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ANALYSIS OF A SPATIALLY INHOMOGENEOUS STOCHASTIC PARTIAL DIFFERENTIAL EQUATION EPIDEMIC MODEL

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Abstract

This work proposes and analyzes a family of spatially inhomogeneous epidemic models. This is our first effort to use stochastic partial differential equations (SPDEs) to model epidemic dynamics with spatial variations and environmental noise. After setting up the problem, existence and uniqueness of solutions of the underlying SPDEs are examined. Then definitions of permanence and extinction are given. Certain sufficient conditions are provided for the permanence and extinction. Our hope is that this paper will open up windows for investigation of epidemic models from a new angle.

Keywords: SIR model, SPDE, mild solution, positivity, extinction, permanence.

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1. Introduction

This work presents an effort of studying stochastic epidemic models, in which spatial in-homogeneity is allowed. The hope is that it will open up a new angle for investigating a large class of epidemic processes. In lieu of the usual stochastic differential equation based formulation considered in the literature, we propose a new

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class of models by using stochastic partial differential equations. This effort largely enriches the class of systems and offers great opportunities both mathematically and practically. Meanwhile, it poses greater challenges.

The epidemic models (compartment models), in which the density functions are spatially homogeneous were introduced in 1927 by Kermack and McKendrick in [24, 25]. The main idea is to partition the population into susceptible, infected, and recovered classes. The dynamics of these classes are given by a system of deterministic differential equations. One of the classical models takes the form

$$\begin{cases} dS(t) = \Big[\Lambda - \mu_S S(t) - \dfrac{\alpha S(t)I(t)}{S(t)+I(t)}\Big]dt \quad t \geq 0, \\ dI(t) = \Big[- (\mu_I + r)I(t) + \dfrac{\alpha S(t)I(t)}{S(t)+I(t)}\Big]dt \quad t \geq 0, \\ dR(t) = \Big[- \mu_R R(t) + rI(t)\Big]dt \quad t \geq 0, \\ S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0, \end{cases}$$

where $S(t)$, $I(t)$, $R(t)$ are the densities of susceptible, infected, and recovered populations, respectively. In the above, Λ is the recruitment rate of the population; μ_S, μ_I, μ_R are the death rates of susceptible, infected and recovered individuals, respectively; α is the infection rate and r is the recovery rate. To simplify the study, it has been noted that the dynamics of recovered individuals have no effect on the disease transmission dynamics. Thus, following the usual practice, the recovered individuals are removed from the formulation henceforth. The SIR models are known to be useful and suited for such diseases as rubella, whooping cough, measles, smallpox, etc. It has also been well recognized that random effect is not avoidable and a population is often subject to random disturbances. Thus, much effort has also been devoted to the investigation of stochastic epidemic models. One popular approach is adding stochastic noise perturbations to the above deterministic models. In recent years, resurgent attention has been devoted to analyzing and designing controls of infectious diseases for host populations; see [1, 4, 6, 14, 17, 23, 20, 26, 36, 37] and references therein.

For the deterministic models, studying the systems from a dynamic system point of view, certain threshold-type results have been found. In accordance with the threshold, the population tends to the disease-free equilibrium or approaches an endemic

equilibrium under certain conditions. It has been a long-time effort to find the critical threshold value for the corresponding stochastic systems. A characterization of the systems using critical threshold was done very recently in [15, 18, 21, 22], in which sufficient and almost necessary conditions were obtained using the idea of Lyapunov exponent so that the asymptotic behavior of the system has been completely classified. Such idea can also be found in the work [16, 28] for related problems.

From another angle, it has been widely recognized that there should be spatial dependence in the model, which will better reflect the spacial variations. In the spatially inhomogeneous case, the epidemic reaction-diffusion system takes the form

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) = k_1\Delta S(t,x) + \Lambda(x) - \mu_1(x)S(t,x) - \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} & \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \dfrac{\partial}{\partial t}I(t,x) = k_2\Delta I(t,x) - \mu_2(x)I(t,x) + \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} & \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 & \text{in } \mathbb{R}^+\times\partial\mathcal{O}, \\ S(x,0) = S_0(x), I(x,0) = I_0(x) & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian with respect to the spatial variable, $\mathcal{O}$ is a bounded domain with C^2 boundary of $\mathbb{R}^l$ ($l \geq 1$), $\partial_\nu S$ denotes the directional derivative with the ν being the outer normal direction on $\partial\mathcal{O}$, and k_1 and k_2 are positive constants representing the diffusion rates of the susceptible and infected population densities, respectively. In addition, $\Lambda(x), \mu_1(x), \mu_2(x), \alpha(x) \in C^2(\mathcal{O})$ are non-negative functions. Recently, the epidemic reaction-diffusion models have been studied in [2, 19, 32, 33, 40] and the references therein. In [36], some results were given for a general epidemic model with reaction-diffusion in terms of basic reproduction numbers. The above models are all noise free. However, random noise perturbations in the environment often inevitably appear. Therefore, a more suitable description requires to consider stochastic epidemic diffusive models. Taking this into consideration, we propose a spatially non-homogeneous model using a system of stochastic partial differential equations given

by

$$\begin{cases} dS(t,x) = \Big[k_1\Delta S(t,x) + \Lambda(x) - \mu_1(x)S(t,x) - \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad +S(t,x)dW_1(t,x) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ dI(t,x) = \Big[k_2\Delta I(t,x) - \mu_2(x)I(t,x) + \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad +I(t,x)dW_2(t,x) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 \qquad\qquad \text{in } \mathbb{R}^+\times\partial\mathcal{O}, \\ S(x,0) = S_0(x), I(x,0) = I_0(x) \qquad\qquad \text{in } \mathcal{O}, \end{cases} \tag{1.2}$$

where $W_1(t,x)$ and $W_2(t,x)$ are $L^2(\mathcal{O},\mathbb{R})$-value Wiener processes, which present the noises in both time and space. We refer the readers to [12] for more details on the $L^2(\mathcal{O},\mathbb{R})$-value Winner process.

Because this is our first work in this direction, we have to settle a number of issues. First, we establish the existence and uniqueness of solutions in the sense of mild solution of the stochastic partial differential equations. Moreover, we examine some long-term behavior of the solutions. These are the main objectives of the current work.

The rest of the paper is arranged as follows. Section 2 gives some preliminary results and also formulates the problem that we wish to study. Section 3 establishes the existence and uniqueness of the solution of the stochastic partial differential equations. Section 4 provides sufficient conditions for the extinction and permanence while Section 5 provides an example. Finally, Section 6 concludes the paper with some further remarks.

2. Preliminary and Formulation

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^l$ (with $l \geq 1$) having C^2 boundary and $H := L^2(\mathcal{O};\mathbb{R})$ be the separable Hilbert space, endowed with the scalar product

$$\langle u,v\rangle_H := \int_{\mathcal{O}} u(x)v(x)dx,$$

and the corresponding norm $|u|_H = \sqrt{\langle u,u\rangle_H}$. We will say $u \geq 0$ if $u(x) \geq 0$ almost everywhere in $\mathcal{O}$. Moreover, we denote by $L^2(\mathcal{O},\mathbb{R}^2)$ the space of all functions $u(x) =$

$\big(u_1(x), u_2(x)\big)$ where $u_1, u_2 \in L^2(\mathcal{O}, \mathbb{R})$, on which the inner product is defined as

$$\begin{aligned}\langle u, v\rangle_{L^2(\mathcal{O},\mathbb{R}^2)} &:= \int_{\mathcal{O}} \big\langle u(x), v(x)\big\rangle_{\mathbb{R}^2} dx = \int_{\mathcal{O}} \big(u_1(x)v_1(x) + u_2(x)v_2(x)\big)dx \\ &= \langle u_1, v_1\rangle_{L^2(\mathcal{O},\mathbb{R})} + \langle u_2, v_2\rangle_{L^2(\mathcal{O},\mathbb{R})},\end{aligned}$$

for all $u, v \in L^2(\mathcal{O}, \mathbb{R}^2)$. Note that $L^2(\mathcal{O}, \mathbb{R}^2)$ is a separable Hilbert space. In what follows, we use u to denote a function that is either real-valued or an $\mathbb{R}^2$-valued. It will be clear from the context. Denote by E the Banach space $C(\overline{\mathcal{O}}; \mathbb{R})$ endowed with the sup-norm

$$|u|_E := \sup_{x\in\overline{\mathcal{O}}} |u(x)|.$$

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ be a complete probability space and $L^p\big(\Omega; C([0,t], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big)$ be the space of all predictable $C(\overline{\mathcal{O}}, \mathbb{R}^2)$-valued processes u in $C\big([0,t], C(\overline{\mathcal{O}}, \mathbb{R}^2)\big)$, $\mathbb{P}$-a.s. with the norm $L_{t,p}$ as follows

$$|u|^p_{L_{t,p}} := \mathbb{E} \sup_{s\in[0,t]} |u(s)|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)},$$

where

$$|u|_{C(\overline{\mathcal{O}},\mathbb{R}^2)} = \Big(\sum_{i=1}^{2} \sup_{x\in\overline{\mathcal{O}}} |u_i(x)|^2\Big)^{\frac{1}{2}} \quad \text{if} \quad u = (u_1, u_2) \in C(\overline{\mathcal{O}}, \mathbb{R}^2).$$

For $\varepsilon > 0, p \geq 1$, denote by $W^{\varepsilon,p}(\mathcal{O}, \mathbb{R}^2)$ the Sobolev-Slobodeckij space (the Sobolev space with non-integer exponent) endowed with the norm

$$|u|_{\varepsilon,p} := |u|_{L^p(\mathcal{O},\mathbb{R}^2)} + \sum_{i=1}^{2} \int_{\mathcal{O}\times\mathcal{O}} \frac{|u_i(x) - u_i(y)|^p}{|x-y|^{\varepsilon p + l}} dxdy.$$

