Higgs bundles – Recent applications

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Introduction

This note is dedicated to introducing Higgs bundles and the Hitchin fibration, with a view towards their appearance within different branches of mathematics and physics, focusing in particular on the role played by the *integrable system* structure carried by their moduli spaces. On a compact Riemann surface Σ of genus $g \geq 2$, Higgs bundles are pairs (E, Φ) where

- E is a holomorphic vector bundle on Σ;
- the Higgs field $\Phi : E \to E \otimes K$ is a holomorphic map, for $K := T^*\Sigma$.

Since their origin in the late 80's in work of Hitchin and Simpson, Higgs bundles manifest as fundamental objects which are ubiquitous in contemporary mathematics, and closely related to theoretical physics. For $G_{\mathbb{C}}$ a complex semisimple Lie group, the *Dolbeault moduli space* of $G_{\mathbb{C}}$-Higgs bundles $\mathcal{M}_{G_{\mathbb{C}}}$ has a hyperkähler structure, and via different complex structures it can be seen as different moduli spaces:

- Via the non-abelian Hodge correspondence developed by Corlette, Donaldson, Simpson and Hitchin, and in the spirit of Uhlenbeck-Yau's work for compact groups, the moduli space is analytically isomorphic as a real manifold to the *De Rahm moduli space* $\mathcal{M}_{dR}$ of flat connections on a smooth complex bundle.
- Via the Riemann-Hilbert correspondence there is a complex analytic isomorphism between the de Rham space and the *Betti moduli space* $\mathcal{M}_B$ of surface group representations $\pi_1(\Sigma) \to G_{\mathbb{C}}$.

Some prominent examples where these moduli spaces appear in mathematics and physics are:

- Through the Hitchin fibration, $\mathcal{M}_{G_{\mathbb{C}}}$ gives examples of hyperkähler manifolds which are *integrable systems*, leading to remarkable applications in physics which we shall discuss later on.
- Building on the work of Hausel and Thaddeus relating Higgs bundles to *Langlands duality*, Donagi and Pantev presented $\mathcal{M}_{G_{\mathbb{C}}}$ as a fundamental example of mirror symmetry.
- Within the work of Kapustin and Witten, Higgs bundles were used to obtain a physical derivation of the *geometric Langlands correspondence* through mirror symmetry. Soon after, Ngô found Higgs bundles to be key ingredients when proving the Fundamental Lemma of the Langlands program, which led him to the Fields Medal a decade ago.

Higgs bundles, and the corresponding Hitchin integrable systems, have been an increasingly vibrant area, and thus there are several expository articles some of which we shall refer to: from the Notices' article "What is a Higgs bundle?" [BGPG07], to several graduate notes on Higgs bundles (e.g., the author's recent [Sch19]), to more advance reviews such as Ngô's 2010 ICM Proceedings article [Châ10]. Hoping to avoid repeating material nicely covered in other reviews, whilst still attempting to inspire the reader to learn more about the subject, we shall take this opportunity to focus on some of the recent work done by leading young members of the community[1].

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[1] As in other similar reviews, the number of references is limited to twenty, and thus we shall refer the reader mostly to survey articles where precise references can be found.

Higgs bundles

Higgs bundles arise as solutions to self-dual Yang-Mills equations, a non-abelian generalization of Maxwell's equations which recurs through much of modern physics. Solutions to Yang-Mills self-duality equations in Euclidean 4d space are called instantons, and when these equations are reduced to Euclidean 3d space by imposing translational invariance in one dimension, one obtains monopoles as solutions. Higgs bundles were introduced by Hitchin in [Hit87a] as solutions of the so-called *Hitchin equations*, the 2-dimensional reduction of the Yang-Mills self-duality equations, given by

$$F_A + [\Phi, \Phi^*] = 0, \qquad \overline{\partial}_A \Phi = 0, \tag{1}$$

where F_A is the curvature of a unitary connection $\nabla_A = \partial_A + \overline{\partial}_A$ associated to a Dolbeault operator $\overline{\partial}_A$ on a holomorphic principal $G_{\mathbb{C}}$ bundle P. The equations give a flat connection

$$\nabla_A + \Phi + \Phi^*, \tag{2}$$

and express the harmonicity condition for a metric in the resulting flat bundle. Concretely, principal $G_{\mathbb{C}}$-Higgs bundles are pairs (P, Φ), where

- P is a principal $G_{\mathbb{C}}$-bundle, and
- Φ a holomorphic section of $\mathrm{ad}(P) \otimes K$.

We shall refer to *classical Higgs bundles* as those described in the Introduction, and consider $G_{\mathbb{C}}$-Higgs bundles in their vector bundle representation: seen as classical Higgs bundles satisfying some extra conditions reflecting the nature of $G_{\mathbb{C}}$, dictated by the need for the (projectively) flat connection to have holonomy in $G_{\mathbb{C}}$. For instance when $G_{\mathbb{C}} = SL(n, \mathbb{C})$, a $G_{\mathbb{C}}$-Higgs bundle (E, Φ) is composed of a holomorphic rank n vector bundle E with trivial determinant $\Lambda^n E \cong \mathcal{O}$, and a Higgs field satisfying $\mathrm{Tr}(\Phi) = 0$, for which we shall write $\Phi \in H^0(\Sigma, \mathrm{End}_0(E) \otimes K)$.

