CENTRAL LIMIT THEOREMS FOR GAPS OF GENERALIZED ZECKENDORF DECOMPOSITIONS

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ABSTRACT. Zeckendorf proved that every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers $\{1, 2, 3, 5, \dots\}$. This has been extended to many other recurrence relations $\{G_n\}$ (with their own notion of a legal decomposition) and to proving that the distribution of the number of summands of an $m \in [G_n, G_{n+1})$ converges to a Gaussian as $n \to \infty$. We prove that for any non-negative integer g the average number of gaps of size g in many generalized Zeckendorf decompositions is $C_\mu n + d_\mu + o(1)$ for constants $C_\mu > 0$ and d_μ depending on g and the recurrence, the variance of the number of gaps of size g is similarly $C_\sigma n + d_\sigma + o(1)$ with $C_\sigma > 0$, and the number of gaps of size g of an $m \in [G_n, G_{n+1})$ converges to a Gaussian as $n \to \infty$. The proof is by analysis of an associated two-dimensional recurrence; we prove a general result on when such behavior converges to a Gaussian, and additionally re-derive other results in the literature.

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1. INTRODUCTION

1.1. **Previous Results.** Zeckendorf [Ze] proved that if the Fibonacci numbers are defined by $F_1 = 1, F_2 = 2$ and $F_{n+1} = F_n + F_{n-1}$, then every integer can be written as a sum of non-adjacent terms. The standard proof is by the greedy algorithm, though combinatorial approaches exist (see [KKMW]). More generally, one can consider other sequences of numbers and rules for a legal decomposition, and ask when a unique decomposition exists, and if it does how the summands are distributed.

There has been much work on these decomposition problems. In this paper we concentrate on the class of positive linear recurrences (see [Al, DDKMV] for signed decompositions, [DDKMMV] for f-decomposition, and [CFHMN1, CFHMN2, CFHMNPX] for some recurrences where the leading term vanishes, which can lead to different limiting behavior).

Definition 1.1. A *positive linear recurrence sequence* (*PLRS*) is a sequence $\{G_n\}$ satisfying

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{1.1}$$

with non-negative integer coefficients c_i with $c_1, c_L, L \geq 1$ and initial conditions $G_1 = 1$ and $G_n \;=\; c_1 G_{n-1} + c_2 G_{n-2} + \cdots + c_{n-1} G_1 + 1$ for $1 \leq n \leq L$.

Informally a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded. We first describe four results on these sequences (see [DG, Ha, Ho, Ke, LT, Len, LT, MW1, MW2, PT, Ste1, Ste2], especially [MW1] for proofs), and then discuss our new work.

Theorem 1.2 (Generalized Zeckendorf Theorem)**.** *Let $\{G_n\}$ be a positive linear recurrence sequence. For each integer $m > 0$, there exists a unique **legal** decomposition*

$$m \;=\; \sum_{i=1}^{N} a_i G_{N+1-i} \tag{1.2}$$

with $a_1 > 0$ and the other $a_i \geq 0$, and one of the following two conditions, which define a legal decomposition, holds.

1. *We have $N < L$ and $a_i = c_i$ for $1 \leq i \leq N$.*
2. *There exists an $s \in \{1, \dots, L\}$ such that $a_1 = c_1, a_2 = c_2, \dots, a_{s-1} = c_{s-1}$ and $a_s < c_s$, $a_{s+1}, \dots, a_{s+\ell} = 0$ for some $\ell \geq 0$, and $\{b_i\}_{i=1}^{N-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is either legal or empty.*

The next result concerns the average number of summands in decompositions, generalizing Lekkerkerker's [Lek] work on this problem for the Fibonacci numbers. Given $\{G_n\}$ a PLRS, we have the legal decomposition

$$m \;=\; \sum_{i=1}^{N} a_i G_{N+1-i} \;=\; G_{i_1} + G_{i_2} + \cdots + G_{i_k} \tag{1.3}$$

for some positive integer $k = a_1 + a_2 + \cdots + a_N$ and $i_1 \geq i_2 \geq \cdots \geq i_k$. The *gaps* in the decomposition of m are the numbers $i_1 - i_2, i_2 - i_3, \dots, i_{k-1} - i_k$ (for example, $101 = F_{10} + F_5 + F_3 + F_1$, and thus has gaps 5, 2, and 2). Throughout this paper we let $\hat{k}(m)$ denote the number of summands of m and $k_g(m)$ the number of gaps of size g in m's decomposition. Let $\hat{K}_n$ be the random variable equal to $\hat{k}(m)$ for an m chosen uniformly from $[G_n, G_{n+1})$, and let $K_{g,n}$ be a random variable equal to $k_g(m)$ for an m chosen uniformly from $[G_n, G_{n+1})$.

Thus $k_g(m)$ is a decomposition of $\hat{k}(m)$, as

$$\hat{k}(m) \;=\; 1 + \sum_{g=0}^{\infty} k_g(m). \tag{1.4}$$

Theorem 1.3 (Generalized Lekkerkerker's Theorem for PLRS)**.** *Let $\{G_n\}$ be a PLRS, let $\hat{K}_n$ be the random variable defined above, and let $\mu_n = \mathbf{E}[\hat{K}_n]$. Then there exist constants $C_\mu > 0$, d_μ, and $\gamma_\mu \in (0,1)$ depending only on L and the c_i's of the recurrence relation such that*

$$\mu_n \;=\; C_\mu n + d_\mu + O(\gamma_\mu^n). \tag{1.5}$$

Theorem 1.4 (Variance is Linear for PLRS)**.** *Let $\{G_n\}$ be a PLRS, let $\hat{K}_n$ be the random variable defined above, and let $\sigma_n^2 = \mathbf{Var}[\hat{K}_n]$. Then there exist constants $C_\sigma > 0$, d_σ, and $\gamma_\sigma \in (0,1)$ depending only on L and the c_i's of the recurrence relation such that*

$$\sigma_n^2 \;=\; C_\sigma n + d_\sigma + O(\gamma_\sigma^n). \tag{1.6}$$

Theorem 1.5 (Gaussian Behavior for Number of Summands in PLRS)**.** *Let $\{G_n\}$ be a PLRS and let $\hat{K}_n$ be the random variable defined above. The mean $\hat{\mu}_n$ and variance $\hat{\sigma}_n^2$ of $\hat{K}_n$ grow linearly in n, and $(\hat{K}_n - \hat{\mu}_n)/\hat{\sigma}_n$ converges weakly to the standard normal $N(0,1)$ as $n \to \infty$.*

Surprisingly, much less has been written on $k_g(m)$ and $K_{g,n}$. We show that similar Central Limit results hold for gaps. The techniques we introduce to prove these results allow us to easily prove some results already in the literature which are often done through tedious technical calculations, which we can bypass.

1.2. **New Results.** Beckwith et al. [BBGILMT] and Bower et al. [BILMT] (see also [DFFHMPP]) explored the distribution of gaps in Generalized Zeckendorf Decompositions arising from PLRS, proving (as $n \to \infty$) exponential decay in the probability of a gap of length g in the decomposition of $m \in [G_n, G_{n+1})$ as g grows and determining that the distribution of the longest gap between summands behaves similarly to what is seen in the distribution of the longest run of heads in tossing a biased coin. We improve on the first result and establish lower order terms (previous work had $O(1)$ instead of d below), then prove the variance has a similar linear behavior, and finally show Gaussian behavior for fixed g.

Theorem 1.6 (Generalized Lekkerkerker's Theorem for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Let $\{G_n\}$ be a PLRS. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is non-trivial (i.e., is not the constant 0) for $n \geq n_0$. Let $\mu_{g,n} = \mathbf{E}[K_{g,n}]$. Then there exists constants $C_{\mu,g} > 0$, $d_{\mu,g}$, and $\gamma_{\mu,g} \in (0,1)$ depending only on g, L, and the c_i's of the recurrence relation such that*

$$\mu_{g,n} \;=\; C_{\mu,g} n + d_{\mu,g} + O(\gamma_{\mu,g}^n). \tag{1.7}$$

Theorem 1.7 (Variance is Linear for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Let $\{G_n\}$ be a PLRS. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is non-trivial (i.e., is not the constant 0) for $n \geq n_0$. Let $\sigma_{g,n}^2 = \mathbf{Var}[K_{g,n}]$. Then there exists constants $C_{\sigma,g} > 0$, $d_{\sigma,g}$, and $\gamma_{\sigma,g} \in (0,1)$ depending only on g, L, and the c_i's of the recurrence relation such that*

$$\sigma_{g,n}^2 \;=\; C_{\sigma,g} n + d_{\sigma,g} + O(\gamma_{\sigma,g}^n). \tag{1.8}$$

These two theorems follow as intermediate results in the proof of the next theorem, which is the main result of this paper. The next theorem proves that we also obtain Gaussian behavior if we fix the gap size and if that gap size occurs; note we have to be careful, as there are never gaps of length 1 between summands in Zeckendorf decompositions arising from Fibonacci numbers, and we must make sure to exclude such cases.

Theorem 1.8 (Gaussian Behavior for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is non-trivial for $n \geq n_0$. The mean $\mu_{g,n}$ and variance $\sigma_{g,n}^2$ of $K_{g,n}$ grow linearly in n, and $(K_{g,n} - \mu_{g,n})/\sigma_{g,n}$ converges weakly to the standard normal $N(0,1)$ as $n \to \infty$.*

Our proof uses the fact that $p_{g,n,k}$, the number of $m \in [G_n, G_{n+1})$ with exactly k gaps of size g, satisfies a homogenous two-dimensional recursion (see §3). We then prove that under certain conditions, the "rows" $\{p_{g,n,k}\}_{k\geq 0}$ of these two-dimensional homogenous recursions converge to a Gaussian (see §2.2). Our proof depends only on the recurrence relation and not the initial conditions of the recursion. This is reasonable because if the number of initial conditions is finite, then as n grows large the recurrence essentially behaves as if all the initial conditions were lumped on a single term. This result should not be surprising, as a specific case is the two dimensional recurrence $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$. This recurrence produces the binomials $\binom{n}{k}$ (or sums of them), and binomials $\binom{n}{\cdot}$ are well known to converge to a normal distribution.

Similar to the work of Miller and Wang [MW1, MW2], we use the method of moments to prove that our random variables converge to Gaussians. More precisely, we prove that the moments of the n^{th} random variable $K_{g,n}$ (or $\hat{K}_n$), $\tilde{\mu}_n(m)$, satisfy

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n(2m)}{\mu_n(2)^m} \;=\; (2m-1)!! \quad \text{and} \quad \lim_{n\to\infty} \frac{\tilde{\mu}_n(2m+1)}{\mu_n(2)^{m+\frac{1}{2}}} \;=\; 0. \tag{1.9}$$

While Miller and Wang use generating functions to directly compute the moments $\tilde{\mu}_n(m)$, we instead compute them recursively (see, for example, Theorem 2.10), which leads to a cleaner computation and could be of use in other investigations.

2. Preliminaries

We first collect some notation we will use throughout the paper, then isolate two technical lemmas on convergence, and then apply these to prove Gaussian behavior for certain two dimensional recurrences. This final result is the basis for the proof of our main result on Gaussian behavior of gaps for a fixed g, Theorem 1.8.

2.1. Notation.

For this paper, all big-Os are taken as $n \to \infty$ unless otherwise specified.

For a polynomial $A(x) = \sum_{k=0}^{d} a_k x^k$, let $\left[x^k\right](A(x)) = a_k$ be the notation for extracting the k^{th} coefficient of A.

For a real number $\lambda_1 > 0$, a polynomial $A(x)$ has the *maximum root property with maximum root* λ_1 if λ_1 is a root of A with multiplicity 1 and all other roots have magnitude strictly less than λ_1.

A sequence of real numbers $\{a_n\}$ *converges exponentially quickly to* a if $\lim_{n\to\infty} a_n = a$ and there exists $\gamma \in (0,1)$ and a constant C such that $|a - a_n| \leq C\gamma^n$ for all n (alternatively, $a_n = a + O(\gamma^n)$).

Let d be a fixed positive integer, and let $\{A_n(x)\}$ be a sequence of degree-d polynomials where $A_n(x) = \sum_{j=0}^{d} a_{j,n} x^j$. We say $\{A_n(x)\}$ *converges exponentially quickly to* $\bar{A}(x) = \sum_{j=0}^{d} \bar{a}_j x^j$ if $\{a_{j,n}\}_{n\in\mathbb{N}}$ converges exponentially quickly to $\bar{a}_j$ for $j = 0, 1, \ldots, d$.

