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Calculus of Variations

On optimal boundary control of Ericksen–Leslie system in dimension two

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Abstract
In this paper, we consider the boundary value problem of a simplified Ericksen–Leslie system in dimension two with non-slip boundary condition for the velocity field u and time-dependent boundary condition for the director field d of unit length. For such a system, we first establish the existence of a global weak solution that is smooth away from finitely many singular times, then establish the existence of a unique global strong solution that is smooth for $t > 0$ under the assumption that the image of boundary data is contained in the hemisphere $\mathbb{S}^2_+$. Finally, we apply these theorems to the problem of optimal boundary control of the simplified Ericksen–Leslie system and show both the existence and a necessary condition of an optimal boundary control.

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Contents

1 Introduction

In this paper, we will consider the boundary value problem of a simplified Ericksen–Leslie system, first proposed by Lin [29], that models the hydrodynamic motion of nematic liquid crystals under the non-slip boundary condition on the velocity field and time-dependent boundary condition on the liquid crystal director field in dimension two. More precisely, for a bounded smooth domain $\Omega \subseteq \mathbb{R}^2$ with $\Gamma = \partial\Omega$ and $0 < T < \infty$, set $Q_T = \Omega \times (0, T)$ and $\Gamma_T = \Gamma \times (0, T)$. We seek a $(\mathbf{u}, \mathbf{d}) : Q_T \mapsto \mathbb{R}^2 \times \mathbb{S}^2$ that solves

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = -\lambda\,\mathrm{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}), \\ \quad \nabla\cdot\mathbf{u} = 0, \\ \mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{d} = \mu(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}), \end{cases} \quad \text{in } Q_T \tag{1.1}$$

subject to the boundary and initial conditions:

$$\mathbf{u}(x,t) = 0,\ \mathbf{d}(x,t) = \mathbf{h}(x,t),\ (x,t) \in \Gamma_T, \tag{1.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x),\ \mathbf{d}|_{t=0} = \mathbf{d}_0(x),\ x \in \Omega, \tag{1.3}$$

where $\mathbf{u}_0 : \Omega \mapsto \mathbb{R}^2$ and $\mathbf{d_0} : \Omega \mapsto \mathbb{R}^3$ are given maps such that $\nabla\cdot\mathbf{u}_0(x) = 0$ and $|\mathbf{d}_0(x)| = 1$ for $x \in \Omega$. Here $\mathbf{u} : Q_T \mapsto \mathbb{R}^2$ represents the fluid velocity field, $P : Q_T \mapsto \mathbb{R}$ represents the pressure function, $\mathbf{d} : \Omega \times (0, T) \mapsto \mathbb{S}^2 = \{y \in \mathbb{R}^3 : |y| = 1\}$ represents the averaged orientation field of the nematic liquid crystal molecules, and ν, μ and λ are three positive constants that represent viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time for the molecular orientation field. The notation $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ stands for the 2×2 matrix whose (i, j)-th entry is given by $\nabla_i\mathbf{d}\cdot\nabla_j\mathbf{d}$ for $1 \le i, j \le 2$.

The system (1.1) is a simplified version of the general Ericksen–Leslie system, which was introduced by Ericksen [10] and Leslie [28] between 1958 and 1968, that represents a macroscopic continuum description of the time evolution of the material under the influence of both the fluid velocity field $\mathbf{u}$ and the macroscopic description of the microscopic orientation configuration field $\mathbf{d}$ of rod-like liquid crystal molecules. Mathematically, (1.1) is a system that strongly couples between the non-homogeneous Navier–Stokes equation and the transported heat flow of harmonic maps to $\mathbb{S}^2$ (see [6] for the Dirichlet problem of harmonic heat flows).

There have been many interesting results on the system (1.1), (1.2), and (1.3), when the boundary data $\mathbf{h}(x,t) = \mathbf{h}(x) : \Gamma_T \mapsto \mathbb{S}^2$ is t-independent. In a series of papers, Lin [29] and Lin and Liu [32,33] initiated the mathematical analysis of (1.1)–(1.3) in 1990's. More precisely, they considered in [32,33] the case of the so-called Ericksen's variable degree of orientation, that replaces the Dirichlet energy $\frac{1}{2}\int_\Omega |\nabla \mathbf{d}|^2\,dx$ for $\mathbf{d} : \Omega \mapsto \mathbb{S}^2$ by the Ginzburg–Landau energy $\int_\Omega (\frac{1}{2}|\nabla \mathbf{d}|^2 + \frac{1}{4\epsilon^2}(1-|\mathbf{d}|^2)^2)\,dx$ for $\mathbf{d} : \Omega \mapsto \mathbb{R}^3$. It has been established by [32,33] the global existence of strong and weak solutions in dimensions 2 and 3 respectively, along with some partial regularity results analogous to [3] on the Navier–Stokes equation. However, since the arguments and crucial estimates in [32,33] depend on the parameter ϵ, it is challenging to utilize such obtained solutions as approximate solutions to (1.1), (1.2), and (1.3) because it remains a very difficult question to study their convergence as ϵ tends to zero. In dimension two, it has been recently shown by Lin et al. [31] and Lin and Wang [35] that (1.1), (1.2), and (1.3) admits a unique global "almost regular" weak solution that is smooth away from finitely many singular times when $\mathbf{h}(= \mathbf{d}_0|_{\partial\Omega}) \in C^{2,\alpha}(\partial\Omega, \mathbb{S}^2)$, see Hong [23], Xu and Zhang [44], Hong and Xin [24], Huang et al. [21], and Lei et al. [30] for related works. In a very recent article, Lin and Wang [37] have proved the existence of a global weak solution to (1.1), (1.2), and (1.3) in dimension 3 when $\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+$, see Huang et al. [22] for related works. The interested reader can also consult the survey article [36] and the papers by Lin and Liu [34], Wang et al. [42], Cavaterra et al. [4], and Wu et al. [43] for related works on the general Ericksen–Leslie system.

Turning to the technically more challenging case of t-dependent boundary data $\mathbf{h} : Q_T \mapsto \mathbb{S}^2$ for $\mathbf{d}$, to the best of our knowledge there has been no previous work addressing (1.1), (1.2), and (1.3) available in the literature yet. For the Ericksen–Leslie system in the case of Ericksen's variable degree of orientation or the so-called Ginzburg–Landau approximate version of (1.1), (1.2), and (1.3), there has been several interesting works by [2,7,8,17,18] extending the main theorems by Lin and Liu [32,33] to t-dependent boundary data for $\mathbf{d}$. In particular, Cavaterra et al. [5] have recently studied the optimal boundary control issue for such a system in dimension 2 (see the books [1,25,41] for more discussions on optimal control of PDEs). The motivation for the study we undertake in this paper is two fold:

(i) We are interested in extending the previous theorems by Lin et al. [31] and establish the theory of global weak and strong solutions of (1.1), (1.2), and (1.3) for t-dependent boundary data of $\mathbf{d}$ in dimension 2. Here the new difficulties arising from the t-dependent boundary data include: (a) the global energy does not necessarily decrease; and (b) the boundary ϵ_0-regularity estimate is much more delicate.

(ii) We plan to employ the existence of a unique, global strong solutions to establish the existence of an optimal boundary control of (1.1), (1.2), and (1.3) and characterize a necessary condition of such an optimal boundary control in a spirit similar to [5]. Here we need to overcome additional difficulties to handle the nonlinearity arising from the constraint $|\mathbf{d}| = 1$.

Now let us briefly set up the boundary control problem. Denote $\mathbf{e}_3 = (0, 0, 1)$, and set

$$\mathbf{H} = \text{ Closure of } \{\mathbf{u} \in C_0^\infty(\Omega, \mathbb{R}^2) : \nabla \cdot \mathbf{u} = 0\} \text{ in } L^2(\Omega, \mathbb{R}^2),$$
$$\mathbf{V} = \text{ Closure of } \{\mathbf{u} \in C_0^\infty(\Omega, \mathbb{R}^2) : \nabla \cdot \mathbf{u} = 0\} \text{ in } H_0^1(\Omega, \mathbb{R}^2),$$
$$H^k(\Omega, \mathbb{S}^2) = \left\{\mathbf{d} \in H^k(\Omega, \mathbb{R}^3) : |\mathbf{d}(x)| = 1 \text{ a.e. } x \in \Omega\right\}, \ k \geq 0.$$

For any given $\beta_i \in \mathbb{R}_+$, $1 \leq i \leq 5$, not all zeroes, and four target maps

$$\mathbf{u}_{Q_T} \in L^2([0, T], \mathbf{H}), \mathbf{d}_{Q_T} \in L^2(Q_T, \mathbb{S}^2), \mathbf{u}_\Omega \in \mathbf{H}, \mathbf{d}_\Omega \in L^2(\Omega, \mathbb{S}^2),$$

our goal is to study the minimization problem of the cost functional

$$\begin{aligned} 2\mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}) = {} & \beta_1 \|\mathbf{u} - \mathbf{u}_{Q_T}\|^2_{L^2(Q_T)} + \beta_2 \|\mathbf{d} - \mathbf{d}_{Q_T}\|^2_{L^2(Q_T)} \\ & + \beta_3 \|\mathbf{u}(T) - \mathbf{u}_\Omega\|^2_{L^2(\Omega)} + \beta_4 \|\mathbf{d}(T) - \mathbf{d}_\Omega\|^2_{L^2(\Omega)} \\ & + \beta_5 \|\mathbf{h} - \mathbf{e}_3\|^2_{L^2(\Gamma_T)}, \end{aligned} \tag{1.4}$$

for any boundary data $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2$ that lies in a bounded, closed, and geometric convex set $\widetilde{\mathcal{U}}_M$ of a function space $\mathcal{U}$, to be specified in the Sect. 4, and $(\mathbf{u}, \mathbf{d})$ is the unique global strong solution to the state problem (1.1), with the boundary condition $(\mathbf{u}, \mathbf{d}) = (0, \mathbf{h})$ and the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ for some fixed maps $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{d}_0 \in H^1(\Omega, \mathbb{S}^2)$, subject to the compatibility condition:

$$\mathbf{h}(x, 0) = \mathbf{d}_0(x), \ x \in \Gamma. \tag{1.5}$$

In order to study both the existence and a necessary condition for an optimal boundary control $\mathbf{h}$ of the cost functional $\mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h})$ over $\widetilde{\mathcal{U}}_M$, we first need to establish the existence of a unique global strong solution to (1.1), (1.2), and (1.3). This turns out to be a challenging task, since the existence theorems by Lin et al. [31] and Huang et al. [22] indicate that the short time smooth solution may develop finite time singularity even for a t-independent boundary value $\mathbf{d}_0$. There are several new difficulties that we need to overcome, when we try to establish Theorem 2.1 extending the main theorem of Lin et al. [31] to (1.1), (1.2), and (1.3) for a t-dependent boundary value $\mathbf{h}$:

(1) the energy $\mathcal{E}(\mathbf{u}, \mathbf{d})(t) = \frac{1}{2} \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(t)$ may grow along the flow, which makes it difficult to estimate the total number of singular times;
(2) the boundary local energy inequality involves contributions by $\mathbf{h}$ that need to be carefully studied; and
(3) the boundary ϵ_0-regularity property is more delicate to establish than the case of t-independent boundary condition.

Under the additional assumption that the initial and boundary values $\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+$ and $\mathbf{h}(\Gamma_T) \subset \mathbb{S}^2_+$, where $\mathbb{S}^2_+ = \{y = (y^1, y^2, y^3) \in \mathbb{S}^2 \mid y^3 \geq 0\}$, we are able to show in Theorem 2.3 that the weak solutions $(\mathbf{u}, \mathbf{d})$ obtained in Theorem 2.1 satisfy both $\mathbf{d} \in L^2([0, T], H^2(\Omega, \mathbb{S}^2_+))$ and the smoothness property that $(\mathbf{u}, \mathbf{d}) \in C^\infty(Q_T) \cap C^\alpha(\overline{\Omega} \times (0, T])$ for any $\alpha \in (0, 1)$. In particular, if, in addition, $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$ and $\mathbf{h} \in H^{\frac{5}{2}, \frac{5}{4}}(\Gamma_T, \mathbb{S}^2_+)$, then $(\mathbf{u}, \mathbf{d})$ is the unique, strong solution to (1.1), (1.2), and (1.3) that enjoys a priori estimate:

$$\|\mathbf{u}\|_{L_t^\infty H_x^1(Q_T)} + \|\mathbf{d}\|_{L_t^\infty H_x^2(Q_T)} + \|\mathbf{u}\|_{L_t^2 H_x^2(Q_T)} + \|\mathbf{d}\|_{L_t^2 H_x^3(Q_T)} \leq C(T).$$

This estimate is established in the Sect. 3, which turns out to be the crucial estimate in order to establish the Fréchet differentiability of the control to state map $\mathcal{S}(\mathbf{h}) = (\mathbf{u}, \mathbf{d})$ over appropriate function spaces, that is to be discussed in the Sect. 4, by which we can obtain a necessary condition of an optimal boundary control $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2_+$.