Assume that $B_{k,1}(t)$ and $B_{k,2}(t)$ with $k = 1, 2, \ldots$, are independent $\{\mathcal{F}_t\}_{t\geq 0}$-adapted one-dimensional Wiener processes. Now, fix an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H and assume that this sequence is uniformly bounded in $L^\infty(\mathcal{O}, \mathbb{R})$, i.e.,

$$C_0 := \sup_{k\in\mathbb{N}} |e_k|_{L^\infty(\mathcal{O},\mathbb{R})} = \sup_{k\in\mathbb{N}} \operatorname*{ess\,sup}_{x\in\mathcal{O}} e_k(x) < \infty.$$

We define the infinite dimensional Wiener processes $W_i(t)$, which are driving noises in equation (1.2) as follows

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{a_{k,i}} B_{k,i}(t) e_k, \quad i = 1, 2,$$

where $\{a_{k,i}\}_{k=1}^{\infty}$ are sequences of non-negative real numbers satisfying

$$a_i := \sum_{k=1}^{\infty} a_{k,i} < \infty, \quad i = 1, 2. \tag{2.1}$$

Let A_1 and A_2 be Neumann realizations of $k_1\Delta$ and $k_2\Delta$ in H, respectively, i.e.,

$$D(A_i) = \Big\{ u \in H \big| \Delta u \in H \text{ and } \partial_\nu u = 0 \text{ on } \partial\mathcal{O} \Big\},$$

$$A_i u = k_i \Delta u, \ \ u \in D(A_i),$$

where the Laplace operator in the above definition is understood in the distribution sense. Then, A_1 and A_2 are infinitesimal generators of analytic semi-groups e^{tA_1} and e^{tA_2} with corresponding Neumann heat kernels, denoted by $p_{\mathcal{O}}^{N,1}(t,x,y), p_{\mathcal{O}}^{N,2}(t,x,y)$, i.e.,

$$(e^{tA_i}u)(x) = \int_{\mathcal{O}} p_{\mathcal{O}}^{N,i}(t,x,y)u(y)dy, \ i = 1, 2,$$

respectively. In addition, if we denote $A := (A_1, A_2)$, the operator defined in $L^2(\mathcal{O}, \mathbb{R}^2)$ by $Au := (A_1u_1, A_2u_2)$ for $u = (u_1, u_2) \in L^2(\mathcal{O}, \mathbb{R}^2)$, then it generates an analytic semigroup e^{tA} with $e^{tA}u = (e^{tA_1}u_1, e^{tA_2}u_2)$. In [13, Theorem 1.4.1], it is proved that the space $L^1(\mathcal{O}, \mathbb{R}^2) \cap L^\infty(\mathcal{O}, \mathbb{R}^2)$ is invariant under e^{tA}, so that e^{tA} may be extended to a non-negative one-parameter semigroup $e^{tA(p)}$ on $L^p(\mathcal{O}; R^2)$, for all $1 \le p \le \infty$. All these semi-groups are strongly continuous and consistent in the sense that $e^{tA(p)}u = e^{tA(q)}u$ for any $u \in L^p(\mathcal{O}, \mathbb{R}^2) \cap L^q(\mathcal{O}, \mathbb{R}^2)$ (see [9]). So, we will suppress the superscript p and denote them by e^{tA} whenever there is no confusion. Moreover, if we consider the part A_i^E of A_i in the space of continuous functions E, it generates an analytic semi-group (see [3, Chapter 2]), which has no dense domain in general. However, since we have assumed that $\mathcal{O}$ has C^2 boundary, in our boundary condition, A_i^E has dense domain in E (see [12, Appendix A.5.2]) and hence, this analytic semi-group is strongly continuous. Finally, we recall some well-known properties of the operators A_i and analytic semi-groups e^{tA_i} for $i = 1, 2$ as follows. For further details, we refer the reader to the monographs [3, 13, 31] and the references therein.

- $\forall u \in H$ then $\int_0^t e^{sA_i}uds \in D(A_i)$ and $A_i(\int_0^t e^{sA_i}uds) = e^{tA_i}u - u$.
- By Green's identity, it can be proved that A_i is symmetric, that A_i is self-adjoint in H, and that $\forall u \in D(A_i)$, $\int_{\mathcal{O}}(A_iu)(x)dx = 0$.

- For any $t > 0, x, y \in \mathcal{O}$,

$$0 \le p_{\mathcal{O}}^{N,i}(t,x,y) \le c_1(t \wedge 1)^{-\frac{l}{2}} e^{-c_2 \frac{|x-y|^2}{t}},$$

for some constant c_1, c_2, which depends on $\mathcal{O}$, but is independent of u, t.

- The semigroup e^{tA} satisfies the fowling properties

$$\left|e^{tA}u\right|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \le c\,|u|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \text{ and } \left|e^{tA}u\right|_{C(\overline{\mathcal{O}},\mathbb{R}^2)} \le c\,|u|_{C(\overline{\mathcal{O}},\mathbb{R}^2)}, \tag{2.2}$$

for some constant c, which depends on $\mathcal{O}$, but is independent of u, t.

- For any $t, \varepsilon > 0$, $p \ge 1$, the semigroup e^{tA} maps $L^p(\mathcal{O},\mathbb{R}^2)$ into $W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$ and $\forall u \in L^p(\mathcal{O},\mathbb{R}^2)$

$$\left|e^{tA}u\right|_{\varepsilon,p} \le c(t \wedge 1)^{-\varepsilon/2}\,|u|_{L^p(\mathcal{O},\mathbb{R}^2)}, \tag{2.3}$$

for some constant c independent of u, t.

Now, we rewrite equation (1.2) as the stochastic differential equation in an infinite dimension space

$$\begin{cases} dS(t) = \Big[A_1 S(t) + \Lambda - \mu_1 S(t) - \dfrac{\alpha S(t) I(t)}{S(t)+I(t)}\Big]dt + S(t)dW_1(t), \\ dI(t) = \Big[A_2 I(t) - \mu_2 I(t) + \dfrac{\alpha S(t) I(t)}{S(t)+I(t)}\Big]dt + I(t)dW_2(t), \\ S(0) = S_0, I(0) = I_0. \end{cases} \tag{2.4}$$

As usual, we say that $\big(S(t), I(t)\big)$ is a mild solution to (2.4), if

$$\begin{cases} S(t) = e^{tA_1}S_0 + \displaystyle\int_0^t e^{(t-s)A_1}\Big(\Lambda - \mu_1 S(s) - \frac{\alpha S(s)I(s)}{S(s)+I(s)}\Big)ds + W_S(t), \\ I(t) = e^{tA_2}I_0 + \displaystyle\int_0^t e^{(t-s)A_2}\Big(-\mu_2 I(s) + \frac{\alpha S(s)I(s)}{S(s)+I(s)}\Big)ds + W_I(t), \end{cases} \tag{2.5}$$

where

$$W_S(t) = \int_0^t e^{(t-s)A_1}S(s)dW_1(s) \text{ and } W_I(t) = \int_0^t e^{(t-s)A_2}I(s)dW_2(s),$$

or in the vector form

$$Z(t) = e^{tA}Z_0 + \int_0^t e^{(t-s)A}F\big(Z(s)\big)ds + \int_0^t e^{(t-s)A}Z(s)dW(s), \tag{2.6}$$

where $Z = (S, I)$, $F(Z) = \big(F_1(Z), F_2(Z)\big) := \Big(\Lambda - \mu_1 S - \dfrac{\alpha SI}{S+I}, -\mu_2 I + \dfrac{\alpha SI}{S+I}\Big)$, and

$$e^{(t-s)A}Z(s)dW(s) := \big(e^{(t-s)A_1}S(s)dW_1(s)\ ,\ e^{(t-s)A_2}I(s)dW_2(s)\big).$$

Because we are modeling the SIR epidemic systems, we are only interested in the positive (≥ 0) solutions. Therefore, we define a "*positive mild solution*" of (2.4) as a mild solution $S(t,x), I(t,x)$ such that $S(t,x), I(t,x) \geq 0$, almost everywhere $x \in \mathcal{O}$, for all $t \geq 0$. Moreover, to have the term $\frac{si}{s+i}$ well defined, we assume that it is equal 0 whenever either $s = 0$ or $i = 0$.

Remark 2.1. The integrals on the right-hand side of (2.5) are understood as Bochner integrals (in the Banach space H) while $W_S(t)$ and $W_I(t)$ are the stochastic integrals (stochastic convolutions). The $S(s)$ (resp. $I(s)$) in the stochastic integrals is understood as multiplication operator, i.e.,

$$S(s)(u) = S(s)u, \quad \forall u \in H.$$

The stochastic integral $\int_0^t e^{(t-s)A_i} U(s) dW_i(s)$ (see [12, Chapter 4] for more details on stochastic integrals) is well-define if the process $U(s)$ satisfies that

$$\int_0^t \sum_{k=1}^{\infty} a_{k,i} \left| e^{(t-s)A_i} U(s) e_k \right|_H^2 ds < \infty.$$

Finally, in the vector form, to simplify notation, we do not write the vectors in the column form. However, the calculations involving vectors are understood as in the usual sense.

To investigate the epidemic models, an important question is whether the infected individual will die out in the long time. That is, the consideration of extinction or permanence. Since the mild solution is used, let us introduce the definitions in the weak sense as follows.

Definition 2.1. A population with density $u(t,x)$ is said to be extinct in the mean if

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E} \int_{\mathcal{O}} u(s,x) dx ds = 0,$$

and that is said to be permanent in the mean if there exists a positive number R_I, is independent of initial conditions of population, such that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \Big(\mathbb{E} \int_{\mathcal{O}} \big(u^2(s,x) \wedge 1\big) dx \Big)^{\frac{1}{2}} ds \geq R_I.$$

Remark 2.2. It is well known that it is fairly difficult to confirm the existence of strong solutions for stochastic partial differential equations (even weak solution); see

[12, Section 6.1]. As an alternative, we shall use the notion of mild solutions. Hence, the convergence in our situation is in the weak sense. Note however, in the deterministic case, in [2, 19, 32, 33, 40], the authors obtained strong solutions of the deterministic reaction-diffusion epidemic models and the convergence is taken in a space such as L^∞, E, or a Sobolev space. In what follows, for convenience, we often suppress the phrase "in the mean" when we refer to extinction and permanence, because we are mainly working with mild solutions.