Example 1. Choosing a square root of K, consider the vector bundle $E = K^{1/2} \oplus K^{-1/2}$. Then, a family of $SL(2, \mathbb{C})$-Higgs bundles (E, Φ_a) parametrized by quadratic differentials $a \in H^0(\Sigma, K^2)$ is given by

$$\left(E = K^{1/2} \oplus K^{-1/2}, \Phi_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \right). \tag{3}$$

One may also consider G-Higgs bundles, for G a real form of $G_{\mathbb{C}}$, which in turn correspond to the Betti moduli space of representations $\pi_1(\Sigma) \to G$. For example, $SL(2, \mathbb{R})$-Higgs bundles are pairs $(E = L \oplus L^*, \Phi)$ for L a line bundle and Φ off diagonal, a family of which is described in *Example 1*.

In order to define a Hausdorff moduli space of Higgs bundles, one needs to incorporate the notion of stability. For this, recall that holomorphic vector bundles E on Σ are topologically classified by their rank $rk(E)$ and their degree $deg(E)$, though which one may define their *slope* as $\mu(E) := deg(E)/rk(E)$. Then, a vector bundle E is *stable* (*or semi-stable*) if for any proper sub-bundle $F \subset E$ one has that $\mu(F) < \mu(E)$ (or $\mu(F) \leq \mu(E)$). It is *polystable* if it is a direct sum of stable bundles whose slope is $\mu(E)$.

One can generalize the stability condition to Higgs bundles (E, Φ) by considering Φ-*invariant subbundles* F of E, vector subbundles $F \subset E$ for which $\Phi(F) \subset F \otimes K$. A Higgs bundle (E, Φ) is said to be *stable* (*semi-stable*) if for each proper Φ-invariant $F \subset E$ one has $\mu(F) < \mu(E)$ (*equiv.* $\leq$). Then, by imposing stability conditions, one can construct the moduli space $\mathcal{M}_{G_{\mathbb{C}}}$ of stable $G_{\mathbb{C}}$-Higgs bundles up to holomorphic automorphisms of the pairs (also denoted $\mathcal{M}_{Dol}$). Going back to Hitchin's equations, one of the most important characterisations of stable Higgs bundles is given in the work of Hitchin and Simpson, and which carries through to more general settings: If a Higgs bundle (E, Φ) is stable and $\deg E = 0$, then there is a unique unitary connection A on E, compatible with the holomorphic structure, satisfying (1).

Finally, Hitchin showed that the underlaying smooth manifold of solutions to (1) is a hyperkähler manifold, with a natural symplectic form ω defined on the infinitesimal deformations $(\dot{A}, \dot{\Phi})$ of a Higgs bundle (E, Φ) by

$$\omega((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_\Sigma \mathrm{tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1), \tag{4}$$

where $\dot{A} \in \Omega^{0,1}(\mathrm{End}_0 E)$ and $\dot{\Phi} \in \Omega^{1,0}(\mathrm{End}_0 E)$. Moreover, he presented a natural way of studying the moduli spaces $\mathcal{M}_{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$-Higgs bundles through what is now called *the Hitchin fibration*, which we shall consider next.

Integrable systems

Given a homogeneous basis $\{p_1, \ldots, p_k\}$ for the ring of invariant polynomials on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$, we denote by d_i the degree of p_i. The *Hitchin fibration*, introduced in [Hit87b], is then given by

$$\begin{array}{rcl} h \ : \ \mathcal{M}_{G_{\mathbb{C}}} & \longrightarrow & \mathcal{A}_{G_{\mathbb{C}}} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}), \\ (E, \Phi) & \mapsto & (p_1(\Phi), \ldots, p_k(\Phi)). \end{array}$$

The map h is referred to as the *Hitchin map*: it is a proper map for any choice of basis and makes the moduli space into an integrable system whose base and fibres have dimension $\dim(\mathcal{M}_{G_{\mathbb{C}}})/2$. In what follows we shall restrict our attention to $GL(n,\mathbb{C})$-Higgs bundles, which are those Higgs bundles introduced in the first paragraph of these notes, and whose Hitchin fibration in low dimension is depicted in Figure 1.

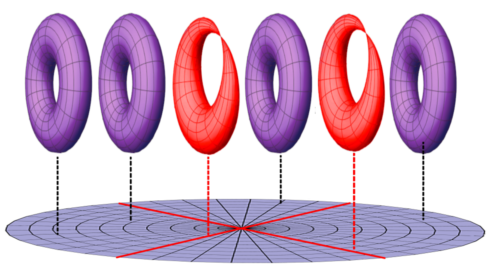

Figure 1: An example of a Hitchin fibration.

The generic or *regular* fibre of the Hitchin fibration — appearing in violet in Figure 1 — is an abelian variety, leading to what is refer to as the *abelianization* of the moduli space of Higgs bundles, and which can be seen geometrically by considering eigenvalues and eigenspaces of the Higgs field. Indeed, a Higgs bundle (E, Φ) defines a ramified cover $\pi : S \to \Sigma$ of the Riemann surface given by its eigenvalues and obtained through its characteristic equation.

$$S = \{\det(\Phi - \eta) = 0\} \subset \mathrm{Tot}K, \tag{5}$$

for η the tautological section of $\pi^* K$. This cover allows one to construct the *spectral data* associated to generic (E, Φ) given by:

- the spectral curve S from (5), generically smooth, defining a generic point in the Hitchin base, since the coefficients of $\{\det(\Phi - \eta) = 0\}$ give a basis of invariant polynomials, and
- a line bundle on S, defining a point in the Hitchin fibre and obtained as the eigenspace of Φ.

