APPROXIMATING REAL-ROOTED AND STABLE POLYNOMIALS, WITH COMBINATORIAL APPLICATIONS

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Abstract. Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial with all roots real and satisfying $x \leq -\delta$ for some $0 < \delta < 1$. We show that for any $0 < \epsilon < 1$, the value of $p(1)$ is determined within relative error ϵ by the coefficients a_k with $k \leq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon \sqrt{\delta}}$ for some absolute constant $c > 0$. As a corollary, we show that if $m_k(G)$ is the number of matchings with k edges in a graph G, then for any $0 < \epsilon < 1$, the total number $M(G) = m_0(G) + m_1(G) + \ldots$ of matchings can be approximated within relative error ϵ in polynomial time from the numbers $m_k(G)$ with $k \leq c\sqrt{\Delta} \ln(v/\epsilon)$, where Δ is the largest degree of a vertex, v is the number of vertices of G and $c > 0$ is an absolute constant. We prove a similar result for polynomials with complex roots satisfying $\Re z \leq -\delta$ and apply it to estimate the number of unbranched subgraphs of G.

1. Introduction and main results

Our main motivation comes from the observation that in some cases, the total number of combinatorial structures of a particular type is determined with high accuracy by the exact number of the structures of the same type but of a small (sometimes, very small) size. We deduce it from some general results on the approximation of the value of a polynomial from its first few lowest coefficients.

Below we talk about approximating some real and complex values up to "relative error ϵ". Given a complex number $a \neq 0$, we say that a complex number $b \neq 0$ approximates a up to (or within) relative error $\epsilon > 0$ if we can write $a = e^z$ and $b = e^w$ for some complex numbers z and w such that $|z - w| \leq \epsilon$.

We prove the following main result.

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(1.1) Theorem. *Suppose that $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a polynomial and all roots x of p are real and satisfy $x \leq -\delta$ for some $0 < \delta < 1$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ, is determined by the coefficients a_k with*

$$k \ \leq \ \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon\sqrt{\delta}},$$

where $c > 0$ is an absolute constant.

In fact, we present a polynomial time algorithm, which, given $a_0, \ldots, a_k$ with k as in the theorem, computes $p(1)$ within relative error ϵ. Theorem 1.1 is a much improved version of the "personal communication" of the author that was referred to in [PR17].

As a prime example, we consider matchings in a given graph. We consider undirected graphs, without loops or multiple edges. Recall that a *matching* in a graph is a collection, possibly empty, of vertex-disjoint edges of the graph. In statistical physics, matchings are known as *monomer-dimer systems*, with edges of the matching corresponding to dimers and vertices not covered by the edges of the matching corresponding to monomers. Matchings have been thoroughly studied for quite some time, see [LP09] for a standard reference.

Given a graph G with $v(G)$ vertices, let $m_k(G)$ be the number of matchings containing exactly k edges and let

$$M(G) = \sum_{k=0}^{v(G)/2} m_k(G)$$

be the total number of matchings in G. The numbers $m_k(G)$ and $M(G)$ were extensively studied, from the statistical (see, for example, [KK98]) and computational (randomized [JS89] and deterministic [B+07]) points of view.

Using the Heilmann-Lieb Theorem [HL72], see also [GG81], we immediately deduce from Theorem 1.1 that the total number of matchings $M(G)$ in a graph is determined with high accuracy by the numbers $m_k(G)$ of matchings with a relatively small number k of edges.

(1.2) Theorem. *For a graph G, let $m_k(G)$ be the number of matchings that contain exactly k edges and let $M(G) = m_0(G) + m_1(G) + \ldots$ be the total number of matchings. Then, for any $0 < \epsilon < 1$, up to relative error ϵ, the number $M(G)$ is determined by the numbers $m_k(G)$ with*

$$k \ \leq \ c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

Again, we have a polynomial time algorithm, which, given $m_k(G)$, with k as in the theorem, produces an estimate of $M(G)$ within relative error ϵ. We note that Patel and Regts [PR17] constructed a polynomial time algorithm for computing $m_k(G)$ with $k = O(\ln v(G))$ provided the largest degree $\Delta(G)$ is fixed in advance (a straightforward enumeration gives only a quasi-polynomial algorithm of $v(G)^{O(\ln v(G))}$ complexity).