From the above definitions we immediately obtain the following useful result.

Lemma 2.1. *Let $\{a_n\}, \{b_n\}$ be sequences that converge exponentially quickly to a and b respectively. Then*

(1) *$\{a_n + b_n\}$ converges exponentially quickly to $a + b$,*
(2) *$\{a_n - b_n\}$ converges exponentially quickly to $a - b$,*

(3) $\{a_n \cdot b_n\}$ *converges exponentially quickly to* $a \cdot b$,
(4) *if* $b_n \neq 0$ *for all* n *and* $b \neq 0$, *then* $\{a_n/b_n\}$ *converges exponentially quickly to* a/b.

We end by recalling some useful results from [BBGILMT].

Theorem 2.2 (Generalized Binet's Formula)**.** *Consider a linear recurrence (not necessarily a positive linear recurrence)*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{2.1}$$

with non-negative integer coefficients c_i *with* $c_1, c_L, L \geq 1$. *Then the characteristic polynomial* $x^L - (c_1 x^{L-1} + c_2 x^{L-2} + \cdots + c_L)$ *has the maximum root property with maximum root* $\lambda_1 > 1$, *and there exist constants such that* $G_n = a_1 \lambda_1^n + O(n^{L-2}\lambda_2^n)$ *with* $|\lambda_2| < \lambda_1$ *the second largest root in absolute value.*

Corollary 2.3. *Consider a linear recurrence (not necessarily a positive linear recurrence)*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{2.2}$$

with non-negative integer coefficients c_i *with* $c_1, c_L, L \geq 1$ *and arbitrary initial conditions. Let* λ_1 *be the maximum root of the characteristic polynomial given by Theorem 2.2. Then, for every fixed positive integer* i, G_{n-i}/G_n *converges to* $1/\lambda_1^i$ *exponentially quickly.*

2.2. **Convergence on non-homogenous linear recurrences with noise.** The following two lemmas follow immediately from the previous definitions and book-keeping, and play a key role in the convergence analysis later. In particular, these two lemmas allow us to pin down the exact behavior of the moments of our random variables $K_{g,n}$ as we prove convergence to the standard normal (see Lemmas 2.11 and 2.12).

Lemma 2.4. *Let* i_0 *be a positive integer. Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers and for each* $1 \leq i \leq i_0$ *let* $\{s_{i,n}\}_{n\in\mathbb{N}}$ *be a sequence of non-negative real numbers such that* $\sum_{i=1}^{i_0} s_{i,n} = 1$ *for all* n. *With slight abuse of notation, suppose also that there exist constants* $\bar{r}$ *and* $\bar{s}_i$ *for* $1 \leq i \leq i_0$, *along with* $\gamma_r, \gamma_s \in (0,1)$ *such that*

$$r_n \;=\; \bar{r} + O(\gamma_r^n), \qquad s_{i,n} \;=\; \bar{s}_i + O(\gamma_s^n). \tag{2.3}$$

Suppose further that the polynomial

$$S(x) \;=\; x^{i_0} - \sum_{i=1}^{i_0} \bar{s}_i x^{i_0-i} \tag{2.4}$$

has the maximum root property with maximum root 1. Let $\{a_n\}_{n\geq n_0}$ *be a sequence with arbitrary initial conditions* $a_{n_0}, \ldots, a_{n_0+i_0-1}$, *and for* $n \geq n_0 + i_0$,

$$a_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n} a_{n-i}\right) + r_n. \tag{2.5}$$

Then there exists a positive integer d *and a real number* $\gamma \in (0,1)$ *such that*

$$a_n \;=\; \frac{\bar{r}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \cdot n + d + O(\gamma^n). \tag{2.6}$$

Roughly speaking, Lemma 2.4 is true because, modulo exponentially small terms, every a_n is a constant plus the weighted average of previous a_{n-i}s, so it should be linear in n.

Proof. It suffices to prove the lemma for $n_0 = 0$. Let $b_n = a_n - \frac{\bar{r}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \cdot n$. Set $\gamma = \max(\gamma_r, \gamma_s)$. Simple manipulations yield

$$\begin{aligned} b_n &= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + r_n - \frac{\sum_{i=1}^{i_0} i \cdot s_{i,n}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \cdot \bar{r} \\ &= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \bar{r} \cdot \left(\frac{r_n}{\bar{r}} - \frac{\sum_{i=1}^{i_0} i \cdot \bar{s}_{i,n}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \right) \\ &= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \bar{r} \cdot ((1 + O(\gamma^n)) - (1 + O(\gamma^n))) \\ &= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + O(\gamma^n). \end{aligned} \tag{2.7}$$

We finish by showing that the sequence b_n converges exponentially quickly to a constant. Simple algebra yields that b_n is bounded (see Appendix A). Let M be an integer such that $|b_n| \le M$ for all n. Then

$$\begin{aligned} b_n - \sum_{i=1}^{i_0} \bar{s}_i b_{n-i} &= b_n - \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \sum_{i=1}^{i_0} (s_{i,n} - \bar{s}_i) b_{n-i} \\ &\le O(\gamma^n) + \sum_{i=1}^{i_0} O(\gamma^n) \cdot b_{n-i} \\ &\le O(\gamma^n) + \sum_{i=1}^{i_0} O(\gamma^n) \cdot M = O(\gamma^n). \end{aligned} \tag{2.8}$$

Thus we can write

$$b_n = \left(\sum_{i=1}^{i_0} \bar{s}_i b_{n-i} \right) + f(n) \tag{2.9}$$

for some function $f : \{i_0, i_0 + 1, \dots\} \to \mathbb{R}$ such that $f(n) = O(\gamma^n)$ as $n \to \infty$. Let $\alpha_f > 0$ be a constant such that $|f(n)| \le \alpha_f \gamma^n$.

From here, the intuition for the finish is as follows. If $f(n) = 0$ for all n, then Theorem 2.2 implies that b_n approaches a constant exponentially quickly. However, since $\gamma < 1$, we have that b_n should still approach a constant exponentially quickly when $f(n) = O(\gamma^n)$.

Let $\{b_n^{(\text{init})}\}_{n \in \mathbb{N}}, \{b_n^{(i_0)}\}_{n \in \mathbb{N}}, \{b_n^{(i_0+1)}\}_{n \in \mathbb{N}}, \dots$ be sequences defined (for $m \ge i_0$) by

$$\begin{aligned} b_n^{(\text{init})} &= \begin{cases} b_n & 0 \le n \le i_0 - 1 \\ \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(\text{init})} & n > i_0 \end{cases} \\ b_n^{(m)} &= \begin{cases} 0 & n < m \\ f(m) & n = m \\ \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(m)} & n > m. \end{cases} \end{aligned} \tag{2.10}$$

By induction, we can verify that

$$b_n = b_n^{(\text{init})} + \sum_{m=i_0}^{\infty} b_n^{(m)} \tag{2.11}$$

for all n (see Appendix B). By the restrictions of s_i, the characteristic polynomials of $\{b_n^{(\text{init})}\}$ and $\{b_n^{(m)}\}$ are equal to $S(x)$ in (2.4) and thus have the maximum root property with maximum root 1. Hence, by generalized Binet's formula $\{b_n^{(\text{init})}\}$ and $\{b_n^{(m)}\}$ all converge to a constant. Suppose that $\{b_n^{(\text{init})}\}$ converges to $\bar{b}^{(\text{init})}$ and $\{b_n^{(m)}\}$ converges to $\bar{b}^{(m)}$ for each $m \geq i_0$. Let $\lambda_2 < 1$ be the second largest magnitude of a root of $S(x)$. Choose $\gamma_1 \in (\max(\gamma, \lambda_2), 1)$. By the generalized Binet's formula

$$b_n^{(\text{init})} - \bar{b}^{(\text{init})} \;=\; O(n^{i_0} \cdot \lambda_2^n) \;\leq\; O(\gamma_1^n), \tag{2.12}$$

so there exists $\alpha_b^{(1)}$ such that

$$\left| b_n^{(\text{init})} - \bar{b}^{(\text{init})} \right| \;\leq\; \alpha_b^{(1)} \gamma_1^n. \tag{2.13}$$

For all m, we can bound $b_n^{(m)}$ similarly. However, note that for all m, $\{b_n^{(m)}/f(m)\}_{n\in\mathbb{N}}$ is the same sequence with the indices shifted. Thus there exists $\alpha_b^{(2)}$ such that

$$b_n^{(m)} - \bar{b}^{(m)} \;\leq\; \alpha_b^{(2)} f(m) \gamma_1^{n-m} \;\leq\; \alpha_b^{(2)} \alpha_f \gamma^m \gamma_1^{n-m}. \tag{2.14}$$

Set $\alpha_b = \max(\alpha_b^{(1)}, \alpha_b^{(2)})$. Then

$$\begin{aligned} |b_n - b| \;&\leq\; \left| b^{(\text{init})} - b_n^{(\text{init})} \right| + \sum_{m=i_0}^{\infty} \left| b^{(m)} - b_n^{(m)} \right| \\ &\leq\; \alpha_b \gamma_1^n + \sum_{m=i_0}^{\infty} \alpha_b \alpha_f \gamma_1^n \left(\frac{\gamma}{\gamma_1} \right)^m \\ &\leq\; \gamma_1^n \left(\alpha_b + \alpha_b \alpha_f \cdot \left(\frac{\gamma}{\gamma_1} \right)^{i_0} \cdot \frac{1}{1 - \frac{\gamma}{\gamma_1}} \right) \;=\; O(\gamma_1^n) \end{aligned} \tag{2.15}$$

as desired. □

The next lemma generalizes Lemma 2.4.

Lemma 2.5. *Let D be a nonnegative integer and let i_0 be a positive integer. Let $\{R_n(x)\}_{n\in\mathbb{N}}$ be a sequence of D degree polynomials with $R_n(x) = \sum_{j=0}^{D} r_{j,n} x^j$. For each $1 \leq i \leq i_0$ let $\{s_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{i=1}^{i_0} s_{i,n} = 1$ for all n. Suppose also that there exist a polynomial $\bar{R}(x) = \bar{r}_D x^D + \bar{r}_{D-1} x^{D-1} + \cdots + \bar{r}_0$ and real numbers $\bar{s}_i$ for $1 \leq i \leq i_0$, along with $\gamma_r, \gamma_s \in (0,1)$, such that, for all $0 \leq j \leq D$ and $1 \leq i \leq i_0$,*

$$r_{j,n} \;=\; \bar{r}_j + O(\gamma_r^n), \qquad s_{i,n} \;=\; \bar{s}_i + O(\gamma_s^n). \tag{2.16}$$

Suppose further that the polynomial, $S(x) = x^{i_0} - \sum_{i=1}^{i_0} \bar{s}_i x^{i_0 - i}$ has the maximum root property with maximum root 1. Let $\{a_n\}_{n \geq n_0}$ be a sequence with arbitrary initial conditions $a_{n_0}, \ldots, a_{n_0+i_0-1}$, and for $n \geq n_0 + i_0$,

$$a_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n} a_{n-i} \right) + R_n(n). \tag{2.17}$$

Then there exists a degree $D+1$ polynomial $Q(x)$ and a $\gamma_1 \in (0,1)$ such that

$$a_n \;=\; Q(n) + O(\gamma_1^n) \tag{2.18}$$

where

$$\left[x^{D+1} \right] (Q(x)) \;=\; \frac{\bar{r}_D}{(D+1) \sum_{i=1}^{i_0} i \cdot \bar{s}_i}. \tag{2.19}$$

In contrast to in Lemma 2.4, a_n is, modulo exponentially small terms, a D degree polynomial in n plus the weighted average of previous a_{n-i}s. Since for any D degree polynomial $A(x)$ the sum $A(1) + A(2) + \cdots + A(n)$ is an $D+1$ degree polynomial in n, we expect a_n to essentially be a $D+1$ degree polynomial in n.

