RECENT PROGRESS ON MOVING BOUNDARY PROBLEMS

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Abstract. We give a brief survey of the recent progress in the area of mathematical well-posedness for moving boundary problems describing fluid-structure interaction between incompressible, viscous fluids and elastic, viscoelastic, and rigid solids.

1. Introduction

In this paper we survey some recent developments and open problems in the mathematical study of moving boundary problems. In particular, the focus is on problems arising from the interaction between incompressible, viscous fluids and elastic, viscoelastic, or rigid solids (also referred to as "structures"). See Fig. 1. Fluid-structure interaction (FSI) problems are ubiquitous in nature, technology and engineering. Examples include the human heart and heart valves interacting with blood flow, biodegradable micro-beads swimming in water to clean up water pollution, a micro camera in the human intestine used for an early colon cancer detection, and design of next generation vascular stents to prop open the clogged arteries, and prevent heart attacks. Numerical simulation and analysis of fluid-structure interaction problems can provide insight into the "invisible" properties of flows and structures, and can be used to advance design of novel technologies and improve the understanding of many biological phenomena.

Interestingly enough, even though the mathematical theory of the motion of bodies in a liquid is one of the oldest and most classical problems in fluid mechanics, mathematicians have only recently become interested in a systematic study of the basic problems related to fluid-structure interaction. One reason for this may be that problems of this type are notoriously difficult to study. In addition to the nonlinearity in the fluid equations (the Navier-Stokes equations) and possibly in the elastic or viscoelastic structure equations, the coupling between the fluid and structure motion may give rise to strong geometric nonlinearities. The study of existence of solutions to the coupled problems must account for the nonlinearities due to the strong energy exchange between the fluid and (elastic) structure motion, and novel compactness arguments have to be designed to deal with such nonlinearities. Due to the fluid domain motion, compactness results holding for *a family of operators* defined on time-dependent function spaces corresponding to moving domains not known *a priori* are needed. The compactness arguments must also account for the fact that the coupled problem involves two sets of equations of different type (parabolic vs. hyperbolic) accounting for the different physics in the problem.

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Making use of the smoothing effects by the fluid viscosity, and capturing its role in keeping the high frequency oscillations of the (hyperbolic) structure under control, is crucial for the existence proofs.

In existence proofs, and in numerical schemes, an additional difficulty is imposed by the incompressibility of the fluid. The main difficulty in existence proofs is related to the construction of divergence-free extensions of fluid velocity to a larger domain containing all the moving domains, and obtaining quantitative estimates of the extensions in terms of the changing geometry. Incompressibility is intimately related to the pressure, and pressure is a major component of the load, i.e., contact force, exerted by the fluid onto the solid. Designing constructive existence proofs and numerical schemes that approximate the load "correctly" is a key ingredient for the stability of constructive solution schemes. In particular, the fluid surrounding the structure affects the structure motion as an extra mass that the structure must displace when moving within a fluid. This has long been known in engineering as the "added mass effect", and its negative impact on the stability of partitioned FSI schemes is a well-known problem for FSI problems for which the fluid and structure have comparable densities, i.e., for which the structure is "light" with respect to the fluid. The added mass is a leading order effect in biofluidic FSI problems, since biological tissues (structures) have density which is approximately the same as that of the surrounding fluid. A failure to account for this effect leads to instabilities in partitioned numerical schemes and to the lack of uniform energy estimates in constructive existence proofs.

The question of global-in-time existence of solutions to moving boundary problems is affected by the so called "no collision" paradox. In addition to the problems related to global existence of solutions inherited from the Navier-Stokes equations, global weak solution existence results for moving boundary problems are typically obtained until contact between solids happens. We survey below the results in this area and mention here that several open questions remain, which would shed light on whether contact of elastic bodies immersed in a viscous incompressible fluid can happen in finite time, and the type of boundary conditions for which the finite-time contact may or may not occur (Navier slip versus no-slip condition).

Nonlinearities in the coupled FSI problem also affect the study of uniqueness of solutions. It is not surprising that uniqueness of weak solutions to the coupled FSI problems is still largely an outstanding open problem, since even in the case of classical 3D Navier-Stokes equations, the uniqueness of the Leray-Hopf weak solutions has not been resolved. However, recent advances in this area are significant, and we summarize those results below.

Thus, the main challenges in the mathematical study of fluid-structure interaction problems can be attributed to the following features of the problem:

(1) *Geometric nonlinearity when fluid and structure are nonlinearly coupled*;
(2) *The problem is of mixed type and defined on moving domains*;
(3) *Incompressibility and the Added Mass Effect*;
(4) *Finite-time contact?*
(5) *Uniqueness.*

To explain the main challenges in more detail, we present a benchmark problem for FSI involving elastic structures, and a benchmark problem for FSI involving rigid solids, and provide a literature review of the recent results.

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