There have been many research articles on the optimal boundary control for parabolic equations, the Navier–Stokes equation, and the Cahn–Hilliard–Navier–Stokes system. See Hömberg et al. [20], Kunisch and Vexler [27], Fattorini and Sritharan [13,14], Fursikov et al. [11,12], Hinze and Kunisch [19], Frigeri et al. [15], Hintermüller and Wedner [26], Colli and Sprekels [9], and Tröltzsch [41], Alekseev et al. [1], and Hinze et al. [25].

Since the exact values of ν, μ and λ don't play a role, we henceforth assume $\nu = \mu = \lambda = 1$.

2 Existence of weak solutions

In this section, we will establish the existence of a global weak solution to (1.1), (1.2), and (1.3). First, let us recall a few notations. For any nonnegative number $k \geq 0$, recall the Sobolev spaces

$$H^k(\Gamma, \mathbb{S}^2) = \left\{ f \in H^k(\Gamma, \mathbb{R}^3) : \ f(x) \in \mathbb{S}^2 \text{ a.e. } x \in \Gamma \right\},$$
$$H^{k,\frac{k}{2}}(\Gamma_T, \mathbb{S}^2) = \left\{ f \in H^{k,\frac{k}{2}}(\Gamma_T, \mathbb{R}^3) : \ f(x,t) \in \mathbb{S}^2, \text{ a.e. } (x,t) \in \Gamma_T \right\},$$

and the dual space of $H^k(\Gamma, \mathbb{R}^3)$, $H^{-k}(\Gamma, \mathbb{R}^3) = (H^k(\Gamma, \mathbb{R}^3))'$. Our first theorem, which is an extension of [31] Theorem 1.3 to time dependent boundary data, states as follows.

Theorem 2.1 *For any $0 < T < \infty$ and any bounded, smooth domain $\Omega \subset \mathbb{R}^2$ with boundary Γ, assume that*

$$\mathbf{h} \in L^2([0,T], H^{\frac{3}{2}}(\Gamma, \mathbb{S}^2)), \ \partial_t \mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T),$$

and $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H} \times H^1(\Omega, \mathbb{S}^2)$ satisfies the compatibility condition (1.5). Then there exists a weak solution $(\mathbf{u}, \mathbf{d}) : Q_T \mapsto \mathbb{R}^2 \times \mathbb{S}^2$ of the system (1.1), with initial and boundary condition (1.2) and (1.3), such that

$$\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{V}), \text{ and } \mathbf{d} \in L^\infty_t H^1_x(Q_T, \mathbb{S}^2). \tag{2.1}$$

Furthermore, there exist $L \in \mathbb{N}$, depending only on $(\mathbf{u}_0, \mathbf{d}_0)$, and $0 < T_1 < \cdots < T_L \leq T$ such that $(\mathbf{u}, \mathbf{d})$ is regular away from $\cup_{i=1}^L \{T_i\}$ in the sense that for any $0 < \alpha < 1$,

$$(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times ((0,T] \setminus \cup_{i=1}^L \{T_i\})) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times ((0,T] \setminus \cup_{i=1}^L \{T_i\})).$$

Moreover, there is a universal constant $\varepsilon_1 > 0$ such that for each singular time T_i, $1 \leq i \leq L$, it holds that

$$\limsup_{t \uparrow T_i} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_r(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t) \geq \varepsilon_1^2, \quad \forall r > 0. \tag{2.2}$$

A few remarks are in the order:

Remark 2.2 (1) Theorem 2.1 was first established by [31] for any time independent boundary data $\mathbf{h}(x,t) = \mathbf{d}_0(x)$, $(x,t) \in \Gamma_T$, with $\mathbf{d}_0 \in C^{2,\beta}(\Gamma, \mathbb{S}^2)$ for some $\beta \in (0,1)$. (2) By the Sobolev embedding theorem, $\mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T)$ and $\partial_t \mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T)$ imply that $\mathbf{h} \in \mathrm{Lip}(\Gamma_T)$. We will present a new proof of the boundary ϵ_0-regularity theorem on (1.1) and (1.2) for any Lipschitz continuous boundary value $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2$, which plays a crucial role in the proof of Theorem 2.1. (3) In contrast with the autonomous boundary condition studied by [31], the system (1.1), (1.2), and (1.3) no longer enjoys the energy dissipation inequality for a time dependent boundary value $\mathbf{h}$. However, under the assumption that both $\mathbf{h}$ and $\partial_t \mathbf{h}$ belong to $L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$, we are able to estimate the growth rate of the energy $\mathcal{E}(\mathbf{u}(t), \mathbf{d}(t))$ by

$$\mathcal{E}(\mathbf{u}(t), \mathbf{d}(t)) \le \exp\big(C \int_0^t \|\partial_t \mathbf{h}\|_{H^{\frac{3}{2}}(\Gamma)}\big)\big[\mathcal{E}(\mathbf{u}_0, \mathbf{d}_0) + C\|(\mathbf{h}, \partial_t \mathbf{h})\|^2_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big]. \tag{2.3}$$

This turns to be sufficient for establishing the finiteness of singular times (4) Concerning the uniqueness issue of the global weak solutions constructed by Theorem 2.1, we establish in a forthcoming paper [16] that there is at most one weak solution in the same class of weak solutions as in Theorem 2.1, if in addition

$$\mathbf{h} \in L^2([0, T], H^{\frac{3}{2}+\delta}(\Gamma, \mathbb{S}^2))$$

holds for some $\delta > 0$.

Applying the maximum principle, we can show that if $\mathbf{d}_0 : \Omega \mapsto \mathbb{S}^2_+$ and $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2_+$, then any weak solution $(\mathbf{u}, \mathbf{d})$ to problem (1.1), (1.2), and (1.3) obtained by Theorem 2.1 satisfies (see Lemma 2.11 below)

$$\mathbf{d}(x, t) : Q_T \mapsto \mathbb{S}^2_+.$$

This, together with Lemma 2.12, ensures that (2.2) never occurs in the interval $(0, T]$. Hence, we obtain the following theorem.

Theorem 2.3 *For any $T > 0$ and a bounded smooth domain $\Omega \subset \mathbb{R}^2$, assume that*

$$\mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T, \mathbb{S}^2_+) \text{ and } \partial_t \mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$$

and $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H} \times H^1(\Omega, \mathbb{S}^2_+)$ satisfies the compatibility condition (1.5). Then there is a weak solution $(\mathbf{u}, \mathbf{d}) : Q_T \mapsto \mathbb{R}^2 \times \mathbb{S}^2_+$ of the system (1.1) with the initial and boundary conditions (1.2) and (1.3) such that

$$\mathbf{u} \in L^\infty([0, T], \mathbf{H}) \cap L^2([0, T], \mathbf{V}),$$
$$\mathbf{d} \in L_t^\infty H_x^1(Q_T, \mathbb{S}^2_+) \cap L_t^2 H_x^2(Q_T, \mathbb{S}^2_+),$$

and $(\mathbf{u}, \mathbf{d}) \in C^\infty(Q_T) \cap C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times (0, T))$ for any $\alpha \in (0, 1)$. In particular, for ε_1 given by Theorem 2.1 there exists $r_0 > 0$ such that

$$\sup_{(x,t)\in\overline{\Omega}\times[0,T]} \int_{\Omega\cap B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(y, t)\, dy < \varepsilon_1^2. \tag{2.4}$$

The smoothness of a weak solution $(\mathbf{u}, \mathbf{d})$ to (1.1) and (1.2) relies on both the interior and boundary ε_1-regularity properties for $(\mathbf{u}, \mathbf{d})$, provided $\mathbf{d} \in L_t^2 H_x^2(Q_T)$ and the condition (2.4) holds. This will be discussed in the following subsection.

2.1 Regularity of weak solutions

In this subsection, we will show both the interior and boundary regularity for weak solutions $(\mathbf{u}, \mathbf{d})$ to (1.1) and (1.2) that satisfies $\mathbf{d} \in L_t^2 H_x^2(Q_T)$ and (2.4).

Theorem 2.4 *For a* $T > 0$ *and a bounded smooth domain* $\Omega \subset \mathbb{R}^2$*, assume* $\mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T, \mathbb{S}^2)$ *and* $\partial_t \mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T, \mathbb{S}^2)$*. If* $(\mathbf{u}, \mathbf{d}) \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{V}) \times L_t^\infty H_x^1(Q_T, \mathbb{S}^2) \cap L_t^2 H_x^2(Q_T, \mathbb{S}^2)$ *is a weak solution of the system* (1.1) *with the boundary condition* (1.2)*, then* $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times [\delta, T]) \cap C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [\delta, T])$ *for any* $\alpha \in (0,1)$ *and* $0 < \delta < T$.

In order to prove Theorem 2.4 and the existence of short time smooth solutions to (1.1), (1.2), and (1.3), we recall the definition of Hölder spaces in Q_T. First, define the parabolic distance in Q_T by

$$\delta(z_1, z_2) = |x_1 - x_2| + \sqrt{|t_1 - t_2|}, \quad z_i = (x_i, t_i) \in Q_T, \ i = 1, 2.$$

For $\alpha \in (0,1]$ and $U \subset Q_T$, a continuous function $f : U \mapsto \mathbb{R}$ belongs to the Hölder space $C^{\alpha, \frac{\alpha}{2}}(U)$, if $\|f\|_{C^{\alpha,\frac{\alpha}{2}}(U)} = \|f\|_{C^0(U)} + \big[f\big]_{C^{\alpha,\frac{\alpha}{2}}(U)} < \infty$, where

$$\big[f\big]_{C^{\alpha,\frac{\alpha}{2}}(U)} = \sup_{z_1, z_2 \in U, z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{\delta(z_1, z_2)^\alpha}.$$

For any positive integer $k \geq 1$, a continuous function $f : U \mapsto \mathbb{R}$ belongs to $C^{k+\alpha, \frac{k+\alpha}{2}}(U)$, if

$$\|f\|_{C^{k+\alpha,\frac{k+\alpha}{2}}(U)} = \sum_{0 \leq r+2s \leq k} \|\partial_t^s \partial_x^r f\|_{C^0(U)} + \big[f\big]_{C^{k+\alpha,\frac{k+\alpha}{2}}(U)} < \infty,$$

where

$$\big[f\big]_{C^{k+\alpha,\frac{k+\alpha}{2}}(U)} = \begin{cases} \displaystyle\sum_{r+2s=k} \big[\partial_t^s \partial_x^r f\big]_{C^{\alpha,\frac{\alpha}{2}}(U)}, & k \text{ is even}, \\ \displaystyle\sum_{r+2s=k} \big[\partial_t^s \partial_x^r f\big]_{C^{\alpha,\frac{\alpha}{2}}(U)} + \sum_{r+2s=k-1} \big[\partial_t^s \partial_x^r f\big]_{C_t^{\frac{1+\alpha}{2}}(U)}, \\ k \text{ is odd}, \end{cases}$$

and

$$\big[f\big]_{C_t^{\frac{1+\alpha}{2}}(U)} = \sup_{(x,t_1),(x,t_2) \in U, t_1 \neq t_2} \frac{|f(x,t_1) - f(x,t_2)|}{|t_1 - t_2|^{\frac{1+\alpha}{2}}}.$$

For $z_0 = (x_0, t_0) \in \Omega \times (0, T]$ and $0 < r < \min\{\sqrt{t_0}, \operatorname{dist}(x_0, \Gamma)\}$, set

$$B_r(x_0) = \big\{x \in \mathbb{R}^2 \mid |x - x_0| \leq r\big\}, \ Q_r(z_0) = B_r(x_0) \times [t_0 - r^2, t_0],$$

and the parabolic boundary of $Q_r(z_0)$ by

$$\partial_p Q_r(z_0) = \big(B_r(x_0) \times \{t_0 - r^2\}\big) \cup \big(\partial B_r(x_0) \times [t_0 - r^2, t_0]\big).$$

Denote $B_r(0)$, $Q_r(0,0)$ and $\partial_p Q_r(0,0)$ by B_r, Q_r and $\partial_p Q_r$ respectively, if $z_0 = (0,0)$. For $f \in L^1(Q_r(z_0))$, denote by

$$f_{z_0,r} = \frac{1}{|Q_r(z_0)|} \int_{Q_r(z_0)} f(x,t),$$

$$f_{x_0,r}(t) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} f(x,t), \ t \in [t_0 - r^2, t_0],$$

as the average of f over $Q_r(z_0)$ and $B_r(x_0)$ respectively.

For $p, q \in (1, \infty)$ and $U \subset Q_T$, define $W^{1,0}_{p,q}(U) = L^q_t W^{1,p}_x(U)$, with the norm

$$\|f\|_{W^{1,0}_{p,q}(U)} = \|f\|_{L^q_t L^p_x(U)} + \|\nabla f\|_{L^q_t L^p_x(U)},$$

and

$$W^{2,1}_{p,q}(U) = \left\{f \in W^{1,0}_{p,q}(U) \mid \nabla^2 f,\ \partial_t f \in L^q_t L^p_x(U)\right\},$$

with the norm

$$\|f\|_{W^{2,1}_{p,q}(U)} = \|f\|_{W^{1,0}_{p,q}(U)} + [f]_{W^{2,1}_{p,q}(U)}$$

where

$$[f]_{W^{2,1}_{p,q}(U)} = \|\nabla^2 f\|_{L^q_t L^p_x(U)} + \|\partial_t f\|_{L^q_t L^p_x(U)}.$$

For $p = q$, denote $L^p(U) = L^p_t L^p_x(U)$ and $W^{r,s}_p(U) = W^{r,s}_{p,p}(U)$.