For classical Higgs bundles, the smooth fibres are Jacobian varieties $\mathrm{Jac}(S)$, and one recovers (E, Φ) up to isomorphism from the data $(S, L \in \mathrm{Jac}(S))$ by taking the direct images $E = \pi_* L$ and $\Phi = \pi_* \eta$.

When considering $G_{\mathbb{C}}$-Higgs bundles, one has to require appropriate conditions on the spectral curve and the line bundle reflecting the nature of $G_{\mathbb{C}}$. This approach originates in the work of Hitchin and of Beauville, Narasimhan and Ramanan (see [Sch19] for references), and we shall describe here an example to illustrate the setting. For $SL(n,\mathbb{C})$-Higgs bundles, the linear term in (5) since $\mathrm{Tr}(\Phi) = 0$, and the generic fibres are isomorphic to Prym varieties $\mathrm{Prym}(S, \Sigma)$ since $\Lambda^n E \cong \mathcal{O}$.

Example 2. For rank two Higgs bundles, we return to the example in the previous page in which the Hitchin fibration is over $H^0(\Sigma, K^2)$, and the Hitchin map is $h : (E, \Phi) \mapsto -\det(\Phi)$. The family (E, Φ_a) gives a section of the Hitchin fibration: a smooth map from the Hitchin base to the fibres, known as *the Hitchin section*. Moreover, this comprises a whole component of the moduli space of real $SL(2,\mathbb{R})$-Higgs bundles inside $\mathcal{M}_{G_{\mathbb{C}}}$, which Hitchin identified with Teichmuller space, and which is now refer to as a *Hitchin component* or *Teichmüller component*. Recall that the Teichmüller space $\mathcal{T}(S)$ of the underlying surface S of Σ is the space of marked conformal classes of Riemannian metrics on S.

In the early 90's Hitchin showed that for any split group G, e.g. for the split form $SL(n,\mathbb{R})$ of $SL(n,\mathbb{C})$, the above components are homeomorphic to a vector space of dimension $dim(G)(2g-2)$ and conjectured that they should parametrize geometric structures. These spaces presented the first family of *higher Teichmüller spaces* within the Betti moduli space of reductive surface group representations $\mathcal{M}_B(G)$, which leads us to applications of Higgs bundles within *Higher Teichmuller Theory* for real forms G of $G_{\mathbb{C}}$.

Higher Teichmüller theory

The moduli spaces of G-Higgs bundles have several connected components. For a split real form G of $G_{\mathbb{C}}$, the Hitchin component of G-Higgs bundles, or equivalently of surface group representations, can be defined as the connected component of the Betti moduli space $\mathcal{M}_B(G)$ containing Fuchsian representations in G, which are representations obtained by composing a discrete and faithful representation $\rho : \pi_1(\Sigma) \to SL(2,\mathbb{R})$ (classically called Fuchsian) with the unique (up to conjugation) irreducible representation $SL(2,\mathbb{R}) \to G$. Moreover, as mentioned before, these representations, called *Hitchin representations*, are considered the first example of higher Teichmüller space for surfaces: a component of the set of representations of discrete groups into Lie groups of higher ranks consisting entirely of discrete and faithful elements. In order to give a geometric description of Hitchin representations, and motivated by dynamical properties, Labourie introduced the notion of *Anosov representations*, which can be thought of as a generalization of convex-co-compact representations to Lie groups G of higher real rank[2].

As beautifully described in Wienhard's ICM Proceedings article [Wie18], building on Labourie's work, higher Teichmüller theory recently emerged as a new field in mathematics, closely related to Higgs bundles (see also [Kas18, Col19]). There are two known families of higher Teichmüller spaces, giving the only known examples of components which consist entirely of Anosov representations for surfaces:

(I) the space of Hitchin representations into a real split simple Lie group G; and

(II) the space of maximal representations into a Hermitian Lie group G.

A representation $\rho : \pi_1(\Sigma) \to G$ is maximal if it maximizes the *Toledo invariant* $T(\rho)$, a topological invariant defined for any simple Lie group G of Hermitian type as

$$\frac{1}{2\pi}\int_{\Sigma} f^*\omega \tag{6}$$

for ω the invariant Kähler form on the Riemannian symmetric space, and $f : \tilde{\Sigma} \to X$ any ρ-equivariant smooth map.

Example 3. The Toledo invariant can be expressed in terms of Higgs bundles. For example for $SL(2,\mathbb{R})$-Higgs bundles $(L \oplus L^*, \Phi)$, the Toledo invariant is $2\deg(L)$ and satisfies $0 \leq |2\deg(L)| \leq 2g-2$. Hence, the family (E, Φ_a) from Example 1 is maximal.

The existence of spaces other than those in **(I)** and **(II)** with similar properties to Teichmüller space is a topic of significant investigation. Expected candidates are spaces of θ-positive representations conjectured by Guichard-Wienhard, some of which were shown to exist via Higgs bundles [AABC$^+$19].

Whilst Anosov representations give a clear link between discrete and faithful representations and geometric structures, there is no known Higgs bundle characterization of Anosov representations, and very little is known about which explicit geometric structures correspond to these spaces. For instance, work of Choi and Goldman shows that the holonomy representations of convex projective structures are the Hitchin representations when $G = PSL(3,\mathbb{R})$.

