DOUBLE BUBBLES ON THE LINE WITH LOG-CONVEX DENSITY f WITH $(\log f)'$ BOUNDED

NAT SOTHANAPHAN

Abstract. We extend results of Bongiovanni et al. [Bo] on double bubbles on the line with log-convex density to the case where the derivative of the log of the density is bounded. We show that the tie function between the double interval and the triple interval still exists but may blow up to infinity in finite time. For the first time, a density is presented for which the blowup time is positive and finite.

1. Introduction

Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density f. Bongiovanni et al. [Bo] show that a perimeter-minimizing double bubble enclosing volumes $V_1 \leq V_2$ is one of the following two configurations of Figure 1:

- A *double interval*: two contiguous intervals in equilibrium enclosing volumes V_1 and V_2;
- A *triple interval*: an interval symmetric about the origin enclosing volume V_1 flanked by two intervals on each side, each enclosing volume $V_2/2$.

They also show that, if $(\log f)'$ is unbounded, there is a tie function $\lambda(V_1)$ such that for $V_2 = \lambda(V_1)$, the double interval and the triple interval tie (have equal perimeter); for $V_2 > \lambda(V_1)$, the triple interval is uniquely perimeter minimizing; and for $V_2 < \lambda(V_1)$, the double interval is uniquely perimeter minimizing up to reflection [Bo, Thm. 4.15].

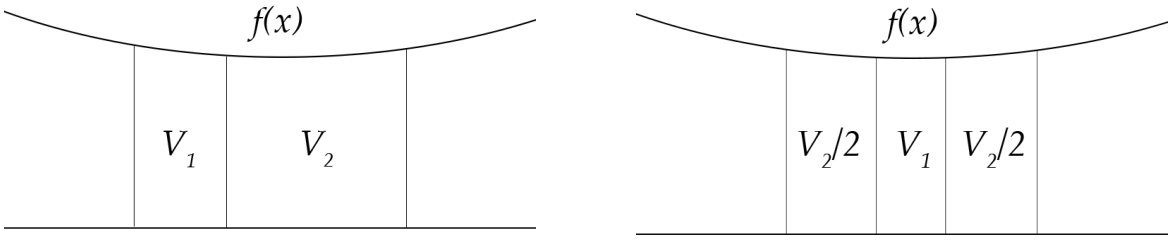

Figure 1. A double interval and a triple interval on the real line.

The goal of this note is to extend this result to the case where $(\log f)'$ is bounded. We show that the tie function λ still exists, but it may "blow up in finite time": it is defined only for $V_1 < V_0$ for some V_0 and approaches infinity as $V_1 \to V_0$. See Figure 2. This proves the conjecture stated at the end of [Bo, Section 4].

Our main result is as follows.

Theorem 1.1. *Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density. There exists a "blowup time" $0 \leq V_0 \leq \infty$ such that, for each $V_1 < V_0$, there is a unique $\lambda(V_1) > V_1$ with the following properties.*