We begin with an interior ε_0-regularity result, whose proof follows exactly from Lin-Lin-Wang [31] Lemma 2.1.

Lemma 2.5 *For any $\alpha \in (0, 1)$, there exists $\varepsilon_0 > 0$ such that for $z_0 = (x_0, t_0) \in Q_T$ and $0 < r < \min\{\sqrt{t_0}, \mathrm{dist}(x_0, \Gamma)\}$, if $(\mathbf{u}, \mathbf{d}) \in W^{1,0}_2(Q_T, \mathbb{R}^2 \times \mathbb{S}^2)$, $P \in W^{1,0}_{\frac{4}{3}}(Q_T)$ is a weak solution to (1.1) satisfying*

$$\int_{Q_r(z_0)} (|\mathbf{u}|^4 + |\nabla \mathbf{d}|^4) \le \varepsilon_0^4, \tag{2.5}$$

then $(\mathbf{u}, \mathbf{d}) \in C^{\alpha, \frac{\alpha}{2}}(Q_{\frac{r}{2}}(z_0), \mathbb{R}^2 \times \mathbb{S}^2)$, and there holds that

$$\begin{aligned}
\left[\mathbf{d}\right]_{C^{\alpha,\frac{\alpha}{2}}(Q_{\frac{r}{2}}(z_0))} &\le C\left(\|\mathbf{u}\|_{L^4(Q_r(z_0))} + \|\nabla \mathbf{d}\|_{L^4(Q_r(z_0))}\right),\\
\left[\mathbf{u}\right]_{C^{\alpha,\frac{\alpha}{2}}(Q_{\frac{r}{2}}(z_0))} &\le C\left(\|\mathbf{u}\|_{L^4(Q_r(z_0))} + \|\nabla \mathbf{d}\|_{L^4(Q_r(z_0))} + \|\nabla P\|_{L^{\frac{4}{3}}(Q_r(z_0))}\right).
\end{aligned}$$

For $r > 0$ and $z_0 = (x_0, t_0)$ with $x_0 \in \Gamma$ and $t_0 > 0$, set

$$B^+_r(x_0) = B_r(x_0) \cap \Omega,\ \ Q^+_r(z_0) = B^+_r(x_0) \times [t_0 - r^2, t_0],$$

and

$$\Gamma_r(x_0) = \partial B^+_r(x_0) \cap \Gamma,\ \ S^+_r(x_0) = \partial B^+_r(x_0) \cap \Omega$$

so that

$$\partial B^+_r(x_0) = \Gamma_r(x_0) \cup S^+_r(x_0),$$

and

$$\partial_p Q^+_r(z_0) = \left(\partial B^+_r(x_0) \times [t_0 - r^2, t_0]\right) \cup \left(B^+_r(x_0) \times \{t_0 - r^2\}\right).$$

If $(x_0, t_0) = (0, 0)$, simply denote

$$B^+_r = B^+_r(0),\ \ Q^+_r = Q^+_r(0, 0),\ \ \Gamma_r = \Gamma_r(0),\ \ S^+_r = S^+_r(0),$$

and

$$\partial B^+_r = \partial B^+_r(0),\ \ \partial_p Q^+_r = \partial_p Q^+_r(0, 0).$$

Next we will establish a corresponding boundary ε_0 regularity for (1.1) and (1.2), which is a highly nontrivial extension of [31] Lemma 2.2, where a time independent boundary data for $\mathbf{d}$ is assumed.

Lemma 2.6 *For any $\alpha \in (0,1)$, $\mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T, \mathbb{S}^2)$ with $\partial_t \mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$, assume that $(\mathbf{u}, \mathbf{d}) \in W_2^{1,0}(Q_T, \mathbb{R}^2 \times \mathbb{S}^2)$, $P \in W_{\frac{4}{3}}^{1,0}(Q_T)$, is a weak solution of (1.1) and (1.2). Then there exist $r_0 \in (0, \sqrt{t_0})$ depending on Γ and $\varepsilon_1 > 0$ depending on α and $\|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}$ such that for any $z_0 = (x_0, t_0) \in \Gamma_T$, if*

$$\int_{Q_{r_0}^+(z_0)} (|\mathbf{u}|^4 + |\nabla \mathbf{d}|^4) \le \varepsilon_1^4, \tag{2.6}$$

then $(\mathbf{u}, \mathbf{d}) \in C^{\alpha, \frac{\alpha}{2}}(Q_{\frac{r_0}{2}}^+(z_0), \mathbb{R}^2 \times \mathbb{S}^2)$, and there holds that

$$\begin{aligned}
\big[\mathbf{d}\big]_{C^{\alpha,\frac{\alpha}{2}}(Q^+_{\frac{r_0}{2}}(z_0))} &\le C\Big[\|\mathbf{u}\|_{L^4(Q^+_{r_0}(z_0))} + \|\nabla \mathbf{d}\|_{L^4(Q^+_{r_0}(z_0))} \\
&\quad + \|(\mathbf{h}, r_0^2 \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_{r_0}(x_0) \times [t_0 - r_0^2, t_0])}\Big],
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\big[\mathbf{u}\big]_{C^{\alpha,\frac{\alpha}{2}}(Q^+_{\frac{r_0}{2}}(z_0))} &\le C\Big[\|\mathbf{u}\|_{L^4(Q^+_{r_0}(z_0))} + \|\nabla \mathbf{d}\|_{L^4(Q^+_{r_0}(z_0))} \\
&\quad + \|\nabla P\|_{L^{\frac{4}{3}}(Q^+_{r_0}(z_0))} \\
&\quad + \|(\mathbf{h}, r_0^2 \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_{r_0}(x_0) \times [t_0 - r_0^2, t_0])}\Big].
\end{aligned} \tag{2.8}$$

Proof The proof of Lemma 2.6 is more delicate than [31] Lemma 2.2, because the boundary value $\mathbf{h}$ is time dependent. Here we will give a detailed argument.

Choosing a sufficiently small $r_0 > 0$ and applying the standard boundary flatten technique, we may, for simplicity, assume that $x_0 = 0$, $t_0 = 1$, $r_0 < 1$ so that

$$\Omega \cap B_{r_0}(0) = \mathbb{R}_+^2 \cap B_{r_0}(0) = B_{r_0}^+, \text{ and } Q_{r_0}^+(0,1) = B_{r_0}^+ \times [1 - r_0^2, 1].$$

First, observe that (2.6) and (1.1) imply

$$\partial_t \mathbf{d} - \Delta \mathbf{d} = |\nabla \mathbf{d}|^2 \mathbf{d} - \mathbf{u} \cdot \nabla \mathbf{d} \in L^2(Q_{r_0}^+).$$

Hence, by the $W_2^{2,1}$-theory on parabolic equations, we have that $\partial_t \mathbf{d} \in L^2(Q^+_{\frac{3r_0}{4}})$, $\mathbf{d} \in L_t^2 H_x^2(Q^+_{\frac{3r_0}{4}})$, and

$$\begin{aligned}
&\big\|(\partial_t \mathbf{d}, \nabla^2 \mathbf{d})\big\|_{L^2(Q^+_{\frac{3r_0}{4}})} \\
&\quad \le C\Big[\big\|(\mathbf{u}, \nabla \mathbf{d})\big\|_{L^4(Q^+_{r_0})} + \big\|(\mathbf{h}, r_0^2 \partial_t \mathbf{h})\big\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_{r_0})}\Big].
\end{aligned} \tag{2.9}$$

For $z_1 = (x_1, t_1) \in \Gamma_{\frac{r_0}{2}} \times [1 - \frac{r_0^2}{4}, 1]$, $0 < r \le \frac{r_0}{4}$, let $\mathbf{d}^1 : Q_r^+(z_1) \mapsto \mathbb{R}^3$ solve

$$\begin{cases}
\partial_t \mathbf{d}^1 - \Delta \mathbf{d}^1 = 0 & \text{in } Q_r^+(z_1), \\
\mathbf{d}^1 = \mathbf{h} & \text{on } \Gamma_r(x_1) \times [t_1 - r^2, t_1], \\
\mathbf{d}^1 = \mathbf{d} & \text{on } \partial_p Q_r^+(z_1) \setminus (\Gamma_r(x_1) \times [t_1 - r^2, t_1]).
\end{cases} \tag{2.10}$$

Then $\mathbf{d}^2 = \mathbf{d} - \mathbf{d}^1 : Q_r^+(z_1) \mapsto \mathbb{R}^3$ solves

$$\begin{cases}
\partial_t \mathbf{d}^2 - \Delta \mathbf{d}^2 = -\mathbf{u} \cdot \nabla \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} & \text{in } Q_r^+(z_1), \\
\mathbf{d}^2 = 0 & \text{on } \partial_p Q_r^+(z_1).
\end{cases} \tag{2.11}$$

From (2.6) and an argument similar to (2.9), we have that $\partial_t \mathbf{d}^2, \nabla^2 \mathbf{d}^2 \in L^2(Q_r^+(z_1))$. Hence, by multiplying $(2.11)_1$ by $-\Delta \mathbf{d}^2$ and integrating over $B_r^+(x_1)$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{B_r^+(x_1)} |\nabla \mathbf{d}^2|^2 + \int_{B_r^+(x_1)} |\Delta \mathbf{d}^2|^2 = \int_{B_r^+(x_1)} (\mathbf{u}\cdot\nabla\mathbf{d} - |\nabla \mathbf{d}|^2\mathbf{d})\cdot \Delta \mathbf{d}^2.$$

Integrating over $t \in [t_1 - r^2, t_1]$ and applying Hölder's inequality, this yields

$$\begin{aligned}
&\|\nabla \mathbf{d}^2\|^2_{L_t^\infty L_x^2(Q_r^+(z_1))} + \|\Delta \mathbf{d}^2\|^2_{L^2(Q_r^+(z_1))}\\
&\le C\big[\||\mathbf{u}||\nabla \mathbf{d}|\|^2_{L^2(Q_r^+(z_1))} + \|\nabla \mathbf{d}\|^4_{L^4(Q_r^+(z_1))}\big]\\
&\le C\big[\|\mathbf{u}\|^2_{L^4(Q_r^+(z_1))} + \|\nabla \mathbf{d}\|^2_{L^4(Q_r^+(z_1))}\big]\|\nabla \mathbf{d}\|^2_{L^4(Q_r^+(z_1))}.
\end{aligned}$$

This, together with the Ladyzhenskaya inequality (see Lemma 2.17 below), yields that

$$\begin{aligned}
&\|\nabla \mathbf{d}^2\|^4_{L^4(Q_r^+(z_1))} + \|\nabla \mathbf{d}^2\|^4_{L_t^\infty L_x^2(Q_r^+(z_1))} + \|\Delta \mathbf{d}^2\|^4_{L^2(Q_r^+(z_1))}\\
&\le C\big[\|\mathbf{u}\|^4_{L^4(Q_r^+(z_1))} + \|\nabla \mathbf{d}\|^4_{L^4(Q_r^+(z_1))}\big]\|\nabla \mathbf{d}\|^4_{L^4(Q_r^+(z_1))}\\
&\le C\varepsilon_1^4\|\nabla \mathbf{d}\|^4_{L^4(Q_r^+(z_1))}.
\end{aligned} \tag{2.12}$$

Now we estimate $\mathbf{d}^1$. First observe that $\mathbf{h}, \partial_t \mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$ implies that $\mathbf{h} \in L_t^\infty H_x^{\frac{3}{2}}(\Gamma_T)$ and[1]

$$\|\mathbf{h}\|^2_{L_t^\infty H_x^{\frac{3}{2}}(\Gamma_T)} \le \Big(\frac{1}{T}\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big)\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}. \tag{2.13}$$

This, combined with $\partial_t \mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$ and the Sobolev embedding theorem, implies that $\mathbf{h} \in C^{\alpha,\frac{\alpha}{2}}(\Gamma_T)$ for any $\alpha \in (0,1)$, and

$$\|\mathbf{h}\|^2_{C^{\alpha,\frac{\alpha}{2}}(\Gamma_T)} \le C(\alpha)\Big(\frac{1}{T}\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big)\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}. \tag{2.14}$$

It follows from (2.10), (2.14) and the boundary regularity theory for parabolic equations that $\mathbf{d}^1 \in C^{\alpha,\frac{\alpha}{2}}(Q^+_{\frac{3r}{4}}(z_1))$ and

$$\begin{aligned}
\|\mathbf{d}^1\|_{C^{\alpha,\frac{\alpha}{2}}(Q^+_{\frac{3r}{4}}(z_1))} &\le C\Big(\|\mathbf{h}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_r^+(z_1))} + \|\nabla \mathbf{d}\|_{L^4(Q_r^+(z_1))}\Big)\\
&\le C\Big(\alpha, T, \varepsilon_1, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big).
\end{aligned} \tag{2.15}$$