Whilst there is no Higgs bundle characterization of Anosov representations[3], Higgs bundles have been an effective tool for describing these structures. This brings us to one of the fundamental problems in modern geometry: the classification of geometric structures admitted by a manifold M. Recall that a model geometry is a pair (G, X) where X is a manifold (*model space*) and G is a Lie group acting transitively on X (*group of symmetries*). A (G, X)-structure on a manifold M is a maximal atlas of coordinate charts on M with values in X such that the transition maps are given by elements of G. Higgs bundles have played a key role describing the closed manifold on which (G, X)-structures live, for example when showing that maximal representations to $PO(2, q)$ give rise to (G, X)-manifolds for at least two choices of X: when X is the space of null geodesics (photons) in a particular Einstein manifold, and when $X = \mathbb{P}(\mathbb{R}^{2+q})$ (e.g. see [Col19]). For an excellent review of geometric structures, see Kassel's ICM Proceedings [Kas18].

[2] For example, for representations in $SL(2,\mathbb{C})$, these are quasi-Fuchsian representations.

[3] Anosov representations are holonomy representations of geometric structures on certain closed manifolds.

Harmonic metrics

Equivariant harmonic maps play an important role in the non-abelian Hodge correspondence mentioned before (and beautifully reviewed in [BGPG07]), and thus we shall dedicate this section to look into some of the advances made in this direction. In our setting, from the work of Corlette and Donaldson, any reductive representation $\rho : \pi_1(\Sigma) \to G_{\mathbb{C}}$ has associated a ρ-equivariant harmonic map f from the universal cover $\tilde{\Sigma}$ of Σ to the corresponding symmetric space of $G_{\mathbb{C}}$, which in turn defines a Higgs bundle (E, Φ). Recall that a map $f : \tilde{\Sigma} \to M$ is called ρ-equivariant if $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$ for all $x \in \tilde{\Sigma}$ and $\gamma \in \pi_1(\Sigma)$. Moreover, though a choice of metric on Σ, one may define the *energy density*

$$e(f) = \frac{1}{2} < df, df >: \tilde{\Sigma} \to \mathbb{R}, \tag{7}$$

which is ρ-invariant and descends to Σ. Then, the *energy* of f is defined as

$$E(f) = \int_{\Sigma} e(f) d\mathrm{Vol}. \tag{8}$$

It only depends on the conformal class, and is finite since Σ is compact. The map f is *harmonic* if it is a critical point of the energy functional $E(f)$ in (8).

Conversely, through the work of Hitchin and Simpson, a polystable Higgs bundle admits a hermitian metric h on the bundle such that the associated Chern connection A solves the Hitchin equations (1), and such a metric is called harmonic. Moreover, the harmonic metric induces a completely reducible representation $\rho : \pi_1(\Sigma) \to G_{\mathbb{C}}$ and a ρ-equivariant harmonic map into the corresponding symmetric space. These two directions together give the celebrated non-abelian Hodge correspondence.

Understanding the geometric and analytic properties of the harmonic maps arising from Hitchin's equations (1) is of significant importance. For instance, one may ask how do those metrics behave at the boundaries of the moduli space, or how do the energy densities of the corresponding harmonic maps at different points of the Hitchin fibration relate to each other (the reader may be interested in the reviews [Li19] and [Fre19], and references therein).

Hitchin's work, the moduli space of Higgs bundles has a natural $\mathbb{C}^*$-action $\lambda \cdot (E, \Phi) = (E, \lambda\Phi)$, whose fixed point sets allow one to study different aspects of the topology and the geometry of the space, as done in [Hit87b] (see also [Ray18, Col19]). Moreover, as shown by Simpson, the fixed points of this action are complex variations of Hodge structure (VHS). Recall that a VHS is a $\mathbb{C}^\infty$ vector bundle V with a Hodge decomposition $V = \bigoplus_{p+q} V^{p,q}$, a rational structure and a flat connection, satisfying the axioms of Griffiths transversality and existence of a polarization.

From the above, one may ask how the energy density of harmonic maps changes along the $\mathbb{C}^*$-flow on the moduli space of Higgs bundles. Whilst this remains a challenging open question in the area, a better understanding might come from the following conjectural picture of Dai-Li described in Figure 2, and through which the harmonic map of a fixed point set of the $\mathbb{C}^*$ action on $\mathcal{M}_{G_{\mathbb{C}}}$ gives rise to two other related harmonic maps.

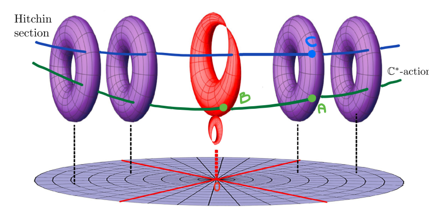

Figure 2: The nilpotent cone in red over the 0, and the points A, B and C lying over the $\mathbb{C}^*$-flow and over the Hitchin section respectively.

A point A within the Hitchin fibration naturally determines two other points: the point B which is the limit of the $\mathbb{C}^*$-flow $\lambda \cdot A$ as $\lambda \to 0$ in the nilpotent cone, and the point C which is the intersection point of the Hitchin fiber containing A and the Hitchin section. Then Dai-Li's conjecture states that the energy densities defined as in Eq.(7) of the corresponding harmonic maps f_A, f_B, f_C satisfy

$$e(f_B) < e(f_A) < e(f_C). \tag{9}$$

As evidence for the above conjecture, one can consider the integral version (through Eq.(8)), for which Hitchin showed that $E(f_B) < E(f_A)$, but where the other corresponding inequality in (9) remains open.