3. Existence and Uniqueness of the Positive Mild Solution

In this section, we shall prove the existence and uniqueness of the positive mild solution of the system as well as its continuous dependence on initial conditions. In what follows, without loss of the generality we can assume $|\mathcal{O}| = 1$, where $|\mathcal{O}|$ is the volume of bounded domain $\mathcal{O}$ in $\mathbb{R}^l$ and the initial values are non-random for simplicity.

Theorem 3.1. *For any initial data $0 \le S_0, I_0 \in E$, there exists a unique positive mild solution $\big(S(t), I(t)\big)$ of (2.4) belongs to $L^p\big(\Omega; C([0,T], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big)$ for any $T > 0, p \ge 1$. Moreover, this solution depends continuously on the initial data.*

Proof. In this proof, the letter c denotes a positive constant whose value may change in different occurrences. We will write the dependence of constant on parameters explicitly if it is essential. First, we rewrite the coefficients by defining f and f^* as follows:

$$f(x,s,i) = \Big(\Lambda(x) - \mu_1(x)s - \frac{\alpha(x)si}{s+i}, -\mu_2(x)i + \frac{\alpha(x)si}{s+i}\Big),\ x \in \mathcal{O},\ (s,i) \in \mathbb{R}^2,$$

and

$$f^*(x,s,i) = f\big(x, s \vee 0, i \vee 0\big).$$

Writing $z = (s,i)$, by noting that as our assumption, the term $\dfrac{si}{s+i}$ will be equal to 0 whenever either $s = 0$ or $i = 0$, it is easy to see that $f^*(x,\cdot,\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is Lipschitz continuous, uniformly in $x \in \mathcal{O}$ so that the composition operator $F^*(z)$ associated with f^*, i.e.,

$$F^*(z)(x) = \big(F_1^*(z)(x), F_2^*(z)(x)\big) := f^*\big(x, z(x)\big), \quad x \in \mathcal{O},$$

is Lipschitz continuous, both in $L^2(\mathcal{O}, \mathbb{R}^2)$ and $C(\overline{\mathcal{O}}, \mathbb{R}^2)$. Now, we consider the

following problem

$$dZ^*(t) = \big[AZ^*(t) + F^*\big(Z^*(t)\big)\big]dt + \big(Z^*(t) \vee 0\big)dW(t), \quad Z^*(0) = Z_0 = (S_0, I_0), \tag{3.1}$$

where $Z^*(t) = \big(S^*(t), I^*(t)\big)$ and $Z^*(t) \vee 0$ is defined by

$$\big(Z^*(t) \vee 0\big)(x) = \big(S^*(t,x) \vee 0, I^*(t,x) \vee 0\big).$$

For any

$$u(t,x) = \big(u_1(t,x), u_2(t,x)\big) \in L^p\big(\Omega; C([0,T], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big),$$

consider the mapping

$$\gamma(u)(t) := e^{tA}Z_0 + \int_0^t e^{(t-s)A}F^*\big(u(s)\big)ds + \varphi(u)(t),$$

where

$$\begin{aligned}\varphi(u)(t) &:= \int_0^t e^{(t-s)A}\big(u(s) \vee 0\big)dW(s)\\ &:= \Big(\int_0^t e^{(t-s)A_1}\big(u_1(s) \vee 0\big)dW_1(s), \int_0^t e^{(t-s)A_2}\big(u_2(s) \vee 0\big)dW_2(s)\Big).\end{aligned}$$

We will prove that γ is a contraction mapping in $L^p\big(\Omega; C([0,T_0], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big)$, for some $T_0 > 0$, and any $p \geq p_0$ for some p_0.

Lemma 3.1. *There exists p_0 such that for any $p \geq p_0$, the mapping φ*

$$\textit{maps } L^p\big(\Omega; C([0,t], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big) \textit{ into itself,}$$

and for any $u = (u_1, u_2), v = (v_1, v_2) \in L^p\big(\Omega; C([0,t], C(\overline{\mathcal{O}}, \mathbb{R}^2))\big)$

$$|\varphi(u) - \varphi(v)|_{L_{t,p}} \leq c_p(t)\,|u - v|_{L_{t,p}}, \tag{3.2}$$

where $c_p(t)$ is some constant satisfying $c_p(t) \downarrow 0$ as $t \downarrow 0$.

Proof. Let p_0 be sufficiently large to satisfy that for any $p \geq p_0$, we can choose simultaneously $\beta, \varepsilon > 0$ such that

$$\frac{1}{p} < \beta < \frac{1}{2} \quad \text{and} \quad \frac{l}{p} < \varepsilon < 2\big(\beta - \frac{1}{p}\big).$$

Now, for any fixed $p \geq p_0$, let β, ε be chosen as above. By using a factorization argument (see e.g., [12, Theorem 8.3]), we have

$$\varphi(u)(t) - \varphi(v)(t) = \frac{\sin \pi\beta}{\pi}\int_0^t (t-s)^{\beta-1}e^{(t-s)A}Y_\beta(u,v)(s)ds,$$

where

$$Y_\beta(u,v)(s)=\int_0^s (s-r)^{-\beta}e^{(s-r)A}\Big(u(r)\vee 0-v(r)\vee 0\Big)dW(r).$$

If

$$\int_0^t |Y_\beta(u,v)(s)|^p_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds<\infty,\ \text{a.s.},$$

then it is easily seen from the properties (2.3) of semi-group e^{tA} and Hölder's inequality that

$$\begin{aligned}
&|\varphi(u)(t)-\varphi(v)(t)|_{\varepsilon,p}\\
&\quad\le c_\beta\int_0^t (t-s)^{\beta-1}\big((t-s)\wedge 1\big)^{-\varepsilon/2}\,|Y_\beta(u,v)|_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds\\
&\quad\le c_{\beta,p}(t)\Big(\int_0^t \big((t-s)\wedge 1\big)^{\frac{p}{p-1}(\beta-\varepsilon/2-1)}ds\Big)^{\frac{p-1}{p}}\Big(\int_0^t |Y_\beta(u,v)(s)|^p_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds\Big)^{\frac{1}{p}}\\
&\quad\le c_{\beta,p}(t)\Big(\int_0^t |Y_\beta(u,v)(s)|^p_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds\Big)^{\frac{1}{p}},\ \text{a.s.},
\end{aligned}\tag{3.3}$$

where $c_{\beta,p}(t)$ is some positive constant, satisfies $c_{\beta,p}(t)\downarrow 0$ as $t\downarrow 0$. Rewriting $Y_\beta(u,v)(s)=\big(Y_{1\beta}(u,v)(s),Y_{2\beta}(u,v)(s)\big)$, where

$$Y_{i\beta}(u,v)(s):=\int_0^s (s-r)^{-\beta}e^{(s-r)A_i}\big(u_i(r)\vee 0-v_i(r)\vee 0\big)dW_i(r),\ i=1,2.$$

Therefore, applying the Burkholder inequality, we obtain that for all $s\in[0,t]$, almost every $x\in\mathcal{O}$,

$$\mathbb{E}\,|Y_{i\beta}(u,v)(s,x)|^p\le c_p\mathbb{E}\Big[\int_0^s (s-r)^{-2\beta}\sum_{k=1}^\infty a_{k,i}\,|M_i(s,r,k,x)|^2\,dr\Big]^{\frac{p}{2}}.$$

where

$$M_i(s,r,k)=e^{(s-r)A_i}\big(u_i(r)\vee 0-v_i(r)\vee 0\big)e_k.$$

In above, we used the notations

$$Y_{i\beta}(u,v)(s,x):=Y_{i\beta}(u,v)(s)(x),\quad M_i(s,r,k,x):=M_i(s,r,k)(x),\ i=1,2.$$

As a consequence,

$$\begin{aligned}
&\mathbb{E}\int_0^t |Y_\beta(u,v)(s)|^p_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds\\
&\quad\le c_p(t)\mathbb{E}\int_0^t\int_{\mathcal{O}}\Big(|Y_{1\beta}(u,v)(s,x)|^p+|Y_{2\beta}(u,v)(s,x)|^p\Big)dxds\\
&\quad\le c_p(t)\int_0^t\mathbb{E}\Big(\int_0^s (s-r)^{-2\beta}(a_1+a_2)\sup_{k\in\mathbb{N}}|M(s,r,k)|^2_{L^\infty(\mathcal{O},\mathbb{R}^2)}\,dr\Big)^{\frac{p}{2}}ds,
\end{aligned}\tag{3.4}$$

where $M(s,r,k) := \big(M_1(s,r,k), M_2(s,r,k)\big)$ and a_1, a_2 are defined in (2.1). Moreover, since the uniformly boundedness property of $\{e_k\}_{k=1}^{\infty}$ and (2.2), we have

$$\sup_{k\in\mathbb{N}} |M(s,r,k)|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \le c\,|u(r)-v(r)|_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,, \tag{3.5}$$

for some constant c independent of s, r, u, v. Combining (3.4) and (3.5) implies that

$$\begin{aligned}
&\mathbb{E}\int_0^t |Y_\beta(u,v)(s)|^p_{L^p(\mathcal{O},\mathbb{R}^2)}\,ds \\
&\quad\le c_p(t)\int_0^t \mathbb{E}\sup_{r\in[0,s]} |u(r)-v(r)|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)} \Big(\int_0^s (s-r)^{-2\beta}dr\Big)^{\frac{p}{2}} ds \\
&\quad\le c_{\beta,p}(t)\int_0^t \mathbb{E}\sup_{r\in[0,s]} |u(r)-v(r)|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,ds \le c_{\beta,p}(t)\,|u-v|^p_{L_{t,p}} < \infty,
\end{aligned} \tag{3.6}$$

where $c_{\beta,p}(t)$ is some positive constant and satisfies $c_{\beta,p}(t)\downarrow 0$ as $t\downarrow 0$. Therefore, the inequality (3.3) holds and as a consequence, $\varphi(u)(t)-\varphi(v)(t)\in W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$. Since $\varepsilon > l/p$, the Sobolev embedding theorem implies that $\varphi(u)(t)-\varphi(v)(t)\in C(\overline{\mathcal{O}},\mathbb{R}^2)$. Finally, (3.3) and (3.6) imply that

$$|\varphi(u)-\varphi(v)|_{L_{t,p}} \le c_p(t)\,|u-v|_{L_{t,p}}$$

for some constant $c_p(t)$, satisfying $c_p(t)\downarrow 0$ as $t\downarrow 0$. The Lemma is proved. □

