INDIRECT REASONING TASK FOR PROSPECTIVE SECONDARY TEACHERS: OPPORTUNITIES AND CHALLENGES

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We describe an instructional module aimed to enhance prospective secondary teachers' (PSTs') subject matter knowledge of indirect reasoning. We focus on one activity in which PSTs had to compare and contrast proof by contradiction and proof by contrapositive. These types of proofs have been shown to be challenging to students at all levels and teachers alike, yet there has been little research on how to support learners in developing this knowledge. Data analysis of 11 PSTs, points to learning opportunities afforded by the module and the PSTs' challenges with indirect reasoning.

Keywords: Teacher Knowledge, Reasoning and Proof, Instructional Activities, Indirect Proof.

Introduction and Theoretical Perspectives

Reasoning and proof have been at the center of substantial research efforts in the last decades, yet, very few studies have focused on indirect reasoning, proof by contradiction and by contrapositive (Stylianides, Stylianides & Weber, 2017). By indirect reasoning, we mean any instance in which a person makes an argument of the form: "it is not possible because otherwise …". Students use indirect reasoning informally, when checking for errors (Thompson, 1996), and formally, when proving by contradiction or by contrapositive. A proof by contrapositive relies on the logical equivalence between a conditional statement S and its contrapositive $(P \Rightarrow Q) \equiv (\sim Q \Rightarrow \sim P)$. A proof by contradiction assumes the negation of S: $(P \wedge \sim Q)$ and proceeds directly to obtain $r \wedge \sim r$ (where r is any statement), which is a contradiction, implying that S is true. Despite the similarities, (e.g., both have $\sim Q$ in the assumption, and contain direct steps), there are also differences, for example, a contradiction can take many forms, but a proof by contrapositive necessarily ends with $\sim P$ (Yopp, 2017).

The studies on indirect proof suggest that students view these types of proof as less convincing and particularly difficult to construct (Harel & Sowder, 1998; Leron, 1985; Antonini & Mariotti, 2008), specifically struggling to construct negations (Lin, Lee, & Wu Yu, 2003). Recently, more research attention has been directed towards studying students' understanding of proof by contrapositive (Yopp, 2017) and proof by contradiction (Brown, 2018). However, almost nothing is known about what types of interventions or pedagogical supports could help to mitigate students' challenges in this area.

Indirect proof is important in both tertiary and secondary mathematics. For example, proof of congruency of alternate interior angles created between parallel lines and a transversal, or proofs of sums and products of rational and irrational numbers are done by contradiction. Thompson (1996) asserts that teachers can support secondary students' understanding of indirect proof. But to do that, teachers themselves, must have robust knowledge of indirect proof and of pedagogical strategies for teaching it. Our review of educational literature found no studies on whether pre-service secondary teachers (PSTs) have such knowledge, or how to support its development.

To address this gap, we designed and systematically studied an instructional module that aimed to enhance PSTs' knowledge of indirect reasoning, as a part of a larger design-based research project, *Mathematical Reasoning and Proof for Secondary Teachers* (Buchbinder &

McCrone, 2018). In this paper we examine the research question: "What challenges and learning opportunities arose from the PSTs' interactions with one of the tasks in this module?"

Methods

Indirect Reasoning Instructional Module (IR Module)

The IR module was the last in the series of four modules in the capstone course, preceded by modules on: direct proof and argument evaluation; conditional statements, and quantification and the role of examples in proving. By the time the PSTs reached the IR module, they had already refreshed and strengthened their knowledge of conditional statements and contrapositive which are critical for indirect reasoning (Buchbinder & McCrone, 2018).

The IR module comprised two in-class activities followed by planning and enacting a lesson in secondary classrooms integrating indirect reasoning. The objectives of the *Indirect Proof Structure* activity were to help PSTs understand the relationships between proof by contradiction and proof by contrapositive. For each of six given proofs, the PSTs were asked to identify the *P* and *Q* of the statement, the assumption and the conclusion of the proof, determine if the proof is by contrapositive or contradiction, and if so, identify the contradiction (see Table 1 for examples). To highlight the differences in the logical structure, we included of a proof by contradiction and by contraposition for the same statement (Statements 4 & 5, Table 1).