Limiting structures

The study of ρ-equivariant harmonic metrics and higher Teichmüller theory through Higgs bundles has received much attention in recent years and brings us to one of the most important conjectures in the area. This conjecture, due to Labourie, states that for each Hitchin representation ρ there is a unique conformal structure X_ρ on the underlying surface S in which the ρ-equivariant harmonic metric is a minimal immersion. In particular, Labourie showed that for all Anosov representations such a conformal structure exists, but the difficultly lies in proving uniqueness — the conjecture has been established only for Lie groups of rank two ([Lab17, Col19]). To understand this problem, one may consider the study of deformations of conformal structures on surfaces and the corresponding harmonic metric.

Some of these deformations can be seen through the hyperkähler structure of the moduli space, by virtue of which it has a $\mathbb{CP}^1$-worth of complex structures labelled by a parameter ξ. Indeed, we can think of a hyperkähler manifolds as a manifold whose tangent space admits an action of three complex structures I, J and K satisfying the quaternionic equations and compatible with a single metric. In our case, I arises from the complex structure I on the Riemann surface Σ, while J is from the complex structure on the group $G_{\mathbb{C}}$. In this setting, one has the following moduli spaces:

- for $\xi = 0$ the space of Higgs bundles,
- for $\xi \in \mathbb{C}^\times$ the space of flat connections[4]
$$\nabla_\xi = \xi^{-1}\Phi + \bar{\partial}_A + \partial_A + \xi\Phi^*; \tag{10}$$
- for $\xi = \infty$ the space of "anti-Higgs bundles".

The hyperkähler metric on Hitchin moduli space is expected to be of type "*quasi-ALG*", which is some expected generalization of ALG. A far reaching open question is the understanding of the behaviour of the metrics at the boundaries of the space, for instance along a path in the Hitchin's base via the limit

$$\lim_{t\to\infty} (\bar{\partial}_A, t\Phi).$$

[4]In particular, for $\xi = 1$ we recover (2).

Almost a decade ago Gaiotto-Moore-Neitzke gave a conjectural description of the hyperkähler metric on $\mathcal{M}_{G_{\mathbb{C}}}$ near infinity, which surprisingly suggests that much of the asymptotic geometry of the moduli space can be derived from the abelian spectral data described before. Recent progress has been made by Mazzeo-Swoboda-Weiss-Witt, Dumas-Neitzke and Fredrickson but the global picture remains open (for a survey of the area, see [Fre19]).

Finally, one further type of limiting structure we would like to mention is that of opers, appearing as certain limits of Higgs bundles in the Hitchin components. To see this, note that for a solution of (1) in the $SL(n,\mathbb{C})$-Hitchin section, one can add a real parameter $R > 0$ to (10) to obtain a natural family of connections

$$\nabla(\xi, R) := \xi^{-1}R\Phi + \bar{\partial}_A + \partial_A + \xi R\Phi^*. \tag{11}$$

Some years ago Gaiotto conjectured that the space of opers (a generalization of projective structures which, like the Hitchin section, is parametrized by the Hitchin base) should be obtained as the $\hbar$-conformal limit of the Hitchin section: taking $R \to 0$ and $\xi \to 0$ simultaneously while holding the ratio $\hbar = \xi/R$ fixed.

The conjecture was recently established for general simple Lie groups by Dumistrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke, who also conjectured that that this oper is the *quantum curve* in the sense of Dumitrescu-Mulase, a quantization of the spectral curve S of the corresponding Higgs bundle by *topological recursion* (see references and details in [Dum18]). More recently, Collier-Wentworth showed that the above conformal limit exists in much more generality and gives a correspondence between (Lagrangian) strata for every stable VHS — and not only the Hitchin components. Specifically, they constructed a generalization of the Hitchin section by considering stable manifolds $\mathcal{W}^0(E_0, \Phi_0)$ arising from each VHS (E_0, Φ_0) given by

$$\{(E, \phi) \in \mathcal{M}_{G_{\mathbb{C}}} \mid \lim_{t\to 0} t \cdot (E, \Phi) = (E_0, \Phi_0)\}. \tag{12}$$

The analog of the Hitchin section is then obtained by parameterizing $\mathcal{W}^0(E_0, \Phi_0)$ with a slice in the space of Higgs bundles through a global slice theorem, analogous to the definition of the Hitchin section.

Correspondences

The appearance of Higgs bundles as parameter spaces for geometric structures is an example of the study of correspondences between solutions to Hitchin's equations (1) and different mathematical objects. In what follows we shall restrict our attention to a few correspondences between Higgs bundles and two classes of mathematical objects: quiver varieties and hyperpolygons (e.g. see references in [Hos18, Ray18]).

Recall that a quiver $Q = (V, A, h, t)$ is an oriented graph, consisting of a finite vertex set V, a finite arrow set A, and head and tail maps $h, t : A \to V$. A Nakajima representation of a quiver Q can be written as families $W := ((W_v), \phi_a, \psi_a)$ for $a \in A$ and $v \in V$, where W_v is a finite dimensional vector space; the map $\phi_a : W_{t(a)} \to W_{h(a)}$ is a linear map for all $a \in A$, and ψ_a is in the cotangent space to $Hom(W_{t(a)}, W_{h(a)})$ at ϕ_a. In particular, a *hyperpolygon* is a representation of the *star-shaped* quiver, an example of which appears in Figure 3.

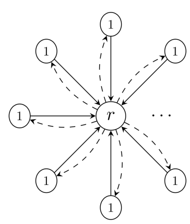

Figure 3: A star-shaped quiver.