For $t \in (0, T]$, let $\mathbf{h}_E(\cdot, t) : \Omega \mapsto \mathbb{R}^3$ be the harmonic extension of $\mathbf{h}(\cdot, t) : \Gamma \mapsto \mathbb{S}^2$. Then we have that

$$\begin{aligned}
\|\mathbf{h}_E\|_{L_t^2 H_x^2(Q_T)} &\le C\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)},\\
\|\partial_t \mathbf{h}_E\|_{L_t^2 H_x^2(Q_T)} &\le C\|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}.
\end{aligned}$$

[1] (2.13) can be obtained by

$$\sup_{0\le t\le T}\|h(t)\|^2_{H_x^{\frac{3}{2}}(\Gamma)} - \frac{1}{T}\int_0^T \|h(s)\|^2_{H_x^{\frac{3}{2}}(\Gamma)}\, ds \le \|\partial_t h\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\|h\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}.$$

Furthermore, (2.13) and (2.14) imply that $\mathbf{h}_E \in L_t^\infty H_x^2(Q_T) \cap C^{\alpha,\frac{\alpha}{2}}(Q_T)$ for any $\alpha \in (0,1)$, and

$$\begin{aligned}
&\max\left\{\left\|\mathbf{h}_E\right\|_{L_t^\infty H_x^2(Q_T)},\ \left\|\mathbf{h}_E\right\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\right\}\\
&\le C(\alpha)\Big(\frac{1}{T}+\|\partial_t\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big)\Big(1+\|\mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big)\\
&\le C\Big(\alpha, T, \|(\mathbf{h},\partial_t\mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\Big).
\end{aligned} \tag{2.16}$$

Observe that $\mathbf{d}^1-\mathbf{h}_E$ solves

$$\partial_t(\mathbf{d}^1-\mathbf{h}_E)-\Delta(\mathbf{d}^1-\mathbf{h}_E) = -\partial_t\mathbf{h}_E \text{ in } Q_r^+(z_1). \tag{2.17}$$

Let $\eta \in C_0^\infty(B_{\frac{3r}{4}}(x_1))$ be a cut-off function of $B_{\frac{r}{2}}(x_1)$, i.e., $0\le\eta\le 1$, $\eta=1$ in $B_{\frac{r}{2}}(x_1)$, and $|\nabla\eta|\le 8r^{-1}$. Multiplying (2.17) by $(\mathbf{d}^1-\mathbf{h}_E)\eta^2$, integrating over $B_r^+(x_1)$, and applying (2.15) and (2.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{B_r^+(x_1)}|\mathbf{d}^1-\mathbf{h}_E|^2\eta^2+2\int_{B_r^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\eta^2\\
&=-2\int_{B_r^+(x_1)}\langle\nabla_{x_i}(\mathbf{d}^1-\mathbf{h}_E),\mathbf{d}^1-\mathbf{h}_E\rangle\nabla_{x_i}\eta^2-2\int_{B_r^+(x_1)}\langle\partial_t\mathbf{h}_E,\mathbf{d}^1-\mathbf{h}_E\rangle\eta^2\\
&\le\int_{B_r^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\eta^2+\int_{B_r^+(x_1)}(4|\mathbf{d}^1-\mathbf{h}_E|^2|\nabla\eta|^2+2|\partial_t\mathbf{h}_E||\mathbf{d}^1-\mathbf{h}_E|)\\
&\le\int_{B_r^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\eta^2+Cr^{2\alpha}+Cr^\alpha\int_{B_r^+(x_1)}|\partial_t\mathbf{h}_E|.
\end{aligned}$$

Integrating this inequality over $t\in[t_1-\frac{r^2}{4},t_1]$ yields

$$\begin{aligned}
\int_{Q_{\frac{r}{2}}^+(z_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2&\le Cr^{2+2\alpha}+Cr^\alpha\int_{Q_r^+(z_1)}|\partial_t\mathbf{h}_E|\\
&\le Cr^{2+2\alpha}+Cr^{2+2\alpha}\|\partial_t\mathbf{h}_E\|_{L_t^2L_x^{\frac{2}{1-\alpha}}(Q_T)}\\
&\le C\big(1+\|\partial_t\mathbf{h}_E\|_{L_t^2H_x^2(Q_T)}\big)r^{2+2\alpha}\\
&\le C\big(1+\|\partial_t\mathbf{h}\|_{L_t^2H_x^{\frac{3}{2}}(\Gamma_T)}\big)r^{2+2\alpha}\le Cr^{2+2\alpha}.
\end{aligned}\tag{2.18}$$

Let $\eta_1\in C_0^\infty(B_{\frac{r}{2}}(x_1))$ be a cut-off function of $B_{\frac{3r}{8}}(x_1)$, i.e. $\eta_1=1$ in $B_{\frac{3r}{8}}(x_1)$, and $|\nabla\eta_1|\le 16r^{-1}$. Multiplying (2.17) by $\Delta(\mathbf{d}^1-\mathbf{h}_E)\eta_1^2$ and integrating over $B_r^+(x_1)$, and using $\partial_t\mathbf{d}^1=\Delta(\mathbf{d}^1-\mathbf{h}_E)$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_r^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\eta_1^2+2\int_{B_r^+(x_1)}|\Delta(\mathbf{d}^1-\mathbf{h}_E)|^2\eta_1^2\\
&\quad=2\int_{B_r^+(x_1)}\big(\langle\partial_t\mathbf{h}_E,\Delta(\mathbf{d}^1-\mathbf{h}_E)\rangle\eta_1^2\\
&\qquad-\langle\partial_t(\mathbf{d}^1-\mathbf{h}_E),\nabla_{x_i}(\mathbf{d}^1-\mathbf{h}_E)\rangle\nabla_{x_i}\eta_1^2\big)\\
&\quad\le\frac{1}{2}\int_{B_r^+(x_1)}|\Delta(\mathbf{d}^1-\mathbf{h}_E)|^2\eta_1^2+8\int_{B_r^+(x_1)}|\partial_t\mathbf{h}_E|^2\eta_1^2\\
&\qquad+\frac{1}{2}\int_{B_r^+(x_1)}|\partial_t\mathbf{d}^1|^2\eta_1^2+8\int_{B_r^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2|\nabla\eta_1|^2\\
&\quad\le\int_{B_r^+(x_1)}|\Delta(\mathbf{d}^1-\mathbf{h}_E)|^2\eta_1^2+C\int_{B_r^+(x_1)}|\partial_t\mathbf{h}_E|^2\\
&\qquad+Cr^{-2}\int_{B_{\frac{r}{2}}^+(x_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2.
\end{aligned}\tag{2.19}
$$

By Fubini's theorem, there exists $t_*\in[t_1-\frac{r^2}{4},t_1-\frac{r^2}{16}]$ such that

$$
\int_{B_{\frac{r}{2}}^+(x_1)\times\{t_*\}}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\le\frac{32}{r^2}\int_{Q_{\frac{r}{2}}^+(z_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^2\le Cr^{2\alpha}.
$$

This, combined with (2.18) and integration of (2.19) over $t\in[t_*,t_1]$, yields

$$
\|\nabla(\mathbf{d}^1-\mathbf{h}_E)\|^2_{L_t^\infty L_x^2(Q^+_{\frac{3r}{8}}(z_1))}+\int_{Q^+_{\frac{3r}{8}}(z_1)}|\Delta(\mathbf{d}^1-\mathbf{h}_E)|^2\le Cr^{2\alpha}.\tag{2.20}
$$

This, combined with the Ladyzhenskaya inequality (see Lemma 2.17 below), implies that

$$
\begin{aligned}
&\int_{Q^+_{\frac{r}{4}}(z_1)}|\nabla(\mathbf{d}^1-\mathbf{h}_E)|^4\\
&\quad\le C\|\nabla(\mathbf{d}^1-\mathbf{h}_E)\|^2_{L_t^\infty L_x^2(Q^+_{\frac{3r}{8}}(z_1))}\Big[\|\nabla(\mathbf{d}^1-\mathbf{h}_E)\|^2_{L_t^\infty L_x^2(Q^+_{\frac{3r}{8}}(z_1))}\\
&\qquad+\int_{Q^+_{\frac{3r}{8}}(z_1)}|\Delta(\mathbf{d}^1-\mathbf{h}_E)|^2\Big]\le Cr^{4\alpha}.
\end{aligned}\tag{2.21}
$$

Since $\mathbf{h}_E\in L_t^\infty H_x^2(Q_T)$, we have that for all $4<p<\infty$,

$$
\begin{aligned}
\int_{Q_r^+(z_1)}|\nabla\mathbf{h}_E|^4&\le r^2\sup_{t\in[t_1-r^2,t_1]}\int_{B_r^+(x_1)}|\nabla\mathbf{h}_E|^4\\
&\le Cr^{4-\frac{8}{p}}\sup_{t\in[t_1-r^2,t_1]}\Big(\int_{B_r^+(x_1)}|\nabla\mathbf{h}_E|^p\Big)^{\frac{4}{p}}\\
&\le C\big(T,\|(\mathbf{h},\partial_t\mathbf{h})\|_{L_t^2H_x^{\frac{3}{2}}(\Gamma_T)}\big)r^{4-\frac{8}{p}}\le Cr^{4-\frac{8}{p}}.
\end{aligned}\tag{2.22}
$$

Without loss of generality, we may only consider $\alpha \in (\frac{1}{2}, 1)$. By choosing $p = \frac{2}{1-\alpha} > 4$, (2.21) and (2.22) imply that

$$\int_{Q^+_{\frac{r}{4}}(z_1)} |\nabla \mathbf{d}^1|^4 \leq Cr^{4\alpha}. \tag{2.23}$$

Putting (2.12) and (2.23) together, we obtain that

$$\int_{Q^+_{\frac{r}{4}}(z_1)} |\nabla \mathbf{d}|^4 \leq Cr^{4\alpha} + C\varepsilon_1^4 \int_{Q^+_r(z_1)} |\nabla \mathbf{d}|^4. \tag{2.24}$$

It is well known that by iterations of (2.24), we can conclude that

$$\begin{aligned}\int_{Q^+_r(z_1)} |\nabla \mathbf{d}|^4 &\leq Cr^{4\alpha} + C\Big(\frac{r}{r_0}\Big)^{4\alpha} \int_{Q^+_{r_0}(z_1)} |\nabla \mathbf{d}|^4 \\ &\leq C(\alpha, \varepsilon_1, r_0)r^{4\alpha}\end{aligned} \tag{2.25}$$

holds for all $z_1 \in \Gamma_{\frac{r_0}{2}} \times [1 - \frac{r_0^2}{4}, 1]$ and $0 < r \leq \frac{r_0}{4}$.

Next we want to estimate $\|\partial_t \mathbf{d}\|_{L^2(Q^+_{\frac{r}{2}}(z_1))}$ for $z_1 \in \Gamma_{\frac{r_0}{2}} \times [1 - \frac{r_0^2}{4}, 1]$ and $0 < r \leq \frac{r_0}{4}$. To do this, we first observe that (2.20), (2.16), (2.12), together with (2.25), imply that

$$\begin{aligned}\int_{Q^+_{\frac{r}{2}}(z_1)} |\Delta \mathbf{d}|^2 &\leq \int_{Q^+_{\frac{r}{2}}(z_1)} |\Delta \mathbf{d}^2|^2 + |\Delta(\mathbf{d}^1 - \mathbf{h}_E)|^2 + |\Delta \mathbf{h}_E|^2 \\ &\leq C(\alpha, \varepsilon_1, r_0)r^{2\alpha}.\end{aligned} \tag{2.26}$$

Hence it follows from the equation of $\mathbf{d}$ that

$$\begin{aligned}\int_{Q^+_{\frac{r}{2}}(z_1)} |\partial_t \mathbf{d}|^2 &\leq C\int_{Q^+_{\frac{r}{2}}(z_1)} (|\mathbf{u}\cdot\nabla \mathbf{d}|^2 + |\Delta \mathbf{d}|^2 + |\nabla \mathbf{d}|^4) \\ &\leq C\left(\int_{Q^+_{\frac{r}{2}}(z_1)} |\mathbf{u}|^4\right)^{\frac{1}{2}}\left(\int_{Q^+_{\frac{r}{2}}(z_1)} |\nabla \mathbf{d}|^4\right)^{\frac{1}{2}} \\ &\quad + C\int_{Q^+_{\frac{r}{2}}(z_1)} (|\Delta \mathbf{d}|^2 + |\nabla \mathbf{d}|^4) \leq C(\alpha, \varepsilon_1, r_0)r^{2\alpha}.\end{aligned} \tag{2.27}$$

Putting (2.25) together with (2.27) and applying Hölder's inequality, we conclude that

$$\frac{1}{r^2}\int_{Q^+_r(z_1)} (|\nabla \mathbf{d}|^2 + r^2|\partial_t \mathbf{d}|^2) \leq Cr^{2\alpha} \tag{2.28}$$

holds for any $z_1 \in \Gamma^+_{\frac{r_0}{2}}(0) \times [1 - \frac{r_0^2}{4}, 1]$ and $0 < r \leq \frac{r_0}{4}$.