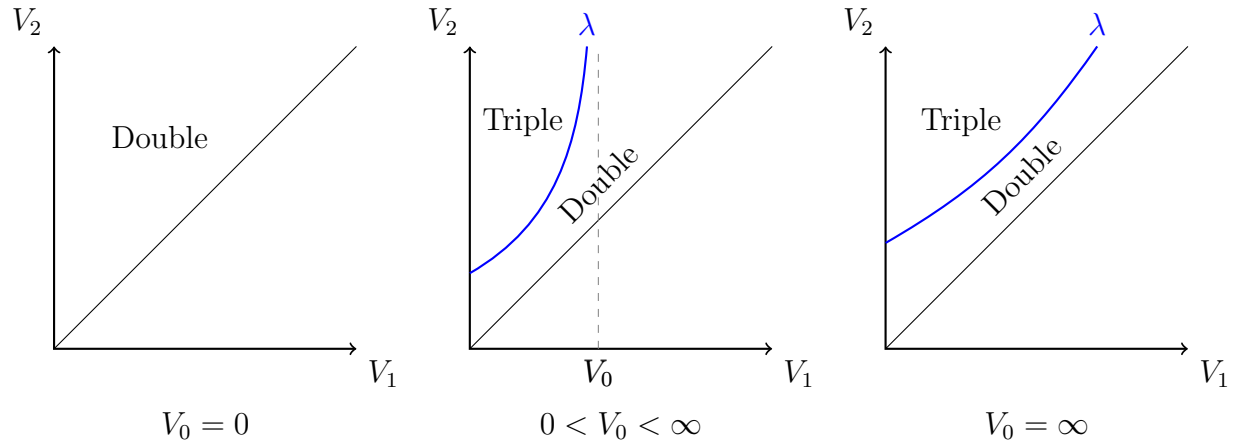

FIGURE 2. Three possibilities for the tie function λ between the double interval and the triple interval: λ not existing, λ blowing up to infinity in finite time, and λ existing for all time. Each case occurs for some symmetric, strictly log-convex, C^1 density (Ex. 3.11).

- *For $V_1 < V_0$ and $V_2 = \lambda(V_1)$, the double interval and the triple interval tie.*
- *For $V_1 < V_0$ and $V_2 > \lambda(V_1)$, the perimeter-minimizing double bubble is uniquely the triple interval.*
- *For either $V_1 \geq V_0$ or $V_1 < V_0$ and $V_1 \leq V_2 < \lambda(V_1)$, the perimeter-minimizing double bubble is uniquely the double interval up to reflection.*

See Figure 2. Moreover, each of the three types of blowup: $V_0 = 0$, $0 < V_0 < \infty$, and $V_0 = \infty$, occurs for some symmetric, strictly log-convex, C^1 density.

Proposition 2.2 gives more properties of the tie function λ. We provide a way to compute the blowup time V_0 in Proposition 3.8, and criteria for when the blowup time is infinity or zero in Corollaries 3.9 and 3.10. Finally, Example 3.11 presents densities exhibiting the three types of blowup. In particular, we show that it is possible for the blowup time to be positive and finite.

This note is organized as follows. Section 2 shows the existence of the blowup time V_0 and tie function λ in Theorem 1.1 by modifying the proof of [Bo, Thm. 4.15].

In Section 3, we derive a formula for the blowup time V_0 in Proposition 3.8. This result is then used to develop criteria for when $V_0 = \infty$ (Cor. 3.9) and $V_0 = 0$ (Cor. 3.10). Finally, we present examples of densities for which $V_0 = 0$, $0 < V_0 < \infty$, and $V_0 = \infty$ in Example 3.11.

2. Existence of Blowup Time

We establish the existence of the blowup time V_0 and tie function λ in Theorem 1.1. Our notation follows Bongiovanni et al. [Bo]. For prescribed volumes $V_1 \leq V_2$, let $\mu(V_1, V_2) = P_3 - P_2$ be the difference of perimeters of the triple interval and the double interval. In Bongiovanni et al. [Bo], Proposition 4.11 requires the extra hypothesis that $(\log f)'$ is unbounded, but Lemmas 4.8, 4.9, and 4.14 and Proposition 4.10 do not and hold for any symmetric, strictly log-convex, C^1 density.

The following quantity is useful in characterizing the blowup time.

Definition 2.1. Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density. For each V_1, define $\mu_\ell(V_1)$ to be

$$\mu_\ell(V_1) = \lim_{V_2 \to \infty} \mu(V_1, V_2),$$

the limit of the difference of perimeters of the triple interval and the double interval as $V_2 \to \infty$.

Notice that μ_ℓ is well defined because μ is strictly decreasing in V_2 [Bo, Lemma 4.9], with $-\infty \leq \mu_\ell < \infty$. Moreover, since μ is strictly increasing in V_1 [Bo, Lemma 4.9], μ_ℓ is nondecreasing.

We can now state the following proposition, which proves the existence of the blowup time V_0 and tie function λ.