For the star-shaped quiver in Figure 3, for which the dimensions of W_v are indicated in each vertex, the cotangent space $T^*\text{Rep}(Q)$ of representations of Q is

$$T^*\left(\bigoplus_{i=1}^{n} \text{Hom}(\mathbb{C}, \mathbb{C}^r)\right) = T^*\left(\text{Hom}(\mathbb{C}^n, \mathbb{C}^r)\right)$$

Konno showed that hyperpolygon spaces are hyperkähler analogs of polygon spaces, which are representation spaces of the star-shaped quivers with simple arrows. Moreover, through the work of Fisher-Rayan, the space of hyperpolygons as in Figure 3 may be identified with a moduli space of certain rank r parabolic Higgs bundles on $\mathbb{P}^1$.

In this setting, one has to puncture $\mathbb{P}^1$ along a positive divisor D and then regard the Higgs field as being valued in $\mathcal{O}(q) = K \otimes \mathcal{O}(D)$, with poles along D and satisfying certain conditions on its residues at the poles. This takes us to a generalization of Higgs bundles on higher genus surfaces obtained by allowing the Higgs field to have poles, leading to the moduli spaces of tame or parabolic Higgs bundles (for logarithmic singularities) initiated by Simpson [Sim92], or of wild Higgs bundles (for higher order poles) initiated by Boalch and Biquard — see references in [AEFS18] to learn more about these other settings. Understanding the more general appearance of parabolic (and wild) Higgs bundles on higher genus Riemann surfaces in terms of hyperpolygons remains an open question.

In a different direction, given a fixed Riemann surface Σ and a homomorphism between two Lie groups $\Psi : G_{\mathbb{C}} \to G'_{\mathbb{C}}$, there is a naturally induced map between the Betti moduli spaces of representations

$$\Psi : \mathcal{M}_B(G_{\mathbb{C}}) \to \mathcal{M}_B(G'_{\mathbb{C}}).$$

It follows that there must be a corresponding induced map between the Higgs bundle moduli spaces, but this does not transfer readily to the induced map on the Hitchin fibrations, in particular since the image might be over the singular locus of the base. When the maps arise through isogenies, together with Bradlow and Branco, the author obtained a description of the map for spectral data in terms of fibre products of spectral curves [Sch19], but of much interest is the understanding of other maps arising in this manner.

Finally, when considering compactifications of the moduli space, one may ask how do the moduli spaces transform when the base Riemann surface Σ changes (for instance, when degenerating the surface Σ as in Figure 4), a question closely related to the relation between Higgs bundles and singular geometry, which we shan't touch upon here — see [AEFS18] for a survey and open problems in this direction.

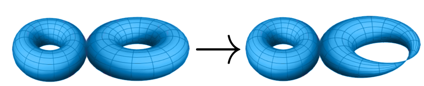

Figure 4: A degeneration of the Riemann surface.

Mirror symmetry and branes

One of the most interesting correspondences of Hitchin systems arises through mirror symmetry. For ${}^L G_{\mathbb{C}}$ the Langlands dual group of $G_{\mathbb{C}}$, there is an identification of the Hitchin basis $\mathcal{A}_{G_{\mathbb{C}}} \simeq \mathcal{A}_{{}^L G_{\mathbb{C}}}$. The two moduli spaces $\mathcal{M}_{G_{\mathbb{C}}}$ and $\mathcal{M}_{{}^L G_{\mathbb{C}}}$ are then torus fibrations over a common base, and through the famous SYZ conjecture, mirror symmetry should manifest as a duality between the spaces of Higgs bundles for Langlands dual groups fibred over the same base via the Hitchin fibration. As first observed by Hausel-Thaddeus for $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$-Higgs bundles, and shown by Donagi and Pantev for general pairs of Langlands dual reductive groups, the non-singular fibres are dual indeed abelian varieties. Kapustin and Witten gave a physical interpretation of this in terms of S-duality, using it as the basis for their approach to the geometric Langlands program.

The appearance of Higgs bundles (and flat connections) within string theory and the geometric Langlands program has led researchers to study the *derived category of coherent sheaves* (B-model) and the *Fukaya category* (A-model) of these moduli spaces. It then becomes fundamental to understand Lagrangian submanifolds of $\mathcal{M}_{G_{\mathbb{C}}}$ (the support of A-branes), and their dual objects (the support of B-branes). By considering the support of branes, we shall refer to a submanifold of a hyperkähler manifold as being of type A or B with respect to each of the complex structures (I, J, K). Hence one may study branes of the four possible types: $(B, B, B), (B, A, A), (A, B, A)$ and (A, A, B), whose dual partner is predicted by Kontsevich's homological mirror symmetry to be:

$$(B, A, A) \longleftrightarrow (B, B, B) \tag{13}$$
$$(A, A, B) \longleftrightarrow (A, A, B) \tag{14}$$
$$(A, B, A) \longleftrightarrow (A, B, A) \tag{15}$$

In view of the SYZ conjecture, it is crucial to obtain the duality between branes within the Hitchin fibration, and in particular between those completely contained within the irregular fibres since the duality has not been established there, and this has remained a very fruitful direction of research for decades. In 2006 Gukov, Kapustin and Witten introduced the first studies of branes of Higgs bundles in relation to the Geometric Langlands Program and electromagnetic duality where the (B, A, A)-branes of real G-Higgs bundles were considered. These branes, which correspond to surface group representations into the real Lie group G, may intersect the regular fibres of the Hitchin fibration in very different ways (see [Sch15, Sch19] for references):

