Constrained convex bodies with extremal affine surface areas *

O. Giladi, H. Huang, C. Schütt and E. M. Werner †

Abstract

Given a convex body $K \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}$, we introduce and study the extremal inner and outer affine surface areas

$$IS_p(K) = \sup_{K' \subseteq K} \big(\mathrm{as}_p(K')\big) \quad \text{and} \quad os_p(K) = \inf_{K' \supseteq K} \big(\mathrm{as}_p(K')\big),$$

where $\mathrm{as}_p(K')$ denotes the L_p-affine surface area of K', and the supremum is taken over all convex subsets of K and the infimum over all convex compact subsets containing K.

The convex body that realizes $IS_1(K)$ in dimension 2 was determined in [3] where it was also shown that this body is the limit shape of lattice polytopes in K. In higher dimensions no results are known about the extremal bodies.

We use a thin shell estimate of [23] and the Löwner ellipsoid to give asymptotic estimates on the size of $IS_p(K)$ and $os_p(K)$. Surprisingly, it turns out that both quantities are proportional to a power of volume.

1 Introduction

F. John proved in [32] that among all ellipsoids contained in a convex body $K \in \mathbb{R}^n$, there is a unique ellipsoid of maximal volume, now called the John ellipsoid of K. Dual to the John ellipsoid is the Löwner ellipsoid, the ellipsoid of minimal volume containing K. These ellipsoids play fundamental roles in asymptotic convex geometry. They are related to the isotropic position, to the study of volume concentration, volume ratio, reverse isoperimetric inequalities, Banach-Mazur distance of normed spaces, and many more, including the hyperplane conjecture, one of the major open problems in asymptotic geometric analysis. We refer to e.g., the books [1, 12] for the details and more information.

In this paper, we introduce the analogue to John's theorem, when volume is replaced by affine surface area. In parallel to John's maximal volume ellipsoid, respectively the minimal volume Löwner ellipsoid, we investigate these convex bodies contained in K, respectively containing K, that have the largest, respectively smallest, L_p-affine surface areas,

$$IS_p(K) = \sup_{K' \subseteq K} \big(\mathrm{as}_p(K')\big) \quad \text{and} \quad os_p(K) = \inf_{K' \supseteq K} \big(\mathrm{as}_p(K')\big). \tag{1.1}$$

*Keywords: Affine surface areas, John and Löwner ellipsoid, thin shell estimate, 2010 Mathematics Subject Classification: 52A20, 52A23, 52A40

†Partially supported by NSF grant DMS-1811146

By compactness and continuity, the supremum and infimum are in fact a maximum and minimum, i.e., $IS_p(K) = as_p(K_0)$ for some convex body $K_0 \subset K$ and $os_p(K) = as_p(K_1)$ for some convex body $K_1 \supset K$.

For $p > 1$, the L_p-affine surface area was introduced by E. Lutwak in his ground breaking paper [38] in the context of the L_p-Brunn-Minkowski theory and in [53] for all other p, (see also [30, 43]). L_1-affine surface area is classical and goes back to W. Blaschke [7]. The definition of L_p-affine surface area is given below in (2.1), where we also list some of its properties. Due to its remarkable properties, this notion is important in many areas of mathematics and applications. We only quote characterizations of L_p-affine surface areas by M. Ludwig and M. Reitzner [36], the L_p-affine isoperimetric inequalities, proved by E. Lutwak [38] for $p > 1$ and for all other p in [60]. The classical case $p = 1$ goes back to W. Blaschke [7]. These inequalities are related to various other inequalities, see e.g., E. Lutwak, D. Yang and G. Zhang [39, 41]. In particular, the affine isoperimetric inequality implies the Blaschke-Santaló inequality and it proved to be the key ingredient in the solution of many problems, see e.g. the books by R. Gardner [17] and R. Schneider [48] and also [31, 35, 37, 54, 55, 56, 60]. Recent developments include extensions to an Orlicz theory, e.g., [18, 28, 35, 61], to a functional setting [13, 14] and to the spherical and hyperbolic setting [5, 6].
Applications of affine surface areas have been manifold. For instance, affine surface area appears in best and random approximation of convex bodies by polytopes, see, e.g., K. Böröczky [8, 9], P. Gruber [21, 22], M. Ludwig [34], M. Reitzner [46, 47] and also [19, 20, 27, 49, 52] and has connections to, e.g., concentration of volume, [15, 35, 41], differential equations [10, 24, 28, 56, 57, 62], and information theory, e.g., [2, 14, 40, 42, 45, 59].

In dimension 2 and for $p = 1$, $IS_1(K)$ was determined exactly by I. Bárány [3]. Moreover, he showed in [3] that the extremal body K_0 of (1.1) is unique and that K_0 is the limit shape of lattice polygons contained in K.

In higher dimensions and for $p \neq 1$, there are no results available on $IS_p(K)$, $os_p(K)$ and related notions $OS_p(K)$ and $is_p(K)$, defined in (2.2) and (2.3) below. We observe that only certain p-ranges are meaningful for the various notions.

We use a thin shell estimate by Guédon and E. Milman [23], see also G. Paouris [44] and Y. T. Lee and S. S. Vempala [33], on concentration of volume to show in our main theorem that $IS_p(K)$ is proportional to a power of the volume $|K|$ of K for fixed p, up to a constant depending only on n. It involves the Euclidean unit ball B_2^n centered at 0, and the isotropic constant L_K^2 of K, defined by

$$nL_K^2 = \min \left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} \|x\|^2 dx : a \in \mathbb{R}^n, T \in GL(n) \right\}. \tag{1.2}$$

Theorem 3.4. *There is a constant $C > 0$ such that for all $n \in \mathbb{N}$, all $0 \leq p \leq n$ and all convex bodies $K \subseteq \mathbb{R}^n$,*

$$\frac{1}{n^{5/6}} \left(\frac{C}{L_K} \right)^{\frac{2np}{n+p}} \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \leq \frac{IS_p(K)}{|K|^{\frac{n-p}{n+p}}} \leq \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}.$$

Equality holds trivially in the right inequality if $p = 0, n$. If $p \neq 0, n$, equality holds in the right inequality iff K is a centered ellipsoid.

Since $\frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}}$, which is asymptotically equivalent to $\frac{c^{\frac{np}{n+p}}}{n^{\frac{n(p-1)-p}{n+p}}}$ with an absolute constant c, the theorem shows that for a fixed p, $IS_p(K)$ is proportional to a power of $|K|$, up to a constant depending on n only.

We use the Löwner ellipsoid of K (e.g., [1, 12] or the survey [26]), to give asymptotic estimates on the size of $os_p(K)$ and $OS_p(K)$, also in terms of powers of $|K|$, in Theorem 3.6. For instance, we show that for $-n < p \leq 0$,

$$\frac{os_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \leq \frac{os_p(K)}{|K|^{\frac{n-p}{n+p}}} \leq n^{n\frac{n-p}{n+p}} \frac{os_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}.$$

Equality holds trivially in the left inequality if $p = 0$. If $p \neq 0$, equality holds in the left inequality iff K is a centered ellipsoid.
If K is centrally symmetric, $n^{n\frac{n-p}{n+p}}$ can be replaced by $n^{n\frac{n-p}{2(n+p)}}$.

We refer to Theorem 3.6 for the details.

2 Background and definitions

Throughout the paper, c, C etc., denote absolute constants that may change from line to line. The center of gravity $g(K)$ of K is defined by

$$g(K) = \frac{1}{|K|} \int_K x \, dx.$$

When the center of gravity of K is at 0, then, for real $p \neq -n$, the L_p-affine surface areas are defined as [38, 43, 53]

$$as_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu(x), \tag{2.1}$$

where $\kappa(x)$ is the (generalized) Gauss-Kronecker curvature at $x \in \partial K$, $N(x)$ is the outer unit normal vector at x to ∂K, the boundary of K, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ which induces the Euclidean norm $\| \cdot \|$. When the center of gravity of K is not at 0, we shift K so that it is. The case $p = 1$ is the classical affine surface area whose definition goes back to Blaschke [7].

We denote by $\mathcal{K}_K$ the collection of all compact convex subsets of K and by $\mathcal{K}^K$ the collection of all compact convex sets containing K.
For $-\infty \leq p \leq \infty$, $p \neq -n$, we then define the *inner and outer maximal affine surface areas* by

$$IS_p(K) = \sup_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big), \quad OS_p(K) = \sup_{C \in \mathcal{K}^K} \big(\mathrm{as}_p(C)\big), \tag{2.2}$$

and the *inner and outer mininal affine surface areas* by

$$is_p(K) = \inf_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big), \quad os_p(K) = \inf_{C \in \mathcal{K}^K} \big(\mathrm{as}_p(C)\big). \tag{2.3}$$

We show in section 3.1 that is_p is identically equal to 0 for all p and all K and that the only meaningful p-range for IS_p is $[0, n]$, for OS_p it is $[n, \infty]$ and for os_p it is $(-n, 0]$.