Proof. We proceed by induction on D, the degree of the polynomials $R_n(x)$. Lemma 2.4 establishes the base case $D = 0$. Now assume $D > 0$ and that the assertion is true for $D-1$. Let $b_n = a_n - C \cdot n^{D+1}$ for $C = \frac{\bar{r}_D}{(D+1)\sum_{i=1}^{i_0} i \cdot \bar{s}_i}$. Straightforward manipulations yield

$$b_n \;=\; \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \sum_{j=0}^{D-1} n^j \cdot \left[\left(\sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} i^{D+1-j} \right) + r_{j,n} \right] + f(n) \tag{2.20}$$

for some function $f(n) \le O(\gamma_0^n)$ for some $\gamma_0 \in (0,1)$ (see Appendix C). The constant C is chosen so that the right side contains an $D-1$ degree polynomial in n, as opposed to an D degree polynomial, which is the case in the recursion for $\{a_n\}$. Let $R_n^*(x) = \sum_{j=0}^{D-1} r_{j,n}^* x^j$ be the polynomial given by

$$\begin{aligned} r_{0,n}^* \;&:=\; \left(\sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1} \binom{D+1}{0} i^{D+1} \right) + r_{0,n} + f(n) \\ r_{j,n}^* \;&:=\; \left(\sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} i^{D+1-j} \right) + r_{j,n} \end{aligned} \tag{2.21}$$

for $1 \le j \le D-1$. Since, as $n \to \infty$, $s_{i,n}$ converges exponentially fast to $\bar{s}_i$, $r_{j,n}$ converges exponentially fast to $\bar{r}_j$, and $f(n)$ converges exponentially fast to 0, we have $r_{j,n}^*$ converges exponentially fast to

$$\lim_{n\to\infty} r_{j,n}^* \;=\; \left(\sum_{i=1}^{i_0} C \bar{s}_i (-1)^{D+1-j} \binom{D+1}{j} i^{D+1-j} \right) + \bar{r}_j. \tag{2.22}$$

for $0 \le j \le D-1$. Writing

$$b_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n} b_{n-i} \right) + R_n^*(n), \tag{2.23}$$

we can apply the induction hypothesis to b_n to obtain a a degree D polynomial $Q^*(x)$ such that $b_n = Q^*(n) + O(\gamma_1^n)$ for some $\gamma_1 \in (0,1)$. Set $Q(x) = Q^*(x) + Cx^{D+1}$. Then $Q(x)$ is a degree $D+1$ polynomial satisfying $a_n = Q(x) + O(\gamma_1^n)$, as desired. □

2.3. **Gaussian Behavior of 2D Recursions.** The result below is the key ingredient in proving Gaussian behavior of gaps.

Theorem 2.6. *Let i_0 and j_0 be positive integers. Let $t_{i,j}$ be real numbers for $1 \le i \le i_0, 0 \le j \le j_0$ such that for all i, $\hat{t}_i := \sum_{j=0}^{j_0} t_{i,j} \ge 0$. Suppose that the polynomial $T(x) = x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0-i}$ has the maximum root property with some maximum root λ_1. Suppose $p_{n,k}$ is a two-dimensional recurrence sequence satisfying, for $n \ge n_0$,*

$$p_{n,k} \;=\; \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} p_{n-i,k-j}. \tag{2.24}$$

Suppose further that $p_{n,k} \geq 0$ for all n and k, $p_{n,k} = 0$ when $n < 0$ or $k < 0$, and finitely many $p_{n,k}$ are nonzero for $n < n_0$. Let X_n be the random variable whose mass function is proportional to $p_{n,k}$ over varying k so that

$$\mathbf{Pr}[X_n = k] \;=\; \frac{p_{n,k}}{\sum_{i=0}^{\infty} p_{n,i}}. \tag{2.25}$$

Let

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}}, \qquad C_\sigma \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot (j-C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} \tag{2.26}$$

be constants, and assume $C_\sigma > 0$. Then there exist constants $d_\mu, d_\sigma \in \mathbb{R}$, and $\gamma_\mu, \gamma_\sigma \in (0,1)$ such that $\mu_n = C_\mu n + d_\mu + O(\gamma_\mu^n)$ and $\sigma_n^2 = C_\sigma n + d_\sigma + O(\gamma_\sigma^n)$. Furthermore, $(X_n - \mu_n)/\sigma_n$ converges weakly to the standard normal $N(0,1)$ as $n \to \infty$.

In this theorem, imagine we have fixed a gap size g and think of $p_{n,k}$ as the number of $m \in [G_n, G_{n+1})$ whose decomposition has exactly k gaps of size g. Under this interpretation, the random variable X_n is be identical to $K_{g,n}$.

We approach this problem using the method of moments, a common method for proving random variables converge in distribution to the standard normal distribution.

Lemma 2.7 (Method of Moments)**.** *Suppose $X_1, X_2, \ldots$ are random variables such that for all integers $m \geq 0$, we have*

$$\lim_{n\to\infty} \mathbf{E}[X_n^{2m}] \;=\; (2m-1)!! \quad \textit{and} \quad \lim_{n\to\infty} \mathbf{E}[X_n^{2m+1}] \;=\; 0. \tag{2.27}$$

Then the sequence $X_1, X_2, \ldots$ converges weakly in distribution to the standard normal $N(0,1)$.

The proof of Theorem 2.6 proceeds by using generating functions to compute the moments of X_n. Let

$$\begin{aligned} P_n(x) \;&:=\; \sum_{k=0}^{\infty} p_{n,k} x^k \\ \Omega_n \;&:=\; P_n(1) = \sum_{k=0}^{\infty} p_{n,k} \\ \tilde{P}_{n,0}(x) \;&:=\; \frac{P_n(x)}{x^{\mu+1}} \\ \tilde{P}_{n,m}(x) \;&:=\; (x\tilde{P}_{n,m-1}(x))' \\ \tilde{\mu}_n(m) \;&:=\; \frac{\tilde{P}_{n,m}(1)}{\Omega_n}. \end{aligned} \tag{2.28}$$

Then it follows from definitions that

$$\begin{aligned} \mu_n \;&=\; \frac{P_n'(1)}{P_n(1)} \\ \tilde{\mu}_n(m) \;&=\; \mathbf{E}[(X_n - \mu_n)^m] \\ \sigma_n^2 \;&=\; \tilde{\mu}_n(2). \end{aligned} \tag{2.29}$$

We now prove several lemmas about the above moments and generating functions. We ultimately obtain a formula in Theorem 2.10 that recursively computes the moments $\tilde{\mu}_n(m)$, which will yield Theorem 2.6.

Lemma 2.8. *For $n \geq n_0$, we have*

$$P_n(x) \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} P_{n-i}(x) x^j. \tag{2.30}$$

Proof. This follows immediately from the definitions:

$$\begin{aligned} P_n(x) \;&=\; \sum_{k=0}^{\infty} p_{n,k} x^k \\ &=\; \sum_{k=0}^{\infty}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} p_{n-i,k-j} x^k \\ &=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} x^j \sum_{k=0}^{\infty} p_{n-i,k-j} x^{k-j} \\ &=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} x^j P_{n-i}(x). \end{aligned} \tag{2.31}$$

□

From the above we immediately deduce the following relations.

Corollary 2.9. *For $n \geq n_0$, we have*

$$\Omega_n \;=\; P_n(1) \;=\; \tilde{P}_{n,0}(1) \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} P_{n-i}(1) \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j}\Omega_{n-i}. \tag{2.32}$$

and

$$\mu_n \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\Omega_{n-i}}{\Omega_n}(\mu_{n-i}+j). \tag{2.33}$$

Theorem 2.10. *For $n \geq n_0$, we have*

$$\tilde{\mu}_n(m) \;=\; \sum_{\ell=0}^{m}\binom{m}{\ell}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{\Omega_{n-i}t_{i,j}}{\Omega_n}\cdot(j+\mu_{n-i}-\mu_n)^{\ell}\cdot\tilde{\mu}_{n-i}(m-\ell). \tag{2.34}$$

Proof. Applying Lemma 2.8, we find

$$\tilde{P}_{n,0}(x) \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j}\tilde{P}_{n-i,0}(x)\cdot x^{j+\mu_{n-i}-\mu_n}. \tag{2.35}$$

By induction, we can establish (see Appendix D)

$$\tilde{P}_{n,m}(x) \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j}\sum_{\ell=0}^{m}\binom{m}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-\ell}(x)\cdot x^{j+\mu_{n-i}-\mu_n}, \tag{2.36}$$

so

$$\begin{aligned}\tilde{\mu}_n(m) \;=\; \frac{\tilde{P}_{n,m}(1)}{\Omega_n} \;&=\; \frac{1}{\Omega_n}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j}\sum_{\ell=0}^{m}\binom{m}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-\ell}(1)\\ &=\; \frac{1}{\Omega_n}\sum_{\ell=0}^{m}\binom{m}{\ell}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j}(j+\mu_{n-i}-\mu_n)^{\ell}\Omega_{n-i}\tilde{\mu}_{n-i}(m-\ell)\\ &=\; \sum_{\ell=0}^{m}\binom{m}{\ell}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{\Omega_{n-i}t_{i,j}}{\Omega_n}\cdot(j+\mu_{n-i}-\mu_n)^{\ell}\cdot\tilde{\mu}_{n-i}(m-\ell),\end{aligned} \tag{2.37}$$

completing the proof. □

Out next goal is to prove.

$$\lim_{n\to\infty}\frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m} \;=\; (2m-1)!! \quad\text{and}\quad \lim_{n\to\infty}\frac{\tilde{\mu}_n(2m+1)}{\tilde{\mu}_n(2)^{m+\frac{1}{2}}} \;=\; 0. \tag{2.38}$$

By Lemma 2.7, these equalities imply Theorem 2.6. To do prove these equalities, we first show μ_n is essentially linear in n. Then we determine for all m the behavior of $\tilde{\mu}_n(m)$, the m^{th} moment of $X_n-\mu_n$, up to an exponentially small term. We prove $\tilde{\mu}_n(m)$ is a degree (at most, if m is odd) $\lfloor m/2\rfloor$ polynomial in n, and for even moments $\tilde{\mu}_n(2m)$ we addtionally compute the leading coefficient of this polynoimal. We rely heavily on Lemmas 2.4 and 2.5 to pin down the polynomial behavior of the moments.

Lemma 2.11. *There exists a real number d_μ and a $\gamma_\mu\in(0,1)$ such that*

$$\mu_n \;=\; C_\mu\cdot n + d_\mu + O(\gamma_\mu^n). \tag{2.39}$$

Proof. Recall

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}}. \tag{2.40}$$

Choose $s_{i,n}=\frac{\Omega_{n-i}}{\Omega_n}\sum_{j=0}^{j_0}t_{i,j}=\frac{\Omega_{n-i}}{\Omega_n}\hat{t}_i$ and $r_n=\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j\cdot\Omega_{n-i}}{\Omega_n}$. Using Lemmas 2.1 and 2.3, we have that, for each i, $s_{i,n}$ converges exponentially quickly to $\bar{s}_i=\frac{1}{\lambda_1^i}\sum_{j=0}^{j_0}t_{i,j}=\hat{t}_i$ and r_n converges exponentially quickly to $\bar{r}=\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}$. By Lemma 2.9, we have

$$\mu_n \;=\; \left(\sum_{i=1}^{i_0}s_{i,n}\mu_{n-i}\right)+r_n. \tag{2.41}$$

Furthermore, the polynomial $S(x):=x^{i_0}-\sum_{i=1}^{i_0}\bar{s}_i x^{i_0-i}$ satisfies $S(x)=T(x/\lambda_1)$, so S has the maximum root property with maximum root 1. Then, by Lemma 2.4, there exist $d_\mu\in\mathbb{R}$ and $\gamma_\mu\in(0,1)$ such that

$$\mu_n \;=\; \frac{\bar{r}}{\sum_{i=1}^{i_0}i\bar{s}_i}\cdot n + d_\mu + O(\gamma_\mu^n) \;=\; C_\mu\cdot n + d_\mu + O(\gamma_\mu^n). \tag{2.42}$$

□

Lemma 2.12. *For each integer $m \geq 0$, there exist $\gamma_{2m}, \gamma_{2m+1} \in (0,1)$ and polynomials Q_{2m} of degree exactly m and Q_{2m+1} of degree at most m such that*

$$\begin{aligned} \tilde{\mu}_n(2m) &= Q_{2m}(n) + O(\gamma_{2m}^n) \\ \tilde{\mu}_n(2m+1) &= Q_{2m+1}(n) + O(\gamma_{2m+1}^n). \end{aligned} \tag{2.43}$$

Furthermore, if $C_{2m} := [x^m]Q_{2m}$ and $C_{2m+1} := [x^m]Q_{2m+1}$, then for all $m \geq 0$, $C_{2m} = (2m-1)!! \cdot C_\sigma^m$ (We take $(-1)!! := 1$).

