_{i\cdot n}\left(\tau_{\mathcal{S}_{1,n}} < \tau_{V'_n}\right) \geq cn^{-1/2}. \tag{45}$$

Thus, the proof of Theorem 2 is complete. □

5 Total Harmonic Measure on Finite Sets

5.1 Upper Bound in Theorem 3

To show the upper bound in (9), without loss of generality we can assume $B \cap L_0 \neq \varnothing$, which implies that $\min_{x\in B}\{x_2\} = 0$. Otherwise, for $x_0 = (x_{1,0}, x_{2,0})$ that has the smallest height in B, define

$$B' = B \cup \{(x_{1,0}, j),\ j = 0, 1, \cdots, x_{2,0} - 1\}.$$

By Proposition 2, we have $H_{B'} \geq H_B$ and $|B'| = |B| + \min_{x\in B}\{x_2\}$. Thus it suffices for us to prove that for any connected and finite B with $B \cap L_0 \neq \varnothing$,

$$H_B \leq C|B|. \tag{46}$$

And we prove (46) inductively. When $|B| = 1$, we have proved the desired upper bound in (17). Suppose we have proved (46) for all connected B with $|B| \leq n$, $B \cap L_0 \neq \varnothing$. Then for a B such that $|B| = n + 1$, $B \cap L_0 \neq \varnothing$, we first show that one can remove one vertex in B and still have a connected subset intersecting L_0. In fact we prove something even stronger:

Lemma 5.1 *For any finite and connected $B \subset \mathbb{Z}^2$ with $|B| \geq 2$, there are always two points $x_1, x_2 \in B$ such that $B \setminus \{x_1\}$ and $B \setminus \{x_2\}$ are both connected.*

Remark 5 With Lemma 5.1, we can make sure that starting from $|B| = n + 1$, $B \cap L_0 \neq \varnothing$, we can remove one point and it will not be in L_0 if $|B \cap L_0| = 1$. Thus the new connected subset still intersects L_0.

Proof Again, we prove this lemma by induction. For $|B| = 2$ or $|B| = 3$, it is easy to check the lemma holds. Now suppose it also holds for all connected $|B| \leq n$. Then from the assumption we also have that
Observation 1: for any connected B such that $|B| \leq n$ and any $x_0 \in B^c$ such that $d(x_0, B) = 1$, where

$$d(x, B) = \inf_{y \in B} \{|x - y|\},$$

there must exists an $x \in B$ such that $B \setminus \{x\}$ is connected while $d(x_0, B \setminus \{x\}) = 1$.
To see this, note that if

$$\left|\{y \in B : \ |x_0 - y| = 1\}\right| \geq 2$$

then removing one point will not change the distance between x and the smaller subset. So either x_1 or x_2 in the inductive assumption is good. Otherwise, let y_0 be the only point in B neighboring x_0. By the inductive assumption we have two points x_1 and x_2 which we can remove, and one of them must not be y_0. Thus we still have an $x \in B$ such that $B \setminus \{x\}$ is connected while $d(x_0, B \setminus \{x\}) = 1$.

With the observation above, now for any connected B such that $|B| = n + 1$, we first choose one point y arbitrarily from B. If $B \setminus \{y\}$ is connected, note that $|B \setminus \{y\}| = n$ and that $d(y, B \setminus \{y\}) = 1$. Our observation above shows that there must be a $y' \in B \setminus \{y\}$ such that $B \setminus \{y, y'\}$ is also connected and

$$d(y, B \setminus \{y, y'\}) = 1.$$

This implies that $B \setminus \{y'\} = B \setminus \{y, y'\} \cup \{y\}$ is connected. And we have found our two "removable" points. Otherwise, if $B \setminus \{y\}$ is not connected, it must have at least two connected components, say B_1 and B_2. □

Remark 6 If $B \setminus \{y\}$ has more than two connected components, just choose two of them arbitrarily.

Let

$$d(A, B) = \inf_{x \in A,\ y \in B} \{|x - y|\}$$

for all finite A and B. Noting that B is connected, we must have $d(B_1, B \setminus B_1) = 1$. But since B_1 is not connected to $B \setminus (B_1 \cup \{y\})$, we also have

$$d\big(B_1, B \setminus (B_1 \cup \{y\})\big) \geq 2.$$

Thus one can now see $d(y, B_1) = 1$ and $d(y, B_2) = 1$. Then note that $|B_1|$ and $|B_2|$ are both less than n. So by Observation 1 we again have there is a $x_1 \in B_1$ such that $B_1 \setminus \{x_1\}$ is connected and that

$$d(y, B_1 \setminus \{x_1\}) = 1,$$

which implies that $(B_1 \setminus \{x_1\}) \cup \{y\}$ is connected,

$$d\big((B_1 \setminus \{x_1\}) \cup \{y\}, B_2\big) = 1,$$

and that $B \setminus \{x_1\} = (B_1 \setminus \{x_1\}) \cup \{y\} \cup B_2$ is connected. The same argument on B_2 also gives that there is a $x_2 \in B_2$ such that $B \setminus \{x_2\}$ is connected. Finally note that B_1 and B_2 are different connected component, which implies that $B_1 \cap B_2 = \varnothing$. So we have $x_1 \neq x_2$ and the proof of this lemma is complete. □

With Lemma 5.1, we continue with the inductive argument for the growth rate of H_B. For any finite and connected B such that $|B| = n + 1$, $B \cap L_0 \neq \varnothing$, there has to be an $x = (x_1, x_2) \in B$ such that $\tilde{B} = B \setminus \{x\}$ is still connected and $\tilde{B} \cap L_0 \neq \varnothing$. By inductive assumption we know that $H_{\tilde{B}} \leq Cn$. So now we can concentrate on comparing $H_{\tilde{B}}$ and H_B.

Since B is finite, for any $v \in B$ sufficiently large N we have

$$H_{B,N}(v) = \sum_{z \in L_N} P_z(\tau_v = \tau_{B \cup L_0}).$$

And thus

$$H_{B,N} = \sum_{v \in B} H_{B,N}(v) = \sum_{z \in L_N} P_z(\tau_B \leq \tau_{L_0})$$

while

$$H_{\tilde{B},N} = \sum_{z \in L_N} P_z(\tau_{\tilde{B}} \leq \tau_{L_0}).$$

Note that for each $z \in L_N$,

$$P_z(\tau_B \leq \tau_{L_0}) - P_z(\tau_{\tilde{B}} \leq \tau_{L_0}) = P_z(\tau_x \leq \tau_{L_0} < \tau_{\tilde{B}}). \tag{47}$$

Moreover, by strong Markov property,

$$P_z(\tau_x \leq \tau_{L_0} < \tau_{\tilde{B}}) = P_z(\tau_x = \tau_{B \cup L_0}) P_x(\tau_{L_0} < \tau_{\tilde{B}}). \tag{48}$$

Combining (47) and (48), we have

$$\begin{aligned} H_{B,N} - H_{\tilde{B},N} &= P_x(\tau_{L_0} < \tau_{\tilde{B}}) \sum_{z \in L_N} P_z(\tau_x = \tau_{B \cup L_0}) \\ &= P_x(\tau_{L_0} < \tau_{\tilde{B}}) H_{B,N}(x). \end{aligned} \tag{49}$$

If $x_2 = 0$, note that in (18) we have $H_{B,N}(x) \leq 1$, which implies that $H_{B,N} - H_{\tilde{B},N} \leq 1$. And for $x_2 \geq 1$, we have by Theorem 1

$$H_{B,N} - H_{\tilde{B},N} \leq C\sqrt{x_2} P_x(\tau_{L_0} < \tau_{\tilde{B}}). \tag{50}$$