Table 1: Three Out of Six Items in the *Indirect Proof Structure* Task

Items	Proof Structure
Statement 2: *Let* a *and* b *be two real numbers. If* $ab = 0$*, then* $a = 0$ *or* $b = 0$ Proof: Suppose that $ab = 0$, and $a \neq 0$ and $b \neq 0$. Since $a \neq 0$ and $b \neq 0$, we can divide both sides of $ab = 0$ by a and by b. This will result in $1 = 0$. Therefore, if $ab = 0$, then $a = 0$ or $b = 0$.	Proof by contradiction P: $a,b \in \mathbb{R}$; $ab = 0$ Q: $a = 0$ or $b = 0$ Assume: $P \wedge \sim Q$ Contradiction: $1 = 0$ Infer: $P \Rightarrow Q$
Statement 4: *For all integers n, if* n^2 *is even, then n is even.* Proof: Suppose n is an integer, such that n^2 is even, and n is not even. Since n is not even, it is odd, that is, $n = 2k + 1$ for some integer k. By substitution and algebra: $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Now, $(2k^2 + 2k)$ is an integer, since the products and the sums of integers are integers. So, $n^2 = 2(integer) + 1$. By definition of odd, n^2 is odd. Therefore, n^2 is both even and odd. Therefore, if n^2 is even, n is even.	Proof by contradiction P: $n \in \mathbb{Z}$; n^2 *is even* Q: *n is odd* Assume: $P \wedge \sim Q$ Contradiction: $P \wedge \sim P$ Infer: $P \Rightarrow Q$
Statement 5: *For all integers n, if* n^2 *is even, then n is even.* Proof: Suppose n is an odd integer. By definition of odd $n = 2k + 1$ for some integer k. By substitution and algebra $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ Now, $(2k^2 + 2k)$ is an integer, since the products and the sums of integers are integers. So, $n^2 = 2(integer) + 1$. By definition of odd, n^2 is odd.	Proof by contrapositive P: $n \in \mathbb{Z}$; n^2 *is even* Q: *n is odd* Assume: $\sim Q$ Obtain: $\sim P$ I.e.: $(\sim Q \Rightarrow \sim P) \Rightarrow (P \Rightarrow Q)$

The PSTs worked on the activity in groups of 3 to 4, followed by a whole class discussion, led by the course instructor, the first author of this paper. Next, the PSTs summarized similarities and differences between proof by contrapositive and by contradiction on posters.

Participants, Data Sources and Analytic Techniques

Participants in the study were 11 PSTs in the capstone course *Mathematical Reasoning and Proof for Secondary Teachers*. All PSTs were in their senior year and had completed proof oriented coursework. Data sources were video recordings of the whole class and of each small groups' work captured with 360° table-top cameras, the PSTs' worksheets and posters. The data were analyzed using open coding (Wiersma & Jurs, 2005) to identify learning opportunities and challenges experienced by the PSTs. Challenges were identified as moments in which the PSTs went back and forth between mathematically correct and incorrect ideas, or verbally expressed their confusion. Learning opportunities were defined as instances where the confusion was resolved in a manner that aligns with conventional mathematical knowledge.

Results

Challenges

Here we identify a few a few challenges that the task elicited.

"A proof by contrapositive is a proof by contradiction." After correctly identifying Statement 4 (Table 1) as proof by contradiction, a group of three PSTs: Jane, Emily and Kelly moved on to Statement 5. They recognized it is the same statement, and correctly identified its P and Q. However, when trying to identify the type of proof a disagreement arose.

> Jane: So, it [the proof] follows the contrapositive, but it ends with a contradiction.
> Emily: This proof is proving ~*Q* implies ~*P*, which is equivalent to *P* implies *Q*.
> Jane: How is it not a contradiction at the very last line?
> Emily: Contradiction is you start with *P* and then you come up with ~*Q* contradicting what you would expect to happen, right? I would expect n^2 to be odd. We assumed ~*Q* so we want ~*P*. It is equivalent to *P* implies *Q*.
> Jane: So this [proof] ends with the false statement? n^2 is odd? I think it's a contradiction.
> Emily: Contrapositive.

Emily correctly identified the proof as a proof by contrapositive. On the other hand, Jane noticed that the proof ends with ~*P* (n^2 is odd) and juxtaposed it with *P* of the given statement (n^2 is even), perceiving it as a contradiction of the form $(P \wedge \sim P)$. In our analysis we identified several instances of this phenomenon, but a more common confusion was the following.

"A proof by contradiction is a proof by contrapositive." The following example comes from a group of four PSTs: Erick, Abby, Phil and Joel, discussing Statement 2.

> Erick: This is probably proof by contrapositive.
> Abby: Contrapositive is $\sim Q \Rightarrow \sim P$ … where's the ~*P* part, though?
> Erick: Right there [points to the paper, see Figure 1], the result would be like 1=0, which I think is kind of the idea that ab can't equal 0.
> Joel: For a contrapositive we would need to start by saying if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$, and prove that it is true. In this case, however, they say "suppose $ab = 0$ and $a \neq 0$ and $b \neq 0$, this is "if *P* and ~*Q*". So there is no contrapositive whatsoever in this proof.
> Erick: I think that they prove the contrapositive true. Because they use the fact that $a \neq 0$ and $b \neq 0$ to show like $ab \neq 0$. […] If you just take out "suppose $ab = 0$", and you get rid of the last line [in the proof], then you have showed that the contrapositive is true. They just put extra things in [the proof] so it would be easier to understand.