Proposition 2.2. *Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density. There exists a "blowup time" $0 \leq V_0 \leq \infty$ such that, for each $V_1 < V_0$, there is a unique $\lambda(V_1) > V_1$ with the following properties.*

- *For $V_1 < V_0$ and $V_2 = \lambda(V_1)$, the double interval and the triple interval tie, and they are the only perimeter-minimizing double bubbles up to reflection.*
- *For $V_1 < V_0$ and $V_2 > \lambda(V_1)$, the perimeter-minimizing double bubble is uniquely the triple interval.*
- *For either $V_1 \geq V_0$ or $V_1 < V_0$ and $V_1 \leq V_2 < \lambda(V_1)$, the perimeter-minimizing double bubble is uniquely the double interval up to reflection.*
- *λ is strictly increasing, C^1, tends to infinity as $V_1 \to V_0$, and tends to a positive limit as $V_1 \to 0$.*

See Figure 2.

Proof. We modify the proof of [Bo, Thm. 4.15]. The idea is that μ_ℓ (Def. 2.1) should be negative in the region where λ is defined. With this in mind, let the blowup time V_0 be

$$V_0 = \sup\{V_1 : \mu_\ell(V_1) < 0\}, \quad 0 \leq V_0 \leq \infty, \tag{2.1}$$

where this quantity is zero if no V_1 satisfies the condition.

For $V_1 < V_0$, we now construct the tie function $\lambda(V_1)$. Because μ_ℓ is nondecreasing, $\mu_\ell(V_1) < 0$. Since $\mu(V_1, V_1) > 0$ [Bo, Prop. 4.10] and μ is strictly decreasing in V_2 [Bo, Lemma 4.9], there is a unique $\lambda(V_1) > V_1$ such that $\mu(V_1, \lambda(V_1)) = 0$. Then $\mu(V_1, V_2) < 0$ for $V_2 > \lambda(V_1)$ and $\mu(V_1, V_2) > 0$ for $V_1 \leq V_2 < \lambda(V_1)$. Because the double interval and the triple interval are the only possible perimeter minimizers [Bo, Prop. 4.6], we have proved the first three items in the case that $V_1 < V_0$.

Consider now the case $V_1 \geq V_0$. We must show that $\mu(V_1, V_2) > 0$ for all $V_2 \geq V_1$. If $V_0 = \infty$, this is trivial. If $V_0 = 0$, then $\mu_\ell(V_1) \geq 0$ for all $V_1 > 0$. Because μ is strictly decreasing in V_2, $\mu(V_1, V_2) > 0$ for all $V_1 \leq V_2$, as desired. Now suppose that $0 < V_0 < \infty$. We claim that $\mu_\ell(V_0) \geq 0$. If $\mu_\ell(V_0) < 0$, then for some $V_2 > V_0$, $\mu(V_0, V_2) < 0$. By continuity of μ, for some $V_0 < V_1 < V_2$, $\mu(V_1, V_2) < 0$, and so $\mu_\ell(V_1) < 0$, contradiction. Thus the claim holds. Becaue μ_ℓ is nondecreasing, $\mu_\ell(V_1) \geq 0$ for all $V_1 \geq V_0$. Then because μ is strictly decreasing in V_2, $\mu(V_1, V_2) > 0$ for all $V_2 \geq V_1 \geq V_0$.

It remains to prove the fourth item. Using exactly the same arguments as in the proof of [Bo, Thm. 4.15], one can show everything except the statement that $\lambda(V_1) \to \infty$ as $V_1 \to V_0$. We now show this last statement. Notice that we can assume $0 < V_0 < \infty$. Suppose to the contrary that λ increases to a finite limit L as $V_1 \to V_0$. Then $\mu(V_1, V_2) < 0$ for all $V_1 < V_0$ and $V_2 > L$. By continuity, $\mu(V_0, V_2) \leq 0$ for all $V_2 > L$, and so because μ is strictly decreasing in V_2, $\mu(V_0, V_2) < 0$ for all $V_2 > L$. Again by continuity, $\mu(V_1, V_2) < 0$ for some $V_1 > V_0$ and $V_2 > L$, implying that $\mu_\ell(V_1) < 0$, contradiction. Therefore, the proposition is proved. □

A question remains: what values can the blowup time V_0 take? Theorem 4.15 of Bongiovanni et al. shows that if $(\log f)'$ is unbounded, then $V_0 = \infty$, and their Example 4.13 gives a density with $V_0 = 0$. In the next section, we will devise a general procedure for determining the value of V_0 and finally present a density for which $0 < V_0 < \infty$.