And since $\tilde{B}$ is connected. There must be a finite nearest neighbor path

$$\tilde{\mathcal{P}}_x = \{x = \tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_{k_n} \in L_0\}$$

connecting x and L_0, where $|\tilde{P}_i - \tilde{P}_{i+1}| = 1$. And since $d(x, L_0) = x_2$, $|x - P_{k_n}| \geq x_2$. Define

$$\tilde{m}_x = \inf\{i : |\tilde{P}_i - x| \geq x_2/2\} - 1$$

and

$$\tilde{\mathcal{Q}}_x = \{\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_{\tilde{m}_x}\}.$$

One can immediately see that $\tilde{Q}_x \subset \tilde{B}$ and that for all sufficiently large x_2,

$$\tilde{Q}_x \subset B(\tilde{P}_1, 2x_2/3).$$

Thus

$$\frac{1}{4} P_x(\tau_{L_0} < \tau_{\tilde{B}}) \le \frac{1}{4} P_x(\tau_{L_0} < \tau_{\tilde{Q}_x}) \le P_{\tilde{P}_1}(\tau_{L_0} < \tau_{\tilde{Q}_x}) \le P_0(\tau_{2x_2/3} < \tau_{\tilde{Q}_x - \tilde{P}_1}).$$

And again by Theorem 1 of Reference [10],

$$P_0(\tau_{2x_2/3} < \tau_{\tilde{Q}_x - \tilde{P}_1}) \le C x_2^{-1/2}. \tag{51}$$

Combining (50) and (51) we have that there is a constant C independent of n, N and x, such that

$$H_{B,N} - H_{\tilde{B},N} \le C.$$

Thus the proof of (9) is complete.

5.2 Lower Bound in Theorem 3

First, (11) is obvious. Now we show the lower bound in (10). Since B is finite, let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a point in B such that

$$\bar{x}_2 = \max_{x \in B}\{x_2\}.$$

Note that by Proposition 2, $H_B \ge H_{\bar{x}}$. It suffices to prove (10) for the single element set $\{\bar{x}\}$. Recall that

$$\begin{aligned} H_{\{\bar{x}\},N} &= E_{\bar{x}}\Big[\text{number of visits to } L_N \text{in } [0, \tau_{\{\bar{x}\}\cup L_0}]\Big] \\ &\ge P_{\bar{x}}(\tau_{L_N} < \tau_{\{\bar{x}\}\cup L_0}) \inf_{z \in L_N} E_z\Big[\text{number of visits to } L_N \text{in } [0, \tau_{\{\bar{x}\}\cup L_0}]\Big] \end{aligned}$$

and that for sufficiently large N and any $z \in L_N$,

$$\begin{aligned} &E_z\Big[\text{number of visits to } L_N \text{in } [0, \tau_{\{\bar{x}\}\cup L_0}]\Big] \\ &\ge E_z\Big[\text{number of visits to } L_N \text{in io } [0, \tau_{L_{x_2}}]\Big] = 4(N - x_2) \ge 2N. \end{aligned}$$

To prove (10) it is sufficient to show that for sufficiently large x_2

$$P_{\bar{x}}(\tau_{L_N} < \tau_{\{\bar{x}\}\cup L_0}) \ge \frac{c\bar{x}_2}{N \log(\bar{x}_2)}. \tag{52}$$

Now let $n_{\bar{x}}$ be the largest odd number less than $\bar{x}_2$. We define $B_1(\bar{x}, n_{\bar{x}})$ be the L_1 ball centered at x with radius n_x. Moreover we define

$$W_{\bar{x}}^{1,+} = \partial B_1(\bar{x}, n_{\bar{x}}) \cap \{(y_1, y_2),\ \ y_2 \ge \bar{x}_2 + (n_{\bar{x}} + 1)/2\}$$

be the upper corner of $\partial B_1(\bar{x}, n_{\bar{x}})$. By symmetry we have

$$P_{\bar{x}}\left(\tau_{\partial B_1(\bar{x}, n_{\bar{x}})} < \tau_{\bar{x}},\ S_{\tau_{\partial B_1(\bar{x}, n_{\bar{x}})}} \in W_{\bar{x}}^{1,+}\right) = \frac{1}{4} P_{\bar{x}}(\tau_{\partial B_1(\bar{x}, n_{\bar{x}})} < \tau_{\bar{x}}).$$

Thus we have

$$\begin{aligned} &P_{\bar{x}}(\tau_{L_N} < \tau_{\{\bar{x}\}\cup L_0}) \\ &\quad \geq P_{\bar{x}}\left(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \tau_{\bar{x}},\ S_{\tau_{\partial B_1(\bar{x},n_{\bar{x}})}} \in W_{\bar{x}}^{1,+}\right) \inf_{y\in W_{\bar{x}}^{1,+}} P_y(\tau_{L_N} < \tau_{L_{\bar{x}_2}}) \\ &\quad \geq \frac{1}{4} P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \tau_{\bar{x}}) \inf_{y\in W_{\bar{x}}^{1,+}} P_y(\tau_{L_N} < \tau_{L_{\bar{x}_2}}). \end{aligned} \tag{53}$$

Then note that for any $y \in W_{\bar{x}}^{1,+}$

$$P_y(\tau_{L_N} < \tau_{L_{\bar{x}_2}}) \geq \frac{n_{\bar{x}}}{2N} \geq \frac{\bar{x}_2}{4N}.$$

Thus it is sufficient for us to prove that

Lemma 5.2 *There is a constant $c > 0$ such that for all sufficiently large $\bar{x}_2$,*

$$P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \tau_{\bar{x}}) \geq \frac{c}{\log(\bar{x}_2)}. \tag{54}$$

Proof For $S_n = (S_{1,n}, S_{2,n})$ to be the simple random walk starting at $\bar{x}$, consider the martingale

$$M_n = (S_{2,n} - \bar{x}_2)^2 - \frac{n}{2}.$$

Note that $M_0 = 0$, so we have

$$E_{\bar{x}}[\tau_{\partial B_1(\bar{x},n_{\bar{x}})}] \leq \sup_{y\in\partial B_1(\bar{x},n_{\bar{x}})} (y_2 - \bar{x}_2)^2 \leq \bar{x}_2^2.$$

Thus

$$P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} \geq x_2^3) \leq \frac{1}{x_2}. \tag{55}$$

On the other hand, for simple random walk in $\mathbb{Z}^2$ it was shown in References [5] and [6] that for sufficiently large x_2,

$$P_{\bar{x}}(\tau_{\bar{x}} > \bar{x}_2^3) = \frac{\pi}{\log(\bar{x}_2^3)} + O\left(\frac{1}{\log^2(\bar{x}_2^3)}\right) \geq \frac{\pi}{6\log(\bar{x}_2)}. \tag{56}$$

Thus note that

$$\begin{aligned} P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \tau_{\bar{x}}) &\geq P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \bar{x}_2^3,\ \tau_{\bar{x}} > \bar{x}_2^3) \\ &= P_{\bar{x}}(\tau_{\bar{x}} > \bar{x}_2^3) - P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} \geq \bar{x}_2^3,\ \tau_{\bar{x}} > \bar{x}_2^3) \\ &\geq P_{\bar{x}}(\tau_{\bar{x}} > \bar{x}_2^3) - P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} \geq \bar{x}_2^3). \end{aligned}$$