It is clear that (2.28), combined with the interior regularity Lemma 2.5 and the parabolic Morrey's decay Lemma (see, e.g., [6]), yields that $\mathbf{d} \in C^{\alpha, \frac{\alpha}{2}}(Q^+_{\frac{r_0}{2}}(z_0), \mathbb{S}^2)$ and the estimate (2.7) holds. On the other hand, the boundary Hölder regularity of $\mathbf{u}$ and the estimate (2.8) can be established exactly as in [31] Lemma 2.2. Thus the proof of Lemma 2.6 is complete. □

Proof of Theorem 2.4 Since $\mathbf{u} \in L^\infty([0,T],\mathbf{H}) \cap L^2([0,T],\mathbf{V})$, it follows from Ladyzhenskaya's inequality that $\mathbf{u} \in L^4(Q_T,\mathbb{R}^2)$. For $\mathbf{d}$, it follows from $\mathbf{d} \in L^2([0,T],H^2(\Omega))$ and $|\mathbf{d}| = 1$ that

$$|\nabla \mathbf{d}|^2 = -\mathbf{d} \cdot \Delta \mathbf{d} \in L^2(Q_T)$$

so that $|\nabla \mathbf{d}| \in L^4(Q_T)$ and $\mathbf{u} \cdot \nabla \mathbf{u} + \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \in L^{\frac{4}{3}}(Q_T)$. From this and Lemma 2.16 below, we conclude that $\nabla P \in L^{\frac{4}{3}}(Q_T)$. By the absolute continuity of L^4-norm of $(\mathbf{u}, \nabla \mathbf{d})$, we can apply both Lemma 2.5 and Lemma 2.6 to show that for any $\alpha \in (0,1)$,

$$(\mathbf{u},\mathbf{d}) \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{\Omega} \times [\delta,T], \mathbb{R}^2 \times \mathbb{S}^2\right)$$

holds for any $0 < \delta < T$. By employing the standard higher order regularity theory, we can get the interior smoothness of $(\mathbf{u},\mathbf{d})$ in Q_T. This completes the proof. □

2.2 Existence of short time smooth solutions

In this subsection, we will establish the existence of a unique short time smooth solution to (1.1)–(1.3) for any smooth initial and boundary data. More precisely, we have

Theorem 2.7 *For any bounded, smooth domain $\Omega \subset \mathbb{R}^2$, $0 < T < \infty$ and $\alpha \in (0,1)$, let $\mathbf{h} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T,\mathbb{S}^2)$, $\mathbf{u}_0 \in C^{2,\alpha}(\overline{\Omega},\mathbb{R}^2)$ with $\nabla \cdot \mathbf{u}_0 = 0$, $\mathbf{d}_0 \in C^{2,\alpha}(\overline{\Omega},\mathbb{S}^2)$ satisfying the compatibility condition (1.5). Then there exist $0 < T_* \le T$ depending on $\|\mathbf{u}_0\|_{C^{2,\alpha}(\Omega)}$, $\|\mathbf{d}_0\|_{C^{2,\alpha}(\Omega)}$ and $\|\mathbf{h}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T)}$ and a unique solution $(\mathbf{u},\mathbf{d})$ to the system (1.1)–(1.3) such that*

$$(\mathbf{u},\mathbf{d}) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_{T_*}},\mathbb{R}^2 \times \mathbb{S}^2).$$

Furthermore, $(\mathbf{u},\mathbf{d}) \in C^\infty(Q_{T_*},\mathbb{R}^2 \times \mathbb{S}^2)$.

Proof While the idea is based on the same contraction mapping principle as in [31], there are several different treatments to handle the contributions of t-dependent boundary data $\mathbf{h}$. For the convenience of readers, we provide the details as follows. Let $\mathbf{h}_P : Q_T \mapsto \mathbb{R}^3$ solve

$$\begin{cases} \partial_t \mathbf{h}_P - \Delta \mathbf{h}_P = 0, & \text{in} \quad Q_T, \\ \mathbf{h}_P = \mathbf{h}, & \text{on} \quad \Gamma_T, \\ \mathbf{h}_P = \mathbf{d}_0, & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{2.29}$$

Also let $\widetilde{\mathbf{u}}_0 : Q_T \mapsto \mathbb{R}^2$ solve

$$\begin{cases} \partial_t \widetilde{\mathbf{u}}_0 - \Delta \widetilde{\mathbf{u}}_0 + \nabla \widetilde{P} = 0, & \text{in} \quad Q_T, \\ \nabla \cdot \widetilde{\mathbf{u}}_0 = 0, & \text{in} \quad Q_T, \\ \widetilde{\mathbf{u}}_0 = 0, & \text{on} \quad \Gamma_T, \\ \widetilde{\mathbf{u}}_0 = \mathbf{u}_0, & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{2.30}$$

For $0 < T_* \le T$ and $K > 0$ to be chosen later, define

$$\begin{aligned} \mathfrak{X}(T_*,K) = \Big\{ &(\mathbf{v},\mathbf{f}) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_{T_*}},\mathbb{R}^2 \times \mathbb{R}^3) : \ (\mathbf{v},\mathbf{f})|_{t=0} = (\mathbf{u}_0,\mathbf{d}_0), \\ &\nabla \cdot \mathbf{v} = 0, \ (\mathbf{v},\mathbf{f})\big|_{\Gamma_{T_*}} = (0,\mathbf{h}), \\ &\left\|(\mathbf{v} - \widetilde{\mathbf{u}}_0, \mathbf{f} - \mathbf{h}_P)\right\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \le K \Big\}, \end{aligned} \tag{2.31}$$

which is equipped with the norm

$$\left\|(\mathbf{v},\mathbf{f})\right\|_{\mathfrak{X}(T_*,K)} := \left\|(\mathbf{v},\mathbf{f})\right\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}, \quad \forall(\mathbf{v},\mathbf{f}) \in \mathfrak{X}(T_*,K).$$

Note that $\left(\mathfrak{X}(T_*,K), \|\cdot\|_{\mathfrak{X}(T_*,K)}\right)$ is a Banach space.

We now define the operator L by letting

$$(\mathbf{u},\mathbf{d}) := L(\mathbf{v},\mathbf{f}) : \mathfrak{X}(T_*,K) \mapsto C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_{T_*}}, \mathbb{R}^2 \times \mathbb{R}^3)$$

be a unique solution to the following non-homogeneous linear system:

$$\begin{cases} \partial_t \mathbf{u} - \Delta\mathbf{u} + \nabla P = -\mathbf{v}\cdot\nabla\mathbf{v} - \nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) & \text{in} \quad Q_{T_*},\\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad Q_{T_*},\\ \partial_t\mathbf{d} - \Delta\mathbf{d} = |\nabla\mathbf{f}|^2\mathbf{f} - \mathbf{v}\cdot\nabla\mathbf{f} & \text{in} \quad Q_{T_*},\\ (\mathbf{u},\mathbf{d}) = (0,\mathbf{h}) & \text{on} \quad \Gamma_{T_*},\\ (\mathbf{u},\mathbf{d}) = (\mathbf{u}_0,\mathbf{d}_0) & \text{in} \quad \Omega\times\{0\}. \end{cases} \tag{2.32}$$

It follows from Lemmas 2.8 and 2.9 below that if we choose a sufficiently small $T_* \in (0,T]$ and a sufficiently large $K > 0$, then $L : \mathfrak{X}(T_*,K) \mapsto \mathfrak{X}(T_*,K)$ is a contractive map so that there is a unique fixed point $(\mathbf{u},\mathbf{d}) \in \mathfrak{X}(T_*,K)$ of L, i.e. $(\mathbf{u},\mathbf{d}) = L(\mathbf{u},\mathbf{d})$. Moreover, it follows from Lemma 2.10 that $|\mathbf{d}| = 1$ in Q_{T_*}. Thus the conclusions of Theorem 2.7 hold, if we can prove Lemmas 2.8, 2.9, and 2.10 below. □

Lemma 2.8 *There exist* $0 < T_* \leq T$ *and* $K > 0$ *such that* $L : \mathfrak{X}(T_*,K) \mapsto \mathfrak{X}(T_*,K)$ *is a bounded operator.*

Proof For any $(\mathbf{v},\mathbf{f}) \in \mathfrak{X}(T_*,K)$, set $(\mathbf{u},\mathbf{d}) = L(\mathbf{v},\mathbf{f})$. Let $C_0 > 0$ denote a constant depending on $\|\mathbf{u}_0\|_{C^{2+\alpha}(\Omega)}$, $\|\mathbf{d}_0\|_{C^{2+\alpha}(\Omega)}$ and $\|\mathbf{h}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T)}$.

By the Schauder theory to the Eq. (2.29), $\mathbf{h}_P$ satisfies

$$\|\mathbf{h}_P\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \leq C\left(\|\mathbf{d}_0\|_{C^{2+\alpha}(\Omega)} + \|\mathbf{h}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_{T_*})}\right), \tag{2.33}$$

Set $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{h}_P$ and $\widetilde{\mathbf{u}} = \mathbf{u} - \widetilde{\mathbf{u}}_0$. Then (2.32) can be rewritten as

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} - \Delta\widetilde{\mathbf{u}} + \nabla P = -\mathbf{v}\cdot\nabla\mathbf{v} - \nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) & \text{in} \quad Q_{T_*},\\ \nabla\cdot\widetilde{\mathbf{u}} = 0 & \text{in} \quad Q_{T_*},\\ \partial_t\widetilde{\mathbf{d}} - \Delta\widetilde{\mathbf{d}} = |\nabla\mathbf{f}|^2\mathbf{f} - \mathbf{v}\cdot\nabla\mathbf{f} & \text{in} \quad Q_{T_*},\\ (\mathbf{u},\widetilde{\mathbf{d}}) = (0,0) & \text{on} \quad \Gamma_{T_*},\\ (\mathbf{u},\widetilde{\mathbf{d}}) = (\mathbf{u}_0,0) & \text{in} \quad \Omega\times\{0\}. \end{cases} \tag{2.34}$$

Assume $K \geq C_0$. By the Schauder theory of parabolic systems, we have

$$\left\|\widetilde{\mathbf{d}}\right\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \leq C\left(\|\mathbf{v}\cdot\nabla\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \||\nabla\mathbf{f}|^2\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\right). \tag{2.35}$$

To estimate the first term in the right-hand side, let $\widetilde{\mathbf{f}} = \mathbf{f} - \mathbf{h}_P$. Then we have

$$\begin{aligned} \left\|\mathbf{v}\cdot\nabla\mathbf{f}\right\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \leq C\Big[&\|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla\widetilde{\mathbf{f}}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|\mathbf{u}_0\cdot\nabla\widetilde{\mathbf{f}}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\ &+\|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla\mathbf{h}_P\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|\mathbf{u}_0\cdot\nabla\mathbf{h}_P\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\Big]. \end{aligned} \tag{2.36}$$

It follows from (2.33) that

$$\|\mathbf{u}_0\cdot\nabla\mathbf{h}_P\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \leq C_0.$$

Since $(\mathbf{v}-\mathbf{u}_0, \widetilde{\mathbf{f}}) = (0,0)$ in $\Omega \times \{0\}$, we have

$$\sum_{k=0}^{2} \|\nabla^k(\mathbf{v}-\mathbf{u}_0)\|_{C^0(Q_{T_*})} \le K T_*^{\frac{\alpha}{2}},$$

and

$$\sum_{k=0}^{2} \|\nabla^k \widetilde{\mathbf{f}}\|_{C^0(Q_{T_*})} = \sum_{k=0}^{2} \|\nabla^k(\mathbf{f}-\mathbf{h}_P)\|_{C^0(Q_{T_*})} \le K T_*^{\frac{\alpha}{2}}.$$

Employing the interpolation inequalities, we have that for any $0 < \delta < 1$,

$$\begin{aligned}
\|\mathbf{v}-\mathbf{u}_0\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le & C\Big(\frac{1}{\delta}\|\mathbf{v}-\mathbf{u}_0\|_{C^0(Q_{T_*})} + \delta\|\mathbf{v}-\mathbf{u}_0\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}\Big)\\
\le & C\Big(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)K,\\
\big\|\big(\widetilde{\mathbf{f}}, \nabla\widetilde{\mathbf{f}}\big)\big\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le & C\Big(\frac{1}{\delta}\|\big(\widetilde{\mathbf{f}}, \nabla\widetilde{\mathbf{f}}\big)\|_{C^0(Q_{T_*})} + \delta\|\widetilde{\mathbf{f}}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}\Big)\\
\le & C\Big(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)K.
\end{aligned}$$

Putting these estimates together, we obtain that

$$\big\|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla\widetilde{\mathbf{f}}\big\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le C T_*^{\frac{\alpha}{2}}\Big(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)K^2.$$

Similarly, we have that

$$\begin{aligned}
&\big\|\mathbf{u}_0\cdot\nabla\widetilde{\mathbf{f}}\big\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla\mathbf{h}_P\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\le C_0\Big(T_*^{\frac{\alpha}{2}} + \delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)K.
\end{aligned}$$

Putting all these estimates together, we obtain that

$$\big\|\mathbf{v}\cdot\nabla\mathbf{f}\big\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le \frac{\sqrt{K}}{4},$$

provided $K = 64C_0^2$, $\delta \le \frac{1}{16(C_0+CK)\sqrt{K}}$, and $T_*^{\frac{\alpha}{2}} = \min\{\frac{1}{2}, \delta^2\}$.