Therefore, for $p\ge p_0$, with sufficiently large p_0, γ maps $L^p\big(\Omega; C([0,t], C(\overline{\mathcal{O}},\mathbb{R}^2))\big)$ into itself. Moreover, by using (2.2) and Lipschitz continuity of F^*, we have

$$\begin{aligned}
&\int_0^t \Big|e^{(t-s)A}\big[F^*\big(u(s)\big)-F^*(v(s))\big]\Big|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,ds \le c\int_0^t \big|\big(u(s)-v(s)\big)\big|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,ds \\
&\quad\le c\int_0^t \sup_{r\in[0,s]}\big|\big(u(r)-v(r)\big)\big|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,ds \le ct\sup_{s\in[0,t]}|u(s)-v(s)|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}\,.
\end{aligned} \tag{3.7}$$

Hence, (3.2) and (3.7) imply that

$$|\gamma(u)-\gamma(v)|_{L_{t,p}} \le c_p(t)\,|u-v|_{L_{t,p}}\,,$$

where $c_p(t)$ is some constant depending on p, t and satisfying $c_p(t)\downarrow 0$ as $t\downarrow 0$. Therefore, for some T_0 sufficient small, γ is a contraction mapping in $L^p\big(\Omega; C([0,T_0], C(\overline{\mathcal{O}},\mathbb{R}^2))\big)$. By a fixed point argument we can conclude that equation (3.1) admits a unique mild solution in $L^p\big(\Omega; C([0,T_0], C(\overline{\mathcal{O}},\mathbb{R}^2))\big)$. Thus, by repeating the above argument in each finite time interval $[kT_0,(k+1)T_0]$, for any $T>0, p\ge p_0$ the equation (3.1) admits a unique mild solution $Z^*(t)=\big(S^*(t),I^*(t)\big)$ in $L^p\big(\Omega; C([0,T], C(\overline{\mathcal{O}},\mathbb{R}^2))\big)$. We proceed to prove the positivity of $S^*(t), I^*(t)$.

Lemma 3.2. *Let* $\big(S^*(t), I^*(t)\big)$ *be the unique mild solution of* (3.1). *Then* $\forall t \in [0,T]$, $S^*(t), I^*(t) \geq 0$ *a.s.*

Proof. Equivalently, $\big(S^*(t), I^*(t)\big)$ is the mild solution of the equation

$$\begin{cases} dS^*(t) = \big[A_1 S^*(t) + F_1\big(S^*(t)\vee 0, I^*(t)\vee 0\big)\big]dt + \big(S^*(t)\vee 0\big)dW_1(t), \\ dI^*(t) = \big[A_2 I^*(t) + F_2\big(S^*(t)\vee 0, I^*(t)\vee 0\big)\big]dt + \big(I^*(t)\vee 0\big)dW_2(t), \\ S^*(0) = S_0, I^*(0) = I_0. \end{cases} \tag{3.8}$$

For $i = 1, 2$, let $\lambda_i \in \rho(A_i)$ be the resolvent set of A_i and $R_i(\lambda_i) := \lambda_i R_i(\lambda_i, A_i)$, with $R_i(\lambda_i, A_i)$ being the resolvent of A_i. For each small $\varepsilon > 0$, $\lambda = (\lambda_1, \lambda_2) \in \rho(A_1)\times\rho(A_2)$, by [27, Proposition 1.3.6], there exists a unique strong solution $S_{\lambda,\varepsilon}(t,x), I_{\lambda,\varepsilon}(t,x)$ of the equation

$$\begin{cases} dS_{\lambda,\varepsilon}(t) = \Big[A_1 S_{\lambda,\varepsilon}(t) + R_1(\lambda_1)F_1\big(\varepsilon\Phi(\varepsilon^{-1}S_{\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}I_{\lambda,\varepsilon}(t))\big)\Big]dt \\ \qquad\qquad + R_1(\lambda_1)\varepsilon\Phi\big(\varepsilon^{-1}S_{\lambda,\varepsilon}(t)\big)dW_1(t), \\ dI_{\lambda,\varepsilon}(t) = \Big[A_2 I_{\lambda,\varepsilon}(t) + R_2(\lambda_2)F_2\big(\varepsilon\Phi(\varepsilon^{-1}S_{\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}I_{\lambda,\varepsilon}(t))\big)\Big]dt \\ \qquad\qquad + R_2(\lambda_2)\varepsilon\Phi\big(\varepsilon^{-1}I_{\lambda,\varepsilon}(t)\big)dW_2(t), \\ S_{\lambda,\varepsilon}(0) = R_1(\lambda_1)S_0, \quad I_{\lambda,\varepsilon}(0) = R_2(\lambda_2)I_0, \end{cases} \tag{3.9}$$

where

$$\Phi(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ 3\xi^5 - 8\xi^4 + 6\xi^3 & \text{if } 0 < \xi < 1, \\ \xi & \text{if } \xi \geq 1, \end{cases}$$

satisfying

$$\begin{cases} \Phi \in C^2(\mathbb{R}), \\ \varepsilon\Phi(\varepsilon^{-1}\xi) \to \xi \vee 0 \quad \text{as } \varepsilon \to 0. \end{cases}$$

Combined with the convergence property in [27, Proposition 1.3.6], we obtain that $\big(S_{\lambda(k),\varepsilon}(t),\, I_{\lambda(k),\varepsilon}(t)\big) \to \big(S^*(t), I^*(t)\big)$ in $L^p\big(\Omega; C([0,T], L^2(\mathcal{O}, \mathbb{R}^2))\big)$ for some sequence $\{\lambda(k)\}_{k=1}^{\infty} \subset \rho(A_1) \times \rho(A_2)$ and $\varepsilon \to 0$.

Now, let

$$g(\xi) = \begin{cases} \xi^2 - \dfrac{1}{6} & \text{if } \xi \le -1, \\ -\dfrac{\xi^4}{2} - \dfrac{4\xi^3}{3} & \text{if } -1 < \xi < 0, \\ 0 & \text{if } \xi \ge 0. \end{cases}$$

Then $g'(\xi) \le 0\ \forall \xi$ and $g''(\xi) \ge 0\ \forall \xi$. Hence, we are to compute $d_t\big(\int_{\mathcal{O}} g(I_{\lambda,\varepsilon}(t,x)dx)\big)$. Since the fact $g'(\xi)\Phi(\xi) = g''(\xi)\Phi(\xi) = 0\ \forall \xi$, by Itô's Lemma [10, Theorem 3.8], we get

$$\begin{aligned} \int_{\mathcal{O}} g\big(I_{\lambda,\varepsilon}(t,x)\big)dx &= k_2 \int_0^t \int_{\mathcal{O}} g'\big(I_{\lambda,\varepsilon}(s,x)\big)\Delta I_{\lambda,\varepsilon}(s,x)dxds \\ &= -k_2 \int_0^t \int_{\mathcal{O}} g''\big(I_{\lambda,\varepsilon}(s,x)\big)\left|\nabla I_{\lambda,\varepsilon}(s,x)\right|^2 dxds \\ &\le 0. \end{aligned}$$

Since $g(\xi) > 0$ for all $\xi < 0$, we conclude that $\forall \lambda \in \rho(A_1) \times \rho(A_2), \varepsilon > 0$ then $I_{\lambda,\varepsilon}(t,x) \ge 0$ for all $t \in [0,T]$, almost everywhere in $\mathcal{O}$. Similarly, we have

$$\begin{aligned} \int_{\mathcal{O}} g\big(S_{\lambda,\varepsilon}(t,x)\big)dx &= \int_0^t \int_{\mathcal{O}} g'\big(S_{\lambda,\varepsilon}(s,x)\big)\Big(k_1\Delta S_{\lambda,\varepsilon}(s,x) + \big(R_1(\lambda_1)\Lambda\big)(x)\Big)dxds \\ &= -k_1 \int_0^t \int_{\mathcal{O}} g''\big(S_{\lambda,\varepsilon}(s,x)\big)\left|\nabla S_{\lambda,\varepsilon}(s,x)\right|^2 dxds \\ &\quad + \int_0^t \int_{\mathcal{O}} g'\big(S_{\lambda,\varepsilon}(s,x)\big)\big(R_1(\lambda_1)\Lambda\big)(x)dxds \\ &\le 0, \end{aligned}$$

where the last inequality above follows from the fact

$$R_1(\lambda_1, A_1) = \int_0^\infty e^{-\lambda_1 t} e^{tA_1} dt$$

preserves positivity. Again, since $g(\xi) > 0$ for all $\xi < 0$, we obtain the positivity of $S_{\lambda,\varepsilon}(t,x)$. Hence, $S^*(t,x), I^*(t,x) \ge 0$ almost everywhere in $\mathcal{O}$ for all $t \in [0,T]$, a.s. □

Completion of the Proof of the Theorem. Since $\big(S^*(t), I^*(t)\big)$ is a unique mild solution of (3.8) and is positive, it is a mild solution of (2.4). Therefore, the equation (2.4) admits a unique positive mild solution $\big(S(t), I(t)\big)$.