For general graphs, the complexity of the algorithm roughly matches that of Bayati et al. [B+07], which estimates $M(G)$ using the correlation decay approach. Although our approach and that of [B+07] look completely different, they are both inspired by the concept of "phase transition" coming from statistical physics; more precisely, two related, but different concepts: ours has to do with the Lee-Yang approach via complex zeros of the "partition function" [YL52], [LY52] while that of [B+07] has to do with correlation decay, cf. [DS87] and [KK98]. Hence the fact that the complexity appears to be roughly the same is not entirely accidental.

The idea of the correlation decay approach is roughly as follows. We consider the set of all matchings in G as a probability space with the uniform measure. Then to compute $M(G)$ it suffices to compute the probability that a vertex, say a, of G is covered by a random matching. It turns out that this event is asymptotically independent on whether the vertices of G sufficiently remote from a, are covered by a random matching and hence the desired probability can be approximated from the local structure of G in the vicinity of a. It follows from [B+07] that up to relative error ϵ the number $M(G)$ can be determined from the local structure of G in k-neighborhoods of the vertices, where $k = O\left(\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon}\right)$. In contrast, Theorem 1.2 states that to approximate $M(G)$ we do not need to know any fine local structure, but just the numbers $m_k(G)$ for those values of k. We also note that to compute $M(G)$ exactly is a #P-hard problem and that there is a fully polynomial *randomized* approximation scheme [JS89], based on the Markov Chain Monte Carlo approach.

Using the Chudnovsky-Seymour extension [CS07] of the Heilmann-Lieb Theorem and the Dobrushin-Shearer bound on the roots of the independence polynomial, see, for example, [SS05], we get another combinatorial application. Recall that a subset of vertices of a graph is called an *independent set* if no two vertices of the subset span an edge of the graph. A graph is called *claw-free* if it does not contain an induced subgraph consisting of a vertex connected to some other three vertices that are pairwise unconnected. We obtain the following result.

(1.3) Theorem. *For a graph G, let $i_k(G)$ be the number of independent sets with exactly k vertices and let $I(G) = i_0(G) + i_1(G) + \ldots$ be the total number of independent sets. Then, for any $0 < \epsilon < 1$, up to relative error ϵ, the number $I(G)$ of a claw-free graph is determined by the numbers $i_k(G)$ with*

$$k \ \leq \ c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

We have a polynomial time algorithm, which, given $i_k(G)$, with k as in the theorem, produces an estimate of $I(G)$ within relative error ϵ. Curiously, while the correlation decay approach of [B+07] is essentially harder in the case of independent sets in claw-free graphs than it is in the case of matchings, our approach is the same in both cases (assuming, of course, the hard work done in [HL72] and [CS07]).

Theorems 1.2 and 1.3 vaguely resemble the "approximate inclusion-exclusion" of [LN90] and [K+96]. The methods, however, look completely different. It would be interesting to find out if there is indeed any connection between our Theorem 1.1 and the results of [LN90] and [K+96].

Next, we consider polynomials $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ with complex roots satisfying $\Re z \leq -\delta$ for some $0 < \delta < 1$ (we call such polynomials "stable"). We allow complex coefficients a_k. We obtain the following result.

(1.4) Theorem. *Suppose that $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is a complex polynomial and all roots z of p satisfy $\Re z \leq -\delta$ for some $0 < \delta < 1$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ, is determined by the coefficients a_k with*

$$k \leq \frac{c}{\delta} \ln \frac{n}{\epsilon\delta},$$

where $c > 0$ is an absolute constant.

We apply Theorem 1.4 to count *unbranched subgraphs*, that is, collections of edges of the graph such that every vertex of the graph is incident to at most two edges from the collection. From Ruelle's Theorem [R99a], [R99b], see also [Wa09], we deduce the following result.

(1.5) Theorem. *For a graph G, let $u_k(G)$ be the number of unbranched subgraphs with exactly k edges and let $U(G) = u_0(G) + u_1(G) + \ldots$ be the total number of unbranched subgraphs. Then, for any $0 < \epsilon < 1$, up to relative error ϵ, the number $U(G)$ is determined by the numbers $u_k(G)$ with*

$$k \leq c\,(\Delta(G))^3 \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

One can easily see that if a non-constant polynomial p satisfies the conditions of Theorems 1.1 or 1.4 then so does its derivative p'. Therefore, in Theorems 1.2, 1.3 and 1.5, we can not only estimate the number of structures of a given type (matchings, independent sets or unbranched subgraphs) by counting structures up to some small size, but also estimate the average size of a structure, the second moment, etc.