By Blaschke's selection theorem, $\mathcal{K}_K$ is compact with respect to the Hausdorff metric. Proposition 3.3 below, proved in [38], shows that the functional $K \mapsto as_p(K)$ is upper semicontinuous with respect to the Hausdorff metric, if $0 \le p \le \infty$, respectively lower semicontinuous if $-n < p \le 0$. We show in Lemma 3.2 that the suprema in (2.2) are in fact maxima for the relevant p-ranges $0 \le p \le n$, respectively, $n \le p \le \infty$,

$$IS_p(K) = \mathrm{as}_p(K_0) \quad \text{and} \quad OS_p(K) = \mathrm{as}_p(K_1)$$

for some convex body $K_0 \subset K$, respectively $K \subset K_1$, and that the second infimum in (2.3) is in fact a minimum for $-n < p \le 0$,

$$os_p(K) = \mathrm{as}_p(K_2),$$

for some K_2 in $\mathcal{K}^K$.

It was shown [38, 53] that for all $p \neq -n$ and for all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathrm{as}_p(T(K)) = |\det(T)|^{\frac{n-p}{n+p}} \ \mathrm{as}_p(K). \tag{2.4}$$

It then follows immediately from the definitions (2.2) and (2.3) that the same holds, replacing as_p with IS_p, OS_p, is_p and os_p.

For a general convex body K in $\mathbb{R}^n$, a particularly useful way to define $as_1(K)$ is the following. For $u \in \mathbb{R}^n$ and $t \ge 0$, define the half-spaces

$$H^+(t,u) = \big\{x \in \mathbb{R}^n \ \big| \ \langle x, u\rangle \ge t\big\}, \quad H^-(t,u) = \big\{x \in \mathbb{R}^n \ \big| \ \langle x, u\rangle \le t\big\}.$$

For a convex body $K \subseteq \mathbb{R}^n$ and $\delta > 0$, the (convex) floating body K_δ was introduced independently by Bárány and Larman [4] and Schütt and Werner [51],

$$K_\delta = \bigcap_{|H^+(t,u)\cap K| \le \delta |K|} H^-(t,u). \tag{2.5}$$

It was shown in [51] that for any convex body K in $\mathbb{R}^n$,

$$\mathrm{as}_1(K) = 2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}} \lim_{\delta\to 0} \frac{|K| - |K_\delta|}{\big(\delta|K|\big)^{\frac{2}{n+1}}}. \tag{2.6}$$

Here, and in what follows, B_2^n denotes the unit Euclidean ball in $\mathbb{R}^n$.

Geometric descriptions in the sense of (2.5) and (2.6) of L_p-affine surface area also exist. We refer to e.g., [29, 52, 53, 58, 60].

3 Main results

Our main results give quantitative estimates for the inner and outer extremal affine surface areas. We observe first that for some p, the values for the extremal affine surface areas can can be given explicitly and the p-ranges can be restricted accordingly in the quantitive estimates of Theorems 3.5 and 3.6 below.

3.1 The relevant p-ranges

(i) **The case $IS_p(K)$**

If $p=0$, then for all K,

$$IS_0(K) = \sup_{K' \in \mathcal{K}_K} \left(\mathrm{as}_0(K') \right) = n \sup_{K' \in \mathcal{K}_K} |K'| = n|K|.$$

If $p=n$, then for all K,

$$IS_n(K) = n|B_2^n|.$$

Indeed, on the one hand, we have by (2.4),

$$IS_n(K) \geq \sup_{\rho B_2^n \in \mathcal{K}_K} \left(\mathrm{as}_n(\rho B_2^n) \right) = \sup_{\rho B_2^n \in \mathcal{K}_K} \left(\mathrm{as}_n(B_2^n) \right) = n|B_2^n|.$$

The equi-affine isoperimetric inequality [38] says that $as_n(K) \leq as_n(B_2^n)$. Therefore,

$$IS_n(K) = \sup_{K' \in \mathcal{K}_K} \left(\mathrm{as}_n(K') \right) \leq \sup_{K' \in \mathcal{K}_K} \left(\mathrm{as}_n(B_2^n) \right) = n|B_2^n|.$$

If $n < p \leq \infty$, then $IS_p(K) = \infty$. This holds as by (2.4),

$$IS_p(K) \geq \sup_{\varepsilon B_2^n \in \mathcal{K}_K} \left(\mathrm{as}_p(\varepsilon B_2^n) \right) = \sup_{\varepsilon} \varepsilon^{n \frac{n-p}{n+p}} \ n|B_2^n| = \infty,$$

since $\frac{n-p}{n+p} < 0$.

If $-n < p < 0$, then for all K, $IS_p(K) = \infty$. Indeed, we have for all polytopes P

$$IS_p(K) \geq \sup_{P \in \mathcal{K}_K} \left(\mathrm{as}_p(P) \right) = \sup_{P \in \mathcal{K}_K} \int_{\partial P} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu(x) = \infty,$$

since $\kappa(x) = 0$ almost everywhere.

If $-\infty \leq p < -n$, then for all K, $IS_p(K) = \infty$. Indeed, as above,

$$IS_p(K) \geq \sup_{\varepsilon B_2^n \in \mathcal{K}_K} \left(\mathrm{as}_p(\varepsilon B_2^n) \right) = \sup_{\varepsilon} \varepsilon^{n \frac{n-p}{n+p}} \ n|B_2^n| = \infty,$$

since $\frac{n-p}{n+p} < 0$.

Conclusion. The relevant p-range for IS_p is $p \in [0, n]$.
We note also that for $p \in [0, n]$,

$$IS_p(B_2^n) = n|B_2^n| = as_p(B_2^n). \tag{3.1}$$

(ii) **The case $OS_p(K)$** .

If $p=n$, then for all K, $OS_n(K) = n|B_2^n|$. Similarly, to (i) above,

$$OS_n(K) \geq \sup_{RB_2^n \in \mathcal{K}^K} \left(\mathrm{as}_n(RB_2^n) \right) = \sup_{RB_2^n \in \mathcal{K}^K} \left(\mathrm{as}_n(B_2^n) \right) = n|B_2^n|$$

and again by the equi-affine isoperimetric inequality,

$$OS_n(K) = \sup_{K' \in \mathcal{K}^K} \left(\mathrm{as}_n(K') \right) \leq \sup_{K' \in \mathcal{K}^K} \left(\mathrm{as}_n(B_2^n) \right) = n|B_2^n|.$$

If $0 \leq p < n$, then, $OS_p(K) = \infty$. This holds as

$$OS_p(K) \geq \sup_{RB_2^n \in \mathcal{K}^K} \left(\mathrm{as}_p(RB_2^n)\right) = \sup_{RB_2^n \in \mathcal{K}^K} R^{\frac{n-p}{n+p}} n|B_2^n|,$$

and R can be made arbitrarily large.

If $-n < p < 0$, then, $OS_p(K) = \infty$.
This holds as we can again take polytopes P that contain K.

If $-\infty \leq p < -n$, then for all K, $OS_p(K) = \infty$.
Let C_ε be a rounded cube centered at 0 containing K and such that each vertex is rounded by replacing it by a Euclidean ball with radius ε. More specifically, C_ε is the convex hull of the 2^n Euclidean balls

$$B_2^n(t \cdot \delta, \epsilon) \qquad\qquad \delta = (\delta_1, \dots, \delta_n)$$

where $\delta_i = \pm 1$ for all $i = 1, \dots, n$ and t is sufficiently big so that the convex hull contains K. The boundary of C_ε contains all the 2^n-tants of the boundary of B_2^n. Therefore, in order to estimate $as_p(C_\varepsilon)$ from below it suffices to restrict the integration over the boundary of C_ε to those 2^n-tants of the boundary of B_2^n. The curvature there equals ε^{-n+1}, while

$$\langle x, N(x)\rangle \leq 2t \cdot \sqrt{n}.$$

Then

$$OS_p(K) \geq as_p(C_\varepsilon) \geq \frac{\varepsilon^{-\frac{n(n-1)}{n+p}}}{(2t\sqrt{n})^{n\frac{p-1}{n+p}}} n|B_2^n|,$$

which can be made arbitrarily large for ε arbitrarily small.

Conclusion. The relevant p-range for OS_p is $p \in [n, \infty]$.
We note also that for $p \in [n, \infty]$,

$$OS_p(B_2^n) = n|B_2^n| = as_p(B_2^n). \tag{3.2}$$

(iii) **The case** $os_p(K)$.