3. Computing Blowup Time

We now seek to compute the blowup time V_0 in Proposition 2.2. We must first define some quantities. As in Bongiovanni et al. [Bo, Lemma 4.2], define the volume coordinate by

$$V = \int_0^x f,$$

where x is the positional coordinate. In particular, f is a strictly log-convex density if and only if f is strictly *convex* in volume coordinate.

Definition 3.1. On $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density f, define

$$L = \lim_{x\to\infty} (\log f)'(x) = \lim_{V\to\infty} f'(V),$$
$$M = \lim_{V\to\infty} f(2V) - 2f(V),$$

where x is the positional coordinate and V the volume coordinate.

Notice that L exists because f' is strictly increasing and M exists because, by taking derivatives, $f(2V) - 2f(V)$ is strictly increasing when $V > 0$. Observe that $0 < L \leq \infty$ and $-\infty < M \leq \infty$.

From now on, we work exclusively with volume coordinates. The following lemma shows a relationship between L and M.

Lemma 3.2. *$L = \infty$ implies $M = \infty$.*

Proof. Let $g(V) = f(2V) - 2f(V)$ for $V > 0$. Because g is strictly increasing, $g \leq M$. Notice that

$$\frac{f(2V)}{2V} - \frac{f(V)}{V} = \frac{g(V)}{2V},$$

so by telescoping,

$$\frac{f(2^n)}{2^n} - f(1) = \sum_{k=0}^{n-1} \frac{g(2^k)}{2^{k+1}} \leq \left(1 - \frac{1}{2^n}\right) M.$$

Since $L = \infty$, $f(V)/V \to \infty$ as $V \to \infty$. So as $n \to \infty$, the left-hand side diverges to infinity, implying that $M = \infty$. □

The following example shows that the converse of Lemma 3.2 is not true.

Example 3.3. Consider $f(V) = V\tan^{-1}V - \log(V^2+1)/2 + 1$. We can easily check that f is symmetric, C^1 with $f'(V) = \tan^{-1}V$, and strictly convex. Therefore $L = \pi/2$, and we can compute that $M = \infty$.

We now define another quantity, which will later be shown to be related to the left endpoint of the double interval.

Definition 3.4. Let f be a symmetric, strictly log-convex, C^1 density on $\mathbb{R}$ with $f'(V)$ bounded. For each V_1, define V^* to be the unique solution to

$$f'(V^*) + f'(V^* + V_1) = -L, \tag{3.1}$$

where L is as in Definition 3.1.

Notice that the left-hand side of (3.1) is strictly increasing in V^*, tends to $-2L$ as $V^* \to -\infty$, and tends to $2L$ as $V^* \to \infty$. So (3.1) has a unique solution, and V^* is well defined.

Let $\widetilde{V}$ be the leftmost endpoint of the double interval. By the equilibrium condition [Bo, Cor. 3.3], $\widetilde{V}$ is the unique solution to

$$f'(\widetilde{V}) + f'(\widetilde{V} + V_1) + f'(\widetilde{V} + V_1 + V_2) = 0. \tag{3.2}$$

Because f' is strictly increasing, we can see that $\widetilde{V}$ is strictly decreasing in both V_1 and V_2. We now characterize V^* as the limit of $\widetilde{V}$ as $V_2 \to \infty$.

Lemma 3.5. *Suppose that $L < \infty$. For a fixed V_1, $\lim_{V_2\to\infty} \widetilde{V} = V^*$.*

Proof. Since $\widetilde{V}$ is strictly decreasing in V_2, the limit $V_\ell = \lim_{V_2\to\infty}\widetilde{V}$ exists (it may be $-\infty$). It remains to show that $V_\ell = V^*$. By [Bo, Lemma 4.8], $\widetilde{V} > -(V_1+V_2)/2$, so $\widetilde{V} + V_1 + V_2 > (V_1+V_2)/2 \to \infty$ as $V_2 \to \infty$. Hence by taking $V_2 \to \infty$ in (3.2),

$$f'(V_\ell) + f'(V_\ell + V_1) + L = 0,$$

where we interpret $f'(-\infty) = -L$ in the case that $V_\ell = -\infty$. Finally, observe that $V_\ell = -\infty$ is not possible due to $L > 0$, so V_ℓ is finite and equals V^*. □

The next lemma collects some properties of V^* as $V_1 \to \infty$.