Finally, we mention the following result implicit in Section 2.2 of [Ba16].

(1.6) Theorem. *Let us fix a connected open set $U \subset \mathbb{C}$ containing 0 and 1. Then there exists a constant $\gamma = \gamma(U) > 0$ such that the following holds. Suppose that*

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad n \geq 2,$$

is a polynomial such that $p(z) \neq 0$ for all $z \in U$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ, is determined by the coefficients a_k with

$$k \leq \gamma \left(\ln n - \ln \epsilon\right).$$

In particular, Lemma 2.2.3 of [Ba16] implies that if U is a δ-neighborhood of the interval $[0,1] \subset \mathbb{C}$ for some $0 < \delta < 1$ (we measure distances by identifying $\mathbb{C} = \mathbb{R}^2$) then one can choose

$$\gamma(U) = e^{O(1/\delta)}.$$

For applications to computing partition functions, see [Ba16]. We can replace the exponential dependence on $1/\delta$ by a polynomial dependence if we assume that $p(z) \neq 0$ for z in the δ-neighborhood of the sector $|\arg z| < \alpha$ for some fixed $\alpha > 0$ and some $\delta > 0$. We briefly discuss this in Section 2 and applications to counting subgraphs with prescribed degrees and independent sets in some graphs in Section 3.

We prove Theorems 1.1 and 1.4 in Section 2 and Theorems 1.2, 1.3 and 1.5 in Section 3.

2. Proofs of Theorems 1.1 and 1.4

We denote the complex plane by $\mathbb{C}$, the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by $\widehat{\mathbb{C}}$ and the open unit disc by $\mathbb{D}$, so that

$$\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}.$$

By c we denote a positive absolute constant, whose value may change from line to line.

We start with a couple of lemmas.

(2.1) Lemma. *Let $h_1, h_2 : \mathbb{C} \longrightarrow \mathbb{C}$ be polynomials of degrees n_1 and n_2 respectively and let*

$$g(z) = \frac{h_1(z)}{h_2(z)}, \quad g : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}},$$

be a rational function. Let $\beta > 1$ be a real number and suppose that

$$h_1(z) \neq 0 \quad \text{and} \quad h_2(z) \neq 0 \quad \text{provided} \quad |z| < \beta,$$

so g has neither zeros no poles in the disc $\beta\mathbb{D} = \{z \in \mathbb{C} : \ |z| < \beta\}$.

Let us choose a branch of

$$f(z) = \ln g(z) \quad \textit{where} \quad |z| \ < \ \beta$$

and let

$$T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k$$

be the Taylor polynomial of degree m of $f(z)$ computed at $z = 0$. Then

$$|f(1) - T_m(1)| \ \leq \ \frac{n_1 + n_2}{\beta^m(\beta - 1)(m + 1)}.$$

Proof. In the case when $g(z)$ is a polynomial (that is, when $h_2(z) \equiv 1$), this is Lemma 2.2.1 of [Ba16]. The proof below in the case of a rational function is very similar.

Let $\alpha_{11}, \ldots, \alpha_{1n_1}$ be the roots of h_1 and let $\alpha_{21}, \ldots, \alpha_{2n_2}$ be the roots of h_2, counting multiplicity. Hence

$$h_1(z) = h_1(0) \prod_{i=1}^{n_1} \left(1 - \frac{z}{\alpha_{1i}}\right) \quad \text{and} \quad h_2(z) = h_2(0) \prod_{j=1}^{n_2} \left(1 - \frac{z}{\alpha_{2j}}\right),$$

where

$$|\alpha_{1i}| \ \geq \ \beta \quad \text{for} \quad i = 1, \ldots, n_1 \quad \text{and} \quad |\alpha_{2j}| \ \geq \ \beta \quad \text{for} \quad j = 1, \ldots, n_2.$$

Then

$$f(z) = f(0) + \sum_{i=1}^{n_1} \ln\left(1 - \frac{z}{\alpha_{1i}}\right) - \sum_{j=1}^{n_2} \ln\left(1 - \frac{z}{\alpha_{2j}}\right),$$

where we choose the branch of the logarithm so that $\ln 1 = 0$.