Now, we prove the second part. For convenience, we use subscripts to indicate the dependence of the solution on initial value. Let $Z_{z_0}(t), Z_{z_0'}(t)$ be the positive mild solutions of (2.6) with the initial condition $Z(0) = z_0$ and $Z(0) = z_0'$, respectively.

That means,

$$Z_{z_0}(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}F^*\big(Z_{z_0}(s)\big)ds + \int_0^t e^{(t-s)A}Z_{z_0}(s)dW(s),$$

and

$$Z_{z_0'}(t) = e^{tA}z_0' + \int_0^t e^{(t-s)A}F^*\big(Z_{z_0'}(s)\big)ds + \int_0^t e^{(t-s)A}Z_{z_0'}(s)dW(s).$$

It implies that

$$\begin{aligned} Z_{z_0}(t) - Z_{z_0'}(t) = e^{tA}z_0 - e^{tA}z_0' + \int_0^t e^{(t-s)A}\Big(F^*\big(Z_{z_0}(s)\big) - F^*\big(Z_{z_0'}(s)\big)\Big)ds \\ + \int_0^t e^{(t-s)A}\Big(Z_{z_0}(s) - Z_{z_0'}(s)\Big)dW(s). \end{aligned}$$

Since (3.3) and (3.6), we can obtain that

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{(s-r)A}\Big(Z_{z_0}(r) - Z_{z_0'}(r)\Big)dW(r)\right|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)} \\ &\qquad \le c_p(t)\int_0^t \mathbb{E}\sup_{r\in[0,s]}\left|Z_{z_0}(r) - Z_{z_0'}(r)\right|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}ds \qquad (3.10) \\ &\qquad \le c_p(t)\int_0^t \left|Z_{z_0} - Z_{z_0'}\right|^p_{L_{s,p}}ds \end{aligned}$$

Therefore, by virtue of (3.7) and (3.10), it is possible to get

$$\left|Z_{z_0} - Z_{z_0'}\right|^p_{L_{t,p}} \le c_p\left|z_0 - z_0'\right|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)} + c_p(t)\int_0^t \left|Z_{z_0} - Z_{z_0'}\right|^p_{L_{s,p}}ds.$$

Hence, it is easy to obtain from Gronwall's inequality that

$$\left|Z_{z_0} - Z_{z_0'}\right|^p_{L_{T,p}} \le c_p(T)\left|z_0 - z_0'\right|^p_{C(\overline{\mathcal{O}},\mathbb{R}^2)}.$$

Therefore, the continuous dependence of the solution on initial values is proved. □

4. Longtime Behavior

This section investigates the properties of the positive mild solution $\big(S(t), I(t)\big)$ of system (2.4) when $t \to \infty$. In particular, we provide the sufficient conditions for the extinction and permanence. For each function $u \in E$, denote

$$u_* = \inf_{x\in\overline{\mathcal{O}}} u(x).$$

Define the number

$$\widehat{R} = \int_{\mathcal{O}} \alpha(x)dx - \int_{\mathcal{O}} \mu_2(x)dx - \frac{a_2}{2}.$$

Theorem 4.1. *If $\Lambda_* > 0$ and $\widehat{R} > 0$, then the infected class is permanent in the sense that for any the initial values $0 \leq S_0, I_0 \in E$ satisfying*

$$\int_{\mathcal{O}} -\ln I_0(x)dx < \infty,$$

we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(\mathbb{E}\int_{\mathcal{O}} \big(I^2(s,x)\wedge 1\big)dx\right)^{\frac{1}{2}} ds \geq R_I,$$

for some $R_I > 0$ independent of initial values.

Proof. To obtain the longtime properties of $\big(S(t), I(t)\big)$, one of tools we use is Itô's formula. Unfortunately, in general the Itô's formula is not valid for the mild solutions. Hence, our idea is to approximate the solution by a sequence of strong solutions when the noise is finite dimensional. First, we assume that $S_0, I_0 \in D(A_i^E)$, where $D(A_i^E)$ is the domain of A_i^E, the part of A_i in E. For each fixed $n \in \mathbb{N}$, let $\overline{S}_n(t,x), \overline{I}_n(t,x)$ be the strong solution (see [12] for more details about strong solutions, weak solutions, and mild solutions) of the following equations

$$\begin{cases} d\overline{S}_n(t,x) = \Big[A_1\overline{S}_n(t,x) + \Lambda(x) - \mu_1(x)\overline{S}_n(t,x) - \dfrac{\alpha(x)\overline{S}_n(t,x)\overline{I}_n(t,x)}{\overline{S}_n(t,x)+\overline{I}_n(t,x)}\Big]dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n} \sqrt{a_{k,1}}e_k(x)\overline{S}_n(t,x)dB_{k,1}(t), \\ d\overline{I}_n(t,x) = \Big[A_2\overline{I}_n(t,x) - \mu_2(x)\overline{I}_n(t,x) + \dfrac{\alpha(x)\overline{S}_n(t,x)\overline{I}_n(t,x)}{\overline{S}_n(t,x)+\overline{I}_n(t,x)}\Big]dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n} \sqrt{a_{k,2}}e_k(x)\overline{I}_n(t,x)dB_{k,2}(t), \\ \overline{S}_n(x,0) = S_0(x), \quad \overline{I}_n(x,0) = I_0(x). \end{cases} \tag{4.1}$$

The existence and uniqueness of the strong solution of (4.1) follow the results in [11] or [12, Section 7.4]. To see that the conditions in these references are satisfied, we note that the semi-groups e^{tA_1}, e^{tA_2} (as well as their restrictions to E) are analytic (see [3, Chapter 2]) and strongly continuous (see [12, Appendix A.5.2]). Moreover, since the characterizations of fractional power of elliptic operators in ([38, Chapter 16] or [12, Appendix A], it is easy to confirm that the coefficients in equation (4.1) satisfies condition (e) in Hypothesis 2 in [11]. Moreover, a detailed argument can be also found in [29, 30].

In addition, since the continuous dependence on parameter ξ of the fixed points of family of uniform contraction mappings $T(\xi)$, by a similar "parameter-dependent contraction mapping" argument, it is easy to obtain that (see [12] or [30, Proposition 4.2]) for any fixed t,

$$\lim_{n\to\infty} \mathbb{E}\left|S(t) - \overline{S}_n(t)\right|_H^2 \to 0,$$

and

$$\lim_{n\to\infty} \mathbb{E}\left|I(t) - \overline{I}_n(t)\right|_H^2 \to 0.$$

To proceed, we state and prove following auxiliary Lemmas.

Lemma 4.1. *Let*

$$\mu_* := \inf_{x\in\overline{\mathcal{O}}} \min\{\mu_1(x), \mu_2(x)\}.$$

If $\mu_* > 0$ *then*

$$\mathbb{E}\int_{\mathcal{O}} \big(S(t,x) + I(t,x)\big)dx \le e^{-\mu_* t}\int_{\mathcal{O}} \big(S_0(x) + I_0(x)\big)dx + \frac{|\Lambda|_E}{\mu_*}.$$

Proof. In view of Itô's formula ([10, Theorem 3.8]), we can obtain

$$\begin{aligned}\mathbb{E}e^{\mu_* t}\int_{\mathcal{O}} \big(\overline{S}_n(t,x) + \overline{I}_n(t,x)\big)dx &\le \int_{\mathcal{O}} \big(S_0(x) + I_0(x)\big)dx + \mathbb{E}\int_0^t e^{\mu_* s}\int_{\mathcal{O}} \Lambda(x)dxds \\ &\le \int_{\mathcal{O}} \big(S_0(x) + I_0(x)\big)dx + \frac{|\Lambda|_E}{\mu_*}e^{\mu_* t}.\end{aligned}$$

Letting $n \to \infty$, we obtain the desired result. □

Now, we are in a position to estimate $\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{S}_n^p(t,x)}dx$ by the following Lemma.

Lemma 4.2. *For any* $p > 0$, *if* $\int_{\mathcal{O}} \frac{1}{S_0^p(x)}dx < \infty$, *there exists* $\widetilde{K}_p > 0$, *which is independent of* n *and initial conditions such that*

$$\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{S}_n^p(t,x)}dx \le e^{-t}\int_{\mathcal{O}} \frac{1}{S_0^p(x)}dx + \widetilde{K}_p.$$