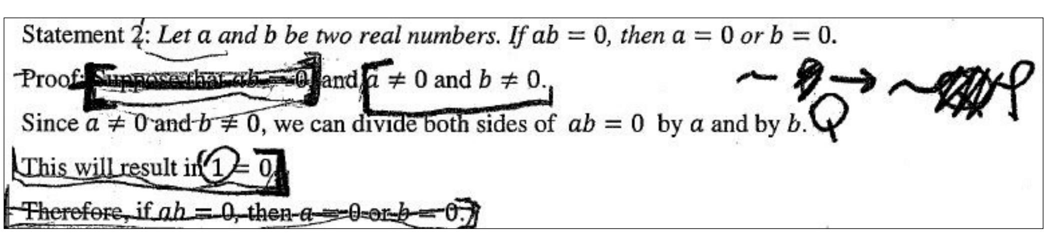
Statement 2: *Let a and b be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.*
~~Proof: Suppose that $ab = 0$~~ and $a \neq 0$ and $b \neq 0$. $\sim Q \to \sim P$
Since ~~$a \neq 0$ and $b \neq 0$~~, we can divide both sides of $ab = 0$ by a and by b.
This will result in 1 = 0.
~~Therefore, if $ab = 0$, then $a = 0$ or $b = 0$.~~

Figure 1: Erick's Worksheet with Crossed Out Portions of the Proof

In this excerpt Erick seems to selectively ignore existing parts of the proof, such as P in the assumption $P \wedge \sim Q$, and the conclusion: therefore, $P \Rightarrow Q$, while mentally inserting a non-existing proof line "1 = 0 implies $ab \neq 0$ which he interprets as $\sim P$ (Fig. 1). Joel's opposition and correct explanation did little to move Erick away from his conviction. It seems that Erick had a pre-existing assumption that a proof by contradiction is essentially a proof by contrapositive and he sought a way to confirm this. Although Erick was the person who most actively expressed his position, the data shows that other PSTs had similar challenges.

Evidence of Learning

Despite these challenges, the task generated multiple learning opportunities for PSTs to clarify and strengthen their knowledge of indirect reasoning, both in the small group and whole class discussions. The culmination of the activity was PSTs creating posters summarizing main features of proof by contradiction and by contrapositive (Figure 2).

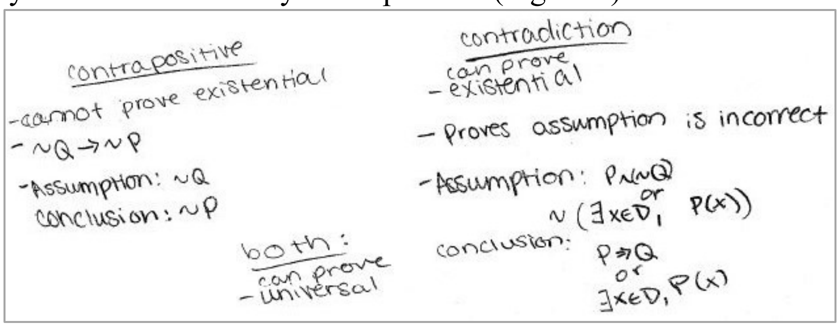

Figure 2: PSTs' Poster Comparing Proof by Contradiction and by Contrapositive

Discussion

The results presented above highlight some of the conceptual challenges and learning opportunities elicited by analyzing the structure of given proofs. These results concur with the literature suggesting that PSTs struggle to distinguish between proofs by contradiction and by contrapositive. We add to the literature by clearly identifying the challenges as confusing these two types of proofs by either: (a) interpreting the conclusion $\sim P$ of a proof by contrapositive as contradicting the hypothesis P in the statement, or (b) interpreting a proof by contradiction as a contrapositive proof due to the presence of $\sim Q$ in the assumption of both types of proofs. The overarching theme underlying these challenges seems to be "selective noticing" of certain aspects of the proof's logical structure, such as fixating on $\sim P$ in $\sim(P \wedge \sim Q)$, while ignoring other critical aspects such as P in the assumption of the proof by contradiction.

Our data illustrate the pedagogical potential of the *Indirect Proof Structure* task to evoke rich conversations that resulted in multiple instances of learning for the PSTs. We attribute this to the careful selection of the types of proofs, the focus on proof comprehension rather than proof production, comparing across multiple proofs, and summarizing the differences on a poster (Brown, 2018; Mejia-Ramos, et al. 2012). Going back to Thompson's (1996) aspiration that teachers support students' understanding of indirect proof, we assert that the *Indirect Proof Structure* task in particular and the IR module in general bear the potential to prepare PSTs to carry this practice into secondary classrooms.

Acknowledgments

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