- *Abelianization — zero-dimensional intersection.* When G is a split real form, the author showed that the (B, A, A)-brane intersects the Hitchin fibration in torsion two points.
- *Abelianization — positive dimensional intersection.* Moreover, we can also show that for other real groups such as $SU(n, n)$, the intersection has positive dimension but may still be described solely in terms of abelian data.
- *Cayley/Langlands type correspondences.* Surprisingly, many spaces of Higgs bundles corresponding to non-abelian real gauge theories do admit abelian parametrizations via auxiliary spectral curves, as shown with Baraglia through Cayley/Langlands type correspondences for the groups $G = SO(p+q, p)$ and $G = Sp(2p+2q, 2p)$.
- *Nonabelianization.* Together with Hitchin we initiated the study of branes which don't intersect the regular locus, through the *nonabelianization of Higgs bundles*, which characterized the branes for $G = SL(n, \mathbb{H})$, $SO(n, \mathbb{H})$ and $Sp(n, n)$ in terms of non-abelian data given by spaces of rank 2 vector bundles on the spectral curves.

Moreover, it has been conjectured (Baraglia-Schaposnik) that the Langlands dual in (13) to the above (B, A, A)-branes should correspond to the (B, B, B)-branes of Higgs bundles with structure group the *Nadler group* [Sch15]. More generally, branes of Higgs bundles have shown to be notoriously difficult to compute in practice, and very few broad classes of examples are known — e.g. see [Sch19] for a partial list of examples. We shall next describe a family of branes defined by the author and Baraglia, obtained by imposing symmetries to the solutions of (1) — see [Sch15] and references therein.

Higgs bundles and 3-manifolds

By considering actions on the Riemann surface Σ and on the Lie group $G_{\mathbb{C}}$, one can induce actions on the moduli space of Higgs bundles and on the Hitchin fibration, and study their fixed point sets. Indeed, for ρ the compact structure of $G_{\mathbb{C}}$, and σ a real form of $G_{\mathbb{C}}$, together with Baraglia we defined the following:

(i) Through the Cartan involution θ of a real form G of $G_{\mathbb{C}}$ one obtains $i_1(\bar{\partial}_A, \Phi) = (\theta(\bar{\partial}_A), -\theta(\Phi))$.

(ii) A real structure $f : \Sigma \to \Sigma$ on Σ induces $i_2(\bar{\partial}_A, \Phi) = (f^*(\rho(\bar{\partial}_A)), -f^*(\rho(\Phi)))$.

(iii) Lastly, by looking at $i_3 = i_1 \circ i_2$, one obtains $i_3(\bar{\partial}_A, \Phi) = (f^*\sigma(\bar{\partial}_A), f^*\sigma(\Phi))$.

The fixed point sets of i_1, i_2, i_3 are branes of type $(B, A, A), (A, B, A)$ and (A, A, B) respectively. The topological invariants can be described using KO, KR and equivariant K-theory [Sch15]. In particular, the fixed points of i_1 give the (B, A, A)-brane of G-Higgs bundles mentioned in the previous section, an example of which appears in Figure 5, and which one can study via the monodromy action on the Hitchin fibration (e.g. see [Sch19]).

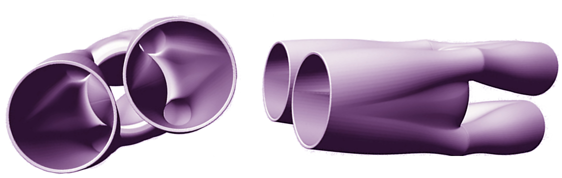

Figure 5: A real slice fixed by i_1 of the moduli space of $SL(2,\mathbb{C})$-Higgs bundles, from two different angles, obtained through Hausel's 3d prints of slices of $\mathcal{M}_{G_{\mathbb{C}}}$.

The fixed points of i_3 are pseudo-real Higgs bundles. To describe the fixed points of the involution i_2, note that a real structure (or anti-conformal map) on a compact connected Riemann surface Σ is an anti-holomorphic involution $f : \Sigma \to \Sigma$. The classification of real structures on compact Riemann surfaces is a classical result of Klein, who showed that all such involutions on Σ may be characterised by two integer invariants (n, a): the number n of disjoint union of copies of the circle embedded in Σ fixed by f; and $a \in \mathbb{Z}_2$ determining whether the complement of the fixed point set has one ($a = 1$) or two ($a = 0$) components, e.g. see Figure 6.

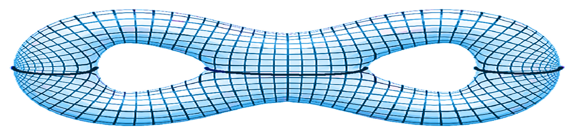

Figure 6: A genus 2 Riemann surface and its fixed point sets under an anti-holomorphic involution with invariants $(n, a) = (3, 0)$.

A real structure f on the Riemann surface Σ induces involutions on the moduli space of representations $\pi_1(\Sigma) \to G_{\mathbb{C}}$, of flat connections and of $G_{\mathbb{C}}$-Higgs bundles on Σ, and the fixed points sets define the (A, B, A)-branes of Higgs bundles in (ii). These branes can be shown to be real integrable systems, given by (possibly singular) Lagrangian fibrations.