Proof. For any $0 < \varepsilon < \frac{p\Lambda_*}{2}$, using Itô's Lemma ([10, Theorem 3.8]) and by direct

calculations, we have

$$
\begin{aligned}
&e^t \int_{\mathcal{O}} \frac{1}{\big(\overline{S}_n(t,x)+\varepsilon\big)^p}dx \\
&\quad = \int_{\mathcal{O}} \frac{1}{\big(S_0(x)+\varepsilon\big)^p}dx + \int_0^t e^s \int_{\mathcal{O}} \frac{1}{\big(\overline{S}_n(s,x)+\varepsilon\big)^p}dxds + \int_0^t e^s \int_{\mathcal{O}} \frac{-p}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}} \\
&\qquad \times \bigg(k_1\Delta\overline{S}_n(s,x) + \Lambda(x) - \mu_1(x)\overline{S}_n(s,x) - \frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}\bigg)dxds \\
&\qquad + \frac{1}{2}\int_0^t e^s \sum_{k=1}^n \int_{\mathcal{O}} \frac{p(p+1)a_{k,1}e_k^2(x)\overline{S}_n^2(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+2}}dxds \\
&\qquad + \sum_{k=1}^n \int_0^t e^s \Big[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-pe_k(x)\overline{S}_n(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}dx\Big]dB_{k,1}(s) \\
&\le \int_{\mathcal{O}} \frac{1}{\big(S_0(x)+\varepsilon\big)^p}dx + \int_0^t e^s \int_{\mathcal{O}} \frac{-pk_1\Delta\overline{S}_n(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}dxds \\
&\qquad + \int_0^t e^s \int_{\mathcal{O}} \frac{p}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}\Big(-\Lambda(x) + \frac{\varepsilon}{p} + \big(|\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2}a_1C_0^2\big) \\
&\qquad \times \overline{S}_n(s,x)\Big)dxds + \sum_{k=1}^n \int_0^t e^s \Big[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-pe_k(x)\overline{S}_n(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}dx\Big]dB_{k,1}(s) \\
&\le \int_{\mathcal{O}} \frac{1}{\big(S_0(x)+\varepsilon\big)^p}dx + \int_0^t \frac{pK_p^{p+1}2^p}{\Lambda_*^p}e^s ds \\
&\qquad + \sum_{k=1}^n \int_0^t e^s \Big[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-pe_k(x)\overline{S}_n(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}dx\Big]dB_{k,1}(s),
\end{aligned}
\tag{4.2}
$$

where $K_p = |\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2}a_1C_0^2$. In the above, we used the following facts

$$
\int_{\mathcal{O}} \frac{-pk_1\Delta\overline{S}_n(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}dx = -p(p+1)k_1\int_{\mathcal{O}} \frac{\big|\nabla\overline{S}_n(s,x)\big|^2}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+2}}dx \le 0 \ \text{ a.s.},
$$

and

$$
\begin{aligned}
&\int_{\mathcal{O}} \frac{p}{(\overline{S}_n(s,x)+\varepsilon)^{p+1}}\Big(-\Lambda(x) + \frac{\varepsilon}{p} + \big(|\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2}a_1C_0^2\big)\overline{S}_n(s,x)\Big)dx \\
&\quad \le \int_{\mathcal{O}} \frac{p}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{p+1}}\Big(-\frac{\Lambda_*}{2} + K_p\overline{S}_n(s,x)\Big)\mathbf{1}_{\{\overline{S}_n(s,x)\ge\frac{\Lambda_*}{2K_p}\}}dx \\
&\quad \le \frac{pK_p^{p+1}2^p}{\Lambda_*^p} \text{ a.s.}
\end{aligned}
$$

Hence, (4.2) implies that $\forall t \geq 0$, $\forall n \in \mathbb{N}$

$$\mathbb{E}\int_{\mathcal{O}} \frac{1}{\left(\overline{S}_n(t,x)+\varepsilon\right)^p}dx \leq e^{-t}\int_{\mathcal{O}} \frac{1}{\left(S_0(x)+\varepsilon\right)^p}dx + e^{-t}\int_0^t \frac{pK_p^{p+1}2^p}{\Lambda_*^p}e^s ds. \tag{4.3}$$

Letting $\varepsilon \to 0$, we have from the monotone convergence theorem that

$$\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{S}_n^p(t,x)}dx \leq e^{-t}\int_{\mathcal{O}} \frac{1}{S_0^p(x)}dx + e^{-t}\int_0^t \frac{pK_p^{p+1}2^p}{\Lambda_*^p}e^s ds. \tag{4.4}$$

The proof of the Lemma is completed. □

Noting that our initial condition are not assumed to satisfy $\int_{\mathcal{O}} \frac{1}{S_0^2(x)}dx < \infty$. However, we will prove that after some finite time, the solutions have the inverse functions that belong to $L^2(\mathcal{O},\mathbb{R})$ as the following Lemma.

Lemma 4.3. *For any $n \in \mathbb{N}$*

$$\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{S}_n^2(4,x)}dx \leq \ell_0,$$

where ℓ_0 depends only initial condition (independent of n).

Proof. By the following facts:

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}} \overline{S}_n(t,x)dx &= \int_{\mathcal{O}} S_0(x)dx + \int_0^t \mathbb{E}\int_{\mathcal{O}} \Big(k_1\Delta\overline{S}_n(s,x) + \Lambda(x) \\ &\qquad - \mu_1(x)\overline{S}_n(s,x) - \frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)\overline{I}_n(s,x)}\Big)dxds \\ &\leq \int_{\mathcal{O}} S_0(x)dx + t\left|\Lambda\right|_E,\end{aligned}$$

and $s^q \leq s+1$, $\forall s \in \mathbb{R} > 0, q \in [0,1]$, it is easy to show that there exists $\ell_1 > 0$ such that

$$\mathbb{E}\int_{\mathcal{O}} \overline{S}_n^q(t,x)dx \leq \ell_1, \text{ for any } t \in [0,1], q \in [0,1], \tag{4.5}$$

where ℓ_1 is independent of n. For any $\varepsilon > 0$, using Itô's Lemma ([10, Theorem 3.8])

again, we have

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}}\big(\overline{S}_n(1,x)+\varepsilon\big)^{\frac{1}{2}}dx\\
&\quad=\int_{\mathcal{O}}\big(S_0(x)+\varepsilon\big)^{\frac{1}{2}}dx+\int_0^1\mathbb{E}\int_{\mathcal{O}}\frac{1}{2\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{1}{2}}}\Big(k_1\Delta\overline{S}_n(s,x)\\
&\qquad+\Lambda(x)-\mu_1(x)\overline{S}_n(s,x)-\frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}\Big)dxds \qquad (4.6)\\
&\qquad-\frac{1}{8}\int_0^1\mathbb{E}\sum_{k=1}^{n}\int_{\mathcal{O}}\frac{a_{k,1}e_k^2(x)\overline{S}_n^2(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{3}{2}}}dxds.\\
&\geq\frac{1}{2}\int_0^1\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{1}{2}}}dxds-N_1\int_0^1\left(\mathbb{E}\int_{\mathcal{O}}\overline{S}_n^{\frac{1}{2}}(s,x)dx\right)ds,
\end{aligned}$$

where

$$N_1=\frac{|\mu_1|_E+|\alpha|_E+\frac{a_1C_0^2}{4}}{2}.$$

In view of (4.5) and (4.6), we have

$$\frac{1}{2}\int_0^1\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{1}{2}}}dxds\leq(1+N_1)\ell_1+\sqrt{\varepsilon},\quad\forall\varepsilon>0,$$

which implies that

$$\int_0^1\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(s,x)\big)^{\frac{1}{2}}}dxds\leq2(1+N_1)\ell_1.$$

or there exists $t_1=t_1(n)\in[0,1]$ such that

$$\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(t_1,x)\big)^{\frac{1}{2}}}dx\leq\frac{2(1+N_1)\ell_1}{\Lambda_*}.$$

Applying Lemma 4.2 and the Markov property of (S_n,I_n), we have

$$\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(t,x)+\varepsilon\big)^{\frac{1}{2}}}dx\leq\ell_2,\ \forall t\in[1,2],$$

for some ℓ_2 independent of n. We again have

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}}\big(\overline{S}_n(2,x)+\varepsilon\big)^{-\frac{1}{2}}dx\\
&\quad=\mathbb{E}\int_{\mathcal{O}}(\overline{S}_n(1,x)+\varepsilon)^{-\frac{1}{2}}dx-\int_1^2\mathbb{E}\int_{\mathcal{O}}\frac{1}{2\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{3}{2}}}\Big(k_1\Delta\overline{S}_n(s,x)\\
&\qquad+\Lambda(x)-\mu_1(x)\overline{S}_n(s,x)-\frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}\Big)dxds\\
&\qquad+\frac{3}{8}\int_1^2\mathbb{E}\sum_{k=1}^{n}\int_{\mathcal{O}}\frac{a_{k,1}e_k^2(x)\overline{S}_n^2(s,x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{3}{2}}}dxds\\
&\leq-\frac{1}{2}\int_1^2\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{3}{2}}}dxds+\mathbb{E}\int_{\mathcal{O}}\big(\overline{S}_n(1,x)+\varepsilon\big)^{-\frac{1}{2}}dx\\
&\qquad+N_2\int_1^2\left(\mathbb{E}\int_{\mathcal{O}}\overline{S}_n^{-\frac{1}{2}}(s,x)dx\right)ds,
\end{aligned}\tag{4.7}$$

where

$$N_2=\frac{|\mu_1|_E+|\alpha|_E+\frac{3a_1C_0^2}{4}}{2}.$$

Thus,

$$\int_1^2\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(s,x)+\varepsilon\big)^{\frac{3}{2}}}dxds\leq\ell_3,$$

for some ℓ_3 depending only on initial conditions. Letting $\varepsilon\to0$, we can obtain that for some $t_2=t_2(n)\in[1,2]$

$$\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\overline{S}_n^{\frac{3}{2}}(t_2,x)}dx\leq\ell_3,$$

which together with Lemma 4.2 implies that

$$\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{\big(\overline{S}_n(t,x)+\varepsilon\big)^{\frac{3}{2}}}dx\leq\ell_4,\ \forall t\in[2,3],$$

where ℓ_4 is some constant independent of n. Keeping this process we can obtain that there exists $t_3=t_3(n)\in[0,4]$, ℓ_5 such that

$$\mathbb{E}\int_{\mathcal{O}}\frac{\Lambda(x)}{(\overline{S}_n(t_3,x))^{\frac{5}{2}}}dx\leq\ell_5.$$

Therefore, it is possible to obtain the existence of two constants $t_4=t_4(n)\in[0,4]$ and ℓ_6 satisfying

$$\mathbb{E}\int_{\mathcal{O}}\frac{1}{\big(\overline{S}_n(t_4,x)\big)^2}dx<\ell_6.$$

The Lemma is proved by applying Lemma 4.2. □

In view of Lemma 4.2 and Lemma 4.3, we have

$$\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{S}_n^2(t,x)}dx \le e^{-t}\ell_0 + \widetilde{K}_2 \ \forall n\in\mathbb{N}, t\ge 4. \tag{4.8}$$