If $p = 0$, then for all K,

$$os_0(K) = \inf_{K' \in \mathcal{K}^K} \left(\mathrm{as}_0(K')\right) = n \inf_{K' \in \mathcal{K}^K} |K'| = n|K|.$$

If $0 < p \leq \infty$ or if $-\infty < p < -n$, then for all K, $os_p(K) = 0$. Indeed, for polytopes $P \in \mathcal{K}^K$, we have for those p-ranges

$$is_p(K) \leq \inf_{P \in \mathcal{K}^K} as_p(P) = 0.$$

Conclusion. The relevant p-range for os_p is $p \in (-n, 0]$.
We note also that for $p \in (-n, 0]$,

$$os_p(B_2^n) = n|B_2^n| = as_p(B_2^n). \tag{3.3}$$

(iv) **The case** $is_p(K)$.

We have that $is_p(K) = 0$ for all p and for all K.
If $0 < p \leq \infty$ or if $-\infty \leq p < -n$ we get for polytopes $P \in \mathcal{K}_K$,

$$is_p(K) \leq \inf_{P \in \mathcal{K}_K} as_p(P) = 0.$$

If $-n < p \leq 0$, then for all K,

$$is_p(K) \leq \inf_{\varepsilon B_2^n \in \mathcal{K}_K} \left(\mathrm{as}_p(\varepsilon B_2^n)\right) = n|B_2^n| \inf_{\varepsilon} \varepsilon^{n\frac{n-p}{n+p}} = 0.$$

Conclusion. There is no interesting p-range for the inner minimal affine surface area is_p.

3.2 Continuity, monotonicity and isoperimetricity

It was proved by Lutwak [38] that for $p \geq 1$, L_p-affine surface area is an upper semicontinuous functional with respect to the Hausdorff metric. In fact, it follows from Lutwak's proof that the same holds for all $0 \leq p < 1$ (aside from the case $p = 0$, which is just volume and hence continuous). For $-n < p \leq 0$, the functional is lower semicontinuous.

Proposition 3.1. *[38] Let $0 \leq p \leq \infty$. Then the functional $K \mapsto \mathrm{as}_p(K)$ is upper semicontinuous with respect to the Hausdorff metric on $\mathbb{R}^n$. For $-n < p \leq 0$, the functional is lower semicontinuous.*

For the proof of the next lemma, we use Proposition 3.1 and the L_p-affine isoperimetric inequalities which were proved by Lutwak [38] for $p > 1$ and for all other p by Werner and Ye [60]. The case $p = 1$ is the classical case.

For $p \geq 0$,

$$\frac{\mathrm{as}_p(K)}{\mathrm{as}_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}, \tag{3.4}$$

and for $-n < p \leq 0$,

$$\frac{\mathrm{as}_p(K)}{\mathrm{as}_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}. \tag{3.5}$$

Equality holds in both inequalities iff K is an ellipsoid. Equality holds trivially in both inequalities if $p = 0$.

Lemma 3.2. *Let K be a convex body in $\mathbb{R}^n$.*

(i) Let $0 \leq p \leq n$. Then there exists a convex body $K_0 \subset K$ such that

$$IS_p(K) = \sup_{C \in \mathcal{K}_K} \left(\mathrm{as}_p(C)\right) = as_p(K_0).$$

(ii) Let $n < p \leq \infty$. Then there exists a convex body $K_0 \supset K$ such that

$$OS_p(K) = \sup_{C \in \mathcal{K}^K} \left(\mathrm{as}_p(C)\right) = as_p(K_0).$$

(iii) Let $-n < p < 0$. Then there exists a convex body $K_0 \supset K$ such that

$$os_p(K) = \inf_{C \in \mathcal{K}^K} \big(\mathrm{as}_p(C)\big) = as_p(K_0).$$

Proof. (i) When $p = 0$, $IS_0(K) = n|K|$ and we take $K_0 = K$ and when $p = n$, $IS_n(K) = n|B_2^n|$ and we can take again $K_0 = K$. Let now $0 < p < n$. By the L_p-affine isoperimetric inequality (3.4), $as_p(K) \leq n \, |K|^{\frac{n-p}{n+p}} \, |B_2^n|^{\frac{2p}{n+p}}$, and in particular, the supremum is finite. There is a sequence $(C_k)_{k\in\mathbb{N}}$ of convex bodies such that for all k, $C_k \subset K$

$$as_p(C_k) + \frac{1}{k} \geq \sup_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big),$$

or

$$\lim_{k\to\infty} as_p(C_k) = \sup_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big).$$

By the Blaschke selection principle, see e.g., [48], there is a subsequence $(C_{k_i})_{i\in\mathbb{N}}$ that converges in Hausdorff distance to a convex set K_0. We claim that K_0 is a convex body in $\mathbb{R}^n$, i.e., K_0 has an interior point. Suppose not. Then $\lim_{i\to\infty} |C_{k_i}| = |K_0| = 0$. By the L_p-affine isoperimetric inequality (3.4),

$$\lim_{i\to\infty} as_p(C_{k_i}) \leq \lim_{i\to\infty} n \, |C_{k_i}|^{\frac{n-p}{n+p}} \, |B_2^n|^{\frac{2p}{n+p}} = 0.$$

Therefore

$$0 = \lim_{i\to\infty} as_p(C_{k_i}) = \sup_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big) > 0,$$

which is a contradiction. The last inequality holds since there is $\rho > 0$ such that a ball with radius ρ is contained in K. By the upper semi continuity of the L_p-affine surface area,

$$\sup_{C \in \mathcal{K}_K} \big(\mathrm{as}_p(C)\big) = \limsup_{i\to\infty} as_p(C_{k_i}) \leq as_p(K_0)$$

and thus $IS_p(K) = as_p(K_0)$.

(ii) We can assume that $g(K) = 0$. There is $R > 0$ such that $K \subset RB_2^n$. For all convex bodies C such that $C \supset K$ and $|C| \geq (nR)^n |B_2^n|$ there is a convex body $\tilde{C}$ such that $\tilde{C} \supset K$, $\tilde{C} \subset nRB_2^n$ and $as_p(C) \leq as_p(\tilde{C})$. We now show the latter. There is an affine map A with determinant 1 and $\rho > 0$ such that ρB_2^n is the ellipsoid of maximal volume is contained in $A(C)$. Then by F. John's theorem

$$\rho B_2^n \subset A(C) \subset n\rho B_2^n.$$

Therefore, $(nR)^n |B_2^n| \leq |C| \leq (n\rho)^n |B_2^n|$ and thus $R \leq \rho$. This yields

$$K \subset RB_2^n \subset \rho \, \frac{R}{\rho} \, B_2^n \subset \frac{R}{\rho} A(C).$$

We pick $\tilde{C} = \frac{R}{\rho} A(C)$. Then $as_p(\tilde{C}) = \left(\frac{R}{\rho}\right)^{n\frac{n-p}{n+p}} as_p(C) \geq as_p(C)$, as $\left(\frac{R}{\rho}\right)^{n\frac{n-p}{n+p}} \geq 1$, as $p > n$. If $C \supset K$ is such that $|C| \leq (nR)^n |B_2^n|$, we proceed as follows. As K is a convex body, there is $r > 0$ such that $rB_2^n \subset K$ and thus $rB_2^n \subset C$. For every $x \in C$, let $x^\perp$ be the hyperplane through the origin and orthogonal to x. We consider the cone with base

$x^\perp \cap rB_2^n$ and apex x. Let h_x denote the height of the cone. Then we have for all $x\in C$ that $\frac{h_x}{n} r^{n-1}|B_2^{n-1}| \leq (nR)^n |B_2^n|$ and thus $C \subset \frac{|B_2^n|}{|B_2^{n-1}|} \ \frac{n^{n+1}R^n}{r^{n-1}} \ B_2^n$.