Lemma 3.6. *Suppose that $L < \infty$ and let V^* be as in Definition 3.4. Then V^* is strictly decreasing in V_1, $\lim_{V_1\to\infty} V^* = -\infty$, and $\lim_{V_1\to\infty}(V^* + V_1) = 0$.*

Proof. The fact that f' is strictly increasing implies that V^* is strictly decreasing in V_1. This and the fact that $V^* \le -V_1$ [Bo, Lemma 4.8] imply that $\lim_{V_1\to\infty} V^* = -\infty$. Now take $V_1 \to \infty$ in (3.1) to obtain $\lim_{V_1\to\infty}(V^* + V_1) = 0$. □

The following proposition gives a way to compute μ_ℓ (Def. 2.1) based on Lemma 3.5.

Proposition 3.7. *Let μ_ℓ be as in Definition 2.1, L and M be as in Definition 3.1, and V^* be as in Definition 3.4. Suppose that $L < \infty$. Then*

$$\mu_\ell(V_1) = 2f\left(\frac{V_1}{2}\right) - f(V^*) - f(V^* + V_1) - V^*L - M, \tag{3.3}$$

where this quantity is finite if $M < \infty$ and equals $-\infty$ if $M = \infty$.

Proof. By Lemma 3.5,

$$\begin{aligned}\mu_\ell(V_1) &= \lim_{V_2\to\infty}(P_3 - P_2)\\ &= \lim_{V_2\to\infty}\left[2f\left(\frac{V_1}{2}\right) + 2f\left(\frac{V_1+V_2}{2}\right) - f(\widetilde{V}) - f(\widetilde{V}+V_1) - f(\widetilde{V}+V_1+V_2)\right]\\ &= 2f\left(\frac{V_1}{2}\right) - f(V^*) - f(V^*+V_1) + \lim_{V_2\to\infty}\left[2f\left(\frac{V_1+V_2}{2}\right) - f(\widetilde{V}+V_1+V_2)\right].\end{aligned}$$

Rewrite the quantity in the last limit as

$$\left[2f\left(\frac{V_1+V_2}{2}\right) - f(V_1+V_2)\right] + \left[f(V_1+V_2) - f(\widetilde{V}+V_1+V_2)\right].$$

The first bracket tends to $-M$ as $V_2\to\infty$. By the Mean Value Theorem, the second bracket equals $-\widetilde{V}f'(V)$ for some $V > \widetilde{V}+V_1+V_2 > (V_1+V_2)/2 \to \infty$ as $V_2\to\infty$ [Bo, Lemma 4.8], so it tends to $-V^*L$ as $V_2\to\infty$. Therefore, μ_ℓ has the desired formula. □

We are now ready to state the proposition computing the blowup time V_0.

Proposition 3.8. *Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density f. Let L and M be as in Definition 3.1 and V^* be as in Definition 3.4. Then the blowup time V_0 of Proposition 2.2 can be computed as follows.*

- *If $L=\infty$ or $M=\infty$, then $V_0=\infty$.*
- *If $L<\infty$ and $M<\infty$, then $V_0<\infty$ and*

$$V_0 = \sup\{V_1 : \mu_\ell(V_1) < 0\} = \inf\{V_1 : \mu_\ell(V_1)\geq 0\}, \tag{3.4}$$

where μ_ℓ has the formula (3.3) *and the* sup *is* 0 *if there is no V_1 satisfying its condition.*

Proof. Bongiovanni et al. [Bo, Prop. 4.11] show that if $L=\infty$, then $V_0=\infty$. So suppose that $L<\infty$. By Proposition 3.7, μ_ℓ is given by (3.3).

The characterization of V_0 in (2.1) shows the first half of (3.4). Then fact that μ_ℓ is nondecreasing implies the second half of (3.4).