Approximating the logarithms by their Taylor polynomials, we obtain

$$\ln\left(1 - \frac{1}{\alpha_{1i}}\right) = -\sum_{k=1}^{m} \frac{1}{k\alpha_{1i}^k} + \eta_{1i} \quad \text{and} \quad \ln\left(1 - \frac{1}{\alpha_{2j}}\right) = -\sum_{k=1}^{m} \frac{1}{k\alpha_{2j}^k} + \eta_{2j},$$

where

$$|\eta_{1i}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{1i}^k} \right| \ \leq \ \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} \ = \ \frac{1}{(m+1)\beta^m(\beta - 1)}$$

for $i = 1, \ldots, n_1$ and, similarly,

$$|\eta_{2j}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{2j}^k} \right| \ \leq \ \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} \ = \ \frac{1}{(m+1)\beta^m(\beta - 1)}$$

for $j = 1, \ldots, n_2$.

Since

$$T_m(1) = -\sum_{i=1}^{n_1} \sum_{k=1}^{m} \frac{1}{k\alpha_{1i}^k} + \sum_{j=1}^{n_2} \sum_{k=1}^{m} \frac{1}{k\alpha_{2j}^k},$$

the proof follows. □

(2.2) Corollary. *For $0 < \epsilon < 1$, under the conditions of Lemma 2.1, we have*

$$|f(1) - T_m(1)| \ \leq \ \epsilon$$

provided

$$m \ \geq \ \frac{c}{\beta - 1} \ln \frac{n_1 + n_2}{\epsilon(\beta - 1)}$$

where $c > 0$ is an absolute constant.

Proof. Follows by Lemma 2.1. □

To compute the value of $T_m(1)$ in Lemma 2.1 and Corollary 2.2, we need to compute the derivatives $f^{(k)}(0)$ for $k = 0, 1, \ldots, m$. This, in turn, reduces to computing the derivatives $g^{(k)}(0)$ for $k = 0, 1, \ldots, m$, as is explained in Section 2.2.2 of [Ba16]. For completeness, we describe the procedure here.

(2.3) Computing $f^{(k)}(0)$ from $g^{(k)}(0)$. We have

$$f'(z) = \frac{g'(z)}{g(z)} \quad \text{from which} \quad g'(z) = f'(z)g(z)$$

and hence

$$g^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0) g^{(j)}(0) \quad \text{for} \quad k = 1, \ldots, m. \tag{2.3.1}$$

Now, (2.3.1) is a triangular system of linear equations in the variables $f^{(k)}(0)$ for $k = 1, \ldots, m$ with diagonal coefficients $g^{(0)}(0) = g(0) \neq 0$, so the matrix of the system is invertible. Given the values of $g(0)$ and $g^{(k)}(0)$ for $k = 1, \ldots, m$, one can compute the values of $f^{(k)}(0)$ for $k = 1, \ldots, m$ in $O(m^2)$ time. This is, of course, akin to computing cumulants of a probability distribution from its moments.

Finally, we employ a rational transformation.

(2.4) Lemma. *For real $0 < \rho < 1$, let*

$$\xi = \xi_\rho = 1 - \sqrt{\frac{\rho}{1 + \rho}}, \quad \beta = \beta_\rho = \xi^{-1} \ \geq \ 1 + \sqrt{\rho}.$$

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{(1 - \xi z)^2} - \rho, \quad \psi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta\mathbb{D} = \{z \in \mathbb{C} : \ |z| \ < \ \beta\}$$

under ψ does not intersect the ray

$$\left\{z \in \mathbb{C}: \ \Im z = 0 \quad \textit{and} \quad \Re z \ \leq \ -\frac{3\rho}{4}\right\}.$$

Proof. Clearly, $\psi(0)=0$ and $\psi(1)=1$. For $z \in \beta\mathbb{D}$, we have $|\xi z|<1$ and hence

$$\arg \frac{1}{1-\xi z} \ < \ \frac{\pi}{2}.$$

Therefore the image of $\beta\mathbb{D}$ under the map

$$z \longmapsto \frac{\rho}{(1-\xi z)^2} \tag{2.4.1}$$

does not contain the non-positive real ray

$$R_- = \{z \in \mathbb{C}: \ \Im z = 0, \ \Re z \ \leq \ 0\}.$$

The real values of the map (2.4.1) on the disc $\beta\mathbb{D}$ are attained when z is real, and are larger than $\rho/4$, which is attained when $z=-\beta$.