Hence, altogether we can assume that the relevant (for the supremum) convex bodies $C \in \mathcal{K}^K$ are contained in $R_0 B_2^n$, where $R_0 = \max\left\{nR, \frac{|B_2^n|}{|B_2^{n-1}|} \ \frac{n^{n+1}R^n}{r^{n-1}}\right\}$. We then proceed as above. By the L_p-affine isoperimetric inequality (3.4), we have for all relevant $C\in\mathcal{K}^K$ that

$$as_p(C) \leq n \ |C|^{\frac{n-p}{n+p}} \ |B_2^n|^{\frac{2p}{n+p}} \leq n \ |K|^{\frac{n-p}{n+p}} \ |B_2^n|^{\frac{2p}{n+p}},$$

and in particular the supremum is finite. Then, as above, there is a sequence $(C_k)_{k\in\mathbb{N}}$ of convex bodies such that we have for all k that $C_k \subset K$ and that

$$as_p(C_k) + \frac{1}{k} \geq \sup_{C\in\mathcal{K}^K} \big(\mathrm{as}_p(C)\big),$$

or

$$\lim_{k\to\infty} as_p(C_k) = \sup_{C\in\mathcal{K}^K} \big(\mathrm{as}_p(C)\big).$$

By the Blaschke selection principle, see e.g., [48], there is a subsequence $(C_{k_i})_{i\in\mathbb{N}}$ that converges in Hausdorff distance to a convex set K_0. K_0 is a convex body as $K\subset C_{k_i}$ for all i and thus $K\subset K_0$. We conclude as in (i).

(iii) The proof is similar to (ii). We include it for completeness. We can again assume that $g(K)=0$ and that there is $R>0$ such that $K\subset RB_2^n$. As in (ii), we claim that for all convex bodies C such that $C\supset K$ and $|C| \geq (nR)^n|B_2^n|$ there is a convex body $\tilde{C}$ such that $\tilde{C}\supset K$, $\tilde{C}\subset nRB_2^n$ and $as_p(C)\geq as_p(\tilde{C})$. We now show this. There is an affine map A with determinant 1 and $\rho>0$ such that ρB_2^n is the ellipsoid of maximal volume is contained in $A(C)$. Then by F. John's theorem

$$\rho B_2^n \subset A(C) \subset n\rho B_2^n.$$

Therefore, $(nR)^n|B_2^n| \leq |C| \leq (n\rho)^n |B_2^n|$ and thus $R\leq\rho$. This yields

$$K \subset RB_2^n \subset \rho \ \frac{R}{\rho} \ B_2^n \subset \frac{R}{\rho} A(C).$$

We pick $\tilde{C} = \frac{R}{\rho}A(C)$. Then $as_p(\tilde{C}) = \left(\frac{R}{\rho}\right)^{n\frac{n-p}{n+p}} as_p(C) \leq as_p(C)$, as $\left(\frac{R}{\rho}\right)^{n\frac{n-p}{n+p}} \leq 1$, as $-n<p<0$. If $C\supset K$ is such that $|C|\leq (nR)^n|B_2^n|$, we proceed as follows. As K is a convex body, there is $r>0$ such that $rB_2^n\subset K$ and thus $rB_2^n\subset C$. For every $x\in C$, consider the cone with base $x^\perp\cap rB_2^n$ and apex x. Let h_x denote the height of the cone. Then we have for all $x\in C$ that $\frac{h_x}{n} r^{n-1}|B_2^{n-1}| \leq (nR)^n|B_2^n|$ and thus $C \subset \frac{|B_2^n|}{|B_2^{n-1}|} \ \frac{n^{n+1}R^n}{r^{n-1}} \ B_2^n$.

Hence, altogether we can assume that the relevant (for the infimum) convex bodies $C \in \mathcal{K}^K$ are contained in $R_0 B_2^n$, where $R_0 = \max\left\{nR, \frac{|B_2^n|}{|B_2^{n-1}|} \ \frac{n^{n+1}R^n}{r^{n-1}}\right\}$. We then proceed as above. By the L_p-affine isoperimetric inequality (3.5), we have for all relevant $C\in\mathcal{K}^K$ that

$$as_p(C) \geq n \ |C|^{\frac{n-p}{n+p}} \ |B_2^n|^{\frac{2p}{n+p}} \geq n \ |K|^{\frac{n-p}{n+p}} \ |B_2^n|^{\frac{2p}{n+p}},$$

and in particular the infimum is finite. Then, as above, there is a sequence $(C_k)_{k\in\mathbb{N}}$ of convex bodies such that $C_k\subset K$ for all k and such that

$$as_p(C_k) \leq \inf_{C\in\mathcal{K}^K} \big(\mathrm{as}_p(C)\big) + \frac{1}{k},$$

for all k, and hence

$$\lim_{k\to\infty} as_p(C_k) = \inf_{C\in\mathcal{K}^K} \big(\mathrm{as}_p(C)\big).$$

By the Blaschke selection principle, see e.g., [48], there is a subsequence $(C_{k_i})_{i\in\mathbb{N}}$ that converges in Hausdorff distance to a convex set K_0. K_0 is a convex body as $K \subset C_{k_i}$ for all i and thus $K \subset K_0$. Again, we conclude as in (i). $\square$

It is natural to ask about the continuity properties of inner and outer maximal, respectively minimal, affine surface areas in the p-ranges that are not already settled by the above considerations.

Proposition 3.3. *Let the set of convex bodies in $\mathbb{R}^n$ be endowed with the Hausdorff metric.*

For $0 \le p \le n$, the functional $K \mapsto IS_p(K)$ is continuous.

For $n \le p \le \infty$, the functional $K \mapsto OS_p(K)$ is continuous.

For $-n < p \le 0$, the functional $K \mapsto os_p(K)$ is continuous.

The next proposition lists affine isoperimetric inequalities and monotonicity properties for the the functionals IS_p, OS_p and os_p.

Proposition 3.4. *Let K be a convex body in $\mathbb{R}^n$.*

(i) Let $0 \le p \le n$. Then $IS_p(K) \le IS_p(B_2^n)\ \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}$.
Equality holds trivially if $p = 0$ or $p = n$.

Let $n \le p \le \infty$. Then $OS_p(K) \le OS_p(B_2^n)\ \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}$.
Equality holds trivially if $p = n$.

Let $-n < p \le 0$. Then $os_p(K) \ge os_p(B_2^n)\ \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}$.
Equality holds trivially if $p = 0$.

For all other p, equality holds in all inequalities iff K is an ellipsoid.

(ii) $p \to \left(\frac{IS_p(K)}{n|K|}\right)^{\frac{n+p}{p}}$ is strictly increasing in $p \in (0, n]$.

$p \to \left(\frac{OS_p(K)}{n|K^\circ|}\right)^{\frac{n+p}{p}}$ is strictly decreasing in $p \in [n, \infty)$.

$p \to \left(\frac{os_p(K)}{n|K^\circ|}\right)^{\frac{n+p}{p}}$ is strictly decreasing in $p \in (-n, 0)$.

3.3 Asymptotic estimates

The next theorems provide estimates for the inner and outer extremal affine surface areas in the p-ranges that are not already settled above. There, L_K is the isotropic constant of K as defined in (1.2).

Theorem 3.5. *There is an absolute constant $C > 0$ such that for all $n \in \mathbb{N}$, all $0 \le p \le n$ and all convex bodies $K \subseteq \mathbb{R}^n$,*

$$\frac{1}{n^{5/6}} \left(\frac{C}{L_K}\right)^{\frac{2np}{n+p}} \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \le \frac{IS_p(K)}{|K|^{\frac{n-p}{n+p}}} \le \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}. \tag{3.6}$$

Equality holds trivially in the right inequality if $p = 0, n$. If $p \neq 0, n$, equality holds in the right inequality iff K is a centered ellipsoid.

By (3.1), $\frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}}$. Therefore, Theorem 3.5 states that

$$\frac{1}{n^{5/6}} \left(\frac{C}{L_K}\right)^{\frac{2np}{n+p}} \le \frac{IS_p(K)}{n\, |B_2^n|^{\frac{2p}{n+p}}\, |K|^{\frac{n-p}{n+p}}} \le 1.$$

Stirling's formula yields that with absolute constants, c_1, c_2,

$$\frac{c_2^{\frac{np}{n+p}}}{n^{\frac{n(p-1)-p}{n+p}}} \le \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}} \le \frac{c_1^{\frac{np}{n+p}}}{n^{\frac{n(p-1)-p}{n+p}}}.$$

As noted, the upper bound is sharp when e.g., K is B_2^n. However, in general we have $IS_p(K) > \mathrm{as}_p(K)$. For example, for the n-dimensional cube B_∞^n centered at 0 with sidelength 2, $\mathrm{as}_p(B_\infty^n) = 0$, but $B_2^n \subseteq B_\infty^n$ and so $IS_p(B_\infty^n) \ge \mathrm{as}(B_2^n) > 0$.

Theorem 3.6. *Let $K \subseteq \mathbb{R}^n$ be a convex body.*

(i) Let $n \le p \le \infty$. Then there are absolute constants c and C such that

$$\max\left\{ n^{-5/6} c^{n\frac{p-n}{p+n}} \left(\frac{C}{L_{(K-s(K))^\circ}}\right)^{\frac{2n^2}{n+p}}, n^{n\frac{n-p}{n+p}} \right\} \frac{OS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \le \frac{OS_p(K)}{|K|^{\frac{n-p}{n+p}}} \le \frac{OS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}, \tag{3.7}$$

where $s(K)$ is the Santaló point of K. Equality holds trivially in the right inequality if $p = n$. If $p \neq n$, equality holds in the right inequality iff K is a centered ellipsoid.