Combining (55) and (56), we have for sufficiently large x_2,

$$P_{\bar{x}}(\tau_{\partial B_1(\bar{x},n_{\bar{x}})} < \tau_{\bar{x}}) \geq \frac{\pi}{6\log(\bar{x}_2)} - \frac{1}{x_2} \geq \frac{\pi}{7\log(\bar{x}_2)}$$

which finished the proof of this lemma. □

With Lemma 5.2, the proof of (10) and thus Theorem 3 is complete. □

5.3 Proof of Theorem 4

Now we show that the total harmonic measure is maximized (up to multiplying a constant) by the vertical line segment V_n over all connected finite subsets with the same cardinality and connected to L_0. And again we do this inductively. By (49), we have

$$H_{V_n,N} - H_{V_{n-1},N} = H_{V_n,N}(y_n)P_{y_n}(\tau_{L_0} < \tau_{V_{n-1}}). \tag{57}$$

According to Theorem 2, we have

$$H_{V_n,N}(y_n) \geq c\sqrt{n}.$$

Noting that

$$P_{y_n}(\tau_{L_0} < \tau_{V_{n-1}}) \geq P_{y_n}(\tau_{L_0} < \tau_{V_n}),$$

it suffices to prove that

$$P_{y_n}(\tau_{L_0} < \tau_{V_n}) \geq \frac{c}{\sqrt{n}}. \tag{58}$$

On the other hand, recall that

$$\mathcal{S}_{1,n} = \partial B_1(i \cdot n, [n/3])$$

and that

$$\mathcal{S}_{1,n}^U = \mathcal{S}_{1,n} \cap \{(x, y), y \geq n\}.$$

We have

$$P_{y_n}(\tau_{L_0} < \tau_{V_n}) \geq P_{y_n}\left(\tau_{\mathcal{S}_{1,n}^U} < \tau_{V_n}\right) \inf_{y \in \mathcal{S}_{1,n}^U} P_y(\tau_{L_0} < \tau_{V_n}).$$

Again by invariant principle, there is a constant $c > 0$ such that for any n and $y \in \mathcal{S}_{1,n}^U$,

$$P_y(\tau_{L_0} < \tau_{V_n}) \geq c.$$

And then by (28) and (29),

$$P_{y_n}\left(\tau_{\mathcal{S}_{1,n}^U} < \tau_{V_n}\right) \geq \frac{c}{\sqrt{n}}.$$

Thus the proof of Theorem 4 is complete. □

6 Construction and Growth Estimate of DLA in $\mathcal{H}$

6.1 Construction of a Growth Model

With the upper bounds of the harmonic measure on the upper half plane, in this section we construct pure growth models which can be used as a dominating process for both the DLA model in $\mathcal{H}$ and the stationary DLA model that will be introduced in a follow up paper. Consider an interacting particle system $\bar{\xi}_t$ defined on $\{0, 1\}^{\mathcal{H}}$ where $\mathcal{H}$ is the upper half plane with 1 standing for a site occupied while 0 for vacant, with transition rates as follows:

(i) For each occupied site $x = (x_1, x_2) \in \mathcal{H}$, if $x_2 > 0$ it will try to give birth to each of its nearest neighbors at a Poisson rate of $\sqrt{x_2}$. If $x_2 = 0$, it will try to give birth to each of its nearest neighbors at a Poisson rate of 1.

(ii) When x attempts to give birth to its nearest neighbors y already occupied, the birth is suppressed.

We prove that an interacting particle system determined by the dynamic above is well-defined.

Proposition 3 *The interacting particle system $\bar{\xi}_t \in \{0, 1\}^{\mathcal{H}}$ satisfying (i) and (ii) is well defined.*

Proof The proof of Proposition 3 uses a similar idea as in Theorem 2.1 of Reference [4]. Although here the transition rates are no longer translation invariant or uniformly bounded, we will show that although with high probability the time increment of each step goes to zero, the summation over the increments diverges. The next idea is very similar to Borel-Cantelli lemma. However, rather than using the result directly, we will have the proof of Borel-Cantelli lemma embedded in our arguments. By doing so, we will be able to make sure the space-time box in each step of our iteration is deterministic and can be explicitly calculated.

Our construction starts with introducing the following families of independent Poisson processes: for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ that are nearest neighbors in $\mathcal{H}$ and $e_{x\to y}$ which is the oriented edge from x to y, let

$$\left\{N_t^{x\to y},\ x, y \in \mathcal{H},\ \|x - y\| = 1\right\}_{t\in[0,\infty)}$$

be a family of independent Poisson processes, where $N_t^{x\to y}$ has intensity $\sqrt{x_2} \vee 1$. Then let

$$\left\{\tilde{N}_t^{x\to y},\ x, y \in \mathcal{H},\ \|x - y\| = 1\right\}_{t\in[0,\infty)}$$

be a family of independent Poisson process independent of N_t with the same intensities. Now consider the space-time combination, $\mathcal{H} \times (-\infty, \infty)$. From each $x \in \mathcal{H}$, we draw a vertical infinite line to represent the double infinite time line at this site. Then for each $e_{x\to y}$, at any time t such that $N_t^{x\to y} = N_{t-}^{x\to y} + 1$, we draw an oriented arrow from (x, t) to (y, t). And at t such that $\tilde{N}_t^{x\to y} = \tilde{N}_{t-}^{x\to y} + 1$, we draw an oriented arrow from $(x, -t)$ to $(y, -t)$. □

Remark 7 Although the construction of our particle system actually only depends on the transitions on the positive time line, by defining the transition for negative t's we are able to have better symmetry on the time reversal and thus formally simplify the proof.

We have an oriented random graph in the space-time combination. Then for any two points (x, t_1) and (x', t_2) with $t_1 < t_2$, we define that (x, t_1) and (x', t_2) are connected or $(x, t_1) \to (x', t_2)$, if there is a (finite) path in the oriented random graph starting from (x, t_1), that goes up vertically and follows the oriented edges ending at (x', t_2). Then

Definition 1 For any $\bar{\xi}_0 \in \{0, 1\}^{\mathcal{H}}$, we define $\bar{\xi}_t$ such that for each $t \geq 0$ and $x \in \mathcal{H}$, $\bar{\xi}_t(x) = 1$ if and only if there is a x' such that $\bar{\xi}_0(x') = 1$ and $(x', 0) \to (x, t)$.

Once we prove that $\bar{\xi}_t$ is well defined, one can check that the conditions (i), (ii) are satisfied. And to show that $\bar{\xi}_t$ is well defined, it suffices to prove that in our oriented random graph, with probability one (x, t) can be connected to only finitely many points $(x', 0)$ so one can determine explicitly whether any of them is occupied in the initial configuration. To be precise, for any $x \in \mathcal{H}$ and any $t, T \geq 0$, define subset

$$R_{t,T}(x) = \{y \in \mathcal{H},\ \text{s.t. } (y, T - t) \to (x, T)\} \tag{59}$$

be the set of all possible ancestors of $\bar{\xi}_T(x)$ at time $T - t$, and we will write $R_T(x)$ in short of $R_{T,T}(x)$. According to the definition, it is easy to see that

$$R_{t_1,T}(x) \subset R_{t_2,T}(x) \tag{60}$$

for all $0 \le t_1 \le t_2$ and $T \ge 0$, and that

$$R_{T_1}(x) \subset R_{T_2}(x) \tag{61}$$

for any $0 \le T_1 \le T_2$. Thus, to show that Definition 1 is self-consistent, we only need to prove that

Lemma 6.1 *With probability one we have $R_T(x)$ is finite for any $T > 0$ and $x \in \mathcal{H}$.*