The idea for the proof is as follows. In the calculation of $\mu_n(m)$ in Theorem 2.10 the coefficients of $\mu_{n-i}(m)$ sum to 1, the coefficients of $\mu_{n-i}(m-1)$ sum to 0, and the coefficients of $\mu_{n-i}(m-2)$ sum to $\binom{2m}{2} \cdot$ (constant). The m^{th} moments can thus be written in the form of (2.17), so we can apply Lemma 2.5 and compute the degrees and leading coefficients appropriately. Because the coefficients of the $(m-1)^{\text{th}}$ moments sum to 0, the degrees of the polynomials increase by one with every two values of m as opposed to every one.

Proof. We proceed by induction on m. The base case $m = 0$ follows from noting that

$$\begin{aligned} \tilde{\mu}_n(0) &= \mathbf{E}[(X_n - \mu_n)^0] = 1 \\ \tilde{\mu}_n(1) &= \mathbf{E}[(X_n - \mu_n)^1] = 0 \end{aligned} \tag{2.44}$$

for all $n \geq n_0$. Now assume the statement is true for $m' \leq m$. That is, there exist γ_0, γ_1, …, $\gamma_{2m-1} \in (0,1)$ and polynomials $Q_0, Q_1, \ldots, Q_{2m-1}$, where Q_k has degree $\lfloor k/2 \rfloor$, such that

$$\begin{aligned} \tilde{\mu}_n(2m-2) &= Q_{2m-2}(n) + O(\gamma_{2m-2}^n) \\ \tilde{\mu}_n(2m-1) &= Q_{2m-1}(n) + O(\gamma_{2m-1}^n). \end{aligned} \tag{2.45}$$

By induction we may assume further that $C_{2m-2} = (2m-3)!! \cdot C_\sigma^{m-1}$. First, we compute $\tilde{\mu}_n(2m)$. Define a sequence of polynomials $\{R_n(x)\}$ via

$$R_n(x) := \sum_{\ell=1}^{2m} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{\Omega_{n-i} t_{i,j}}{\Omega_n} \cdot (j + \mu_{n-i} - \mu_n)^\ell \cdot Q_{2m-\ell}(x-i). \tag{2.46}$$

Furthermore, set

$$s_{i,n} = \frac{\Omega_{n-i}}{\Omega_n} \sum_{j=0}^{j_0} t_{i,j} \quad \text{and} \quad \bar{s}_i = \frac{1}{\lambda_1^i} \sum_{j=0}^{j_0} t_{i,j}. \tag{2.47}$$

Then

$$\tilde{\mu}_n(2m) = \sum_{i=1}^{i_0} s_{i,n} \tilde{\mu}_{n-i}(2m) + R_n(n). \tag{2.48}$$

Note that $R_n(x)$ is the sum of finitely many polynomials that, by Lemma 2.1, converges exponentially quickly. Thus $R_n(x)$ converges exponentially quickly to

$$\bar{R}(x) := \sum_{\ell=1}^{2m} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i} \cdot (j - C_\mu i)^\ell \cdot Q_{2m-\ell}(x-i). \tag{2.49}$$

Furthermore, we have $\deg \bar{R}(x) \le m-1$ since each $R_n(x)$ has degree at most $m-1$. We can compute the leading coefficient of R using (2.49) to get

$$\begin{aligned}
[x^{m-1}](\bar{R}(x)) &= \sum_{\ell=1}^{2m}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\binom{2m}{\ell}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell\cdot[x^{m-1}](Q_{2m-\ell}(x-i)) \\
&= \sum_{\ell=1}^{2}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\binom{2m}{\ell}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell\cdot C_{2m-\ell} \\
&= C_{2m-2}\cdot\binom{2m}{2}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2 + C_{2m-1}\cdot 2m\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i) \\
&= C_{2m-2}\cdot\binom{2m}{2}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2 + C_{2m-1}\cdot 2m\cdot 0 \\
&= C_{2m-2}\cdot\binom{2m}{2}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2. \qquad (2.50)
\end{aligned}$$

Recalling

$$C_\sigma = \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} = \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2}{\sum_{i=1}^{i_0} i\cdot\bar{s}_i}, \qquad (2.51)$$

we have

$$[x^{m-1}](\bar{R}(x)) = C_{2m-2}\cdot\binom{2m}{2}\cdot C_\sigma\cdot\left(\sum_{i=1}^{i_0} i\cdot\bar{s}_i\right). \qquad (2.52)$$

By Lemma 2.5, there exists a degree $\deg\bar{R}(x)+1$ polynomial $Q_{2m}(x)$ with x^m coefficient C_{2m} and a $\gamma_{2m}\in(0,1)$ such that

$$\mu_n(2m) = Q_{2m}(n) + O(\gamma_{2m}^n) \qquad (2.53)$$

and

$$C_{2m} = \frac{C_{2m-2}\cdot\binom{2m}{2}\cdot C_\sigma\cdot\left(\sum_{i=1}^{i_0} i\cdot\bar{s}_i\right)}{m\cdot\sum_{i=1}^{i_0} i\cdot\bar{s}_i} = C_{2m-2}\cdot(2m-1)\cdot C_\sigma. \qquad (2.54)$$

By the inductive hypothesis, we conclude $C_{2m} = (2m-1)!!\cdot C_\sigma^m$. By our technical assumption, $C_\sigma\neq 0$, so $C_{2m}\neq 0$ and thus the degree of Q_{2m} is exactly m.

We can perform the same computation to show that the $\tilde{\mu}_n(2m+1)$ can be expressed as the sum of an m^{th} degree polynomial in n and an exponentially small term. To see this, define a sequence of polynomials $\{R_n(x)\}$ via

$$R_n(x) := \sum_{\ell=1}^{2m+1}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{\Omega_{n-i}t_{i,j}}{\Omega_n}\cdot(j+\mu_{n-i}-\mu_n)^\ell\cdot Q_{2m+1-\ell}(x-i). \qquad (2.55)$$

Just as for the $2m^{\text{th}}$ moments, set

$$s_{i,n} = \frac{\Omega_{n-i}}{\Omega_n}\sum_{j=0}^{j_0} t_{i,j}. \qquad (2.56)$$

Then

$$\tilde{\mu}_n(2m+1) \;=\; \sum_{i=1}^{i_0} s_{i,n}\tilde{\mu}_{n-i}(2m+1) + R_n(n). \tag{2.57}$$

Note that $R_n(x)$ is the sum of finitely many polynomials that, by Lemma 2.1, converge exponentially quickly. Thus $R_n(x)$ converges exponentially quickly to

$$\bar{R}(x) \;:=\; \sum_{\ell=1}^{2m+1}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell \cdot Q_{2m+1-\ell}(x-i). \tag{2.58}$$

Furthermore, we have $\deg \bar{R}(x) \le m-1$. Indeed, Q_{2m} has degree m, so to show that $\deg \bar{R}(x) \le m-1$, we simply need to show that the coefficient of x^m is 0. Indeed, looking at the x^m coefficients of (2.58) gives

$$\begin{aligned}[x^m](\bar{R}(x)) \;&=\; \sum_{\ell=1}^{2m}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\binom{2m}{\ell}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell\cdot[x^m](Q_{2m+1-\ell}(x-i)) \\ &=\; C_{2m}\cdot\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} 2m\cdot\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^1 \;=\; C_{2m}\cdot 2m\cdot 0 \;=\; 0. \end{aligned}\tag{2.59}$$

Again, applying Lemma 2.5 gives that there exists a degree $\deg \bar{R}(x)+1$ polynomial $Q_{2m+1}(x)$ such that $\tilde{\mu}_n(2m+1) = Q_{2m+1}(n) + O(\gamma_{2m+1}^n)$. Since $\deg \bar{R}(x)+1 \le m$, this completes the induction. □

Proof of Theorem 2.6. Lemma 2.11 proves the first part of Theorem 2.6. Lemma 2.12 implies that $\sigma_n^2 = \tilde{\mu}_n(2) = Q_2(n) + O(\gamma_2^n)$. Writing $Q_2(n) = C_\sigma n + d_\sigma$ for some $d_\sigma \in \mathbb{R}$, we have $\sigma_n^2 = C_\sigma n + d_\sigma + O(\gamma_2^n)$, proving the second part of Theorem 2.6. We finish the proof of Theorem 2.6 by noting that (2.38) is an immediate consequence of Lemma 2.12. □

3. Gap Theorems

3.1. **Gap Recurrence.** We start by finding a recurrence relation for an $m \in [G_n, G_{n+1})$ having exactly k gaps of size g.

Lemma 3.1. *Let $\{G_n\}$ be a positive linear recurrence satisfying*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L}. \tag{3.1}$$

Let $k_g(m)$ denote the number of gaps of size g in the Zeckendorf Decomposition of m. Slightly abusing notation (reusing the letter p), let

$$p_{g,n,k} \;:=\; |\{m \in [G_n, G_{n+1}) : k_g(m) \;=\; k\}|. \tag{3.2}$$

Define $d_0 = 0$ and $d_i = c_1 + c_2 + \cdots + c_i$ for $1 \le i \le L$ and set $c_i^* = c_i$ for $1 \le i < L$ and $c_L^* = c_L - 1$. Then there exists $n_0 = L + g$ and $k_0 = d_L$ such that, for $n \ge n_0, k \ge k_0, g \ge 2$,

we have

$$\begin{aligned}
p_{0,n,k} &= \sum_{i=1}^{L}\sum_{j=1}^{c_i-1} p_{0,n-i,k-(d_{i-1}-(i-1)+(j-1))} + \sum_{i=1}^{L} p_{0,n-i,k-(d_{i-1}-(i-1))} \\
p_{1,n,k} &= p_{1,n-1,k} + \sum_{i=1}^{L}(c_i-1)p_{1,n-i,k-(i-1)} + \sum_{i=2}^{L} p_{1,n-i,k-(i-2)} \\
&\quad + \sum_{i=1}^{L}(c_i-1)\left(\left(p_{1,n-i,k-i} - p_{1,n-i,k-(i-1)}\right) - \left(p_{1,n-i-1,k-i} - p_{1,n-i-1,k-(i-1)}\right)\right) \\
p_{g,n,k} &= \sum_{i=1}^{L} c_i p_{g,n-i,k} + \sum_{i=1}^{L} c_i^* \left(\left(p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}\right) - \left(p_{g,n-i-g,k-1} - p_{g,n-i-g,k}\right)\right).
\end{aligned} \tag{3.3}$$

Proof. Define

$$q_{g,n,k} := \left|\{m \in [1, G_n) : k_g(m) = k\}\right| = \sum_{i=1}^{n-1} p_{g,i,k}; \tag{3.4}$$

thus while $p_{g,n,k}$ is the number of m in $[G_n, G_{n+1})$ such that $k_g(m) = k$, $q_{g,n,k}$ is the corresponding quantity for integers in $[1, G_n)$. Set $H_{n,0} = 0$ and $H_{n,i} = \sum_{i'=1}^{i} c_{i'} G_{n+1-i'}$ so that, for all n, $H_{n,L} = G_{n+1}$. Let

$$Z = \{(i,j) \in \mathbb{Z}^2 : 0 \le i \le L-1, 0 \le j \le c_i - 1, (i,j) \ne (0,0)\}. \tag{3.5}$$

For $n \in \mathbb{N}$ and $(i,j) \in Z$, let $I_{n,i,j} = [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i})$ be an interval of integers. The $c_1 + c_2 + \cdots + c_L - 1$ intervals $\{I_{n,i,j} : (i,j) \in Z\}$ form a partition of $[G_n, G_{n+1})$, and the sequential order of these intervals is equal to their lexicographical order by (i,j). For each $(i,j) \in Z$, we can express $|\{m \in I_{n,i,j} : k_g(m) = k\}$ in terms of $p_{g,n,k}$ and $q_{g,n,k}$ with smaller values of n by case work on whether the smallest term in $H_{n,i} + jG_{n-i}$ (either G_{n+1-i} or G_{n-i} depending on whether $j = 0$) is part of a gap of size g:

$$\begin{aligned}
\left|\{m \in I_{n,i,0} : k_0(m) = k\}\right| &= q_{0,n-i,k-(d_i-i)} \\
\left|\{m \in I_{n,i,0} : k_1(m) = k\}\right| &= q_{1,n-i,k-(i-1)} \\
\left|\{m \in I_{n,i,0} : k_g(m) = k\}\right| &= q_{g,n-i,k} + p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k} \\
\left|\{m \in I_{n,i,j} : k_0(m) = k\}\right| &= q_{0,n-i,k-(d_i-i+(j-1))} \\
\left|\{m \in I_{n,i,j} : k_1(m) = k\}\right| &= q_{1,n-i,k-i} + p_{g,n-i-1,k-(i+1)} - p_{g,n-i-1,k-i} \\
\left|\{m \in I_{n,i,j} : k_g(m) = k\}\right| &= q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k}
\end{aligned} \tag{3.6}$$

(see Appendix E.1 for details). These formulas are clean because the number of size g gaps in an $m = H_{n,i} + jG_{n-i} + m' \in I_{n,i,j}$ is simply the number of size g gaps in $H_{n,i} + jG_{n-i}$ plus the number of size g gaps in m' plus possibly one more gap between the two decompositions. By definition, for $g \ge 0$ we have

$$p_{g,n,k} = \sum_{(i,j)\in Z} \left|\{m \in I_{n,i,j} : k_g(m) = k\}\right|. \tag{3.7}$$

From this equation, we can substitute from (3.6), plug in the result for $p_{g,n,k}$ and $p_{g,n-1,k}$ in the expression $p_{g,n,k} - p_{g,n-1,k}$, use the identity $q_{g,n,k} - q_{g,n-1,k} = p_{g,n-1,k}$, and apply straightforward manipulations to obtain the desired result (see Appendix E.2 for calculations). □

3.2. **Proof of Gap Theorems.** Lemma 3.1 allows us to apply Theorem 2.6 to the distribution of the number of fixed sized gaps. The proof is essentially verifying that the conditions of Theorem 2.6 are met by our gap recurrences.