Noting that both ℓ_0 and $\widetilde{K}_2$ are independent of n; and ℓ_0 may depend on initial point but $\widetilde{K}_2$ is independent. By Itô's Lemma ([10, Theorem 3.8]) again and similar calculations in the process of getting (4.3) we have

$$\begin{aligned}
\mathbb{E}\int_{\mathcal{O}}\overline{I}_n(t,x)dx \ge& \mathbb{E}\int_{\mathcal{O}}\ln\big(\overline{I}_n(t,x)+\varepsilon\big)dx\\
=&\int_{\mathcal{O}}\ln\big(I_0(x)+\varepsilon\big)dx + \int_0^t\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{I}_n(s,x)+\varepsilon}\Big(k_2\Delta\overline{I}_n(s,x)\\
&-\mu_2(x)\overline{I}_n(s,x)+\frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}dxds\Big)\\
&-\frac{1}{2}\int_0^t\mathbb{E}\sum_{k=1}^n\int_{\mathcal{O}}\frac{a_{k,2}\overline{I}_n^2(s,x)e_k^2(x)}{\big(\overline{I}_n(s,x)+\varepsilon\big)^2}dxds\\
\ge&\int_{\mathcal{O}}\ln\big(I_0(x)+\varepsilon\big)dx-\big(\frac{a_2}{2}+|\mu_2|_E\big)t,\ \forall n\in\mathbb{N},\forall t>0, 0<\varepsilon<1.
\end{aligned}$$

As a consequence

$$\mathbb{E}\int_{\mathcal{O}}\overline{I}_n(t,x)dx \ge \mathbb{E}\int_{\mathcal{O}}\ln\overline{I}_n(t,x)dx \ge \int_{\mathcal{O}}\ln I_0(x)dx-\big(\frac{a_2}{2}+|\mu_2|_E\big)t>-\infty,\ \forall t>0. \tag{4.9}$$

That means

$$\mathbb{P}\big\{\overline{I}_n(t,x)>0 \text{ almost everywhere in } \mathcal{O}\big\}=1,\ \forall n\in\mathbb{N},\forall t>0. \tag{4.10}$$

On the other hand, combining Itô's Lemma and basic calculations implies that

$$\begin{aligned}
0\ge\mathbb{E}\int_{\mathcal{O}}\ln\frac{\overline{I}_n(t,x)+\varepsilon}{1+\overline{I}_n(t,x)}dx \ge& \int_{\mathcal{O}}\ln\frac{I_0(x)+\varepsilon}{1+I_0(x)}dx+\widehat{R}t\\
&-\int_0^t\mathbb{E}\int_{\mathcal{O}}\Big(\frac{\alpha(x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}+\frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\big(\overline{S}_n(s,x)+\overline{I}_n(s,x)\big)\big(\overline{I}_n(s,x)+1\big)}\Big)dxds\\
&-\int_0^t\mathbb{E}\int_{\mathcal{O}}\frac{\alpha(x)\varepsilon}{\overline{I}_n(s,x)+\varepsilon}dxds,\ \forall t>0, n\in\mathbb{N}, 0<\varepsilon<1.
\end{aligned}$$

Thus, $\forall t>0, n\in\mathbb{N}$, $0<\varepsilon<1$

$$\begin{aligned}
&\int_0^t\mathbb{E}\int_{\mathcal{O}}\Big(\frac{\alpha(x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}+\frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\big(\overline{S}_n(s,x)+\overline{I}_n(s,x)\big)\big(\overline{I}_n(s,x)+1\big)}\Big)dxds\\
&\quad\ge\mathbb{E}\int_{\mathcal{O}}\ln\frac{I_0(x)+\varepsilon}{1+I_0(x)}dx+\widehat{R}t-|\alpha|_E\int_0^t\mathbb{E}\int_{\mathcal{O}}\frac{\varepsilon}{\overline{I}_n(s,x)+\varepsilon}dxds.
\end{aligned} \tag{4.11}$$

Let $\varepsilon \to 0$ and using (4.10) and (4.11) we have

$$\begin{aligned}\int_0^t \mathbb{E}\int_{\mathcal{O}} \Big(\frac{\alpha(x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)} + \frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\big(\overline{S}_n(s,x)+\overline{I}_n(s,x)\big)\big(\overline{I}_n(s,x)+1\big)}\Big)dxds \\ \geq \int_{\mathcal{O}} \ln\frac{I_0(x)}{1+I_0(x)}dx + \widehat{R}t,\ \forall t>0, n\in\mathbb{N}.\end{aligned} \tag{4.12}$$

We have the following estimates:

$$\begin{aligned}|\alpha|_E \left(\mathbb{E}\int_{\mathcal{O}} \frac{\overline{I}_n^2(s,x)}{\big(1+\overline{I}_n(s,x)\big)^2}dx\right)^{\frac{1}{2}} &\geq \mathbb{E}\int_{\mathcal{O}} \frac{\alpha(x)\overline{I}_n(s,x)}{1+\overline{I}_n(s,x)}dx \\ &\geq \mathbb{E}\int_{\mathcal{O}} \frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\big(\overline{S}_n(s,x)+\overline{I}_n(s,x)\big)\big(\overline{I}_n(s,x)+1\big)}dx,\end{aligned}$$

and

$$\begin{aligned}&|\alpha|_E \left(\mathbb{E}\int_{\mathcal{O}} \frac{\overline{I}_n^2(s,x)}{\big(1+\overline{I}_n(s,x)\big)^2}dx\right)^{\frac{1}{2}} \left(\mathbb{E}\int_{\mathcal{O}} \left(\frac{1}{\overline{S}_n(s,x)}+1\right)^2 dx\right)^{\frac{1}{2}} \\ &\geq \mathbb{E}\int_{\mathcal{O}} \frac{\alpha(x)\overline{I}_n(s,x)}{1+\overline{I}_n(s,x)}\left(\frac{1}{\overline{S}_n(s,x)}+1\right)dx \geq \mathbb{E}\int_{\mathcal{O}} \frac{\alpha(x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)}dx,\end{aligned}$$

since

$$\frac{1+I}{S+I} = \frac{1}{S+I} + \frac{I}{S+I} \leq \frac{1}{S}+1.$$

Therefore, after some basic estimates, we can get from (4.12) that

$$\begin{aligned}&\int_4^t |\alpha|_E \left(\mathbb{E}\int_0 \frac{\overline{I}_n^2(s,x)}{\big(1+\overline{I}_n(s,x)\big)^2}dx\right)^{\frac{1}{2}} \left(1+\left(\mathbb{E}\int_{\mathcal{O}} \left(\frac{1}{\overline{S}_n(s,x)}+1\right)^2 dx\right)^{\frac{1}{2}}\right) ds \\ &\geq \int_4^t \mathbb{E}\int_{\mathcal{O}} \Big(\frac{\alpha(x)\overline{I}_n(s,x)}{\overline{S}_n(s,x)+\overline{I}_n(s,x)} + \frac{\alpha(x)\overline{S}_n(s,x)\overline{I}_n(s,x)}{\big(\overline{S}_n(s,x)+\overline{I}_n(s,x)\big)\big(\overline{I}_n(s,x)+1\big)}\Big)dxds \\ &\geq \int_{\mathcal{O}} \ln\frac{I_0(x)}{1+I_0(x)}dx + \widehat{R}t - 8|\alpha|_E,\end{aligned}$$

which together with (4.8) leads to

$$\begin{aligned}&\int_4^t |\alpha|_E \left(\mathbb{E}\int_0 \frac{\overline{I}_n^2(s,x)dx}{\big(1+\overline{I}_n(s,x)\big)^2}\right)^{\frac{1}{2}} \left(2\sqrt{e^{-s}\ell_0}+2\widetilde{K}_2^{\frac{1}{2}}+3\right) ds \\ &\geq \int_{\mathcal{O}} \ln\frac{I_0(x)}{1+I_0(x)}dx - 8|\alpha|_E + \widehat{R}t.\end{aligned}$$

Letting $n\to\infty$ yields

$$\begin{aligned}&\int_4^t |\alpha|_E \left(\mathbb{E}\int_{\mathcal{O}} \frac{I^2(s,x)}{\big(1+I(s,x)\big)^2}dx\right)^{\frac{1}{2}} \left(2\sqrt{e^{-s}\ell_0}+2\widetilde{K}_2^{\frac{1}{2}}+3\right) ds \\ &\geq \int_{\mathcal{O}} \ln\frac{I_0(x)}{1+I_0(x)}dx - 8|\alpha|_E + \widehat{R}t,\end{aligned} \tag{4.13}$$

which is easily followed by

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\mathbb{E}\int_{\mathcal{O}}\frac{I^2(s,x)}{\big(1+I(s,x)\big)^2}dx\right)^{\frac{1}{2}}ds\geq\frac{\widehat{R}}{|\alpha|_E\left(2\widetilde{K}_2^{\frac{1}{2}}+3\right)}.$$

As a consequence,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\mathbb{E}\int_{\mathcal{O}}\big(I^2(s,x)\wedge 1\big)dx\right)^{\frac{1}{2}}ds\geq R_I>0,$$

where R_I is independent of initial points. The proof of the theorem is completed by using dense property of $D(A_i^E)$ in E and continuous dependence on initial data of the solution. In more detailed, since constants $\widetilde{K}_2,\widehat{R}$ are independents of initial points, the estimates (4.8) and (4.13) still hold for the solution starting from arbitrary initial points $S_0,I_0\in E$ with $\int_{\mathcal{O}}-\ln I_0(x)dx<\infty$. □

Theorem 4.2. *For any nonnegative initial data $S_0,I_0\in E$, if*

$$(\mu_2-\alpha)_*=\inf_{x\in\overline{\mathcal{O}}}\big(\mu_2(x)-\alpha(x)\big)>0,\tag{4.14}$$

then the infected class will be extinct with exponential rate.