Proof Let

$$\text{Rad}_{t,T}(x) = \sup_{y \in R_{t,T}(x)} |x - y|$$

be the radius of $R_{t,T}(x)$ and $\text{Rad}_T(x) = \text{Rad}_{T,T}(x)$. By (61), it is sufficient to prove that for each given $T > 0$ and $x \in \mathcal{H}$ we have

$$\text{Rad}_T(x) < \infty \tag{62}$$

almost surely. Then, we can take all rational numbers of T's and all $x \in \mathcal{H}$ which are both countable to get the lemma. Moreover, note that to show $P(\text{Rad}_T(x) < \infty) = 1$, it suffices to prove that for any $\epsilon > 0$,

$$P(\text{Rad}_T(x) < \infty) > 1 - \epsilon. \tag{63}$$

For any given T and $t \ge 0$ and $x = (x^{(1)}, x^{(2)}) \in \mathcal{H}$, note that $R_{t,T}(x)$ is the collection of all x' such that $(x', T - t)$ is connected to (x, T). And for $(x', T - t)$ and (x, T) to be connected, there must be a path between them, i.e., there must be a sequence of times $T - t \le t_1 < t_2 < \cdots < t_n \le T$ and $x' = x_0, x_1, x_2, \cdots, x_n = x$ which is a nearest neighbor sequence in $\mathcal{H}$ such that

$$N_{t_i}^{x_{i-1} \to x_i} = N_{t_i -}^{x_{i-1} \to x_i} + 1$$

if $t_i \ge 0$, or

$$\tilde{N}_{-t_i}^{x_{i-1} \to x_i} = \tilde{N}_{-t_i -}^{x_{i-1} \to x_i} + 1$$

if $t_i < 0$, for all $i = 1, 2, \cdots, n$. Thus it is easy to see that for any nearest neighbor path $x_0, x_1, x_2, \cdots, x_n = x$ in $\mathcal{H}$, it is open between $T - t$ and T in our oriented random graph only if there is at least one transition at each edge along the path during this time interval. Thus we have

$$\begin{aligned}
&P(\text{Rad}_{t,T}(x) \ge n) \\
&\quad \le P(\exists \text{ an open path in } [T - t, T] \text{ending at } x \text{with length } n) \\
&\quad \le \sum_{x_0, x_1, x_2, \cdots, x_n \in \mathcal{P}_{x,n}} \prod_{i=1}^{n} P\left(N_T^{x_{i-1} \to x_i} - N_{T-t}^{x_{i-1} \to x_i} \ge 1\right) \\
&\quad = \sum_{x_0, x_1, x_2, \cdots, x_n \in \mathcal{P}_{x,n}} \prod_{i=1}^{n} \left[1 - e^{-t\sqrt{x_{i-1}^{(2)}}}\right] \\
&\quad \le t^n \sum_{x_0, x_1, x_2, \cdots, x_n \in \mathcal{P}_{x,n}} \sqrt{\prod_{i=1}^{n} x_{i-1}^{(2)}}
\end{aligned} \tag{64}$$

where $\mathcal{P}_{x,n}$ is the collection of all nearest neighbor paths in $\mathcal{H}$ of length n ending at x, and $x_{i-1}^{(2)}$ stands for the $y-$coordinate of x_{i-1}. $\square$

Remark 8 Without loss of generality, the inequalities above is written for $0 \le t \le T$. By symmetry the same hold for $t > 0$ and $T < 0$. Note that even when $T > 0$ and $T - t < 0$, the total number of transitions of $N_s^{x_{i-1}\to x_i}$ in $s \in [0, T]$ plus the total number of transitions of $\tilde{N}_s^{x_{i-1}\to x_i}$ in $s \in [0, t - T]$ is again a Poisson random variable with intensity $t\sqrt{x_{i-1}^{(2)}}$. So (64) still holds.

Then note that $|\mathcal{P}_{x,n}| \le 4^n$ and that for each $x_0, x_1, x_2, \cdots, x_n = x \in \mathcal{P}_{x,n}$, we have

$$x_{n-i}^{(2)} \le x^{(2)} + i,\ \ i = 0, 1, 2, \cdots, n.$$

Thus, we have

$$\sqrt{\prod_{i=1}^{n} x_{i-1}^{(2)}} \le \sqrt{\prod_{i=1}^{n}(x^{(2)} + i)}$$

which implies that

$$P(\mathrm{Rad}_{t,T}(x) \ge n) \le (4t)^n \sqrt{\prod_{i=1}^{n}(x^{(2)} + i)}. \tag{65}$$

Now for each $\gamma \in (0, 1/2)$, define

$$M_\gamma = \sum_{k=0}^{\infty} k^{2/(1-\gamma)} 2^{-k^{\gamma/(1-\gamma)}} < \infty.$$

Now for any $\epsilon > 0$ let

$$N_1 = \left\lfloor \frac{4\left(x^{(2)}\right)^{\gamma}}{1-\gamma} \right\rfloor$$

and

$$\delta_1 = t_1 = \frac{\epsilon}{64 M_\gamma \sqrt{x^{(2)} + N_1}}.$$

By (65), we have

$$P(\mathrm{Rad}_{t_1,T}(x) \ge N_1) \le \frac{\epsilon}{M_\gamma} 2^{-N_1} \le \frac{\epsilon}{M_\gamma} k_1^{2/(1-\gamma)} 2^{-k_1^{\gamma/(1-\gamma)}} \tag{66}$$

where

$$k_1 = \lfloor \left(x^{(2)}\right)^{1-\gamma} \rfloor. \tag{67}$$

The last inequality in (66) follows from

$$N_1 = \left\lfloor \frac{4\left(x^{(2)}\right)^{\gamma}}{1-\gamma} \right\rfloor \ge \left(x^{(2)}\right)^{\gamma} \ge k_1^{\gamma/(1-\gamma)}.$$

Then on the event

$$E_1 = A_1 = \{\mathrm{Rad}_{t_1,T}(x) < N_1\},$$

we define

$$
\begin{aligned}
x^{(2),2} &= x^{(2)} + N_1, \\
N_2 &= \left\lfloor \frac{4\left(x^{(2),2}\right)^{\gamma}}{1-\gamma} \right\rfloor, \\
\delta_2 &= \frac{\epsilon}{64 M_{\gamma} \sqrt{x^{(2),2} + N_2}},
\end{aligned}
$$

and

$$
t_2 = t_1 + \delta_2.
$$

Then define event

$$
A_2 = \bigcap_{y \in B(x, x^{(2),2} - x^{(2)} - 1)} \{\mathrm{Rad}_{\delta_2, T-t_1}(y) < N_2\}.
$$

One can first see that by the same calculation as in (66)

$$
\begin{aligned}
P(A_2) &\geq 1 - \sum_{y \in B(x, x^{(2),2} - x^{(2)} - 1)} P(\mathrm{Rad}_{\delta_2, T-t_1}(y) \geq N_2) \\
&\geq 1 - \left(x^{(2),2}\right)^2 \frac{\epsilon}{16 M_{\gamma}} 2^{-k_2^{\gamma/(1-\gamma)}}
\end{aligned} \tag{68}
$$

where

$$
k_2 = \lfloor \left(x^{(2),2}\right)^{1-\gamma} \rfloor.
$$

Moreover, we have

$$
\left(x^{(2),2}\right)^2 = \left[(x^{(2),2})^{1-\gamma} \right]^{2/(1-\gamma)}
$$

while

$$
(x^{(2),2})^{1-\gamma} \leq 2k_2.
$$

Thus

$$
\begin{aligned}
P(A_2) &\geq 1 - \left(2k_2\right)^{2/(1-\gamma)} \frac{\epsilon}{16 M_{\gamma}} 2^{-k_2^{\gamma/(1-\gamma)}} \\
&\geq 1 - \frac{\epsilon}{M_{\gamma}} k_2^{2/(1-\gamma)} 2^{-k_2^{\gamma/(1-\gamma)}}.
\end{aligned} \tag{69}
$$