(i) Let $-n < p \le 0$. Then

$$\frac{os_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \le \frac{os_p(K)}{|K|^{\frac{n-p}{n+p}}} \le n^{n\frac{n-p}{n+p}} \frac{os_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}. \tag{3.8}$$

Equality holds trivially in the left inequality if $p = 0$. If $p \neq 0$, equality holds in the left inequality iff K is a centered ellipsoid.

If $p = n$, then the maximum in the lower bound of (i) is achieved for the second term and is 1. If $p = \infty$, the maximum is achieved for the first term and it is equal to $n^{-5/6}c^n$. If K is centrally symmetric, $n^{n\frac{n-p}{n+p}}$ can be replaced by $n^{n\frac{n-p}{2(n+p)}}$.

3.4 Relation to quermassintegrals

Finally, we turn to the relation of the extremal affine surface areas to quermassintegrals. While some of the (trivial) extremal affine surface areas are quermassintegrals, we will see that in general this is not the case.

Given a convex body $K \subseteq \mathbb{R}^n$ and $t \geq 0$, the Steiner formula (see, for example [48]) says that there exist non-negative numbers $W_0(K), \dots, W_n(K)$, such that

$$|K + t\, B_2^n| = W_0(K) + \binom{n}{1} W_1(K)t + \binom{n}{2} W_2(K)t^2 + \cdots + W_n(K)t^n.$$

The numbers $W_0(K), \dots, W_n(K)$ are called the quermassintegrals. In particular, $W_0(K) = |K|$ and $W_n(K) = |B_2^n|$. Therefore, by section 3.1, $IS_0(K) = os_0(K) = n|K| = nW_0(K)$ and $IS_n(K) = OS_n(K) = n|B_2^n| = nW_n(K)$ are (multiples of) quermassintegrals. However, as shown in the next proposition, in general the extremal affine surface areas are not (multiples, or powers of) quermassintegrals.

We only treat the cases IS_1, os_{-1} and OS_{n^2}. The other relevant p-cases are treated similarly.

Proposition 3.7. *(i) If $\beta > 0$, then IS_1^β and os_{-1} are not equal to W_i, for any $0 \leq i \leq n$, and if $\beta < 0$, then $OS_{n^2}^\beta$ is not equal to W_i, for any $0 \leq i \leq n$.*

(ii) The quantities IS_1, os_{-1} and OS_{n^2} are not a linear combination of quermassintegrals. In particular, those quantities are not valuations.

Remark 3.1. From [50] it is known that affine surface area is a valuation, that is, for every $K, L \subseteq \mathbb{R}^n$ convex,

$$\mathrm{as}_1(K \cap L) + \mathrm{as}_1(K \cup L) = \mathrm{as}_1(K) + \mathrm{as}_1(L).$$

It is also known by Hadwiger's characterization theorem [25], that every continuous rigid motion invariant valuation on the set of convex bodies is a linear combination of quermassintegrals. Thus, Proposition 3.7 (ii) shows in particular that IS_1, os_{-1} and OS_{n^2} are not valuations.

4 Proofs

Proof of Proposition 3.3. By section (3.1) (i), $IS_0(K) = n|K|$ is just volume, which is continuous and $IS_0(K) = n|B_2^n|$, which is constant and hence continuous. Thus for $IS_p(K)$ we only need to consider $p \in (0, n)$. We may assume that 0 is the center of gravity of K, that is,

$$\int_K x\, dx = 0.$$

Hence, there exists $\rho > 0$ such that $\rho B_2^n \subseteq K$. Let $\{K_l\}_{l=1}^{\infty}$ be a sequence of convex bodies, all having center of gravity at the origin, that converges to K in the Hausdorff metric. That is, for every $\varepsilon > 0$, there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$,

$$K_l \subseteq K + \varepsilon B_2^n \quad \text{and} \quad K \subseteq K_l + \varepsilon B_2^n.$$

If $\varepsilon > 0$ is sufficiently small, then we can assume that for all $l \geq l_0$, $\frac{\rho}{10} B_2^n \subseteq K_l$. Thus, for all $l \geq l_0$,

$$K_l \subseteq K + \varepsilon B_2^n \subseteq K + \frac{\varepsilon}{\rho} K = \left(1 + \frac{\varepsilon}{\rho}\right) K, \tag{4.1}$$

and

$$K \subseteq K_l + \varepsilon B_2^n \subseteq K_l + \frac{10\varepsilon}{\rho} K_l = \left(1 + \frac{10\varepsilon}{\rho}\right) K_l. \tag{4.2}$$

Hence,

$$\left(1 + \frac{\varepsilon}{\rho}\right)^{n\frac{n-p}{n+p}} IS_p(K) = IS_p\left(\left(1 + \frac{\varepsilon}{\rho}\right) K\right) \overset{(4.1)}{\geq} IS_p(K_l),$$

and

$$\left(1 + \frac{10\varepsilon}{\rho}\right)^{n\frac{n-p}{n+p}} IS_p(K_l) = IS_p\left(\left(1 + \frac{10\varepsilon}{\rho}\right) K_l\right) \overset{(4.2)}{\geq} IS_p(K).$$

In the last two lines above, we have also used (2.4), resp. the remark after it. Altogether, for all $l \geq l_0$,

$$\left(1 + \frac{\varepsilon}{\rho}\right)^{-n\frac{n-p}{n+p}} IS_p(K_l) \leq IS_p(K) \leq \left(1 + \frac{10\varepsilon}{\rho}\right)^{n\frac{n-p}{n+p}} IS_p(K_l).$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

Continuity for outer maximal affine surface area OS_p and outer minimal surface area os_p is treated similarly. $\square$

For the proof of Proposition 3.4 and Theorem 3.5, we use the above quoted L_p-affine isoperimetric inequalities.

Proof of Proposition 3.4. (i) When $0 < p \leq n$ and $K' \subseteq K$, we use (3.4) and (3.1),

$$\begin{aligned} IS_p(K) &= \sup_{K' \in \mathcal{K}_K} \left(\text{as}_p(K')\right) \leq \sup_{K' \in \mathcal{K}_K} \text{as}_p(B_2^n) \left(\frac{|K'|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \leq \text{as}_p(B_2^n) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \\ &= IS_p(B_2^n) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}. \end{aligned}$$

From the equality characterization of (3.4) it follows that equality holds iff K is an ellipsoid.
Similarly, we get for OS_p when $p \in (n, \infty]$, also using (3.2),

$$\begin{aligned} OS_p(K) &= \sup_{K' \in \mathcal{K}^K} \left(\text{as}_p(K')\right) \leq \sup_{K' \in \mathcal{K}^K} \text{as}_p(B_2^n) \left(\frac{|K'|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \leq \text{as}_p(B_2^n) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \\ &= OS_p(B_2^n) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}. \end{aligned}$$

From the equality characterization of (3.4) it follows that equality holds iff K is an ellipsoid.
In the same way, using (3.5) and (3.3) when $-n < p < 0$, we have

$$\begin{aligned} os_p(K) &= \inf_{K' \in \mathcal{K}^K} \left(\mathrm{as}_p(K')\right) \geq \mathrm{as}_p(B_2^n) \inf_{K' \in \mathcal{K}^K} \left(\frac{|K'|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \\ &\geq os_p(B_2^n) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}. \end{aligned}$$

Equality characterization follows from the equality characterization of (3.5).