The proof now follows. □

Now we are ready to prove Theorem 1.1.

(2.5) Proof of Theorem 1.1. Let $\rho = 4\delta/3$ and let $\psi=\psi_\rho: \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be the corresponding rational transformation of Lemma 2.4. We consider the composition

$$g(z) = p\left(\psi(z)\right).$$

Clearly, $g(0)=p(0)$ and $g(1)=p(1)$. Let

$$\xi = 1 - \sqrt{\frac{\rho}{1+\rho}} \quad \text{and} \quad \beta = \xi^{-1} \ \geq \ 1 + \sqrt{\frac{4\delta}{3}},$$

as in Lemma 2.4. Since the image $\psi\left(\beta\mathbb{D}\right)$ does not intersect the ray

$$R = \{z \in \mathbb{C}: \ \Im z = 0 \quad \text{and} \quad \Re z \ \leq \ -\delta\},$$

we conclude that

$$g(z) \neq 0 \quad \text{provided} \quad |z| \ < \ \beta.$$

For some polynomials $h_1(z)$ and $h_2(z)$, we can write

$$g(z) = \frac{h_1(z)}{h_2(z)} \quad \text{where} \quad h_2(z) = (1-\xi z)^{2n} \quad \text{and} \quad \deg h_2(z) \ \leq \ 2n.$$

Let us choose a branch of

$$f(z) = \ln g(z) \quad \text{for} \quad |z| < \beta$$

and let $T_m(z)$ be the Taylor polynomial of f of degree m, computed at $z = 0$. From Corollary 2.2, we have

$$|T_m(1) - f(1)| = |T_m(1) - \ln p(1)| \ \leq \ \epsilon,$$

as long as

$$m \ \geq \ \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon\sqrt{\delta}}$$

for some absolute constant $c > 0$.

It remains to show how to compute the values $f^{(k)}(0)$ for $k = 0, \ldots, m$ from the coefficients a_k, $k = 0, 1, \ldots, m$, of the polynomial p. First, we compute the values $g^{(k)}(0)$ for $k = 0, \ldots, m$. To that effect, let

$$p_{[m]}(z) = \sum_{k=0}^{m} a_k z^k$$

be the truncation of the polynomial p and let

$$\psi_{[m]}(z) = \rho \sum_{k=1}^{m} (k+1) \xi^k z^k$$

be the truncation of the Taylor series expansion of $\psi(z)$ in the disc $\beta\mathbb{D}$. Note that since $\psi(0) = 0$, the constant term of the expansion is 0. We then compute the composition

$$p_{[m]}\left(\psi_{[m]}(z)\right)$$

and discard the terms of degree higher than m. A fast way to do it is by Horner's method, successively computing

$$\left(\ldots\left(\left(a_m \psi_{[m]}(z) + a_{m-1}\right) \psi_{[m]}(z)\right) + \ldots\right) \psi_{[m]}(z) + a_0$$

and discarding monomials of degree higher than m on each step. This gives us the Taylor polynomial of degree m of $g(z)$, computed at $z = 0$. We then compute the derivatives $f^{(k)}(0)$ as in Section 2.3. $\square$

To prove Theorem 1.4, we use a different (simpler) rational transformation.

(2.6) Lemma. *For real* $0 < \rho < 1$, *let*

$$\xi = \xi_\rho = \frac{1}{1+\rho}, \quad \beta = \beta_\rho = \xi^{-1} = 1 + \rho.$$

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{1-\xi z} - \rho, \quad \psi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta\mathbb{D} = \{z \in \mathbb{C} : \ |z| \ < \ \beta\}$$

under ψ does not intersect the half-plane

$$\{z \in \mathbb{C} : \quad \Re\, z \ \leq \ -\rho\}.$$

Proof. Clearly, $\psi(0) = 0$ and $\psi(1) = 1$. For $z \in \beta\mathbb{D}$, we have $|\xi z| < 1$ and hence

$$\arg \frac{1}{1-\xi z} \ < \ \frac{\pi}{2}.$$

Therefore, the image of $\beta\mathbb{D}$ under the map

$$z \longmapsto \frac{1}{1-\xi z}$$

does not intersect the half-plane $\Re\, z \leq 0$.