We can estimate the second term in the right-hand side of (2.35) by

$$\begin{aligned}
&\||\nabla\mathbf{f}|^2\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\le \||\nabla\widetilde{\mathbf{f}}|^2\widetilde{\mathbf{f}}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + 2\||\nabla\widetilde{\mathbf{f}}||\nabla\mathbf{h}_P||\widetilde{\mathbf{f}}|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad + \||\nabla\widetilde{\mathbf{f}}|^2|\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \||\nabla\mathbf{h}_P|^2|\widetilde{\mathbf{f}}|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad + 2\||\nabla\widetilde{\mathbf{f}}||\nabla\mathbf{h}_P||\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \||\nabla\mathbf{h}_P|^2|\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}.
\end{aligned} \tag{2.37}$$

It is easy to see that

$$\begin{aligned}
&\||\nabla\mathbf{h}_P|^2|\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le C_0,\\
&\||\nabla\widetilde{\mathbf{f}}|^2\widetilde{\mathbf{f}}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \le C T_*^{\alpha}\Big(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)K^3,
\end{aligned}$$

and

$$\||\nabla\widetilde{\mathbf{f}}||\nabla\mathbf{h}_P||\widetilde{\mathbf{f}}|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \leq C_0 T_*^{\frac{\alpha}{2}}\big(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta}\big)K^3.$$

Similarly, we have that

$$\begin{aligned}
&\||\nabla\widetilde{\mathbf{f}}|^2|\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \||\nabla\mathbf{h}_P|^2|\widetilde{\mathbf{f}}|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad + 2\||\nabla\widetilde{\mathbf{f}}||\nabla\mathbf{h}_P||\mathbf{h}_P|\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\leq C_0 T_*^{\frac{\alpha}{2}}(T_*^{\frac{\alpha}{2}} + \delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta})K^2.
\end{aligned}$$

Substituting these estimates into (2.37), we get that

$$\begin{aligned}
\||\nabla\mathbf{f}|^2\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} &\leq C_0 T_*^{\frac{\alpha}{2}}(T_*^{\frac{\alpha}{2}} + \delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta})K^2 + C T_*^{\alpha}(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta})K^3 + C_0\\
&\leq \frac{\sqrt{K}}{4},
\end{aligned}$$

provided $K = 64C_0^2$, $\delta \leq \frac{1}{32(C_0+C)K^{\frac{5}{2}}}$ and $T_*^{\frac{\alpha}{2}} = \min\{\frac{1}{2}, \delta^2\}$. Hence

$$\|\widetilde{\mathbf{d}}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \leq \frac{\sqrt{K}}{2}. \tag{2.38}$$

By the Schauder theory for non-homogeneous, non-stationary Stokes equations $(2.34)_1$, we have that

$$\begin{aligned}
\|\mathbf{u} - \widetilde{\mathbf{u}}_0\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \leq C\big[&\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&+ \|\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\big].
\end{aligned} \tag{2.39}$$

For the first term of the right-hand side of (2.39), we have

$$\begin{aligned}
&\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\leq C\|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla(\mathbf{v}-\mathbf{u}_0)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|(\mathbf{v}-\mathbf{u}_0)\cdot\nabla\mathbf{u}_0\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad + \|\mathbf{u}_0\cdot\nabla(\mathbf{v}-\mathbf{u}_0)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|\mathbf{u}_0\cdot\nabla\mathbf{u}_0\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\leq C T_*^{\frac{\alpha}{2}}(\delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta})K^2 + C_0(T_*^{\frac{\alpha}{2}} + \delta + \frac{T_*^{\frac{\alpha}{2}}}{\delta})K + C_0 \leq \frac{\sqrt{K}}{4},
\end{aligned}$$

provided $K = 64C_0^2$, $\delta \leq \frac{1}{16(C_0+CK)\sqrt{K}}$ and $T_*^{\frac{\alpha}{2}} = \min\{\frac{1}{2}, \delta^2\}$.

For the second term in the right hand side of (2.39), we have

$$
\begin{aligned}
&\|\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad\le C\big[\|\nabla\cdot(\nabla\widetilde{\mathbf{d}}\odot\nabla\widetilde{\mathbf{d}})\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\nabla\cdot(\nabla\mathbf{h}_P\odot\nabla\widetilde{\mathbf{d}})\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\qquad+\|\nabla\cdot(\nabla\widetilde{\mathbf{d}}\odot\nabla\mathbf{h}_P)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\nabla\cdot(\nabla\mathbf{h}_P\odot\nabla\mathbf{h}_P)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\big]\\
&\quad\le C_0\|\widetilde{\mathbf{d}}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+T_*^{\frac{\alpha}{2}}K\|\widetilde{\mathbf{d}}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+C_0\\
&\quad\le C_0\sqrt{K}+CT_*^{\frac{\alpha}{2}}K^2+C_0\le\frac{K}{2},
\end{aligned}
$$

provided $K=64C_0^2$ and $T_*^{\frac{\alpha}{2}}\le\min\{1,\frac{1}{8(1+CK^{\frac{1}{2}})K}\}$. Here we have used (2.38).

These two inequalities, together with (2.38) and (2.39), imply that

$$
\|\mathbf{u}-\mathbf{u}_0\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{d}-\mathbf{h}_P\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}\le K,
$$

and hence $L:\mathfrak{X}(T_*,K)\mapsto\mathfrak{X}(T_*,K)$ is bounded. The proof of Lemma 2.8 is completed. □

Lemma 2.9 *There exist a sufficiently large $K>0$ and a sufficiently small $T_*>0$ such that $L:\mathfrak{X}(T_*,K)\mapsto\mathfrak{X}(T_*,K)$ is a contractive map.*

Proof For $i=1,2$, and any given $(\mathbf{v}_i,\mathbf{f}_i)\in\mathfrak{X}(T_*,K)$, let $(\mathbf{u}_i,\mathbf{d}_i)\in\mathfrak{X}(T_*,K)$ be defined by

$$
(\mathbf{u}_i,\mathbf{d}_i)=L(\mathbf{v}_i,\mathbf{f}_i).
$$

Set

$$
(\mathbf{u},\mathbf{d},P,\mathbf{v},\mathbf{f})=(\mathbf{u}_1-\mathbf{u}_2,\mathbf{d}_1-\mathbf{d}_2,P_1-P_2,\mathbf{v}_1-\mathbf{v}_2,\mathbf{f}_1-\mathbf{f}_2).
$$

Then $(\mathbf{u},\mathbf{d})$ solves

$$
\begin{cases}
\partial_t\mathbf{u}-\Delta\mathbf{u}+\nabla P=\mathbf{G}, & \text{in}\quad Q_{T_*},\\
\nabla\cdot\mathbf{u}=0, & \text{in}\quad Q_{T_*},\\
\partial_t\mathbf{d}-\Delta\mathbf{d}=\mathbf{H}, & \text{in}\quad Q_{T_*},\\
(\mathbf{u},\mathbf{d})=(0,0), & \text{on}\quad \Gamma_{T_*}\cup\Omega\times\{0\},
\end{cases}
$$

where

$$
\mathbf{G}=-\mathbf{v}\cdot\nabla\mathbf{v}_1-\mathbf{v}_2\cdot\nabla\mathbf{v}-\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}_1+\nabla\mathbf{d}_2\odot\nabla\mathbf{d}),
$$

and

$$
\mathbf{H}=|\nabla\mathbf{f}_1|^2\mathbf{f}+\langle\nabla(\mathbf{f}_1+\mathbf{f}_2),\nabla\mathbf{f}\rangle\mathbf{f}_2-\mathbf{v}\cdot\nabla\mathbf{f}_1+\mathbf{v}_2\cdot\nabla\mathbf{f}.
$$

From Lemma 2.8, we have that

$$
\sum_{i=1}^{2}\left(\|\mathbf{u}_i-\mathbf{u}_0\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{d}_i-\mathbf{h}_P\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}\right)\le K.
$$

As in Lemma 2.8, we can apply the Schauder theory to get

$$
\begin{aligned}
&\|\mathbf{d}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}\le C\|\mathbf{H}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad\le C\||\nabla\mathbf{f}_1|^2\mathbf{f}+\langle\nabla(\mathbf{f}_1+\mathbf{f}_2),\nabla\mathbf{f}\rangle\mathbf{f}_2-\mathbf{v}\cdot\nabla\mathbf{f}_1+\mathbf{v}_2\cdot\nabla\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\\
&\quad\le C(K^2+K)(\|\mathbf{v}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\nabla\mathbf{f}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}),
\end{aligned}
\tag{2.40}
$$

and

$$\begin{aligned}\|\mathbf{u}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} &\le C\|\mathbf{G}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \\ &\le C\|\mathbf{v}\cdot\nabla\mathbf{v}_1+\mathbf{v}_2\cdot\nabla\mathbf{v}+\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}_1+\nabla\mathbf{d}_2\odot\nabla\mathbf{d})\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} \\ &\le CK(\|\mathbf{v}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\nabla\mathbf{v}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{d}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}). \end{aligned} \tag{2.41}$$

Hence, it follows from (2.40) and (2.41) that

$$\begin{aligned}\|L(\mathbf{v}_1,\mathbf{f}_1)-L(\mathbf{v}_2,\mathbf{f}_2)\|_{\mathfrak{X}(T_*,K)} &= \|\mathbf{u}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{d}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} \\ &\le CK^3\Big[\delta(\|\mathbf{v}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{f}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}) \\ &\quad+\frac{1}{\delta}(\|\mathbf{v}\|_{C^0(Q_{T_*})}+\|\mathbf{f}\|_{C^0(Q_{T_*})})\Big] \\ &\le CK^3\Big(\delta+\frac{T_*^{\frac{\alpha}{2}}}{\delta}\Big)(\|\mathbf{v}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{f}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}) \\ &\le \frac{1}{2}(\|\mathbf{v}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}+\|\mathbf{f}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}) \\ &\le \frac{1}{2}\|(\mathbf{v}_1,\mathbf{f}_1)-(\mathbf{v}_2,\mathbf{f}_2)\|_{\mathfrak{X}(T_*,K)},\end{aligned}$$

provided δ and T_* are sufficiently small. Thus we obtain that $L:\mathfrak{X}(T_*,K)\mapsto\mathfrak{X}(T_*,K)$ is a contractive map. This completes the proof of Lemma 2.9. □

Lemma 2.10 *For a bounded smooth domain $\Omega\subset\mathbb{R}^2$ and $0<T<\infty$, let $\mathbf{u}\in W_2^{2,1}(Q_T,\mathbb{R}^2)$ with $\nabla\cdot\mathbf{u}=0$, $\mathbf{h}\in C^{\alpha,\frac{\alpha}{2}}(\Gamma_T,\mathbb{S}^2)$, and $\mathbf{d}_0\in C^{2+\alpha}(\Omega;\mathbb{S}^2)$. If $\mathbf{d}\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T,\mathbb{R}^3)$ is a solution of (1.1)$_3$–(1.2)–(1.3), then it holds that*

$$|\mathbf{d}|=1\quad \textit{in } Q_T.$$

Proof Multiplying (1.1)$_3$ by $\mathbf{d}$, we get that

$$\partial_t(|\mathbf{d}|^2-1)+\mathbf{u}\cdot\nabla(|\mathbf{d}|^2-1)=\Delta(|\mathbf{d}|^2-1)+2|\nabla\mathbf{d}|^2(|\mathbf{d}|^2-1).$$

Set $g=|\mathbf{d}|^2-1$ and $g^+=\max\{g,0\}$. We have that

$$\begin{cases}\partial_t g^+-\Delta g^+=-\mathbf{u}\cdot\nabla g^++2|\nabla\mathbf{d}|^2g^+, & \text{in} \quad Q_T,\\ g^+=\max\{|\mathbf{h}|^2-1,0\}=0 & \text{on} \quad \Gamma_T,\\ g^+=\max\{|\mathbf{d}_0|^2-1,0\}=0 & \text{on} \quad \Omega\times\{0\}.\end{cases} \tag{2.42}$$

Multiplying (2.42) with g^+, integrating the resulting equation over Ω, and using the fact that $\nabla\cdot\mathbf{u}=0$, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(g^+)^2+\int_\Omega|\nabla g^+|^2 &= -\frac{1}{2}\int_\Omega\mathbf{u}\cdot\nabla(g^+)^2+2\int_\Omega|\nabla\mathbf{d}|^2(g^+)^2\\ &=2\int_\Omega|\nabla\mathbf{d}|^2(g^+)^2.\end{aligned}$$

Integrating over $[0,t]$ for $0<t\le T$, and employing the fact $\|\nabla\mathbf{d}\|_{L^\infty(Q_T)}<\infty$ and the Gronwall inequality, we obtain that

$$g^+=0\quad \text{in } Q_T.$$

This implies $|\mathbf{d}| \le 1$ in Q_T. Similarly, we can show that $|\mathbf{d}| \ge 1$ in Q_T. Hence $|\mathbf{d} = 1$ in Q_T. This completes the proof. $\square$

In order to construct the existence of global strong solutions to (1.1)–(1.3), we also need the following Lemma.