(ii) It was shown in [60] (see also [45]) that the function $p \to \left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}}$ is strictly increasing in $p \in (0, \infty)$. Therefore we get for $0 < p < q \leq n$,

$$\begin{aligned} \left(\frac{IS_p(K)}{n|K|}\right)^{\frac{n+p}{p}} &= \frac{\sup_{K' \in \mathcal{K}_K} \left(\mathrm{as}_p(K')^{\frac{n+p}{p}}\right)}{(n|K|)^{\frac{n+p}{p}}} < \frac{\sup_{K' \in \mathcal{K}_K} (n|K'|)^{\frac{n}{p} - \frac{n}{q}} \ \left(\mathrm{as}_q(K')^{\frac{n+q}{q}}\right)}{(n|K|)^{\frac{n+p}{p}}} \\ &\leq \frac{(n|K|)^{\frac{n}{p} - \frac{n}{q}}}{(n|K|)^{\frac{n+p}{p}}} \left(\sup_{K' \in \mathcal{K}_K} \mathrm{as}_q(K')\right)^{\frac{n+q}{q}} = \left(\frac{IS_q(K)}{n|K|}\right)^{\frac{n+q}{q}}. \end{aligned}$$

It was also shown in [60] (see also [45]) that the function $p \to \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}$ is strictly decreasing in $p \in (0, \infty)$. Therefore we get for $n \leq p < q < \infty$,

$$\begin{aligned} \left(\frac{OS_p(K)}{n|K^\circ|}\right)^{n+p} &= \frac{\sup_{K' \in \mathcal{K}^K} \left(\mathrm{as}_p(K')^{n+p}\right)}{(n|K^\circ|)^{n+p}} > \frac{\sup_{K' \in \mathcal{K}^K} \left(n|K^{'\circ}|\right)^{p-q} \ \left(\mathrm{as}_q(K')^{n+q}\right)}{(n|K^\circ|)^{n+p}} \\ &\geq \frac{\left(\sup_{K' \in \mathcal{K}^K} \mathrm{as}_q(K')\right)^{n+q}}{(n|K^\circ|)^{n+q}} = \left(\frac{OS_q(K)}{n|K^\circ|}\right)^{n+q} \end{aligned}$$

and for $-n \leq p < q \leq 0$,

$$\begin{aligned} \left(\frac{os_p(K)}{n|K^\circ|}\right)^{n+p} &= \frac{\inf_{K' \in \mathcal{K}^K} \left(\mathrm{as}_p(K')^{n+p}\right)}{(n|K^\circ|)^{n+p}} > \frac{\inf_{K' \in \mathcal{K}^K} \left(n|K^{'\circ}|\right)^{p-q} \ \left(\mathrm{as}_q(K')^{n+q}\right)}{(n|K^\circ|)^{n+p}} \\ &\geq \frac{\left(\inf_{K' \in \mathcal{K}^K} \mathrm{as}_q(K')\right)^{n+q}}{(n|K^\circ|)^{n+q}} = \left(\frac{OS_q(K)}{n|K^\circ|}\right)^{n+q} \end{aligned}$$

□

In part of the proof below it is most convenient to work with a body which is in isotropic position. A body $K \subseteq \mathbb{R}^n$ is said to be in isotropic position if $|K| = 1$ and there exists $L_K > 0$ such that for all $\theta \in \mathbb{S}^{n-1}$,

$$\int_K \langle x, \theta \rangle dx = 0, \qquad \int_K \langle x, \theta \rangle^2 dx = L_K^2.$$

Here and in what follows, $\mathbb{S}^{n-1}$ denotes the unit Euclidean sphere in $\mathbb{R}^n$. It is known that for every convex body $K \subseteq \mathbb{R}^n$, there exists $T : \mathbb{R}^n \to \mathbb{R}^n$ affine and invertible such that TK is isotropic. See for example [12] for this and other facts on isotropic position used here.

Proof of Theorem 3.5. The upper bound, together with the equality characterizations, follows immediately from Proposition 3.4 (i).

Now we turn to the lower bound in the case (i). As noted above, $IS_p(TK) = \det(T)^{\frac{n-p}{n+p}} IS_p(K)$ for any invertible linear map T. Therefore, to prove the lower bound for $0 < p < \infty$, it is sufficient to consider K in isotropic position. Let L_K be the isotropic constant of K. By the thin shell estimate of O. Guédon and E.Milman [23] (see also [16, 33, 44]), we have with universal constants c and C, that for all $t \geq 0$,

$$\left| K \cap \left\{ x \in \mathbb{R}^n : \left| \|x\| - L_K\sqrt{n} \right| < tL_K\sqrt{n} \right\} \right| > 1 - C\exp(-cn^{1/2}\min(t^3,\, t)).$$

Taking $t = O(n^{-1/6})$, there is a a new universal constant $c > 0$ such that for all $n \in \mathbb{N}$,

$$\left| K \cap \left\{ x \in \mathbb{R}^n : \left| \|x\| - L_K\sqrt{n} \right| < cL_K n^{1/3} \right\} \right| \geq \frac{1}{2}. \tag{4.3}$$

This set consists of all $x \in K$ for which

$$L_K\left(n^{1/2} - cn^{1/3}\right) < \|x\| < L_K\left(n^{1/2} + cn^{1/3}\right).$$

We consider those $n \in \mathbb{N}$ for which $n^{1/6} > c$.
We will truncate the above set. For $i = 0,\, 1,\, 2,\, \dots k_n = \lfloor n \log_2 \frac{n^{1/2}+cn^{1/3}}{n^{1/2}-cn^{1/3}} \rfloor$, consider the sets

$$L_i := K \cap \{x \in \mathbb{R}^n \,:\, 2^{i/n}(L_K(n^{1/2} - cn^{1/3})) < \|x\| \leq 2^{(i+1)/n}(L_K(n^{1/2} - cn^{1/3}))\}.$$

Then

$$2^{\frac{k_n}{n}} \leq 2^{\log_2 \frac{n^{1/2}+cn^{1/3}}{n^{1/2}-cn^{1/3}}} = \frac{n^{1/2}+cn^{1/3}}{n^{1/2}-cn^{1/3}}$$

and thus

$$K \cap \left\{ x \in \mathbb{R}^n : \left| \|x\| - L_K\sqrt{n} \right| < cL_K n^{1/3} \right\} \subset \cup_{i=0}^{k_n} L_i. \tag{4.4}$$

Moreover, with a new absolute constant C_0,

$$k_n \leq n\log_2 \frac{n^{1/2}+cn^{1/3}}{n^{1/2}-cn^{1/3}} = n\log_2 \frac{1+cn^{-1/6}}{1-cn^{-1/6}} \leq\ C_0\ n^{5/6}.$$

By (4.3) and (4.4), there exists $i_0 \in \{1,\, 2,\, \dots,\, \lfloor C_0 n^{5/6} \rfloor\}$ such that

$$|L_{i_0}| \geq \frac{1}{2\,\lfloor C_0 n^{5/6} \rfloor}. \tag{4.5}$$

We set $R = 2^{i_0/n}(L_K(n^{1/2} - cn^{1/3}))$. In particular, we have

$$L_{i_0} = K \cap \{x \in \mathbb{R}^n \,:\, R < \|x\| \leq 2^{1/n} R\}.$$

Let

$$O = \left\{ \theta \in S^{n-1} \,:\, \rho_K(\theta) > R \right\}, \text{ and } S_O = \{r\theta \,:\, \theta \in O \,\text{and}\, r \in [0,\, R]\} \subset K,$$

where $\rho_K(\theta) = \max\{r \geq 0 \,:\, r\theta \in K\}$ is the radial function of K.

Now we claim that

$$L_{i_0} \subset 2^{1/n} S_O. \tag{4.6}$$

Indeed, let $y \in L_{i_0}$. We express $y = r\theta$ in polar coordinates. By definition, we have $R < r < 2^{1/n}R$ and $r\theta \in K$. Thus, $\rho_K(\theta) \geq r > R$ and hence $\theta \in O$. Therefore, $r\theta \in 2^{1/n}S_O$ because $r \in [0,\, 2^{1/n}R]$. By (4.5) and (4.6) we conclude that

$$|S_O| \geq \left(2^{-1/n}\right)^n |L_{i_0}| \geq \frac{1}{4\,\lfloor C_0 n^{5/6} \rfloor}. \tag{4.7}$$

Now, we consider $as_p\left(K \cap RB_2^n\right)$. For $\theta \in O$, $R\theta$ is a boundary point of $K \cap RB_2^n$. Thus,

$$\begin{aligned} as_p\left(K \cap RB_2^n\right) &\geq \int_{RO} \frac{\kappa^{\frac{p}{n+p}}}{\langle x,\, N(x)\rangle^{\frac{n(p-1)}{n+p}}}\, d\mu(x) = \int_{RO} \frac{R^{-(n-1)\frac{p}{n+p}}}{R^{\frac{n(p-1)}{n+p}}}\, d\mu(x) \\ &= \mu(RO)\left(\frac{1}{R}\right)^{\frac{(n-1)p+n(p-1)}{n+p}} = \mu(RO)\left(\frac{1}{R}\right)^{\frac{2np}{n+p}-1}, \end{aligned}$$

where μ is the surface area measure of RS^{n-1}. We can compare surface area and volume,

$$\frac{\mu(RO)\cdot R}{n} = |S_O|.$$

Hence,

$$\begin{aligned} as_p\left(K \cap RB_2^n\right) &\geq \left(\frac{1}{R}\right)^{\frac{2np}{n+p}-1} \frac{n}{R}|S_O| = \left(\frac{1}{R}\right)^{\frac{2np}{n+p}} n\,|S_O| \\ &\geq \left(\frac{1}{R}\right)^{\frac{2np}{n+p}} \frac{n}{4\,\lfloor C_0 n^{5/6} \rfloor}. \end{aligned}$$

Since $R \leq 2\sqrt{n}L_K$, this finishes the proof for the lower bound. □

Proof of Theorem 3.6. The upper bound of (i) and the lower bound of (ii), together with the equality characterizations, follow immediately from Proposition 3.4 (i).