Then note that for any $x \geq 1$, we have by calculus

$$
\left(x^{1-\gamma} + 4\right)^{1/(1-\gamma)} > x + \frac{4}{1-\gamma} x^{\gamma}
$$

while

$$
\begin{aligned}
\left(x^{1-\gamma} + 1\right)^{1/(1-\gamma)} &\leq x + \frac{1}{1-\gamma} \left(x^{1-\gamma} + 1\right)^{\gamma/(1-\gamma)} \\
&\leq x + \frac{1}{1-\gamma} \left(2x^{1-\gamma}\right)^{\gamma/(1-\gamma)} \\
&< x + \frac{2}{1-\gamma} x^{\gamma} \\
&< x + \left\lfloor \frac{4x^{\gamma}}{1-\gamma} \right\rfloor.
\end{aligned}
$$

We have that

$$\left(x^{(2),2}\right)^{1-\gamma} = \left(x^{(2)} + N_1\right)^{1-\gamma} \in \left(\left(x^{(2)}\right)^{1-\gamma} + 1, \left(x^{(2)}\right)^{1-\gamma} + 4\right) \tag{70}$$

and that

$$k_2 = \lfloor \left(x^{(2),2}\right)^{1-\gamma} \rfloor \in [k_1 + 1, k_1 + 4).$$

Using exactly the same argument on

$$x^{(2),3} = x^{(2),2} + N_2,$$

and

$$k_3 = \lfloor \left(x^{(2),3}\right)^{1-\gamma} \rfloor,$$

we have

$$k_3 \in [k_2 + 1, k_2 + 4).$$

Then we note that the event A_1 depends only on the transitions within $B(x, N_1) \times [T - t_1, T]$, while the event A_2 depends only on the transitions within $B(x, N_1 + N_2) \times [T - t_2, T - t_1]$. By the independence of increment in a Poisson processes, we have that A_1 is independent of A_2, and thus for $E_2 = A_1 \cap A_2$,

$$P(E_2) = P(A_1)P(A_2) \geq \left(1 - \frac{\epsilon}{M_\gamma} k_1^{2/(1-\gamma)} 2^{-k_1^{\gamma/(1-\gamma)}}\right) \cdot \left(1 - \frac{\epsilon}{M_\gamma} k_2^{2/(1-\gamma)} 2^{-k_2^{\gamma/(1-\gamma)}}\right).$$

Finally, recalling the definition of $\mathrm{Rad}_{t_1,T}$, one can immediately have on E_2

$$\mathrm{Rad}_{t_2,T}(x) < x^{(2),3} - x^{(2)} < \infty.$$

Repeating such iteration, i.e., for all $n \geq 2$ let

$$\begin{aligned}
x^{(2),n} &= x^{(2),n-1} + N_{n-1}, \\
N_n &= \left\lfloor \frac{4\left(x^{(2),n}\right)^{\gamma}}{1-\gamma} \right\rfloor, \\
\delta_n &= \frac{\epsilon}{64 M_\gamma \sqrt{x^{(2),n} + N_n}}, \\
t_n &= t_{n-1} + \delta_n, \\
A_n &= \bigcap_{y \in B(x, x^{(2),n} - x^{(2)} - 1)} \{\mathrm{Rad}_{\delta_n, T - t_{n-1}}(y) < N_n\},
\end{aligned}$$

and

$$E_n = E_{n-1} \cap A_n.$$

Consider

$$E_\infty = \bigcap_{n=1}^{\infty} A_n.$$

Under E_∞ we have for any $n \geq 1$,

$$\mathrm{Rad}_{t_n,T}(x) < x^{(2),n+1} - x^{(2)} < \infty. \tag{71}$$

At the same time

$$t_n = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{\epsilon}{64 M_\gamma \sqrt{x^{(2),i} + N_i}} = \frac{\epsilon}{64 M_\gamma} \sum_{i=1}^{n} \frac{1}{\sqrt{x^{(2),i+1}}}. \tag{72}$$

Moreover, by (70) we have for each i

$$\left(x^{(2),i}\right)^{1-\gamma} = \left(x^{(2),i-1} + N_{i-1}\right)^{1-\gamma} \in \left(\left(x^{(2),i-1}\right)^{1-\gamma} + 1, \left(x^{(2),i-1}\right)^{1-\gamma} + 4\right) \tag{73}$$

which together implies that

$$\left(x^{(2),i}\right)^{1-\gamma} \le \left(x^{(2)}\right)^{1-\gamma} + 4i. \tag{74}$$

Combining (72)–(74) we have

$$t_n \ge \frac{\epsilon}{64 M_\gamma} \sum_{i=1}^{n} \left[\left(x^{(2)}\right)^{1-\gamma} + 4i\right]^{-1/(2-2\gamma)}. \tag{75}$$

Recalling that $\gamma \in (0, 1/2)$, $1/(2-2\gamma) < 1$, which implies that the series in (75) is divergent. So for any $T \ge 0$ there is a $n(T, \gamma, \epsilon) < \infty$ such that for all $n \ge n(T, \gamma, \epsilon)$, $t_n \ge T$,

$$\mathrm{Rad}_T(x) \subset \mathrm{Rad}_{t_n,T}(x).$$

Thus we have on the event E_∞, $\mathrm{Rad}_T(x) < \infty$. On the other hand, Noting that by the independence increment of Poisson processes, we have $A_1, A_2, \cdots$ gives a sequence of independent events, and that according to our iteration for each i

$$P(A_i) \ge 1 - \frac{\epsilon}{M_\gamma} k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}} > 1 - \epsilon \tag{76}$$

with

$$k_i \in [k_{i-1} + 1, k_{i-1} + 4). \tag{77}$$

Thus for sufficiently small ϵ such that for all $x \in (0, \epsilon)$, $\log(1-x) \ge -2x$ and any $n \ge 1$, we have

$$P(E_n) - 1 \ge \log\left(P(E_n)\right) = \log\left(\prod_{i=1}^{n} P(A_i)\right) \ge 2 \sum_{i=1}^{n} [P(A_i) - 1].$$

By (76),

$$P(E_n) - 1 \ge -\frac{2\epsilon}{M_\gamma} \sum_{i=1}^{n} k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}}.$$

Then noting that by (77) $k_i \ge k_{i-1} + 1$ and the fact that all k_i's are integers by definition, we have

$$\sum_{i=1}^{n} k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}} \le M_\gamma$$

and thus

$$P(E_n) \ge 1 - 2\epsilon. \tag{78}$$

Note that the right hand side of (78) is independent of n. We have $P(E_\infty) \geq 1 - 2\epsilon$. And since ϵ is arbitrarily chosen, $P(\text{Rad}_T(x) < \infty) = 1$ which completes the proof of Lemma 6.1. □

Thus the proof of Proposition 3 is complete. □

With the proof of Proposition 3, one can easily apply the technique of Poisson thinning to define the following particle system where time is slowed down in-homogeneously and define a dominating process for the future stationary DLA model i.e., we can consider the slower interacting particle system $\tilde{\xi}_t$ defined on $\{0, 1\}^{\mathcal{H}}$ with transition rates as follows:

(i)' For each occupied site $x = (x_1, x_2) \in \mathcal{H}$ at time $t \geq 0$, it will try to give birth to each of its nearest neighbors at a Poisson rate of $\frac{\sqrt{x_2}}{\sqrt{t+1}}$.

(ii)' When x attempts to give birth to its nearest neighbors y already occupied, the birth is suppressed.