Lemma 2.11 *For $T > 0$ and a bounded smooth domain $\Omega \subset \mathbb{R}^2$, for a given $\mathbf{u} \in W_2^{2,1}(Q_T, \mathbb{R}^2)$ with $\nabla \cdot \mathbf{u} = 0$, $\mathbf{h} \in C^{\alpha,\frac{\alpha}{2}}(\Gamma_T, \mathbb{S}^2))$ and $\mathbf{d}_0 \in C^{2+\alpha}(\Omega, \mathbb{S}^2)$, let $\mathbf{d} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [0,T], \mathbb{S}^2)$ solve (1.1)$_3$–(1.2)–(1.3). If $\mathbf{d}_0^3 \ge 0$ in Ω and $\mathbf{h}_3 \ge 0$ on Γ_T, then*

$$\mathbf{d}^3(x,t) \ge 0 \quad \text{in } Q_T.$$

Here $\mathbf{d}^3$ denotes the third component of $\mathbf{d}$.

Proof The proof is similar to that of Lemma 2.10. For the convenience of readers, we sketch it here. Set $\mathbf{d}_-^3 = \min\{\mathbf{d}^3, 0\}$. Then

$$\begin{cases} \partial_t \mathbf{d}_-^3 - \Delta \mathbf{d}_-^3 = -\mathbf{u} \cdot \nabla \mathbf{d}_-^3 + |\nabla \mathbf{d}|^2 \mathbf{d}_-^3, & \text{in} \quad Q_T, \\ \mathbf{d}_-^3 = 0, & \text{on} \quad \partial_p Q_T. \end{cases}$$

Multiplying this equation by $\mathbf{d}_-^3$, integrating over Ω, and applying $\nabla \cdot \mathbf{u} = 0$, we obtain

$$\frac{d}{dt} \int_\Omega |\mathbf{d}_-^3|^2 + \int_\Omega |\nabla \mathbf{d}_-^3|^2 = 2 \int_\Omega |\nabla \mathbf{d}|^2 |\mathbf{d}_-^3|^2 \le C \int_\Omega |\mathbf{d}_-^3|^2.$$

Hence by the Gronwall inequality we have

$$\int_\Omega |\mathbf{d}_-^3(t)|^2 \le e^{Ct} \int_\Omega |(\mathbf{d}_0^3)_-|^2 = 0, \ \forall t \in [0,T].$$

This implies that $\mathbf{d}^3 \ge 0$ in Q_T. $\square$

In the process to obtain global strong solutions, we also need the following elementary Lemma.

Lemma 2.12 *If $\omega \in C^\infty(\mathbb{S}^2, \mathbb{S}_+^2)$ is a harmonic map, then ω must be a constant map.*

Proof Recall that ω solves the harmonic map equation:

$$\Delta_{\mathbb{S}^2} \omega + |\nabla_{\mathbb{S}^2} \omega|^2 \omega = 0 \text{ on } \mathbb{S}^2. \tag{2.43}$$

Here $\nabla_{\mathbb{S}^2}$ and $\Delta_{\mathbb{S}^2}$ denote the gradient and Laplace operator on $\mathbb{S}^2$ respectively. Integrating the equation over $\mathbb{S}^2$ yields

$$0 = \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} \omega^3 + |\nabla_{\mathbb{S}^2} \omega|^2 \omega^3)\, d\sigma = \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} \omega|^2 \omega^3 \, d\sigma.$$

Since $\omega^3 \ge 0$ on $\mathbb{S}^2$, this implies that

$$|\nabla_{\mathbb{S}^2} \omega|^2 \omega^3 \equiv 0 \text{ on } \mathbb{S}^2.$$

There are two cases that we need to consider:
(a) If there exists $p_0 \in \mathbb{S}^2$ such that $\omega^3(p_0) > 0$, then there exists $\delta_0 > 0$ such that

$$\nabla_{\mathbb{S}^2} \omega = 0 \text{ in } B_{\delta_0}(p_0) \cap \mathbb{S}^2,$$

and hence $\omega \equiv p_1 \in \mathbb{S}^2_+$ in $B_{\delta_0}(p_0) \cap \mathbb{S}^2$. This, combined with the unique continuation property, yields that $\omega \equiv p_1$ on $\mathbb{S}^2$.
(b) If $\omega^3 \equiv 0$ on $\mathbb{S}^2$, then we must have $\omega(\mathbb{S}^2) \subset \partial\mathbb{S}^2_+ \equiv \mathbb{S}^1 \subset \mathbb{R}^2$. In this case, we can write $\omega = e^{i\phi}$ for a smooth function $\phi \in C^\infty(\mathbb{S}^2)$. Direct calculations imply that ω is a harmonic map to $\mathbb{S}^1$ if and only if ϕ is a harmonic function on $\mathbb{S}^2$. Hence by the maximum principle we conclude that ϕ is a constant. Hence $\omega = e^{i\phi}$ is also a constant on $\mathbb{S}^2$. □

2.3 A priori estimates on energy and pressure

In this subsection, we will provide some basic estimates on both the energy and the pressure. First, we have the following generalized global energy inequality.

Lemma 2.13 *For $T > 0$, $\mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T, \mathbb{S}^2))$, $\partial_t \mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T)$, and $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H} \times H^1(\Omega, \mathbb{S}^2)$, suppose $\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{V})$, $\mathbf{d} \in L^\infty_t H^1_x(Q_T, \mathbb{S}^2)$ and $P \in L^{\frac{4}{3}}_t W^{1,\frac{4}{3}}_x(Q_T)$ is a weak solution to the system (1.1)–(1.3). Then there exists $C > 0$ depending only on Ω such that for any $t \in (0, T]$,*

$$\begin{aligned}
&\int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t) + \int_{Q_t} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2) \\
&\quad \le \psi(t)\Big[\int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) + C \int_0^t \|(\mathbf{h}, \partial_t \mathbf{h})(\tau)\|^2_{H^{\frac{3}{2}}(\Gamma)}\, d\tau\Big],
\end{aligned} \tag{2.44}$$

where

$$\psi(t) = \exp\Big(C \int_0^t \|\partial_t \mathbf{h}(\tau)\|_{H^{\frac{3}{2}}(\Gamma)}\, d\tau\Big).$$

Proof Let $\mathbf{h}_E \in L^2_t H^2_x(Q_T, \mathbb{R}^3)$ be the harmonic extension of $\mathbf{h}$, i.e., for all $t \in (0, T]$,

$$\begin{cases} \Delta \mathbf{h}_E(\cdot, t) = 0 & \text{in} \quad \Omega, \\ \mathbf{h}_E(\cdot, t) = \mathbf{h}(\cdot, t) & \text{on} \quad \Gamma. \end{cases}$$

Then we have that $\mathbf{h}_E, \partial_t \mathbf{h}_E \in L^2_t H^2_x(Q_T)$, and

$$\begin{cases} \|\mathbf{h}_E\|_{L^2_t H^2_x(Q_T)} \le C\|\mathbf{h}_E\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}, \\ \|\partial_t \mathbf{h}_E\|_{L^2_t H^2_x(Q_T)} \le C\|\partial_t \mathbf{h}_E\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}. \end{cases} \tag{2.45}$$

Multiplying $(1.1)_1$ by $\mathbf{u}$, integrating over Ω, and using (1.2), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 + \int_\Omega |\nabla \mathbf{u}|^2 = -\int_\Omega \langle \mathbf{u} \cdot \nabla \mathbf{d}, \Delta \mathbf{d} \rangle, \tag{2.46}$$

Multiplying $(1.1)_3$ by $\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}$ and integrating over Ω yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \mathbf{d}|^2 + \int_\Omega |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 \\
&\quad = \int_\Omega \langle \mathbf{u} \cdot \nabla \mathbf{d}, \Delta \mathbf{d} \rangle + \int_\Gamma \langle \frac{\partial \mathbf{d}}{\partial \nu}, \partial_t \mathbf{h} \rangle,
\end{aligned} \tag{2.47}$$

where ν is the outward unit normal vector of Γ.

Adding (2.46) with (2.47), we have that

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2)+\int_\Omega (|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2)=\int_\Gamma \langle \frac{\partial \mathbf{d}}{\partial \nu}, \partial_t \mathbf{h}\rangle.$$

Now we estimate the right hand side as follows

$$\int_\Gamma \left\langle \frac{\partial \mathbf{d}}{\partial \nu}, \partial_t \mathbf{h}\right\rangle = \int_\Gamma \left\langle \frac{\partial(\mathbf{d}-\mathbf{h}_E)}{\partial \nu}, \partial_t \mathbf{h}_E\right\rangle + \int_\Gamma \left\langle \frac{\partial \mathbf{h}_E}{\partial \nu}, \partial_t \mathbf{h}_E\right\rangle = I + II. \tag{2.48}$$

It is easy to see that

$$\begin{aligned}|II| &\le C\|\frac{\partial \mathbf{h}_E}{\partial \nu}\|_{H^{\frac{1}{2}}(\Gamma)}\|\partial_t \mathbf{h}\|_{H^{-\frac{1}{2}}(\Gamma)}\\ &\le C\big(\|\mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}+\|\partial_t \mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}\big).\end{aligned}$$

While, by the second Green identity, we have

$$\begin{aligned}I &= \int_\Gamma \big\langle \mathbf{d}-\mathbf{h}_E, \frac{\partial}{\partial \nu}(\partial_t \mathbf{h}_E)\big\rangle\\ &\quad + \int_\Omega \langle \Delta(\mathbf{d}-\mathbf{h}_E), \partial_t \mathbf{h}_E\rangle - \int_\Omega \langle \mathbf{d}-\mathbf{h}_E, \Delta(\partial_t \mathbf{h}_E)\rangle\\ &= \int_\Omega \langle \Delta \mathbf{d}, \partial_t \mathbf{h}_E\rangle = \int_\Omega \langle \Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}, \partial_t \mathbf{h}_E\rangle - \int_\Omega \langle |\nabla \mathbf{d}|^2\mathbf{d}, \partial_t \mathbf{h}_E\rangle\\ &\le \frac{1}{2}\int_\Omega |\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2 + C\big(\int_\Omega |\partial_t \mathbf{h}_E|^2 + \|\partial_t \mathbf{h}_E\|_{L^\infty(\Omega)}\int_\Omega |\nabla \mathbf{d}|^2\big).\end{aligned}$$

For any $t \in (0, T)$, it follows from Sobolev's embedding theorem that

$$\|\partial_t \mathbf{h}_E(\cdot,t)\|_{L^\infty(\Omega)} \le C\|\partial_t \mathbf{h}_E(\cdot,t)\|_{H^2(\Omega)} \le C\|\partial_t \mathbf{h}(\cdot,t)\|_{H^{\frac{3}{2}}(\Gamma)},$$

while

$$\int_\Omega |\partial_t \mathbf{h}_E(\cdot,t)|^2 \le \|\partial_t \mathbf{h}_E(\cdot,t)\|^2_{H^2(\Omega)} \le C\|\partial_t \mathbf{h}(\cdot,t)\|^2_{H^{\frac{3}{2}}(\Gamma)}.$$

Substituting these two estimates into I and then adding the resulting inequality with II, we obtain that

$$\begin{aligned}\int_\Gamma \langle \frac{\partial \mathbf{d}}{\partial \nu}, \partial_t \mathbf{h}\rangle &\le \frac{1}{2}\int_\Omega |\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2 + C\big(\|\mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}+\|\partial_t \mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}\big)\\ &\quad + C\|\partial_t \mathbf{h}(\cdot,t)\|_{H^{\frac{3}{2}}(\Gamma)}\int_\Omega |\nabla \mathbf{d}|^2.\end{aligned}$$

Putting this estimate into (2.48), we achieve

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{1}{2}(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2)+\frac{1}{2}\int_\Omega (|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2)\\ &\quad \le C\big(\|\mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}+\|\partial_t \mathbf{h}\|^2_{H^{\frac{3}{2}}(\Gamma)}\big) + C\|\partial_t \mathbf{h}(\cdot,t)\|_{H^{\frac{3}{2}}(\Gamma)}\int_\Omega |\nabla \mathbf{d}|^2.\end{aligned}$$

Integrating this inequlaity over $[0, t]$ and applying the Gronwall's inequality yields (2.44). This completes the proof. □

Next we will establish both interior and boundary generalized local energy inequalities for the system (1.1)–(1.3). More precisely,

Lemma 2.14 *For $T > 0$, assume $\mathbf{u} \in L^\infty([0,T],\mathbf{H}) \cap L^2([0,T],\mathbf{V})$, $\mathbf{d} \in L_t^\infty H_x^1(Q_T,\mathbb{S}^2) \cap L_t^2 H_x^2(Q_T,\mathbb{S}^2)$, and $P \in L_t^{\frac{4}{3}} W_x^{1,\frac{4}{3}}(Q_T)$ is a weak solution to the system (1.1)–(1.3). Then, for any nonnegative $\phi \in C_0^\infty(\Omega)$ and $0 < s < t \le T$, it holds that*

$$\begin{aligned}
&\int_\Omega \phi(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2)(\cdot,t) + 2\int_s^t\int_\Omega \phi(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2\mathbf{d}|^2)\\
&\quad\le \int_\Omega \phi(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot,s)\\
&\qquad + \int_s^t\int_\Omega |\nabla\phi|(|\mathbf{u}|^3+2|\nabla\mathbf{u}||\mathbf{u}|+2|P-P_\Omega||\mathbf{u}|+|\nabla\mathbf{d}|^2|\mathbf{u}|+2|\partial_t\mathbf{d}||\nabla\mathbf{d}|),
\end{aligned}\tag{2.49}$$

where $P_\Omega = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega P$ is the average of P over Ω.