For the other estimates, we will rely on the dual body of K. Recall that the Santaló point $s(K)$ of a convex body K is the unique point $s(K)$ for which the origin is the barycenter of $(K - s(K))^\circ$. Without loss of generality, we may assume $s(K) = 0$ and K° is in isotropic position.

Following the proof of Theorem 3.5, there exists $R \in [\frac{1}{2}\sqrt{n}L_{K^\circ}, 2\sqrt{n}L_{K^\circ}]$ such that

$$O := \{\theta \in S^{n-1} \,:\, \rho_{K^\circ}(\theta) > R\}$$

satisfies

$$\mu(RO) \geq \frac{n}{R}\frac{1}{Cn^{5/6}},$$

where μ is the surface area measure of RS^{n-1}.

We consider the following convex hull $\mathrm{conv}\{R^2K,\, RB_2^n\}$. Recall that the support function of a convex body L is $h_L(\theta) := \max_{x\in L}\langle x, \theta\rangle$. Furthermore, we have the identity $h_L(\theta) = \frac{1}{\rho_{L^\circ}(\theta)}$. Thus, for $\theta \in O$

$$h_{R^2K}(\theta) = R^2 h_K(\theta) = R^2 \frac{1}{\rho_{K^\circ}(\theta)} < R.$$

Therefore, $R\theta \in \partial(\text{conv}\{R^2K,\, RB_2^n\})$. Now, we have

$$OS_p(R^2K) \geq as_p(\text{conv}\{R^2K,\, RB_2^n\}) \geq \int_{RO} \frac{R^{-(n-1)\frac{p}{n+p}}}{R^{\frac{n(p-1)}{n+p}}} d\mu\,(x) = \frac{n}{Cn^{5/6}} \left(\frac{1}{R}\right)^{\frac{2np}{n+p}}.$$

Using the fact that $|K^\circ| = 1$ and the volume product estimate $|L||L^\circ| \geq c^n |B_2^n|^2$ from Theorem 1 of [11], we have

$$|R^2K| \geq c^n R^{2n} |B_2^n|^2$$

for some constant $c > 0$.

Alltogether we conclude

$$\frac{OS_p(R^2K)}{|R^2K|^{\frac{n-p}{n+p}}} \geq \frac{n}{Cn^{5/6}} \left(\frac{1}{R}\right)^{\frac{2np}{n+p}} (c^n R^{2n} |B_2^n|^2)^{\frac{p-n}{n+p}} = \frac{n}{Cn^{5/6}} \left(\frac{1}{R}\right)^{\frac{2n^2}{n+p}} (c^n |B_2^n|^2)^{\frac{p-n}{n+p}}.$$

With the identity $\frac{OS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}}$, we obtain

$$\frac{OS_p(R^2K)}{|R^2K|^{\frac{n-p}{n+p}}} \geq \frac{OS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \frac{1}{n^{5/6}} c^{n\frac{p-n}{p+n}} \left(\frac{C}{L_{K^\circ}}\right)^{\frac{2n^2}{n+p}}.$$

Furthermore, we can derive a different bound using Löwner position. We will assume that K is in Löwner position, i.e., the Löwner ellipsoid $L(K)$, which is the ellipsoid of minimal volume containing K, is the Euclidean ball $\frac{|L(K)|}{|B_2^n|} B_2^n$. We also have that

$$K \subset L(K) \subset n\, K, \tag{4.8}$$

and that for a 0-symmetric convex body K,

$$K \subset L(K) \subset \sqrt{n}\, K. \tag{4.9}$$

(i) We get with (2.4), (3.2) and (4.8),

$$OS_p(K) \geq as_p(L(K)) = \left(\frac{|L(K)|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} n\, |B_2^n| \geq n^{n\frac{n-p}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} OS_p(B_2^n),$$

which finishes the lower estimate of (i).

(ii) Similarly, we get with (2.4), (3.2) and (4.8),

$$os_p(K) \leq as_p(L(K)) = \left(\frac{|L(K)|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} n\, |B_2^n| \leq n^{n\frac{n-p}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} os_p(B_2^n).$$

In the 0 -symmetric case we use (4.9) to get the estimate with $n^{n\frac{n-p}{2(n+p)}}$ instead of $n^{n\frac{n-p}{n+p}}$.
□

Proof of Proposition 3.7. We only give the proofs for IS_1. The proofs for os_{-1} and OS_{n^2} are the same with the obvious modifications.

(i) To prove the first assertion, note that by (the remark after) (2.4), IS_1^β is homogeneous of degree $\frac{\beta n(n-1)}{n+1}$. Also, it is known that W_i is homogeneous of degree $n-i$. Hence, if $IS_1^\beta = W_i$ for some i, then $\frac{\beta n(n-1)}{n+1} \in \mathbb{N}$ and in particular $\beta \in \mathbb{Q}$. On the other hand, it is known that $W_i(B_2) = |B_2^n|$. Thus, we must also have

$$|B_2^n| = IS_p^\beta(B_2^n) \overset{(*)}{=} n^\beta |B_2^n|^\beta,$$

where in $(*)$ we used (3.1). Therefore, we have $|B_2^n|^{\frac{1-\beta}{\beta}} \in \mathbb{N}$. Now, it is known that

$$|B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = \begin{cases} \frac{\pi^{\frac{n}{2}}}{(n/2)!} & 2 \mid n, \\ \frac{2((n-1)/2)!(4\pi)^{\frac{n-1}{2}}}{n!} & 2 \nmid n. \end{cases}$$

In other words, for every $n \in \mathbb{N}$, we have $|B_2^n| = Q_n \pi^{\frac{n}{2}}$ or $|B_2^n| = Q_n \pi^{\frac{n-1}{2}}$, where $Q_n \in \mathbb{Q}$. Therefore, if $|B_2^n|^{\frac{1-\beta}{\beta}} \in \mathbb{N}$ with $\beta \in \mathbb{Q}$, that would imply that π is an algebraic number, which is not the case. This proves the first assertion.

(ii) Suppose that IS_1 is a linear combination of quermassintegrals. Then, for K given, there exist λ_i, $0 \le i \le n$, not all of them equal to 0, such that $IS_1(K) = \sum_{i=0}^n \lambda_i W_i(K)$. The respective homogeneity properties then imply that for all $\alpha \in \mathbb{R}$,

$$\alpha^{n\frac{n-1}{n+1}} IS_1(K) = \sum_{i=0}^n \lambda_i \alpha^{n-i} W_i(K),$$

and in particular, for $K = B_2^n$, that for all $\alpha \in \mathbb{R}$,

$$n\ \alpha^{n\frac{n-1}{n+1}} = \sum_{i=0}^n \lambda_i \alpha^{n-i} = \lambda_0 \alpha^n + \lambda_1 \alpha^{n-1} + \cdots + \lambda_n. \tag{4.10}$$

Letting $\alpha = 0$ in (4.10) shows that $\lambda_n = 0$. This means that for all $\alpha \in \mathbb{R}$,

$$n\ \alpha^{n\frac{n-1}{n+1}} = \sum_{i=0}^{n-1} \lambda_i \alpha^{n-i} = \lambda_0 \alpha^n + \lambda_1 \alpha^{n-1} + \cdots + \lambda_{n-1}\alpha.$$

Differentiation gives

$$n\left(n\frac{n-1}{n+1}\right)\alpha^{n\frac{n-1}{n+1}-1} = n\lambda_0\alpha^{n-1} + (n-1)\lambda_1\alpha^{n-2} + \cdots + \lambda_{n-1}. \tag{4.11}$$

Letting $\alpha = 0$ in (4.11) shows that $\lambda_{n-1} = 0$. We continue differentiating till the largest $k \in \mathbb{N}$ for which the exponent $n\frac{n-1}{n+1} - k$ of α on the left hand side of the equality is strictly larger than 0. We can take $k = n-2$ and get that $\lambda_n = \lambda_{n-1} = \cdots = \lambda_2 = 0$. Thus equality (4.10) reduces to the following: there exist λ_0 and λ_1 such that for all $\alpha \in \mathbb{R}$,

$$\frac{n}{\alpha^{\frac{n-1}{n+1}}} = \lambda_0 \alpha + \lambda_1,$$

which is not possible. The proof is therefore complete. □

Acknowledgement: We would like to thank the referee for all the helpful comments.