Proofs of Theorems 1.6, 1.7, and 1.8. Let $k_g(m)$ denote the number of gaps of size g in the Zeckendorf Decomposition of m, let

$$p_{g,n,k} \ := \ |\{m \in [G_n, G_{n+1}) : k_g(m) = k\}| \tag{3.8}$$

and let $i_0 = L + g$, $j_0 = d_L$. By Lemma 3.1, for every $g \geq 0$, there exist $t_{i,j}$ for $1 \leq i \leq L + g$ and $0 \leq j \leq d_L$ such that for $n > i_0$

$$p_{g,n,k} \ = \ \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} p_{g,n-i,k-j}. \tag{3.9}$$

Define $\hat{t}_i = \sum_{j=0}^{j_0} t_{i,j} = c_i$. By (3.3), $\hat{t}_i = c_i$ for $1 \leq i \leq L$ and $\hat{t}_i = 0$ for $L < i \leq i_0$. To see this, note that in each recursive formula of (3.3) the terms of the form $p_{g,n-x,y_1} - p_{g,n-x,y_2}$ contribute 0 to $\sum_{j=0}^{j_0} t_{x,j}$, and for each $0 \leq i \leq L-1$ the remaining coefficients of $p_{g,n-i-1,k}$ (over varying k) sum to c_{i+1}. Thus the polynomial

$$T(x) = x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0 - i} = x^{i_0 - L} \left(x^L - \sum_{i=1}^{L} c_i x^{L-i} \right) \tag{3.10}$$

has the maximum root property with some maximum root $\lambda_1 > 1$ by Theorem 2.2. As $p_{g,n,k}$ counts something that is well defined when $n \geq 1$ and $k \geq 0$, we have $p_{g,n,k} \geq 0$ for all n, k and $p_{g,n,k} = 0$ for $n < 0$ or $k < 0$. Also, there are finitely many pairs (n, k) with $n \leq i_0$ such that $p_{g,n,k} \neq 0$, as $p_{g,n,k} = 0$ for all $k \geq n$, since no $m \in [G_n, G_{n+1})$ can have a gap greater than n. Lastly, for every g, if the random variable $K_{g,n}$ is nontrivial then the $t_{i,j}$ satisfy

$$C_\mu \ := \ \frac{\sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j} \cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j} \cdot i}{\lambda_1^i}}, \quad C_\sigma \ := \ \frac{\sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i} \cdot (j - C_\mu i)^2}{\sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j} \cdot i}{\lambda_1^i}}. \tag{3.11}$$

To prove each of $C_\mu > 0$ and $C_\sigma > 0$, we split into cases on whether $g = 0$, $g = 1$, or $g \geq 2$, for each case substituting into (3.11), and performing standard manipulations (see Appendix F). Putting these observations together, the proofs follow by applying Theorem 2.6. □

4. Lekkerkerker and Gaussian Summands

We show the power of Theorem 2.6 by reproving Theorems 1.3, 1.4, and 1.5. We borrow from the proof given by Miller and Wang [MW1] the recursion established for $p_{n,k}$, the number of $m \in [G_n, G_{n+1})$ with exactly k summands. This recursion is extracted as (4.1) from generating functions in [MW1]. Their arguments quickly show the mean and variance grow linearly in n, but a lot of technical calculations are needed to show the coefficients of n are positive (which is a key ingredient in the proof of the Gaussian behavior). See [CFHMNPX] for another approach, which bypasses the difficulties through an elementary argument involving conditional probabilities, or [B-AM] for a proof through Markov processes.

Similar to §3.2, the proof is essentially verifying that the conditions of Theorem 2.6 are met by the summands recursion given by Miller and Wang.

Proofs of Theorems 1.3, 1.4, and 1.5. Let $p_{n,k}$ be the number of $m \in [G_n, G_{n+1})$ with exactly k summands. Then $\mathbf{Pr}[\hat{K}_n = k] = \frac{p_{n,k}}{\sum_{k=0}^{\infty} p_{n,k}}$. Again, $p_{n,k} \geq 0$ for all n, k, $p_{n,k} = 0$ for all $n < 0$ and $k < 0$, and $p_{n,k} > 0$ for finitely many pairs with $n < L$ as $p_{n,k} = 0$ for all

$k > n \cdot \max_i(c_i)$, since each m has, for each $a \in \{1, \dots, n\}$, at most $\max_i(c_i)$ copies of G_a in each decomposition.

Define $d_i = c_1 + c_2 + \cdots + c_i$ for $1 \le i \le L$. By Proposition 3.1 from [MW2], $p_{n,k}$ satisfies, for $n \ge L$ and $k \ge d_L$,

$$p_{n,k} \;=\; \sum_{i=1}^{L} \sum_{j=d_{m-1}}^{d_m-1} p_{n-i,k-j}. \tag{4.1}$$

For $1 \le i \le L$, $0 \le j < d_L$, set $t_{i,j}$ to be 1 if $d_{i-1} \le j < d_i - 1$ and 0 otherwise. Defining $\hat{t}_i := \sum_{j=0}^{d_L-1} t_{i,j}$ gives $\hat{t}_i = c_i$, and the polynomial

$$T(x) \;=\; x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0-i} \;=\; x^{i_0-L}\left(x^L - \sum_{i=1}^{L} c_i x^{L-i}\right) \tag{4.2}$$

has the maximum root property with some maximum root $\lambda_1 > 1$ by Theorem 2.2. Lastly, since all the $t_{i,j}$ are nonnegative and $t_{n-L,k-(d_L-1)} = 1$ with $k - (d_L - 1) > 0$, (2.26) tells us

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;\ge\; \frac{\frac{k-(d_L-1)}{\lambda_1^{n-L}}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;>\; 0. \tag{4.3}$$

Since $t_{1,0} = 1$, we have

$$C_\sigma \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i}\cdot (j - C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;\ge\; \frac{\frac{t_{1,0}}{\lambda_1^1}\cdot (0 - C_\mu 1)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;>\; 0. \tag{4.4}$$

Thus we can apply Theorem 2.6, implying the theorems. □

5. Further Work and Open Questions

We end with a few natural questions for future work.

(1) Are there other two-dimensional recurrences to which we can apply our central limit type result? The second named author is currently investigating two dimensional sequences and associated notions of legality with colleagues. These lead to recurrence relations, though the resulting sequences do not have unique decomposition.
(2) Can one remove the constraint that every coefficient c_i must be positive and obtain the same results? Notice that with negative constraints one loses some of the interpretations for the algebra.
(3) What is the rate at which $K_{g,n}$ converges to a normal distribution?

APPENDIX A. b_n IS BOUNDED IN LEMMA 2.4

Since $s_{i,n} = \bar{s}_i + O(\gamma^n)$, there exist $\alpha_1, \alpha_2 > 0, n_0 \in \mathbb{N}$ such that, for all $n \geq n_0 + i_0$,

$$\begin{aligned}
|b_n| \;&=\; \left|\sum_{i=1}^{i_0}(\bar{s}_i + O(\gamma^n))b_{n-i} + O(\gamma^n)\right| \\
&\leq\; \sum_{i=1}^{i_0}(\bar{s}_i + \alpha_1\gamma^n)|b_{n-i}| + \alpha_2\gamma^n \\
&\leq\; (1 + i_0\alpha_1\gamma^n)\left|\max_{1\leq i\leq i_0} b_{n-i}\right| + \alpha_2\gamma^n \\
&\leq\; (1 + (i_0\alpha_1 + \alpha_2)\gamma^n)\left|\max\left(1, \max_{1\leq i\leq i_0} b_{n-i}\right)\right| \\
&\leq\; e^{(i_0\alpha_1+\alpha_2)\gamma^n}\left|\max\left(1, \max_{1\leq i\leq i_0} b_{n-i}\right)\right|. \qquad \text{(A.1)}
\end{aligned}$$

Let $B = \max_{0\leq i<i_0} |b_{n_0+i}|$. We prove by induction that

$$|b_n| \;\leq\; (B+1)e^{(i_0\alpha_1+\alpha_2)\left(\gamma^{n_0}+\gamma^{n_0+1}+\cdots+\gamma^n\right)}. \qquad \text{(A.2)}$$

For $n < n_0 + i_0$, we have

$$|b_n| < (B+1) \;\leq\; (B+1)e^{(i_0\alpha_1+\alpha_2)(\gamma^{n_0}+\gamma^{n_0+1}+\cdots+\gamma^n)}. \qquad \text{(A.3)}$$

Now assume $n \geq n_0 + i_0$, and suppose the assertion is true for $n' < n$. Then, by (A.1),

$$\begin{aligned}
|b_n| \;&\leq\; e^{(i_0\alpha_1+\alpha_2)\gamma^n}\left|\max\left(1, \max_{1\leq i\leq i_0} b_{n-i}\right)\right| \\
&\leq\; e^{(i_0\alpha_1+\alpha_2)\gamma^n}\left|\max\left(1, \max_{1\leq i\leq i_0} (B+1)e^{(i_0\alpha_1+\alpha_2)(\gamma^{n_0}+\gamma^{n_0+1}+\cdots+\gamma^{n-i})}\right)\right| \\
&=\; e^{(i_0\alpha_1+\alpha_2)\gamma^n}(B+1)e^{(i_0\alpha_1+\alpha_2)(\gamma^{n_0}+\gamma^{n_0+1}+\cdots+\gamma^{n-1})} \\
&\leq\; (B+1)e^{(i_0\alpha_1+\alpha_2)(\gamma^{n_0}+\gamma^{n_0+1}+\cdots+\gamma^n)} \qquad \text{(A.4)}
\end{aligned}$$

completing the induction. Thus we have

$$|b_n| \;\leq\; (B+1)e^{(i_0\alpha_1+\alpha_2)\cdot\frac{\gamma^{n_0}}{1-\gamma}}, \qquad \text{(A.5)}$$

so the sequence $\{b_n\}$ is bounded.

APPENDIX B. DECOMPOSITION OF b_n INTO SIMILAR SEQUENCES 2.4

We prove by induction that

$$b_n \;=\; b_n^{(\text{init})} + \sum_{m=i_0}^{\infty} b_n^{(m)}. \qquad \text{(B.1)}$$

For $n < i_0$, we have

$$b_n \;=\; b_n^{(\text{init})} \;=\; b_n^{(\text{init})} + \sum_{m=i_0}^{\infty} b_n^{(m)}. \qquad \text{(B.2)}$$

Suppose the statement is true for $n' < n$. Using the fact that $f(n) = b_n^{(n)}$ and the recursive definitions for $b_n^{(\text{init})}$ and $b_n^{(m)}$, we obtain

$$\begin{aligned}
b_n &= f(n) + \left(\sum_{i=1}^{i_0} \bar{s}_i b_{n-i}\right) \\
&= f(n) + \sum_{i=1}^{i_0} \bar{s}_i \left(b_{n-i}^{(\text{init})} + \sum_{m=i_0}^{\infty} b_{n-i}^{(m)}\right) \\
&= f(n) + \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(\text{init})} + \sum_{i=1}^{i_0}\sum_{m=i_0}^{\infty} \bar{s}_i b_{n-i}^{(m)} \\
&= f(n) + \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(\text{init})} + \sum_{m=i_0}^{n-1}\sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(m)} + \sum_{m=n}^{\infty}\sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(m)} \\
&= b_n^{(n)} + b_n^{(\text{init})} + \sum_{m=i_0}^{n-1} b_n^{(m)} + \sum_{m=n}^{\infty}\sum_{i=1}^{i_0} \bar{s}_i \cdot 0 \\
&= b_n^{(\text{init})} + \sum_{m=i_0}^{n} b_n^{(m)} \\
&= b_n^{(\text{init})} + \sum_{m=i_0}^{n} b_n^{(m)} + \sum_{m=n+1}^{\infty} b_n^{(m)} \\
&= b_n^{(\text{init})} + \sum_{m=i_0}^{\infty} b_n^{(m)}, \qquad \text{(B.3)}
\end{aligned}$$

completing the induction.