Proof. First, we define the linear operator $J:H\mapsto\mathbb{R}$ as following

$$\forall u\in H,Ju:=\int_{\mathcal{O}}u(x)dx.$$

By the properties of e^{tA_i}, $\forall u\in H,J(e^{tA_i}u-u)=0$ or $Ju=Je^{tA_i}u,\forall i=1,2$.

Now, as in the definition of mild solution, we have

$$I(t)=e^{tA_2}I_0+\int_0^t e^{(t-s)A_2}\Big(-\mu_2I(s)+\frac{\alpha S(s)I(s)}{S(s)+I(s)}\Big)ds+\int_0^t e^{(t-s)A_2}I(s)dW_2(s).$$

Hence, applying the operator J to both sides, using the properties of operator J and stochastic convolution (see [12, Proposition 4.15]), we obtain

$$\begin{aligned}\int_{\mathcal{O}}I(t,x)dx=&\int_{\mathcal{O}}I_0(x)dx+\int_0^t\int_{\mathcal{O}}\Big(-\mu_2(x)I(s,x)+\frac{\alpha(x)S(s,x)I(s,x)}{S(s,x)+I(s,x)}\Big)dxds\\&+\int_0^t J(e^{(t-s)A_2}I(s))dW_2(s),\end{aligned}$$

where $J(e^{(t-s)A_2}I(s))$ in the stochastic integral is understood as the process taking values in spaces of linear operator from H to $\mathbb{R}$, that is defined by

$$J(e^{(t-s)A_2}I(s))u:=\int_{\mathcal{O}}\Big(e^{(t-s)A_2}I(s)u\Big)(x)dx\quad\forall u\in H.$$

Since (2.1), it is easy to see that these integrals are well-defined. By taking the expectation on both sides and using the properties of stochastic integral [10, Proposition 2.9],

$$\mathbb{E}\int_{\mathcal{O}} I(t,x)dx = \int_{\mathcal{O}} I_0(x)dx + \mathbb{E}\int_0^t \int_{\mathcal{O}} \Big(-\mu_2(x)I(s,x) + \frac{\alpha(x)S(s,x)I(s,x)}{S(s,x)+I(s,x)}\Big)dxds$$

As a consequence,

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}} I(t,x)dx - \mathbb{E}\int_{\mathcal{O}} I(s,x)dx &= \int_s^t \mathbb{E}\int_{\mathcal{O}} \Big(-\mu_2(x)I(r,x) + \frac{\alpha(x)S(r,x)I(s,x)}{S(r,x)+I(r,x)}\Big)dxdr \\ &\le -(\mu_2-\alpha)_* \int_s^t \mathbb{E}\int_{\mathcal{O}} I(r,x)dxdr \end{aligned} \tag{4.15}$$

Hence, we can obtain the following estimate for the upper Dini derivative

$$\frac{d}{dt^+}\mathbb{E}\int_{\mathcal{O}} I(t,x)dx \le -(\mu_2-\alpha)_*\mathbb{E}\int_{\mathcal{O}} I(t,x)dx,\ \forall t \ge 0.$$

Since $(\mu_2-\alpha)_* > 0$, we can get that $\mathbb{E}\int_{\mathcal{O}} I(t,x)dx$ converges to 0 with exponential rate as $t\to\infty$. Hence, it easy to claim that the infected class goes extinct. □

Theorem 4.3. *Suppose that $W_2(t)$ is a space-independent Brownian motion with covariance $a_2 t$. For any nonnegative initial data $S_0, I_0 \in E$, if*

$$(\mu_2-\alpha)_* + \frac{a_2}{2} := \inf_{x\in\overline{\mathcal{O}}}\big(\mu_2(x)-\alpha(x)\big) + \frac{a_2}{2} > 0, \tag{4.16}$$

then when $p>0$ be sufficiently small that

$$R_p := (\mu_2-\alpha)_* + \frac{(1-p)a_2}{2} > 0,$$

we have

$$\limsup_{t\to\infty} \frac{\ln \mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p}{t} \le -pR_p < 0.$$

Proof. Since $W_2(t)$ is a space-independent Brownian motion, as the arguments in proof of Theorem 4.1, the mild solution $I(t)$ is also the solution in the strong sense if $I_0 \in D(A_i^E)$. Hence, with initial value in $D(A_i^E)$, we have

$$\int_{\mathcal{O}} I(t,x) = \int_0^t \int_{\mathcal{O}} \Big(-\mu_2(x)I(s,x) + \frac{\alpha(x)S(s,x)I(s,x)}{S(s,x)+I(s,x)}\Big)dxds + \int_0^t \int_{\mathcal{O}} I(s,x)dW_2(s)$$

By Itô's formula, we obtain that

$$\begin{aligned}
&\Big(\int_{\mathcal{O}} I(t,x)dx\Big)^p \\
&= \int_s^t \bigg(p\Big(\int_{\mathcal{O}} I(r,x)dx\Big)^{p-1} \int_{\mathcal{O}} \Big(-\mu_2(x)I(r,x) + \frac{\alpha(x)S(r,x)I(r,x)}{S(r,x)+I(r,x)}\Big)dx\bigg)\,dr \\
&+ \int_s^t p(1-p)\frac{a_2}{2}\left(\int_{\mathcal{O}} I(r,x)dx\right)^p dr + \int_s^t \left(\int_{\mathcal{O}} I(r,x)dx\right)^p dW_2(r) \\
&\le -pR_p \int_s^t \left(\int_{\mathcal{O}} I(r,x)dx\right)^p dr + \int_s^t \left(\int_{\mathcal{O}} I(r,x)dx\right)^p dW_2(r).
\end{aligned}$$

Since $\mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p < \infty$, we have

$$\mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p = \mathbb{E}\left(\int_{\mathcal{O}} I(s,x)dx\right)^p - pR_p \int_s^t \mathbb{E}\left(\int_{\mathcal{O}} I(r,x)dx\right)^p dr$$

which easily derives that

$$\frac{d}{dt^+}\mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p \le -pR_p\mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p.$$

An application of the differential inequality shows

$$\mathbb{E}\left(\int_{\mathcal{O}} I(t,x)dx\right)^p \le e^{-pR_p t}\left(\int_{\mathcal{O}} I(0,x)dx\right)^p, \tag{4.17}$$

for any $t \ge 0$ and initial values in $D(A_i^E)$. Since $D(A_i^E)$ is dense in E, (4.17) holds for each fixed t and any initial values in E. Then the desired result can be obtained. □

5. An Example

In this section, to demonstrate our results, we consider an example when the processes driving noise processes in equation (1.2) are standard Brownian motions and the recruitment rate, the death rates, the infection rate, and the recovery rate are

independent of space variable. Precisely, we consider the following equation

$$\begin{cases} dS(t,x) = \Big[k_1\Delta S(t,x) + \Lambda - \mu_1 S(t,x) - \dfrac{\alpha S(t,x)I(t,x)}{S(t,x)+I(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad +\sigma_1 S(t,x)dB_1(t) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ dI(t,x) = \Big[k_2\Delta I(t,x) - \mu_2 I(t,x) + \dfrac{\alpha S(t,x)I(t,x)}{S(t,x)+I(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad +\sigma_2 I(t,x)dB_2(t) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 \qquad \text{in} \quad \mathbb{R}^+\times\partial\mathcal{O}, \\ S(x,0) = S_0(x), I(x,0) = I_0(x) \qquad \text{in} \quad \overline{\mathcal{O}}, \end{cases} \tag{5.1}$$

where $\Lambda, \mu_1, \mu_2, \alpha$ are positive constants, and $B_1(t)$, $B_2(t)$ are independent standard Brownian motions. As we obtained above, for any initial values $0 \le S_0, I_0 \in E$, (5.1) has unique positive mild solution $S(t,x), I(t,x) \ge 0$. Moreover, the long-time behavior of the system is shown as the following theorem.

Theorem 5.1. *Let $S(t,x), I(t,x)$ be the positive mild solution (in fact also in the strong sense) of equation* (5.1).

(i) *For any non-negative initial values $S_0, I_0 \in E$, if $\alpha < \mu_2 + \dfrac{\sigma_2^2}{2}$, then the infected individual is extinct.*

(ii) *For the initial values $0 \le S_0, I_0 \in E$ satisfy*

$$\int_{\mathcal{O}} -\ln I_0(x)dx < \infty.$$

If $\alpha > \mu_2 + \dfrac{\sigma_2^2}{2}$, then the infected class is permanent.

Remark 5.1. As in Theorem 5.1, the sufficient condition for permanence is almost necessary condition. It is similar to the result for SIS reaction-epidemic model, which is shown in [33, Theorem 1.2].

6. Concluding Remarks

Being possibly among one of the first papers working on spatially inhomogeneous stochastic partial differential equation epidemic models, we hope that our effort will

provide some insights for subsequent study and investigation. For possible future study, we mention the following topics.

- First, there is a growing interest to use the so-called regime-switching stochastic models in various applications; see [39] for the treatment of switching diffusion models, in which both continuous dynamics and discrete events coexist. Such switching diffusion models have gained popularity with applications range from networked control systems to financial engineering. For instance, in a financial market model, one may use the random switching process to model the mode of the market (bull and bear). Such a random switching process can be built into the SPDE models considered here. The switching is used to reflect different random environment that are not reflected from the SPDE part of the model.
- Second, instead of systems driven by Brownian motions, we may consider systems driven by Lévy process; some recent work can be seen in [5]. One could work with SPDE models driven by Lévy processes. The recent work on switching jump diffusions [8] may also be adopted to the SPDE models.
- Finally, in terms of the mathematical development, various estimates about longtime properties were given in average norm although the solution is in the better space E. Our effort in the future will be to obtain stochastic regularity of the solution by using the methods in [7, 34, 35] so that it is possible to provide estimates in the sup-norm ($|\cdot|_E$). Nevertheless, some mathematical details need to carefully worked out. The result in turn, will be of interests for people working on real data. Some other properties such as strictly positivity of the solutions and sharper conditions for extinction and permanence are worthy of consideration.

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