For $\tilde{\xi}_t$ we have

Corollary 1 *The interacting particle system $\tilde{\xi}_t \in \{0, 1\}^{\mathcal{H}}$ satisfying (i)' and (ii)' is well defined.*

Proof We construct $\tilde{\xi}_t$ with the same families of Poisson processes. Recall that in the proof of Proposition 3, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathcal{H}$ with $|x - y| = 1$ and $e_{x\to y}$ which is the the oriented edge from x to y,

$$\left\{N_t^{x\to y},\ x, y \in \mathcal{H},\ |x - y| = 1\right\}$$

is a family of independent Poisson process with intensity of $N_t^{x\to y}$ equals to $\sqrt{x_2}$. Moreover, for each $e_{x\to y}$, we define $\{U_n^{x\to y}\}_{n=1}^{\infty}$ be a i.i.d. sequence of random variables uniform on $[0, 1]$. And we make the sequences for different edges independent of each other and also independent of the Poisson processes previously defined.

Now consider the space-time combination, $\mathcal{H} \times [0, \infty)$. From each $x \in \mathcal{H}$, we draw a vertical infinite half line to represent the time line at this site. Then for each $e_{x\to y}$, at any time t such that $N_t^{x\to y} = n = N_{t-}^{x\to y} + 1$, we draw an oriented arrow from (x, t) to (y, t) if $U_n^{x\to y} < 1/\sqrt{t+1}$. Thus we have another oriented random graph in the space-time combination which is a subset of the one we have for $\bar{\xi}_t$. By Proposition 3 we can see the following particle system is well defined. □

Definition 2 For any $\tilde{\xi}_0 \in \{0, 1\}^{\mathcal{H}}$, we define $\tilde{\xi}_t$ such that for each $t \geq 0$ and $x \in \mathcal{H}$, $\tilde{\xi}_t(x) = 1$ if and only if there is a x' such that $\bar{\xi}_0(x') = 1$ and $(x', 0)$ is connected to (x, t) in the new smaller oriented random graph.

□

6.2 Proof of Theorem 5

By Theorem 1 we have seen that for any B, $x \in B \setminus L_0$ and any $\vec{e} = x \to y$ with $\|x - y\| = 1$,

$$H_B(\vec{e}) \leq H_B(x) \leq C\sqrt{x_2}$$

for some $C > 1$. Moreover, by (18), if $x_2 = 0$,

$$H_B(\vec{e}) \leq H_B(x) \leq 1.$$

We construct our DLA model on $\mathcal{H}$ as follows: First, recall that

$$\left\{N_t^{x\to y},\ x, y \in \mathcal{H},\ \|x-y\|=1\right\}$$

is a family of independent Poisson processes, such that the intensity of $N_t^{x\to y}$ equals to $\sqrt{x_2}$ and that $\{U_n^{x\to y}\}_{n=1}^{\infty}$ is an i.i.d. sequence of random variables uniform on [0, 1] independent of the Poisson processes. Let $\bar{A}_0 = \{(0,0)\}$, and for any $t > 0$,

- If there is an $\vec{e} = x \to y$ such that $x \in \bar{A}_{t-}$ and $y \notin \bar{A}_{t-}$, where $N_t^{x\to y} = n$ and $N_{t-}^{x\to y} = n-1$, we let $\bar{A}_t = \bar{A}_{t-} \cup \{y\}$ if

$$U_n^{x\to y} \le \frac{H_{\bar{A}_{t-}}(\vec{e})}{C\sqrt{x_2}}.$$

- Otherwise, $\bar{A}_t = \bar{A}_{t-}$.

To prove Theorem 5, we first need to show

Lemma 6.2 *For each time t, $\bar{A}_t$ above is with probability 1 well defined and finite.*

Proof To prove Lemma 6.2, we construct an event with probability one such that $\bar{A}_t$ is well defined and finite on this event. For any $x \in \mathcal{H}$ and any $0 \le t < T$, define subset

$$I_{t,T}(x) = \{y \in \mathcal{H},\ \text{s.t. } (x,t) \to (y,T)\} \tag{79}$$

and let

$$\mathcal{I}_{t,T}(x) = \sup_{y\in I_{t,T}(x)} \|x-y\|.$$

Following exactly the same argument as in Lemma 6.1, we have with probability one

$$\mathcal{I}_{0,Ct}(0) < \infty.$$

Under $\{\mathcal{I}_{0,Ct}(0) < \infty\}$ one can easily put all of the finite Poisson transitions within the space time box $\mathcal{I}_{0,t}(0) \times [0,t]$ in order and construct $\bar{A}_t$ explicitly over finite steps. Moreover, by definition we can always have $\bar{A}_t \subset I_{0,t}(0)$ thus $\bar{A}_t$ is finite. □

Let $A_t = \bar{A}_{Ct}$, then it is easy to check A_t has the same dynamic as in Theorem 5 while being almost surely well defined and finite at the same time. Now, to finish the proof of Theorem 5, one may again follow the argument as in Lemma 6.1. □

Remark 9 The proof of Theorem 6 actually contains all that is needed in the proof of Theorem 5 (and more). Thus we will not present the details of basically the same thing for a third time.

6.3 Proof of Theorem 6

To prove (14), since we have $A_t = \bar{A}_{Ct} \subset I_{0,Ct}(0)$, it is sufficient to show for any $t \ge 0$ and integer $m \ge 1$

$$E\left[\mathcal{I}_{0,t}(0)^m\right] < \infty. \tag{80}$$

The proof here is similar to the one for Lemma 6.1. However, since some more delicate estimates on the upper bounds of probabilities are needed, we still provide a detailed proof for the completeness of this paper.

Recall (64), we have for any t and n,

$$
\begin{aligned}
&P(\mathcal{I}_{0,t}(0) \geq n)\\
&\quad \leq P(\exists \text{ an open path in } [0,t] \text{starting at } 0 \text{with length } n)\\
&\quad \leq \sum_{x_0,x_1,x_2,\cdots,x_n \in \mathcal{P}_{n,0}} \prod_{i=1}^{n} P(N_T^{x_{i-1}\to x_i} - N_{T-t}^{x_{i-1}\to x_i} \geq 1)\\
&\quad = (1-e^{-t}) \sum_{x_0,x_1,x_2,\cdots,x_n \in \mathcal{P}_{n,0}} \prod_{i=1}^{n-1} \left[1 - e^{-t\sqrt{x_i^{(2)}}}\right]\\
&\quad \leq t^n \sum_{x_0,x_1,x_2,\cdots,x_n \in \mathcal{P}_{n,0}} \sqrt{\prod_{i=1}^{n} x_i^{(2)}}.
\end{aligned} \tag{81}
$$

Here we use $\mathcal{P}_{n,0}$ to denote the collection of all nearest neighbor paths starting at 0 with length n. Then note that $|\mathcal{P}_{n,0}| \leq 4^n$ and that for each $0 = x_0, x_1, x_2, \cdots, x_n \in \mathcal{P}_{n,0}$, we have

$$
x_i^{(2)} \leq i,\ \ i = 0, 1, 2, \cdots, n.
$$

Thus, we have

$$
\sqrt{\prod_{i=1}^{n} x_i^{(2)}} \leq \sqrt{n!}
$$

which implies that

$$
P(\mathcal{I}_{0,t}(0) \geq n) \leq (4t)^n \sqrt{n!}. \tag{82}
$$

Now for each $\gamma \in (0, 1/2)$, define

$$
M_\gamma = \sum_{k=0}^{\infty} k^{2/(1-\gamma)} 2^{-k^{\gamma/(1-\gamma)}} < \infty.
$$