References

[1] S. Artstein-Avidan, A. Giannopoulos and V. Milman, *Asymptotic Geometric Analysis*, Mathematical Surveys and Monographs **202**, (2015).

[2] S. Artstein-Avidan, B. Klartag, C. Schütt and E.M. Werner, *Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality*, J. Functional Analysis **262** (2012) 4181–4204.

[3] I. Bárány, *Affine perimeter and limit shape*, J. Reine Angew. Math. **484** (1988) 71–84.

[4] I. Bárány and D.G. Larman, *Convex bodies, economic cap coverings, random polytopes*, Mathematika **35** (1988) 274–291.

[5] F. Besau and E.M. Werner, *The spherical convex floating body*, Adv. Math. **301** (2016) 867–901.

[6] F. Besau and E.M. Werner, *The floating body in real space forms*, Journal of Differential Geometry **110** (2018) 187–220.

[7] W. Blaschke, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer Verlag, Berlin, (1923).

[8] K. Böröczky, Jr., *Polytopal approximation bounding the number of k-faces*, J. Approx. Theory **102** (2000) 263–285.

[9] K. Böröczky, Jr., *Approximation of general smooth convex bodies*, Adv. Math. **153** (2000) 325–341.

[10] K. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The Logarithmic Minkowski Problem*, J. Amer. Math. Soc. **26** (2013) 831–852 .

[11] J. Bourgain, V. Milman, *New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$*, Invent. Math. **88** (1987) 319-340

[12] S. Brazitikos, A. Giannopoulos, P. Valettas and B.H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs **196**, American Mathematical Society, Providence, RI, (2014).

[13] U. Caglar, M. Fradelizi, O. Guédon, J. Lehec, C. Schütt and E.M. Werner, *Functional versions of L_p-affine surface area and entropy inequalities*, Int. Math. Res. Not. IMRN **4** (2016) 1223–1250.

[14] U. Caglar and E.M. Werner, *Divergence for s-concave and log concave functions*, Adv. Math. **257** (2014) 219–247.

[15] B. Fleury, O. Guédon, and G. Paouris, *A stability result for mean width of L_p-centroid bodies*, Adv. in Math. **214** (2007) 865–877.

[16] B. Fleury, *Concentration in a thin Euclidean shell for log-concave measures*, J. Funct. Anal. **259** (2010) 832–841.

[17] R.J. Gardner, *Geometric tomography*, Second edition. Encyclopedia of Mathematics and its Applications, 58. Cambridge University Press, Cambridge (2006).

[18] R.J. Gardner, D. Hug, W. Weil, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*, J. Differential Geom. **97**(2014) 427–476.

[19] J. Grote and E.M. Werner, *Approximation of smooth convex bodies by random polytopes*, Electron. J. Probab. **23**, article 9 (2018).

[20] J.Grote, Ch. Thäle and E.M. Werner, *Surface area deviation between smooth convex bodies and polytopes*, preprint, arxiv

[21] P. M. Gruber, *Approximation of convex bodies.* Convexity and its Applications, Birkhäuser, Basel, (1983) 131–162.

[22] P. M. Gruber, *Aspects of approximation of convex bodies*, In: *Handbook of Convex Geometry.* Elsevier, North-Holland, (1993) 319–345.

[23] O. Guédon and E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct. Anal. **21** (2011) 1043–1068.

[24] C. Haberl and F. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009) 1–26.

[25] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin-Göttingen-Heidelberg, (1957).

[26] M. Henk. *Löwner-John ellipsoids.* Doc. Math., ExtraVol. ISMP, 95–106 (2012).

[27] S.D. Hoehner, C. Schütt and E.M. Werner *The Surface Area Deviation of the Euclidean Ball and a Polytope*, J. Theor. Probab. **31**, 244–267 (2018).

[28] Y. Huang, E. Lutwak, D. Yang, G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), 325–388.

[29] H. Huang, B. Slomka and E.M. Werner, *Ulam Floating bodies*, to appear in J. London Math. Society.

[30] D. Hug, *Contributions to affine surface area*, Manuscripta Mathematica **91** (1996) 283–301.

[31] M.N. Ivaki and A. Stancu, *Volume preserving centro-affine normal flows*, Comm. Anal. Geom. **21** (2013) 671–685.

[32] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, Interscience, New York (1948) 187–204.

[33] Y.T. Lee and S.S. Vempala, *Stochastic localization + Stieltjes barrier = tight bound for log-sobolev* , Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (New York, NY, USA), STOC 2018, ACM, (2018), 11221129.

[34] M. Ludwig, *Asymptotic approximation of smooth convex bodies by general polytopes*, Mathematika **46** (1999) 103–125.

[35] M. Ludwig, *General affine surface areas*, Adv. Math. 224 (2010) 2346–2360.

[36] M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations.* Annals of Math. **172** (2010) 1223–1271.

[37] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995) 227–246.

[38] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996) 244–294.

[39] E. Lutwak, D. Yang and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom., **56** (2000)111–132.

[40] E. Lutwak, D. Yang and G. Zhang, *The Cramer-Rao inequality for star bodies* Ann. Probab., **32** (2004) 757–774.

[41] E. Lutwak, D. Yang and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002)17–38.

[42] E. Lutwak, D. Yang and G. Zhang, *Moment-entropy inequalities* Duke Math. J. **112** (2002) 59–81.

[43] M. Meyer and E.M. Werner, *On the p-affine surface area.* Adv. Math. **152** (2000) 288–313.

[44] G. Paouris, *Concentration of mass in convex bodies* Geom. Funct. Analysis **16** (2006) 1021–1049.

[45] G. Paouris and E.M. Werner, *Relative entropy of cone measures and L_p centroid bodies*, Proc. Lond. Math. Soc. (3) **104** (2012) 253–286.

[46] M. Reitzner, *Random points on the boundary of smooth convex bodies*, Trans. Amer. Math. Soc. **354** (2002) 2243–2278.

[47] M. Reitzner, *Random polytopes*, In: *New Perspectives in Stochastic Geometry*, Oxford University Press (2010).

[48] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge (2013).

[49] C. Schütt, *The convex floating body and polyhedral approximation*, Israel J. Math. **73** (1991) 65–77.

[50] C. Schütt, *On the affine surface area*, Proc. Amer. Math. Soc. **118** (1993) 1213–1218.

[51] C. Schütt and E.M. Werner, *The convex floating body*, Math. Scand. **66** (1990) 275–290.

[52] C. Schütt and E.M. Werner, *Polytopes with vertices chosen randomly from the boundary of a convex body*, GAFA Seminar Notes, Lecture Notes in Mathematics **1807**, Springer-Verlag (2003) 241–422.

[53] C. Schütt and E.M. Werner, *Surface bodies and p-affine surface area*, Adv. Math. **187** (2004) 98–145.

[54] F. E. Schuster and T. Wannerer, *GL(n) contravariant Minkowski valuations*, Trans. Amer. Math. Soc. **364** (2012) 815–826.

[55] A. Stancu, *The discrete planar L_0-Minkowski problem*, Adv. Math. **167** (2002) 160–174.

[56] N.S. Trudinger and X.J. Wang, *Affine complete locally convex hypersurfaces.* Invent. Math. **150** (2002) 45–60.

[57] N.S. Trudinger, X.J Wang, *Boundary regularity for the Monge-Ampere and affine maximal surface equations*, Ann. of Math. **167** (2008) 993–1028.

[58] E.M. Werner, *The p-affine surface area and geometric interpretations*, Rend. Circ. Mat. Palermo (2) Suppl. **70** (2002) 367–382.

[59] E.M. Werner, *Rényi divergence and L_p-affine surface area for convex bodies*, Adv. Math., **230** (2012) 1040–1059.

[60] E.M. Werner and D. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218** (2008) 762–780.

[61] D. Ye, *New Orlicz affine isoperimetric inequalities*, J. Mathematical Analysis and Applications bf 427 (2015) 905–929.

[62] Y. Zhao, *On L_p -affine surface area and curvature measures*, Int. Math. Res. Not. IMRN **5** (2016) 1387–1423.

Ohad Giladi
School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, NSW 2308, Australia
ohad.giladi@newcastle.edu.au

Han Huang
Department of Mathematics
Georgia Institute of Technology, Atlanta, Georgia
hhuang421@math.gatech.edu

Carsten Schütt
Mathematisches Seminar
Universität Kiel
Germany
schuett@math.uni-kiel.de

Elisabeth M. Werner
Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106, U. S. A.
elisabeth.werner@case.edu

Université de Lille 1
UFR de Mathématique
59655 Villeneuve d'Ascq, France