It remains to show that, still assuming $L<\infty$, $V_0=\infty$ if and only if $M=\infty$. If $M=\infty$, then $\mu_\ell(V_1)=-\infty$ by (3.3), and so $V_0=\infty$. Now suppose that $M<\infty$. By (3.3), we can write

$$\mu_\ell(V_1) = \left[2f\left(\frac{V_1}{2}\right) - f(V_1)\right] + [f(-V_1) - f(V^*)] - f(V^*+V_1) - V^*L - M.$$

Take $V_1\to\infty$ and apply Lemma 3.6. The first bracket tends to $-M$. The second bracket is $(-V_1-V^*)f'(V)$ for some V, which tends to 0 because $V^*+V_1\to 0$ as $V_1\to\infty$ and f' is bounded. Finally, $f(V^*+V_1)\to f(0)$ and $V^*L\to-\infty$. Hence $\mu_\ell\to\infty$ as $V_1\to\infty$, showing that $V_0<\infty$. □

From Proposition 3.8, we obtain two corollaries stating conditions for when $V_0=0$ and $V_0=\infty$.

Corollary 3.9. *Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density. Let L and M be as in Definition 3.1. Then $V_0=\infty$ if and only if $M=\infty$. In particular, $L=\infty$ implies $V_0=\infty$.*

Proof. By Proposition 3.8, $V_0 = \infty$ if and only if $L = \infty$ or $M = \infty$. This is equivalent to $M = \infty$ by Lemma 3.2. □

Corollary 3.10. *Consider $\mathbb{R}$ with a symmetric, strictly log-convex, C^1 density f. Let L and M be as in Definition 3.1. Then $V_0 = 0$, that is, the double interval is uniquely perimeter minimizing for all prescribed volumes if and only if $L < \infty$ and*

$$2f(0) - 2f(V) + VL - M \geq 0, \quad \text{where } V = (f')^{-1}(L/2). \tag{3.5}$$

Proof. By Corollary 3.9, $L = \infty$ implies $V_0 = \infty$. So suppose that $L < \infty$. By Proposition 3.8, $V_0 = 0$ if and only if $\mu_\ell(V_1) \geq 0$ for all $V_1 > 0$. Since μ_ℓ is nondecreasing, this is equivalent to $\lim_{V_1 \to 0} \mu_\ell(V_1) \geq 0$. From (3.1), $V^* \to -(f')^{-1}(L/2)$ as $V_1 \to 0$. Hence by (3.3), $\lim_{V_1 \to 0} \mu_\ell(V_1)$ equals the left-hand side of (3.5), proving the corollary. □

Finally, the following example shows densities with the three types of blowup: $V_0 = 0$, $0 < V_0 < \infty$, and $V_0 = \infty$. In particular, it shows that the case $0 < V_0 < \infty$ is indeed possible.

Example 3.11. All densities below are symmetric, C^1, and strictly convex in volume coordinate.

- [Bo, Ex. 4.13]. Consider $f(V) = |V| + e^{-|V|}$. We can compute $L = 1$, $M = 0$, and $(f')^{-1}(L/2) = \log 2$. By Corollary 3.10, we can check that $V_0 = 0$.
- Consider $f(V) = \sqrt{V^2 + 1} - 1/2$. Then $L = 1$, $M = 1/2$, and $(f')^{-1}(L/2) = 1/\sqrt{3}$. Corollaries 3.9 and 3.10 imply that $0 < V_0 < \infty$.
- Consider the Borell density $f(x) = e^{x^2}$. Then $(\log f)'(x) = 2x$ is unbounded, so $L = \infty$. By Corollary 3.9, $V_0 = \infty$.

With this example, we have completed the proof of Theorem 1.1.

Acknowledgements

I would like to thank Frank Morgan for checking the details and giving suggestions that improve the exposition of this paper.

References

[Bo] Eliot Bongiovanni, Alejandro Diaz, Leonardo Di Giosia, Jay Habib, Arjun Kakkar, Lea Kenigsberg, Dustin Ping, Dylanger Pittman, Nat Sothanaphan, Weitao Zhu, Double bubbles on the line with log-convex density, Anal. Geom. Metric Spaces 6 (2018), 64-88.

MSC2010: 49Q10

Key words and phrases: double bubble, density, isoperimetric

Courant Institute of Mathematical Sciences, New York University, New York, NY, 10003, USA

E-mail address: natsothanaphan@gmail.com