Appendix C. Computing b_n in Lemma 2.5

Our goal this section is to simplify (2.17) using the substitution $a_n - b_n = C \cdot n^{D+1}$ where $C = \frac{\bar{r}_D}{(D+1)\sum_{i=1}^{i_0} i \cdot \bar{s}_i}$. We compute a recursive formula for b_n to obtain a linear combination of smaller b_{n-i}s plus a polynomial in n. For any choice of C, the resulting coefficient of n^{D+1} in this polynomial is 0, but for our specific choice of C, the n^D term also disappears, so the remaining polynomial has degree at most $D-1$.

Let $\gamma_0 \in (\max(\gamma_r, \gamma_s), 1)$. Since $a_n = \sum_{i=1}^{i_0} s_{i,n} a_{n-i} + \sum_{j=0}^{D} r_{j,n} n^j$, we can write

$$\begin{aligned}
b_n &= a_n - C \cdot n^{D+1} \\
&= a_n - \sum_{i=1}^{i_0} C s_{i,n} n^{D+1} \\
&= \sum_{i=1}^{i_0} s_{i,n} a_{n-i} + \sum_{j=0}^{D} r_{j,n} n^j. \qquad \text{(C.1)}
\end{aligned}$$

Substituting $a_n - b_n = C \cdot n^{D+1}$ gives

$$b_n = \left(\sum_{i=1}^{i_0} s_{i,n}\left(b_{n-i} + C(n-i)^{D+1}\right)\right) + \sum_{j=0}^{D} r_{j,n} n^j - \sum_{i=1}^{i_0} C s_{i,n} n^{D+1}. \qquad \text{(C.2)}$$

We now expand $(n-i)^{D+1}$ to get

$$\begin{aligned} b_n &= \left(\sum_{i=1}^{i_0} s_{i,n}\left(b_{n-i} + C\cdot\sum_{j=0}^{D+1}(-1)^{D+1-j}\binom{D+1}{j}n^j i^{D+1-j}\right)\right) + \sum_{j=0}^{D} r_{j,n}n^j - \sum_{i=1}^{i_0} Cs_{i,n}n^{D+1} \\ &= \sum_{i=1}^{i_0} s_{i,n}b_{n-i} + \left(\sum_{j=0}^{D+1}\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j}n^j i^{D+1-j}\right) + \sum_{j=0}^{D} r_{j,n}n^j - \sum_{i=1}^{i_0} Cs_{i,n}n^{D+1}. \end{aligned} \tag{C.3}$$

As $\sum_{i=1}^{i_0} Cs_{i,n}$ is the coefficient of n^{D+1} in the binomial expansion, we can cancel to get

$$b_n = \sum_{i=1}^{i_0} s_{i,n}b_{n-i} + \left(\sum_{j=0}^{D}\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j}n^j i^{D+1-j}\right) + \sum_{j=0}^{D} r_{j,n}n^j. \tag{C.4}$$

We can also pull out the n^D terms of the binomial expansions to get

$$\begin{aligned} b_n &= \sum_{i=1}^{i_0} s_{i,n}b_{n-i} + \left(\sum_{j=0}^{D-1}\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j}n^j i^{D+1-j}\right) \\ &\quad + \sum_{j=0}^{D} r_{j,n}n^j - \sum_{i=1}^{i_0} Cs_{i,n}(D+1)n^D i \\ &= \sum_{i=1}^{i_0} s_{i,n}b_{n-i} + \left(\sum_{j=0}^{D-1}\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j}n^j i^{D+1-j}\right) \\ &\quad + \sum_{j=0}^{D-1} r_{j,n}n^j + r_{D,n}n^D - C(D+1)n^D\sum_{i=1}^{i_0} s_{i,n}i. \end{aligned} \tag{C.5}$$

Now we substitute the value of C in. Note that C is chosen so that the coefficient of n^D becomes $O(\gamma_0^n)$. This will happen as $\frac{\sum_{i=1}^{i_0} i\cdot s_{i,n}}{\sum_{i=1}^{i_0} i\cdot \bar{s}_i}$ and $\frac{r_{D,n}}{\bar{r}_D}$ are of the form $1+O(\gamma_s^n)$ and $1+O(\gamma_r^n)$

respectively. We find

$$\begin{aligned}
b_n &= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \left(\sum_{j=0}^{D-1} \sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} n^j i^{D+1-j} \right) \\
&\quad + \sum_{j=0}^{D-1} r_{j,n} n^j + r_{D,n} n^D - \frac{\bar{r}_D \cdot n^D}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \sum_{i=1}^{i_0} s_{i,n} \cdot i \\
&= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \left(\sum_{j=0}^{D-1} \sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} n^j i^{D+1-j} \right) \\
&\quad + \sum_{j=0}^{D-1} r_{j,n} n^j - \bar{r}_D n^D \left(\frac{\sum_{i=1}^{i_0} i \cdot s_{i,n}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} - \frac{r_{D,n}}{\bar{r}_D} \right) \\
&= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \left(\sum_{j=0}^{D-1} \sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} n^j i^{D+1-j} \right) \\
&\quad + \sum_{j=0}^{D-1} r_{j,n} n^j - \bar{r}_D n^D \cdot O(\max(\gamma_r, \gamma_s)^n) \\
&= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \left(\sum_{j=0}^{D-1} \sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} n^j i^{D+1-j} \right) \\
&\quad + \sum_{j=0}^{D-1} r_{j,n} n^j + O(\gamma_0^n) \\
&= \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \sum_{j=0}^{D-1} n^j \cdot \left[\left(\sum_{i=1}^{i_0} C s_{i,n} (-1)^{D+1-j} \binom{D+1}{j} i^{D+1-j} \right) + r_{j,n} \right] + O(\gamma_0^n).
\end{aligned} \tag{C.6}$$

Appendix D. Recursive formula for $\tilde{P}_{n,m}(x)$ (Equation (2.36))

We wish to prove

$$\tilde{P}_{n,m}(x) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=0}^{m} \binom{m}{\ell} (j + \mu_{n-i} - \mu_n)^\ell \tilde{P}_{n-i,m-\ell}(x) \cdot x^{j+\mu_{n-i}-\mu_n}. \tag{D.1}$$

The base case $m = 0$ is given by (2.35). Now let $m \geq 1$, and suppose

$$\tilde{P}_{n,m-1}(x) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (j + \mu_{n-i} - \mu_n)^\ell \tilde{P}_{n-i,m-1-\ell}(x) \cdot x^{j+\mu_{n-i}-\mu_n}. \tag{D.2}$$

Then

$$\begin{aligned}
\tilde{P}_{n,m}(x) &= (x\tilde{P}_{n,m-1}(x))' \\
&= \left(x\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-1-\ell}(x)\cdot x^{j+\mu_{n-i}-\mu_n}\right)' \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\left(x\tilde{P}_{n-i,m-1-\ell}(x)\cdot x^{j+\mu_{n-i}-\mu_n}\right)' \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\Bigg[\left(\tilde{P}_{n-i,m-\ell}(x)\cdot x^{j+\mu_{n-i}-\mu_n}\right) \\
&\quad +\left(x\tilde{P}_{n-i,m-1-\ell}(x)\cdot(j+\mu_{n-i}-\mu_n)x^{j+\mu_{n-i}-\mu_n-1}\right)\Bigg] \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-\ell}(x)x^{j+\mu_{n-i}-\mu_n} \\
&\quad +\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell+1}\tilde{P}_{n-i,m-1-\ell}(x)x^{j+\mu_{n-i}-\mu_n} \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-\ell}(x)x^{j+\mu_{n-i}-\mu_n} \\
&\quad +\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}t_{i,j}\sum_{\ell=1}^{m}\binom{m-1}{\ell-1}(j+\mu_{n-i}-\mu_n)^{\ell}\tilde{P}_{n-i,m-\ell}(x)x^{j+\mu_{n-i}-\mu_n}. \qquad \text{(D.3)}
\end{aligned}$$

Taking out the $\ell = 0$ term from the first sum and the $\ell = m$ term from the latter one, and pairing the remaining terms by common ℓ, we obtain

$$\begin{aligned}
\tilde{P}_{n,m}(x) &= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=1}^{m-1} \left(\binom{m-1}{\ell-1} + \binom{m-1}{\ell} \right) (j+\mu_{n-i}-\mu_n)^{\ell} \tilde{P}_{n-i,m-\ell}(x) x^{j+\mu_{n-i}-\mu_n} \\
&\quad + \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \binom{m-1}{0} (j+\mu_{n-i}-\mu_n)^{0} \tilde{P}_{n-i,m-0}(x) x^{j+\mu_{n-i}-\mu_n} \\
&\quad + \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \binom{m-1}{m-1} (j+\mu_{n-i}-\mu_n)^{m} \tilde{P}_{n-i,m-m}(x) x^{j+\mu_{n-i}-\mu_n} \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=1}^{m-1} \binom{m}{\ell} (j+\mu_{n-i}-\mu_n)^{\ell} \tilde{P}_{n-i,m-\ell}(x) x^{j+\mu_{n-i}-\mu_n} \\
&\quad + \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \binom{m}{0} (j+\mu_{n-i}-\mu_n)^{0} \tilde{P}_{n-i,m-0}(x) x^{j+\mu_{n-i}-\mu_n} \\
&\quad + \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \binom{m}{m} (j+\mu_{n-i}-\mu_n)^{m} \tilde{P}_{n-i,m-m}(x) x^{j+\mu_{n-i}-\mu_n} \\
&= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=0}^{m} \binom{m}{\ell} (j+\mu_{n-i}-\mu_n)^{\ell} \tilde{P}_{n-i,m-\ell}(x) x^{j+\mu_{n-i}-\mu_n} \qquad \text{(D.4)}
\end{aligned}$$

as desired. This completes the induction.

APPENDIX E. DETAILS FOR RECURSIVELY COUNTING GAPS IN LEMMA 3.1

E.1. **Computing $|\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_g(m) = k\}|$.** In this section, we compute a recursive formula for $|\{m \in I_{n,i,j} : k_g(m) = k\}|$. Note that the number of size g gaps in an $m = H_{n,i} + jG_{n-i} + m' \in I_{n,i,j}$ is simply the number of size g gaps in $H_{n,i} + jG_{n-i}$ plus the number of size g gaps in m' plus possibly one more gap between the two decompositions. This observation gives a clean way to produce the desired recursions. We need to be careful in our case work as the number of size g gaps in $H_{n,i} + jG_{n-i}$ varies depending on whether g is 0, 1, or at least 2, and the the existence of the gap between the smallest term in the decomposition of $H_{n,i} + jG_{n-i}$ and the largest term in the decomposition of m' depends on whether j is nonzero.