Now for any $\epsilon > 0$ let

$$
N_1 = \left\lfloor \frac{4m}{1-\gamma} \right\rfloor \geq 4m
$$

and

$$
\delta_1 = t_1 = \frac{\epsilon}{64 M_\gamma \sqrt{N_1}}.
$$

By (82), we have

$$
\begin{aligned}
P(\mathcal{I}_{0,t_1}(0) \geq N_1) &\leq \left(\frac{\epsilon}{16 M_\gamma \sqrt{N_1}}\right)^{N_1} \sqrt{N_1!}\\
&\leq \frac{\epsilon^{4m}}{M_\gamma} 16^{-N_1} \leq \frac{\epsilon^{4m}}{M_\gamma} k_1^{2/(1-\gamma)} 2^{-k_1^{\gamma/(1-\gamma)}}
\end{aligned} \tag{83}
$$

where $k_1 = 1$. Then on the event

$$
E_1 = A_1 = \{\mathcal{I}_{0,t_1}(0) < N_1\},
$$

we define

$$x^{(2),2} = 1 + N_1,$$
$$N_2 = \left\lfloor \frac{4m\left(x^{(2),2}\right)^{\gamma}}{1-\gamma} \right\rfloor \geq 4m,$$
$$\delta_2 = \frac{\epsilon}{64 M_\gamma \sqrt{x^{(2),2} + N_2}},$$

and

$$t_2 = t_1 + \delta_2.$$

Then define event

$$A_2 = \bigcap_{y \in B(0, x^{(2),2}-1)} \{\mathcal{I}_{t_1,t_2}(y) < N_2\}.$$

One can first see that by the same calculation as in (66)

$$\begin{aligned} P(A_2) &\geq 1 - \sum_{y \in B(0,x^{(2),2}-1)} P(\mathcal{I}_{t_1,t_2}(y) \geq N_2) \\ &\geq 1 - 4\left(x^{(2),2}\right)^2 \left(\frac{\epsilon}{16 M_\gamma \sqrt{x^{(2),2}+N_2}}\right)^{N_2} \sqrt{\prod_{j=x^{(2),2}-1}^{x^{(2),2}+N_2-2} j} \\ &\geq 1 - \left(x^{(2),2}\right)^2 \frac{\epsilon^{4m}}{16 M_\gamma} 2^{-k_2^{\gamma/(1-\gamma)}} \end{aligned} \tag{84}$$

where

$$k_2 = \lfloor \left(x^{(2),2}\right)^{1-\gamma} \rfloor.$$

The last inequality in (84) results from

$$N_2 = \left\lfloor \frac{4m\left(x^{(2),2}\right)^{\gamma}}{1-\gamma} \right\rfloor \geq \left(x^{(2),2}\right)^{\gamma} \geq k_1^{\gamma/(1-\gamma)}.$$

Moreover, we have

$$\left(x^{(2),2}\right)^2 = \left[\left(x^{(2),2}\right)^{1-\gamma}\right]^{2/(1-\gamma)}$$

while

$$\left(x^{(2),2}\right)^{1-\gamma} \leq 2k_2.$$

Thus

$$\begin{aligned} P(A_2) &\geq 1 - \left(2k_2\right)^{2/(1-\gamma)} \frac{\epsilon^{4m}}{16 M_\gamma} 2^{-k_2^{\gamma/(1-\gamma)}} \\ &\geq 1 - \frac{\epsilon^{4m}}{M_\gamma} k_2^{2/(1-\gamma)} 2^{-k_2^{\gamma/(1-\gamma)}}. \end{aligned} \tag{85}$$

Then note that for any $x \geq 1$, we have by calculus

$$\left(x^{1-\gamma} + 4m\right)^{1/(1-\gamma)} > x + \frac{4m}{1-\gamma} x^{\gamma}$$

while

$$\begin{aligned}\left(x^{1-\gamma}+1\right)^{1/(1-\gamma)} &\le x+\frac{1}{1-\gamma}\left(x^{1-\gamma}+1\right)^{\gamma/(1-\gamma)}\\ &\le x+\frac{1}{1-\gamma}\left(2x^{1-\gamma}\right)^{\gamma/(1-\gamma)}\\ &< x+\frac{2}{1-\gamma}x^{\gamma}\\ &< x+\left\lfloor\frac{4mx^{\gamma}}{1-\gamma}\right\rfloor.\end{aligned}$$

We have that

$$\left(x^{(2),2}\right)^{1-\gamma}=\left(1+N_1\right)^{1-\gamma}=\left(1+\left\lfloor\frac{4m\cdot 1^{\gamma}}{1-\gamma}\right\rfloor\right)^{1-\gamma}\in\left(2,1+4m\right) \tag{86}$$

and that

$$k_2=\lfloor\left(x^{(2),2}\right)^{1-\gamma}\rfloor\in(k_1+1,k_1+4m)$$

since $k_1=1$. Then using exactly the same argument on

$$x^{(2),3}=x^{(2),2}+N_2,$$

and

$$k_3=\lfloor\left(x^{(2),3}\right)^{1-\gamma}\rfloor,$$

we have

$$\begin{aligned}\left(x^{(2),3}\right)^{1-\gamma}=\left(x^{(2),2}+N_2\right)^{1-\gamma}&=\left(x^{(2),2}+\left\lfloor\frac{4m\left(x^{(2),2}\right)^{\gamma}}{1-\gamma}\right\rfloor\right)^{1-\gamma}\\ &\in\left((x^{(2),2})^{1-\gamma}+1,(x^{(2),2})^{1-\gamma}+4m\right)\end{aligned} \tag{87}$$

and thus

$$k_3\in(k_2+1,k_2+4m).$$

Then we note that the event A_1 depends only on the transitions within $B(0,N_1)\times[0,t_1]$, while event A_2 depends only on the transitions within $B(0,N_1+N_2)\times[t_1,t_2]$. By the independence of Poisson process increments, we have that A_1 is independent of A_2, and thus for $E_2=A_1\cap A_2$,

$$P(E_2)=P(A_1)P(A_2)\ge\left(1-\frac{\epsilon^{4m}}{M_\gamma}k_1^{2/(1-\gamma)}2^{-k_1^{\gamma/(1-\gamma)}}\right)\cdot\left(1-\frac{\epsilon^{4m}}{M_\gamma}k_2^{2/(1-\gamma)}2^{-k_2^{\gamma/(1-\gamma)}}\right).$$

Finally, recalling the definition of $\mathcal{I}_{0,t}$, one can immediately have on E_2

$$\mathcal{I}_{0,t_2}(0)<x^{(2),3}<\infty.$$

Repeat the iteration above, i.e., for all $n\ge 3$ let

$$\begin{aligned}x^{(2),n}&=x^{(2),n-1}+N_{n-1},\\ N_n&=\left\lfloor\frac{4m\left(x^{(2),n}\right)^{\gamma}}{1-\gamma}\right\rfloor,\end{aligned}$$

$$\begin{aligned} \delta_n &= \frac{\epsilon}{64M_\gamma\sqrt{x^{(2),n}+N_n}}, \\ t_n &= t_{n-1}+\delta_n, \\ A_n &= \bigcap_{y\in B(0,x^{(2),n}-1)} \{\mathcal{I}_{t_{n-1},t_n}(y) < N_n\}, \end{aligned}$$

and

$$E_n = E_{n-1} \cap A_n.$$

Consider

$$E_\infty^\epsilon = \bigcap_{n=1}^\infty A_n.$$

Under E_∞^ϵ we have for any $n \geq 1$,

$$\mathcal{I}_{0,t_n}(x) < x^{(2),n+1} < \infty. \tag{88}$$

At the same time

$$t_n = \sum_{i=1}^n \delta_i = \sum_{i=1}^n \frac{\epsilon}{64M_\gamma\sqrt{x^{(2),i}+N_i}} = \frac{\epsilon}{64M_\gamma}\sum_{i=1}^n \frac{1}{\sqrt{x^{(2),i+1}}}. \tag{89}$$