The proof now follows. □

(2.7) Proof of Theorem 1.4. We define the transformation $\psi = \psi_\delta$ as in Lemma 2.6, consider the composition $g(z) = p(\psi(z))$ and proceed as in the proof on Theorem 1.1 in Section 2.5 with straightforward modifications. □

(2.8) Remark: approximating $p'(1)$. It follows from Rolle's Theorem that if $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a non-constant polynomial satisfying the conditions of Theorem 1.1 then $p'(x) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}$ also satisfies the conditions of Theorem 1.1. Similarly, it follows from the Gauss-Lucas Theorem that if a non-constant polynomial $p(z)$ satisfies the conditions of Theorem 1.4, then so does $p'(z)$.

(2.9) Possible ramifications. To prove Theorem 1.6, we construct an auxiliary polynomial $\psi(z)$ such that for some $\beta > 1$ the image $\psi(\beta\mathbb{D})$ lies in U and, additionally, $\psi(0) = 0$ and $\psi(1) = 1$. Then the proof proceeds as in Section 2.5. Such a polynomial ψ can be found from the Riemann Conformal Mapping Theorem (this was pointed out to the author by P. Etingof [Et19]). To that end, let us choose an open simply connected set $U' \subset U$ containing 0 and 1 and such that the ϵ-neighborhood of U' lies in U for some $\epsilon > 0$. It suffices to construct an analytic map $\phi : \beta'\mathbb{D} \longrightarrow U'$ for some $\beta' > 1$ such that $\phi(0) = 0$ and $\phi(1) = 1$. Then using the Runge Theorem (see, for example, Chapter VIII of [Co73]), we approximate ϕ

uniformly and sufficiently close by a polynomial $\widehat{\psi}$ on the closed disc $\beta\overline{\mathbb{D}}$ for some $1 < \beta < \beta'$, so that the polynomial

$$\psi(z) = \frac{\widehat{\psi}(z) - \widehat{\psi}(0)}{\widehat{\psi}(1) - \widehat{\psi}(0)}$$

satisfies the required property. To construct a map ϕ as above, using the Riemann Theorem (see, for example, Chapter VII of [Co73]), we construct a conformal isomorphism $\widehat{\phi} : \mathbb{D} \longrightarrow U'$ such that $\widehat{\phi}(0) = 0$. Let $\zeta \in \mathbb{D}$ satisfy $\widehat{\phi}(\zeta) = 1$. Combining $\widehat{\phi}$ with a rotation of the disc, if necessary, we can assume that $0 < \zeta < 1$ is a real number. We let $\beta' = \zeta^{-1}$ and define $\phi : \beta'\mathbb{D} \longrightarrow U'$ by

$$\phi(z) = \widehat{\phi}(\zeta z).$$

Suppose now that $p(z)$ is a polynomial of degree n such that $p(z) \neq 0$ whenever z lies in the δ-neighborhood of the sector

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : \quad |\arg z| \leq \alpha\}$$

for some fixed $\alpha > 0$ and some $\delta > 0$. In this case, for $0 < \rho < 1$ define

$$\xi = \xi_\rho = 1 - \left(\frac{\rho}{1+\rho}\right)^{\pi/2\alpha}, \quad \beta = \beta_\rho = \xi^{-1}\left(1 - \frac{1}{2}\left(\frac{\rho}{1+\rho}\right)^{\pi/2\alpha}\right) \ > \ 1$$

and consider the (well-defined) map

$$\psi(z) = \psi_\rho(z) = \rho\,(1 - \xi z)^{-2\alpha/\pi} - \rho \quad \text{for} \quad |z| < \beta.$$

We observe that $\psi(0) = 0$, $\psi(1) = 1$ and one can show that by choosing ρ small enough, we can make sure that the image of the disc $\beta\mathbb{D}$ lies in a prescribed neighborhood of the sector $|\arg z| \leq \alpha$. We then can use a sufficiently accurate polynomial approximation $\tilde{\psi}$ to ψ to show that the value of $p(1)$, up to relative error ϵ, is determined by the lowest

$$\frac{c}{\delta^{\pi/2\alpha}} \ln \frac{n}{\epsilon\delta^{\pi/2\alpha}}$$

coefficients of p, where $c > 0$ is an absolute constant.