From a representation theoretic point of view, one may ask which interesting representations these branes correspond to, a question closely related to the understanding of *which representations of* $\pi_1(\Sigma)$ *extend to* $\pi_1(M)$ *for* M *a 3-manifold whose boundary is* Σ. Whilst this question in its full generality remains an important open problem, we can consider some particular cases in which the answer becomes clear from the perspective of Higgs bundles. For this, as seen in [Sch15] and references therein, we considered the 3-manifolds

$$M = \frac{\Sigma \times [-1, 1]}{(x, t) \mapsto (f(x), -t)}, \tag{16}$$

for which $\partial M = \Sigma$ (e.g. a handle body). In this setting, together with Baraglia, we were able to show that a connection solving the Hitchin equations (1) on Σ extends over M given in (16) as a flat connection if and only if the Higgs bundle (E, Φ) is fixed by i_2 and the class $[E] \in \tilde{K}^0_{\mathbb{Z}_2}(\Sigma)$ in reduced equivariant K-theory is trivial. That is, the Higgs bundles which will extend are only those whose vector bundle is preserved by the lift of the involution i_2, and for which the action of i_2 over the fibres of E is trivial when restricted to each fixed circles.

Global topology

The computation of topological invariants of Higgs bundle moduli spaces has received vast attention from researchers who have tackled this problem with a diverse set of mathematical tools (see references in [Hau13, Ray18]). One of the central questions considered for Higgs bundles and their generalizations is what the Poincaré polynomial of the space is. A useful fact is that the total space of the Hitchin fibration deformation retracts onto the nilpotent cone $h^{-1}(0)$ via the gradient flow of the moment map of the $\mathbb{C}^*$-action introduced in the harmonic metrics section. The cohomology ring localises to the fixed-point locus inside $h^{-1}(0)$: as first seen by Morse-theoretic methods in the work of Hitchin, the Poincaré series that generates the Betti numbers of the rational cohomology $H^\bullet(\mathcal{M}_{G_\mathbb{C}}, \mathbb{Q})$ is a weighted sum of the Poincaré series of the connected components of the fixed-point locus.

Example 4. As shown by Hitchin, for the family of $SL(2, \mathbb{C})$-Higgs bundles in *Example 1*, when the genus of Σ is $g = 2$, the Poincaré series is

$$1 + t^2 + 4t^3 + 2t^4 + 34t^5 + 2t^6. \tag{17}$$

Using Morse theory, it has only been possible to compute Poincaré polynomials for low rank groups, and extending this to higher rank has been a challenging open problem for some time. More recently, interesting alternative techniques have been used to access the higher-rank Poincaré polynomials by Mozgovoy, Schiffmann, Mellit, and others. One may further ask about the structure of the ring $H^\bullet(\mathcal{M}_{G_\mathbb{C}}, \mathbb{Q})$ itself: for instance Heinloth recently proved that the intersection pairing in the middle dimension for the smooth moduli space vanishes in all dimensions for $G_\mathbb{C} = PGL(n, \mathbb{C})$; and Cliff-Nevins-Shen proved that that the Kirwan map from the cohomology of the moduli stack of G-bundles to the moduli stack of semistable G-Higgs bundles fails to be surjective.

One of the most important cohomological conjectures in the area is de-Cataldo-Hausel-Migliorini's *P=W conjecture*, which gives a correspondence between the weight filtration and the perverse filtration on the cohomology of $\mathcal{M}_B$ and $\mathcal{M}_{Dol}$, respectively, obtained via non-abelian Hodge theory. Only certain special cases are known, e.g., for rank 2 Higgs bundles, shown by de-Cataldo-Hausel-Migliorini's (see [Hau13]), and for certain moduli spaces of wild Higgs bundles, proven recently by Shen-Zhang and Szabo.

Inspired by the SYZ conjecture mentioned before, Hausel-Thaddeus conjectured that mirror moduli spaces of Higgs bundles present an agreement of appropriately defined Hodge numbers:

$$h^{p,q}(\mathcal{M}_{G_\mathbb{C}}) = h^{p,q}(\mathcal{M}_{{}^L G_\mathbb{C}}). \tag{18}$$

Very recently, the first proof of this conjecture was established for the moduli spaces of $SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$-Higgs bundles by Groechenig-Wyss-Ziegler in [GWZ17], where they established the equality of stringy Hodge numbers using p-adic integration relative to the fibres of the Hitchin fibration, and interpreted canonical gerbes present on these moduli spaces as characters on the Hitchin fibres.

Further combinatorial properties of $\mathcal{M}_{G_\mathbb{C}}$ can be glimpsed through their twisted version, consisting of Higgs bundles (E, Φ) on Σ with $\Phi : E \to E \otimes \mathcal{L}$, where Σ now has any genus, L is a line bundle with $\deg(L) > \deg(K)$, but without any punctures or residues being fixed. The corresponding moduli spaces carry a natural $\mathbb{C}^*$-action but but are not hyperkähler and there is no immediate relationship to a character variety. Hence, there is no obvious reason for the Betti numbers to be invariant with regards to the choice of $deg(E)$, which in the classical setting would follow from non-abelian Hodge theory. However, the independence holds in direct calculations of the Betti numbers in low rank, and was recently shown for $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$-Higgs bundles by Groechenig-Wyss-Ziegler in [GWZ17]. This suggests that some topological properties of Hitchin systems are independent of the hyperkähler geometry (see references in [Hau13, Ray18] for more details).

Finally, it should be mentioned that an alternative description of the Hitchin fibration can be given through Cameral data, as introduced by Donagi and Gaitsgory, and this perspective presents many advantages, in particular when considering correspondences arising from mirror symmetry and Langlands duality, as those mentioned in previous sections studied by Donagi-Pantev.

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