Moreover, by (87) we have for each i

$$\left(x^{(2),i}\right)^{1-\gamma} = \left(x^{(2),i-1}+N_{i-1}\right)^{1-\gamma} \in \left(\left(x^{(2),i-1}\right)^{1-\gamma}+1, \left(x^{(2),i-1}\right)^{1-\gamma}+4m\right) \tag{90}$$

which together implies that

$$\left(x^{(2),i}\right)^{1-\gamma} \leq 4im. \tag{91}$$

Combining (89)–(91) we have

$$t_n \geq \frac{\epsilon}{64M_\gamma}\sum_{i=1}^n (4im)^{-1/(2-2\gamma)}. \tag{92}$$

Recalling that $\gamma \in (0, 1/2)$, $1/(2-2\gamma) < 1$, which implies that the series in (92) is divergent. So for any $t \geq 0$ there is an $n_0 < \infty$ such that for all $n \geq n_0$, $t_n \geq t$, and that $t_{n_0-1} < t$.

$$\mathcal{I}_{0,t}(0) \leq \mathcal{I}_{0,t_{n_0}}(0).$$

And by (88) and (91), on E_∞^ϵ,

$$\mathcal{I}_{0,t_{n_0}}(0) < x^{(2),n+1} \leq [4m(n_0+1)]^{1/(1-\gamma)}.$$

Thus we have on the event E_∞^ϵ,

$$\mathcal{I}_{0,t}(0) \leq [4m(n_0+1)]^{1/(1-\gamma)} \leq (8m)^{1/(1-\gamma)} \cdot n_0^{1/(1-\gamma)} \tag{93}$$

On the other hand,

$$t \geq t_{n_0-1} \geq \frac{\epsilon}{64M_\gamma}\sum_{i=1}^{n_0-1} (4m\cdot i)^{-1/(2-2\gamma)} \geq \frac{c\epsilon}{64M_\gamma(4m)^{1/(2-2\gamma)}} n_0^{(1-2\gamma)/(2-2\gamma)}. \tag{94}$$

Combining (93) and (94), we have on the event E_∞^ϵ there is a constant $C_{m,\gamma}$ depending on m and γ but independent of ϵ such that

$$\mathcal{I}_{0,t}(0) \le C_{m,\gamma}\epsilon^{-2/(1-\gamma)}t^{2/(1-2\gamma)}. \tag{95}$$

Note that by the independence of Poisson processes increments, we have that $A_1, A_2, \cdots$ gives a sequence of independent events. And according to (84) and the construction in our iteration, we have for each i

$$P(A_i) \ge 1 - \frac{\epsilon^{4m}}{M_\gamma} k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}} > 1-\epsilon \tag{96}$$

with

$$k_i = \lfloor \left(x^{(2),i}\right)^{1-\gamma} \rfloor \in [k_{i-1}+1, k_{i-1}+4m). \tag{97}$$

Thus for sufficiently small ϵ such that for all $x \in (0,\epsilon)$, $\log(1-x) \ge -2x$ and any $n \ge 1$, we have

$$P(E_n) - 1 \ge \log\left(P(E_n)\right) = \log\left(\prod_{i=1}^n P(A_i)\right) \ge 2\sum_{i=1}^n [P(A_i)-1].$$

By (96),

$$P(E_n) - 1 \ge -\frac{2\epsilon^{4m}}{M_\gamma}\sum_{i=1}^n k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}}.$$

Then noting that by (97) $k_i \ge k_{i-1}+1$ and the fact that all k_i's are integers by definition, we have

$$\sum_{i=1}^n k_i^{2/(1-\gamma)} 2^{-k_i^{\gamma/(1-\gamma)}} \le M_\gamma$$

and thus

$$P(E_n) \ge 1 - 2\epsilon^{4m}. \tag{98}$$

Note that the right hand side of (98) is independent of n. We have $P(E_\infty^\epsilon) \ge 1-2\epsilon^{4m}$. Now let

$$\epsilon_j = \left(\frac{1}{j}\right)^{\frac{1-\gamma}{2m}}.$$

Then we have for each j sufficiently large,

$$\begin{aligned} P\left(\frac{\mathcal{I}_{0,t}(0)^m}{(C_{m,\gamma})^m t^{2m/(1-2\gamma)}} > \epsilon_j^{-2m/(1-\gamma)}\right) &= P\left(\mathcal{I}_{0,t}(0) > C_{m,\gamma}\epsilon_j^{-2/(1-\gamma)} t^{2/(1-2\gamma)}\right) \\ &\le 1 - P(E_\infty^{\epsilon_j}) \le 2\epsilon_j^{4m} \end{aligned} \tag{99}$$

and thus

$$P\left(\frac{\mathcal{I}_{0,t}(0)^m}{(C_{m,\gamma})^m t^{2m/(1-2\gamma)}} > j\right) \le 2\left(\frac{1}{j}\right)^{2(1-\gamma)}. \tag{100}$$

Noting that $\gamma < 1/2$ and thus $2(1-\gamma) > 1$,

$$\sum_{j=1}^{\infty} P\left(\frac{\mathcal{I}_{0,t}(0)^m}{(C_{m,\gamma})^m t^{2m/(1-2\gamma)}} > j\right) < \infty$$

which implies that $E[\mathcal{I}_{0,t}(0)^m] < \infty$. □

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References

1. Antunović, T., Procaccia, E.B.: Stationary eden model on cayley graphs. Ann. Appl. Probab. **27**(1), 517–549 (2017)
2. Benjamini, I., Yadin, A.: Upper bounds on the growth rate of diffusion limited aggregation. arXiv preprint arXiv:1705.06095 (2017)
3. Berger, N., Kagan, J .J., Procaccia, E .B.: Stretched IDLA. ALEA Lat. Am. J. Probab. Math. Stat. **11**(1), 471–481 (2014)
4. Durrett, R.: Ten lectures on particle systems. In: Lectures on probability theory (Saint-Flour, 1993), volume 1608 of Lecture Notes in Mathematics, pp. 97–201. Springer, Berlin (1995)
5. Dvoretzky, A., Erdös, P.: Some problems on random walk in space. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 353–367. University of California Press, Berkeley and Los Angeles (1951)
6. Erdös, P., Taylor, S .J.: Some problems concerning the structure of random walk paths. Acta Math. Acad. Sci. Hung. **11**, 137–162 (1960). (unbound insert)
7. Kesten, H.: Hitting probabilities of random walks on Z^d. Stoch. Process. Appl. **25**(2), 165–184 (1987)
8. Kesten, H.: How long are the arms in DLA? J. Phys. A **20**(1), L29–L33 (1987)
9. Kesten, H.: Upper bounds for the growth rate of DLA. Physics A **168**(1), 529–535 (1990)
10. Lawler, G.F., Limic, V.: The beurling estimate for a class of random walks. Electron. J. Probab. **9**(27), 846–861 (2004)
11. Meakin, P.: Diffusion-controlled deposition on fibers and surfaces. Phys. Rev. A **27**(5), 2616 (1983)
12. Procaccia, E. B., Zhang, Y.: On sets of zero stationary harmonic measure. arXiv preprint arXiv:1711.01013 (2017)
13. Procaccia, E. B., Zhang, Y.: Two dimensional stationary dla. in preparation (2018)
14. Procaccia, E. B., Ye, J., Zhang, Y.: Stationary harmonic measure as the scaling limit of truncated harmonic measure. arXiv preprint arXiv:1811.04793 (2018)
15. Voss, R. F.: Fractals in nature: from characterization to simulation. In: The science of fractal images, pp. 21–70. Springer (1988)
16. Witten, T.A., Sander, L.M.: Diffusion-limited aggregation. Phys. Rev. B (3) **27**(9), 5686–5697 (1983)