3. Proofs of Theorems 1.2, 1.3 and 1.5

(3.1) Proof of Theorem 1.2. Given a graph G, we define its *matching polynomial* by

$$p(x) = 1 + \sum_{k=1}^{v(G)/2} m_k(G) x^k.$$

The Heilmann-Lieb Theorem [HL72], see also [GG81], asserts that the roots x of $p(x)$ are real and satisfy $x \leq -\delta$ for

$$\delta = \frac{1}{4\,(\Delta - 1)} \quad \text{for} \quad \Delta = \max\{\Delta(G), 2\}.$$

The proof now follows from Theorem 1.1. $\square$

(3.2) Proof of Theorem 1.3. Given a graph G, we define its *independence polynomial* by

$$p(x) = 1 + \sum_{k=1}^{v(G)} i_k(G) x^k.$$

Chudnovsky and Seymour proved [CS07] that if G is claw-free, then the roots of $p(x)$ are necessarily non-positive real, see also [Be18] for an alternative proof and an extension of the result. On the other hand, the Dobrushin-Shearer bound, cf. [SS05], states that the roots z of the independence polynomial of any graph G satisfy

$$|z| \ \geq \ \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} = \frac{1}{\Delta e}\left(1 + O\left(\frac{1}{\Delta}\right)\right) \quad \text{as} \quad \Delta \longrightarrow \infty,$$

where $\Delta = \max\{2, \Delta(G)\}$. The proof now follows from Theorem 1.1. $\square$

(3.3) Proof of Theorem 1.5. Given a graph G, we define its *unbranched subgraph polynomial* by

$$p(z) = 1 + \sum_{k=1}^{v(G)} u_k(G) z^k.$$

Ruelle proved [R99a], [R99b], see also [Wa09], that all roots z of $p(z)$ satisfy

$$\Re\, z \ \leq \ -\frac{2}{\Delta(\Delta-1)^2} \quad \text{for} \quad \Delta = \max\{2, \Delta(G)\}.$$

The proof now follows from Theorem 1.4. $\square$

(3.4) Estimating averages. We note that the value $\frac{p'(1)}{p(1)}$ is interpreted the average number of edges in a matching in Theorem 1.2, the average number of vertices in an independent set in Theorem 1.3 and the average number of edges in a unbranched subgraph in Theorem 1.5. It follows from Remark 2.8 that we can estimate the averages within relative error $\epsilon > 0$ by inspecting the matchings, independent sets and unbranched subgraphs of pretty much the same size as prescribed by Theorems 1.2, 1.3 and 1.5, though with different absolute constants. Similarly, by computing $\frac{p''(1)}{p(1)}$ we can estimate the second moment, etc.

(3.5) Possible ramifications. For each vertex w of a graph G, let us choose a set A_w of allowable degrees of subgraphs. Wagner proved [Wa09] that if $0 \in A_w \subset \{0, 1, 2\}$ for all vertices w, then the corresponding subgraph counting polynomial is non-zero in the sector

$$S_{\pi/3} = \left\{ z \in \mathbb{C} : \ |\arg z| < \frac{\pi}{3} \right\}$$

and is also non-zero in a δ-neighborhood of $z = 0$ for some $\delta = \Omega(1/\Delta(G))$. Using the approach sketched in Section 2.9, one can show that within relative error $\epsilon > 0$, the total number of such subgraphs is determined by the numbers of subgraphs with

$$k = (\Delta(G))^{O(1)} \ln \frac{v(G)}{\epsilon} \tag{3.5.1}$$

edges.

In [Be18], Bencs shows that the independence polynomials of graphs satisfying some weakening of the claw-free condition do not have roots in a sector $|\arg z| < \alpha$. For such graphs, the total number of independent sets is determined, within prescribed relative error, by the numbers of independent sets of a small size k as in (3.5.1).

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