For $j \geq 0$, the decomposition of any element of $I_{n,i,j} = [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i})$ begins with the decomposition of $H_{n,i} + jG_{n-i}$, and the decomposition of $H_{n,i} + jG_{n-i}$ contains only gaps of sizes 0 and 1 by definition of $H_{n,i}$. In particular, when $j = 0$, the decomposition of $H_{n,i}$ contains $d_i - i$ gaps of size 0 and $i - 1$ gaps of size 1. When $j \geq 1$, the decomposition of

$H_{n,i} + jG_{n-i}$ contains $d_i - i + (j-1)$ gaps of size 0 and i gaps of size 1. For $i \geq 1$, this gives

$$\begin{aligned}
&|\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_0(m) = k\}| \\
&\quad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_0(m) = k - (d_i - i)\}| \\
&|\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_1(m) = k\}| \\
&\quad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_1(m) = k - (i-1)\}| \\
&|\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_g(m) = k\}|, \qquad \text{(E.1)}
\end{aligned}$$

and for $i \geq 0, j \geq 1$, we have

$$\begin{aligned}
&|\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_0(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_0(m) = k - (d_i - i + (j-1))\}| \\
&|\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_1(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_1(m) = k - i\}| \\
&|\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_g(m) = k\}|. \qquad \text{(E.2)}
\end{aligned}$$

We can further push (E.1) by noting that, for $m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i})$, the decomposition of m begins with G_{n+1-i}, and furthermore G_{n+1-i} is not a part of a gap of size 0 or 1. Thus

$$\begin{aligned}
&|\{m \in I_{n,i,0} : k_0(m) = k\}| \\
&\qquad = |\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_0(m) = k\}| \\
&\qquad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_0(m) = k - (d_i - i)\}| \\
&\qquad = |\{m \in [0, G_{n-i}) : k_0(m) = k - (d_i - i)\}| \\
&\qquad = q_{0,n-i,k-(d_i-i)} \\
&|\{m \in I_{n,i,0} : k_1(m) = k\}| \\
&\qquad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_1(m) = k - (i-1)\}| \\
&\qquad = |\{m \in [0, G_{n-i}) : k_1(m) = k - (i-1)\}| \\
&\qquad = q_{1,n-i,k-(i-1)}. \qquad \text{(E.3)}
\end{aligned}$$

Similarly, the decomposition of any $m \in [G_{n-i}, 2G_{n-i})$ begins with G_{n-i}, and G_{n-i} is not a part of a gap of size 0, so we have

$$\begin{aligned}
&|\{m \in I_{n,i,j} : k_0(m) = k\}| \\
&\qquad = |\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_0(m) = k\}| \\
&\qquad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_0(m) = k - (d_i - i + (j-1))\}| \\
&\qquad = |\{m \in [0, G_{n-i}) : k_0(m) = k - (d_i - i + (j-1))\}| \\
&\qquad = q_{0,n-i,k-(d_i-i+(j-1))}. \qquad \text{(E.4)}
\end{aligned}$$

For $g \geq 2$ we have

$$\begin{aligned}
&|\{m \in I_{n,i,0} : k_g(m) = k\}| \\
&\quad = |\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n+1-i}, G_{n+1-i} + G_{n+1-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n+1-i} + G_{n+1-i-g}, G_{n+1-i} + G_{n+2-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n+1-i} + G_{n+2-i-g}, G_{n+1-i} + G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [0, G_{n+1-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n+1-i-g}, G_{n+2-i-g}) : k_g(m) = k-1\}| \\
&\qquad + |\{m \in [G_{n+2-i-g}, G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [0, G_{n-i}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n+1-i-g}, G_{n+2-i-g}) : k_g(m) = k-1\}| \\
&\qquad - |\{m \in [G_{n+1-i-g}, G_{n+2-i-g}) : k_g(m) = k\}| \\
&\quad = q_{g,n-i,k} + p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}, \qquad \text{(E.5)}
\end{aligned}$$

where the third equality comes from noting that for $m \in [G_{n+1-i}, G_{n+1-i} + G_{n-i})$, G_{n+1-i} is part of a gap of size g in the decomposition of m if and only if $m \in [G_{n+1-i} + G_{n+1-i-g}, G_{n+1-i} + G_{n+2-i-g})$. Using the same argument, we obtain, for $g \geq 1$ and $j \geq 1$,

$$\begin{aligned}
&|\{m \in [G_{n-i}, 2G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, G_{n-i} + G_{n-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n-i} + G_{n-i-g}, G_{n-i} + G_{n+1-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n-i} + G_{n+1-i-g}, 2G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [0, G_{n-i-g}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n-i-g}, G_{n+1-i-g}) : k_g(m) = k-1\}| \\
&\qquad + |\{m \in [G_{n+1-i-g}, G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [0, G_{n-i}) : k_g(m) = k\}| \\
&\qquad + |\{m \in [G_{n-i-g}, G_{n+1-i-g}) : k_g(m) = k-1\}| \\
&\qquad - |\{m \in [G_{n-i-g}, G_{n+1-i-g}) : k_g(m) = k\}| \\
&\quad = q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k}. \qquad \text{(E.6)}
\end{aligned}$$

Combining with (E.2), we have, for $g \geq 2$,

$$\begin{aligned}
&|\{m \in I_{n,i,j} : k_1(m) = k\}| \\
&\quad = |\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_1(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_1(m) = k-i\}| \\
&\quad = q_{1,n-i,k-i} + p_{1,n-i-1,k-i-1} - p_{1,n-i-1,k-i} \\
&|\{m \in I_{n,i,j} : k_g(m) = k\}| \\
&\quad = |\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_g(m) = k\}| \\
&\quad = |\{m \in [G_{n-i}, 2G_{n-i}) : k_g(m) = k\}| \\
&\quad = q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k}. \qquad \text{(E.7)}
\end{aligned}$$

This establishes all six equalities that we desire.

E.2. **Computing $p_{g,n,k}$.** This section uses careful bookkeeping to produce homogenous two dimensional recursive formulas for $p_{g,n,k}$ using (3.6).

Recall that for $g \geq 0$ we have

$$\begin{aligned} p_{g,n,k} &= \sum_{(i,j)\in Z} |\{m \in I_{n,i,j} : k_g(m) = k\}| \\ &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} |\{m \in [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i}) : k_g(m) = k\}| \\ &\quad + \sum_{i=1}^{L-1} |\{m \in [H_{n,i}, H_{n,i} + G_{n-i}) : k_g(m) = k\}|. \end{aligned} \tag{E.8}$$

Substituting from (3.6), we have (for $g \geq 2$)

$$\begin{aligned} p_{0,n,k} &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} q_{0,n-i,k-(d_i-i+(j-1))} + \sum_{i=1}^{L-1} q_{0,n-i,k-(d_i-i)} \\ p_{1,n,k} &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} \left(q_{1,n-i,k-i} + p_{1,n-i-1,k-i-1} - p_{1,n-i-1,k-i}\right) + \sum_{i=1}^{L-1} q_{1,n-i,k-(i-1)} \\ &= \sum_{i=0}^{L-1} (c_{i+1}-1)\left(q_{1,n-i,k-i} + p_{1,n-i-1,k-i-1} - p_{1,n-i-1,k-i}\right) + \sum_{i=1}^{L-1} q_{1,n-i,k-(i-1)} \\ p_{g,n,k} &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} \left(q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k}\right) \\ &\quad + \sum_{i=1}^{L-1} \left(q_{g,n-i,k} + p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}\right) \\ &= \sum_{i=0}^{L-1} (c_{i+1}-1)\left(q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k}\right) \\ &\quad + \sum_{i=1}^{L-1} \left(q_{g,n-i,k} + p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}\right). \end{aligned} \tag{E.9}$$

Substituting for $p_{g,n,k}$ and $p_{g,n-1,k}$ and using $q_{g,n,k} - q_{g,n-1,k} = p_{g,n-1,k}$, we obtain for $g = 0$

$$\begin{aligned} p_{0,n,k} - p_{0,n-1,k} &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} p_{0,n-i-1,k-(d_i-i+(j-1))} + \sum_{i=1}^{L-1} p_{0,n-i-1,k-(d_i-i)} \\ p_{0,n,k} &= \sum_{i=0}^{L-1} \sum_{j=1}^{c_{i+1}-1} p_{0,n-i-1,k-(d_i-i+(j-1))} + \sum_{i=0}^{L-1} p_{0,n-i-1,k-(d_i-i)}. \\ &= \sum_{i=1}^{L} \sum_{j=1}^{c_i-1} p_{0,n-i,k-(d_{i-1}-(i-1)+(j-1))} + \sum_{i=1}^{L} p_{0,n-i,k-(d_{i-1}-(i-1))}. \end{aligned} \tag{E.10}$$

Similarly for $g = 1$ we obtain

$$\begin{aligned} p_{1,n,k} - p_{1,n-1,k} \;=\; & \sum_{i=0}^{L-1}(c_{i+1}-1)\Big[p_{1,n-i-1,k-i} + (p_{1,n-i-1,k-i-1} - p_{1,n-i-1,k-i}) \\ & - (p_{1,n-i-2,k-i-1} - p_{1,n-i-2,k-i})\Big] + \sum_{i=1}^{L-1} p_{1,n-i-1,k-(i-1)}. \end{aligned} \tag{E.11}$$

Thus

$$\begin{aligned} p_{1,n,k} \;=\; & p_{1,n-1,k} + \sum_{i=0}^{L-1}(c_{i+1}-1)p_{1,n-i-1,k-i} + \sum_{i=1}^{L-1} p_{1,n-i-1,k-(i-1)} \\ & + \sum_{i=0}^{L-1}(c_{i+1}-1)\left((p_{1,n-i-1,k-i-1} - p_{1,n-i-1,k-i}) - (p_{1,n-i-2,k-i-1} - p_{1,n-i-2,k-i})\right) \\ =\; & p_{1,n-1,k} + \sum_{i=1}^{L}(c_i-1)p_{1,n-i,k-(i-1)} + \sum_{i=2}^{L} p_{1,n-i,k-(i-2)} \\ & + \sum_{i=1}^{L}(c_i-1)\left(\left(p_{1,n-i,k-i} - p_{1,n-i,k-(i-1)}\right) - \left(p_{1,n-i-1,k-i} - p_{1,n-i-1,k-(i-1)}\right)\right), \end{aligned} \tag{E.12}$$

and for $g \geq 2$ we have

$$\begin{aligned} p_{g,n,k} - p_{g,n-1,k} \;=\; & \sum_{i=0}^{L-1}(c_{i+1}-1)\Big[p_{g,n-i-1,k} + (p_{g,n-i-g,k-1} - p_{g,n-i-g,k}) \\ & - (p_{g,n-i-g-1,k-1} - p_{g,n-i-g-1,k})\Big] + \sum_{i=1}^{L-1}\Big[p_{g,n-i-1,k} \\ & + (p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k})\Big]. \end{aligned} \tag{E.13}$$

Thus

$$\begin{aligned} p_{g,n,k} \;=\; & \sum_{i=0}^{L-1} c_{i+1}p_{g,n-i-1,k} \\ & + \sum_{i=0}^{L-1}(c_{i+1}-1)\left((p_{g,n-i-g,k-1} - p_{g,n-i-g,k}) - (p_{g,n-i-g-1,k-1} - p_{g,n-i-g-1,k})\right) \\ & + \sum_{i=1}^{L-1}\left((p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k})\right) \\ =\; & \sum_{i=1}^{L} c_i p_{g,n-i,k} + \sum_{i=1}^{L} c_i^* \left((p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k})\right) \end{aligned} \tag{E.14}$$

as desired.

APPENDIX F. PROVING $C_\mu > 0$ AND $C_\sigma > 0$ IN SECTION 3.2

In this section we prove $C_\mu > 0$ and $C_\sigma > 0$ in (3.11). We first verify that the denominators of C_μ and C_σ are positive. Because the numerators of C_μ and C_σ are linear in the $t_{i,j}$, we obtain expressions for their numerators directly from the gap recurrences in (3.3). As the recurrences for the cases $g = 0$, $g = 1$, and $g \geq 2$ are different, we check that the numerators of C_μ and C_σ are positive for each case separately. Each case is dealt with using standard methods.

Recall

$$\begin{aligned} C_\mu &:= \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} \\ C_\sigma &:= \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}}. \end{aligned} \tag{F.1}$$

Let

$$\begin{aligned} C_\mu^* &:= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}, \\ C_\sigma^* &:= \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2 \end{aligned} \tag{F.2}$$

be the numerators of C_μ, C_σ in (F.1), respectively. Note that the denomatators of C_μ and C_σ are both always positive as

$$\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i} \;=\; \sum_{i=1}^{i_0}\frac{\hat{t}_i\cdot i}{\lambda_1^i} \;>\; 0 \tag{F.3}$$

since $\hat{t}_i = c_i > 0$ for $1 \leq i \leq L$ and $\hat{t}_i = 0$ for $L < i < i_0$. Thus it suffices to prove $C_\mu^* > 0$ and $C_\sigma^* > 0$ when $K_{g,n}$ is nontrivial.

We first prove $C_\mu^* > 0$. Since C_μ^* is linear in $t_{i,j}$, (F.1) tells us we can obtain C_μ^* by replacing every instance of $p_{g,n-x,k-y}$ in the recurrence relations of (3.3) with y/λ_1^x.

Suppose $g = 0$. If $c_i = 1$ for all $1 \leq i < L$ and c_L is 1 or 2, then no m has gaps of size 0 in the decomposition, so the random variable $K_{g,n}$ is trivial. Otherwise, $c_i \geq 2$ for some $i < L$. In this case, $d_{L-1} - (L-1) > 0$. Thus, evaluating C_μ^* gives

$$p_{0,n,k} \;=\; \sum_{i=1}^{L}\sum_{j=1}^{c_i-1} p_{0,n-i,k-(d_{i-1}-(i-1)+(j-1))} + \sum_{i=1}^{L} p_{0,n-i,k-(d_{i-1}-(i-1))}, \tag{F.4}$$

so

$$\begin{aligned} C_\mu^* \;&=\; \sum_{i=1}^{L}\sum_{j=1}^{c_i-1}\frac{k-(d_{i-1}-(i-1)+(j-1))}{\lambda_1^i} + \sum_{i=1}^{L}\frac{k-(d_{i-1}-(i-1))}{\lambda_1^i} \\ &>\; \sum_{i=1}^{L}\sum_{j=1}^{c_i-1}\frac{0}{\lambda_1^i} + \sum_{i=1}^{L}\frac{0}{\lambda_1^i} \;=\; 0. \end{aligned} \tag{F.5}$$

Now suppose $g = 1$. If $c_1 = c_2 = 1$ and $L = 2$ (i.e., $\{G_n\}$ is the Fibonaccis), then $K_{1,n} = 0$ is trivial, so we can assume otherwise. Recall

$$\begin{aligned} p_{1,n,k} \;=\; & p_{1,n-1,k} + \sum_{i=1}^{L}(c_i - 1)p_{1,n-i,k-(i-1)} + \sum_{i=2}^{L} p_{1,n-i,k-(i-2)} \\ & + \sum_{i=1}^{L}(c_i - 1)\left(\left(p_{1,n-i,k-i} - p_{1,n-i,k-(i-1)}\right) - \left(p_{1,n-i-1,k-i} - p_{1,n-i-1,k-(i-1)}\right)\right). \end{aligned} \tag{F.6}$$

Note, when we perform the substitution to obtain C_μ^*, any expression of the form

$$(p_{1,n-(x-1),k-y} - p_{1,n-(x-1),k-(y-1)}) - (p_{1,n-x,k-y} + p_{1,n-x,k-(y-1)}) \tag{F.7}$$

becomes

$$\frac{y-(y-1)}{\lambda_1^{x-1}} - \frac{y-(y-1)}{\lambda_1^{x}} \;=\; \frac{\lambda_1 - 1}{\lambda_1^{x}} \;>\; 0. \tag{F.8}$$

Thus (F.6) gives that when $g = 1$,

$$C_\mu^* \;=\; \frac{0}{\lambda_1^1} + \sum_{i=1}^{L}(c_i - 1)\frac{(i-1)}{\lambda_1^i} + \sum_{i=2}^{L}\frac{i-2}{\lambda_1^i} + \sum_{i=1}^{L}(c_i - 1)\frac{\lambda_1 - 1}{\lambda_1^{i+1}} \;>\; 0. \tag{F.9}$$

To see that the sum is in fact positive, note first that every summand is nonnegative. Furthermore, if any c_i is greater than 1, the last sum is strictly positive. Otherwise, all the c_i's are 1, in which case $L \geq 3$ since the sequence is not the Fibonaccis. Then the second to last sum will be strictly positive.

Lastly, assume $g \geq 2$. Recall

$$p_{g,n,k} \;=\; \sum_{i=1}^{L} c_i p_{g,n-i,k} + \sum_{i=1}^{L} c_i^* \left((p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k})\right). \tag{F.10}$$

Performing the same substitution gives

$$C_\mu^* \;=\; \sum_{i=1}^{L} c_i \cdot 0 + \sum_{i=1}^{L} c_i^* \cdot \frac{\lambda_1 - 1}{\lambda_1^{i+g}} \;>\; 0 \tag{F.11}$$

as $c_i^* > 0$ for some i by definition. This proves that for any g, we have $C_\mu^* > 0$, so for any g we also have $C_\mu > 0$.

We can similarly casework on g to prove $C_\sigma^* > 0$ when $K_{g,n}$ is nontrivial. As before C_σ^* linear in the $t_{i,j}$, so by (F.1) we can obtain C_σ^* by replacing every instance of $p_{g,n-x,k-y}$ in (3.3) with $(y - C_\mu x)^2/\lambda_1^x$. This produces an expression for C_σ^* that we prove is positive using standard techniques.

First suppose $g = 0$. Recall

$$p_{0,n,k} \;=\; \sum_{i=1}^{L}\sum_{j=1}^{c_i - 1} p_{0,n-i,k-(d_{i-1}-(i-1)+(j-1))} + \sum_{i=1}^{L} p_{0,n-i,k-(d_{i-1}-(i-1))}. \tag{F.12}$$

By considering $i = 1$ and $j = 1$ in the double sum of (F.12), we have $t_{1,0} = 1$. Since $t_{i,j} \geq 0$ for all i, j, we have $C_\sigma^* = 0$ if and only if $j - C_\mu i = 0$ for all i, j satisfying $t_{i,j} \neq 0$. But this is impossible as $t_{1,0} = 1$ implies $C_\mu = 0$, and we already showed $C_\mu > 0$ when $K_{g,n}$ is nontrivial. Thus $C_\sigma^* > 0$ when $g = 0$ and $K_{0,n}$ is nontrivial.

Now suppose $g = 1$. Recall

$$\begin{aligned} p_{1,n,k} \;=\; & p_{1,n-1,k} + \sum_{i=1}^{L}(c_i - 1)p_{1,n-i,k-(i-1)} + \sum_{i=2}^{L} p_{1,n-i,k-(i-2)} \\ & + \sum_{i=1}^{L}(c_i - 1)\left(\left(p_{1,n-i,k-i} - p_{1,n-i,k-(i-1)}\right) - \left(p_{1,n-i-1,k-i} - p_{1,n-i-1,k-(i-1)}\right)\right). \end{aligned} \tag{F.13}$$

Again, if $c_1 = c_2 = 1$ and $L = 2$ (i.e., $\{G_n\}$ is the Fibonaccis), then $K_{1,n} = 0$ is trivial, so we can assume otherwise. Substituting for C_σ^* gives

$$\begin{aligned} C_\sigma^* \;=\; & \frac{(0 - C_\mu)^2}{\lambda_1^i} + \sum_{i=1}^{L}(c_i - 1)\frac{(i - 1 - C_\mu i)^2}{\lambda_1^i} + \sum_{i=2}^{L}\frac{(i - 2 - C_\mu i)^2}{\lambda_1^i} \\ & + \sum_{i=1}^{L}(c_i - 1)\left(\frac{(i - C_\mu i)^2}{\lambda_1^i} - \frac{(i - 1 - C_\mu i)^2}{\lambda_1^i} - \frac{(i - C_\mu(i+1))^2}{\lambda_1^{i+1}} + \frac{(i - 1 - C_\mu(i+1))^2}{\lambda_1^{i+1}}\right) \\ = \; & \frac{C_\mu^2}{\lambda_1^i} + \sum_{i=2}^{L}\frac{(i - 2 - C_\mu i)^2}{\lambda_1^i} + \sum_{i=1}^{L}(c_i - 1)\left(\frac{(i - C_\mu i)^2}{\lambda_1^i} - \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1^{i+1}}\right) \end{aligned} \tag{F.14}$$

By an earlier argument $C_\mu > 0$, so we simplify to get

$$C_\sigma^* \;>\; \sum_{i=1}^{L}(c_i - 1)\left(\frac{(i - C_\mu i)^2}{\lambda_1^i} - \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1^{i+1}}\right). \tag{F.15}$$

Now we show that for all $i \geq 1$ we have

$$(i - C_\mu i)^2 - \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1} \;\geq\; 0. \tag{F.16}$$

If $2i - 1 - 2C_\mu(i+1) \geq 0$, then since $\lambda_1 > 1$, we obtain

$$\begin{aligned} (i - C_\mu i)^2 - & \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1} \\ & \geq\; (i - C_\mu i)^2 - (2i - 1 - 2C_\mu(i+1)) \\ & =\; (i - 1 - C_\mu i)^2 + 2C_\mu \;>\; 0. \end{aligned} \tag{F.17}$$

Otherwise, we have $2i - 1 - 2C_\mu(i+1) < 0$ so

$$(i - C_\mu i)^2 - \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1} \;>\; (i - C_\mu i)^2 \;\geq\; 0. \tag{F.18}$$

These two cases allow us to conclude

$$\begin{aligned} C_\sigma^* \;>\; & \sum_{i=1}^{L}\frac{c_i - 1}{\lambda_1^i}\left((i - C_\mu i)^2 - \frac{2i - 1 - 2C_\mu(i+1)}{\lambda_1}\right) \\ \geq\; & \sum_{i=1}^{L}\frac{c_i - 1}{\lambda_1^i}\cdot 0 \;=\; 0 \end{aligned} \tag{F.19}$$

as desired.

Finally suppose $g = 2$. Recall

$$p_{g,n,k} = \sum_{i=1}^{L} c_i p_{g,n-i,k} + \sum_{i=1}^{L} c_i^* \left((p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k}) \right). \tag{F.20}$$

Substituting for C_σ^* as before gives

$$\begin{aligned} C_\sigma^* &= \sum_{i=1}^{L} c_i \cdot \frac{(0 - C_\mu i)^2}{\lambda_1^i} \\ &\quad + \sum_{i=1}^{L} c_i^* \left(\frac{(1 - C_\mu(i+g-1))^2 - (0 - C_\mu(i+g-1)^2)}{\lambda_1^{i+g-1}} - \frac{(1 - C_\mu(i+g))^2 - (0 - C_\mu(i+g)^2)}{\lambda_1^{i+g}} \right) \\ &= \sum_{i=1}^{L} c_i \cdot \frac{(C_\mu i)^2}{\lambda_1^i} + \sum_{i=1}^{L} c_i^* \left(\frac{1 - 2C_\mu(i+g-1)}{\lambda_1^{i+g-1}} - \frac{1 - 2C_\mu(i+g)}{\lambda_1^{i+g}} \right) \\ &= \sum_{i=1}^{L} \frac{c_i}{\lambda_1^{i+g}} \cdot \left(\lambda_1^g (C_\mu i)^2 + \frac{c_i^*}{c_i} \left(\lambda_1(1 - 2C_\mu(i+g-1)) - (1 - 2C_\mu(i+g)) \right) \right). \end{aligned} \tag{F.21}$$

Since $\lambda_1 > 1$, the coefficient of i in $\lambda_1(1 - 2C_\mu(i+g-1)) - (1 - 2C_\mu(i+g))$ is negative, so it is minimized when $i = L$. Thus if $\lambda_1(1 - 2C_\mu(L+g-1)) - (1 - 2C_\mu(L+g)) \geq 0$, (F.21) tells us $C_\sigma^* > 0$. Thus we may assume $\lambda_1(1 - 2C_\mu(L+g-1)) - (1 - 2C_\mu(L+g)) < 0$, Since $c_i^*/c_i \leq 1$ with equality if and only if $i \neq L$, we can simplify (F.21) to get

$$C_\sigma^* > \sum_{i=1}^{L} \frac{c_i}{\lambda_1^{i+g}} \cdot \left(\lambda_1^g (C_\mu i)^2 + \left(\lambda_1(1 - 2C_\mu(i+g-1)) - (1 - 2C_\mu(i+g)) \right) \right). \tag{F.22}$$

Using standard techniques (such as plugging into Mathematica), one can show that $x^y(zw)^2 + (x(1 - 2z(w+y-1)) - (1 - 2z(w+y)) \geq 0$ for all $x \geq 1, y \geq 2, z \geq 0, w \geq 1$, and substituting $x = \lambda_1, y = g, z = C_\mu, w = i$ gives

$$\begin{aligned} C_\sigma^* &> \sum_{i=1}^{L} \frac{c_i}{\lambda_1^{i+g}} \cdot \left(\lambda_1^g (C_\mu i)^2 + \left(\lambda_1(1 - 2C_\mu(i+g-1)) - (1 - 2C_\mu(i+g)) \right) \right) \\ &\geq \sum_{i=1}^{L} \frac{c_i}{\lambda_1^{i+g}} \cdot 0 = 0. \end{aligned} \tag{F.23}$$

as desired.

For every g and every sequence for which $K_{g,n}$ is nontrivial, we've proven $C_\sigma^* > 0$, so we can conclude $C_\sigma > 0$ is all of these cases.

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