Proof This proof is exactly same as that of [31] Lemma 4.2. For reader's convenience, we sketch it here. Multiplying $(1.1)_1$ by $\mathbf{u}\phi$ and integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2\phi + \int_\Omega |\nabla\mathbf{u}|^2\phi\\
&= \int_\Omega (\frac{1}{2}|\mathbf{u}|^2\mathbf{u} - \langle\nabla\mathbf{u},\mathbf{u}\rangle + (P-P_\Omega)\mathbf{u} + \frac{1}{2}|\nabla\mathbf{d}|^2\mathbf{u})\cdot\nabla\phi - \langle\mathbf{u}\cdot\nabla\mathbf{d},\Delta\mathbf{d}\rangle\phi.
\end{aligned}\tag{2.50}$$

On the other hand, multiplying $(1.1)_3$ by $-(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})\phi$ and integrating over Ω implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mathbf{d}|^2\phi + \int_\Omega |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\phi\\
&= \int_\Omega \langle\mathbf{u}\cdot\nabla\mathbf{d},\Delta\mathbf{d}\rangle\phi + \langle\partial_t\mathbf{d},\nabla\mathbf{d}\rangle\cdot\nabla\phi.
\end{aligned}\tag{2.51}$$

Adding (2.50) with (2.51), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)\phi + 2\int_\Omega (|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2)\phi\\
&\quad\le \int_\Omega |\nabla\phi|(|\mathbf{u}|^3 + 2|\nabla\mathbf{u}||\mathbf{u}| + 2|P-P_\Omega||\mathbf{u}| + |\nabla\mathbf{d}|^2|\mathbf{u}| + 2|\partial_t\mathbf{d}||\nabla\mathbf{d}|).
\end{aligned}$$

(2.49) follows by integrating this inequality over $[s,t]$. □

Next we will state the local generalized boundary energy inequality, whose proof is more delicate than [31] Lemma 4.3.

Lemma 2.15 *For $T > 0$, $\mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T,\mathbb{S}^2))$, $\partial_t\mathbf{h} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)$ and $(\mathbf{u}_0,\mathbf{d}_0) \in \mathbf{H}\times H^1(\Omega,\mathbb{S}^2)$, assume $\mathbf{u} \in L^\infty([0,T],\mathbf{H}) \cap L^2([0,T],\mathbf{V})$, $\mathbf{d} \in L_t^\infty H_x^1(Q_T,\mathbb{S}^2) \cap L_t^2 H_x^2(Q_T,\mathbb{S}^2)$, and $P \in L_t^{\frac{4}{3}} W_x^{1,\frac{4}{3}}(Q_T)$ is a weak solution to the system (1.1)–(1.3). There exists $r_0 = r_0(\Gamma) > 0$ such that for any $x_0 \in \Gamma$, $0 < r \le r_0$, $0 < s < t \le T$, if*

$0 \le \phi \in C_0^\infty(B_r(x_0))$ *then*

$$\begin{aligned}
&\int_{B_r^+(x_0)} \phi(|\mathbf{u}|^2+|\nabla\widehat{\mathbf{d}}|^2)(\cdot,t) + \int_s^t \int_{B_r^+(x_0)} \phi(|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2) \\
&\le \int_{B_r^+(x_0)} \phi(|\mathbf{u}|^2+|\nabla\widehat{\mathbf{d}}|^2)(\cdot,s) + \int_s^t \int_{B_r^+(x_0)} (|\nabla\mathbf{d}|^2|\partial_t\mathbf{h}_E| + |\partial_t\mathbf{h}_E|^2)\phi \\
&\quad + \int_s^t \int_{B_r^+(x_0)} |\nabla\phi|[|\mathbf{u}|(|\mathbf{u}|^2 + |\nabla\mathbf{u}| + |P-P_\Omega| + |\nabla\mathbf{d}|^2) + |\partial_t\widehat{\mathbf{d}}||\nabla\widehat{\mathbf{d}}|],
\end{aligned} \tag{2.52}$$

where $\mathbf{h}_E(\cdot,t)$ *is the harmonic extension of* $\mathbf{h}(\cdot,t)$ *for* $0 < t \le T$, *and* $\widehat{\mathbf{d}} = \mathbf{d} - \mathbf{h}_E$.

Proof Multiplying $(1.1)_1$ by $\mathbf{u}\phi$, integrating over $B_r^+(x_0)$, and using $\mathbf{u}\phi = 0$ on $\partial B_r^+(x_0)$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{B_r^+(x_0)} |\mathbf{u}|^2\phi + \int_{B_r^+(x_0)} |\nabla\mathbf{u}|^2\phi \\
&= \int_{B_r^+(x_0)} (\frac{1}{2}|\mathbf{u}|^2\mathbf{u} - \langle\nabla\mathbf{u},\mathbf{u}\rangle + (P-P_\Omega)\mathbf{u} + \frac{1}{2}|\nabla\mathbf{d}|^2\mathbf{u})\cdot\nabla\phi \\
&\quad - \langle\mathbf{u}\cdot\nabla\mathbf{d}, \Delta\mathbf{d}\rangle\phi.
\end{aligned} \tag{2.53}$$

Let $\widehat{\mathbf{d}} = \mathbf{d} - \mathbf{h}_E$. Then

$$\partial_t\widehat{\mathbf{d}}\phi = 0 \text{ on } \partial B_r^+(x_0),$$

and

$$\begin{aligned}
&-\int_{B_r^+(x_0)} \langle\partial_t\mathbf{d}, \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\rangle\phi = -\int_{B_r^+(x_0)} \langle\partial_t\mathbf{d}, \Delta\mathbf{d}\rangle\phi \\
&= -\int_{B_r^+(x_0)} \langle\partial_t\widehat{\mathbf{d}} + \partial_t\mathbf{h}_E, \Delta\mathbf{d}\rangle\phi \\
&= -\int_{B_r^+(x_0)} \langle\partial_t\widehat{\mathbf{d}}, \Delta\widehat{\mathbf{d}}\rangle\phi - \int_{B_r^+(x_0)} \langle\partial_t\mathbf{h}_E, \Delta\mathbf{d}\rangle\phi \\
&= \frac{1}{2}\frac{d}{dt}\int_{B_r^+(x_0)} |\nabla\widehat{\mathbf{d}}|^2\phi + \int_{B_r^+(x_0)} \langle\partial_t\widehat{\mathbf{d}}, \nabla\widehat{\mathbf{d}}\rangle\cdot\nabla\phi - \int_{B_r^+(x_0)} \langle\partial_t\mathbf{h}_E, \Delta\mathbf{d}\rangle\phi.
\end{aligned}$$

Hence, after multiplying $(1.1)_3$ by $-(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})\phi$ and integrating over $B_r^+(x_0)$, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{B_r^+(x_0)} |\nabla\widehat{\mathbf{d}}|^2\phi + \int_{B_r^+(x_0)} |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\phi \\
&= \int_{B_r^+(x_0)} [\langle\mathbf{u}\cdot\nabla\mathbf{d}, \Delta\mathbf{d}\rangle\phi - \langle\partial_t\widehat{\mathbf{d}}, \nabla\widehat{\mathbf{d}}\rangle\cdot\nabla\phi + \langle\partial_t\mathbf{h}_E, \Delta\mathbf{d}\rangle\phi] \\
&= \int_{B_r^+(x_0)} [\langle\mathbf{u}\cdot\nabla\mathbf{d}, \Delta\mathbf{d}\rangle\phi - \langle\partial_t\widehat{\mathbf{d}}, \nabla\widehat{\mathbf{d}}\rangle\cdot\nabla\phi + \langle\partial_t\mathbf{h}_E, \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\rangle\phi] \\
&\quad - \int_{B_r^+(x_0)} \langle\partial_t\mathbf{h}_E, |\nabla\mathbf{d}|^2\mathbf{d}\rangle\phi.
\end{aligned} \tag{2.54}$$

It is readily seen that (2.52) follows by adding (2.53) with (2.54) and applying Hölder's inequality. The proof of Lemma 2.15 is now complete. □

We also need the following Lemma on the estimate of pressure function P that is assumed in both Lemmas 2.5 and 2.6.

Lemma 2.16 *For $T > 0$, assume $\mathbf{u} \in L^\infty([0,T],\mathbf{H}) \cap L^2([0,T],\mathbf{V})$, $\mathbf{d} \in L_t^\infty H_x^1(Q_T,\mathbb{S}^2) \cap L_t^2 H_x^2(Q_T,\mathbb{S}^2)$, and $P \in L_t^{\frac{4}{3}} W_x^{1,\frac{4}{3}}(Q_T)$ is a weak solution to the system (1.1)–(1.3). Then it holds that for any $0 < t \leq T$,*

$$\max\left\{\|\nabla P\|_{L^{\frac{4}{3}}(Q_t)}, \|P - P_\Omega\|_{L_t^{\frac{4}{3}} L_x^4(Q_t)}\right\}$$
$$\leq C\left(\|\mathbf{u}\|_{L^4(Q_t)}\|\nabla \mathbf{u}\|_{L^2(Q_t)} + \|\nabla \mathbf{d}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}\|_{L^2(Q_t)}\right).$$

2.4 Proof of Theorem 2.1

In this subsection, we will establish the existence of a global weak solution to (1.1)–(1.3). Let us first recall the following version of Ladyzhenskaya's inequality (see Struwe [40] Lemma 3.1).

Lemma 2.17 *There exist $M_0 > 0$ and $r_0 > 0$ depending only on Ω such that for any $T > 0$, if $f \in L^\infty([0,T],L^2(\Omega)) \cap L^2([0,T],H^1(\Omega))$ then for $r \in (0,r_0)$ it holds that for any $0 < t \leq T$*

$$\int_{Q_t} |f|^4 \leq M_0 \sup_{(x,t)\in Q_t} \int_{\Omega\cap B_r(x)} |f|^2 \Big(\int_{Q_t} |\nabla f|^2 + \frac{1}{r^2}\int_{Q_t} |f|^2\Big).$$

Next we will show a lower bound estimate of the lift span of the short time smooth solutions in terms of the local energy profile of the initial and boundary data. More precisely, we have

Lemma 2.18 *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $0 < T < +\infty$, $\mathbf{u}_0 \in C^{2,\alpha}(\overline{\Omega},\mathbb{R}^2)$, $\mathbf{d}_0 \in C^{2,\alpha}(\overline{\Omega},\mathbb{S}^2)$, and $\mathbf{h} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T,\mathbb{S}^2)$ satisfy (1.5). Let $\varepsilon_0 > 0$ be the smaller constant given by Lemmas 2.5 and 2.6. Then there exist $0 < \varepsilon_1 < \varepsilon_0$ and*

$$0 < \theta_0 = \theta_0\big(\varepsilon_1, \|(\mathbf{u}_0,\nabla \mathbf{d}_0)\|_{L^2(\Omega)}, \|(\mathbf{h},\partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big)$$

such that if $0 < r_0 < \varepsilon_1^4$ satisfies

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{2r_0}(x)} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \leq \varepsilon_1^2,$$

then there exist $T_0 \geq \theta_0 r_0^2$ and a unique solution

$$(\mathbf{u},\mathbf{d}) \in C^\infty(Q_{T_0},\mathbb{R}^2\times\mathbb{S}^2) \cap C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T_0],\mathbb{R}^2\times\mathbb{S}^2)$$

to the system (1.1)–(1.3). Furthermore, it holds that

$$\sup_{(x,t)\in\overline{\Omega}\times[0,T_0]} \int_{\Omega\cap B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot,t) \leq 2\varepsilon_1^2. \tag{2.55}$$

Proof Since $\mathbf{h} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T,\mathbb{S}^2)$ and $(\mathbf{u}_0,\mathbf{d}_0) \in C^{2,\alpha}(\overline{\Omega},\mathbb{R}^2\times\mathbb{S}^2)$, Theorem 2.7 implies that there exist $0 < T_0 \leq T$ and a unique smooth solution

$$(\mathbf{u},\mathbf{d}) \in C^\infty(Q_{T_0},\mathbb{R}^2\times\mathbb{S}^2) \cap C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T_0],\mathbb{R}^2\times\mathbb{S}^2)$$

to the system (1.1)–(1.3). Let $0 < t_0 \le T_0$ be the maximal time such that

$$\sup_{0\le t\le t_0} \sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t) \le 2\varepsilon_1^2. \tag{2.56}$$

Then we must have that

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t_0) = 2\varepsilon_1^2.$$

In what follows, we will estimate the lower bound of t_0. Without loss of generality, we may assume $t_0 \le r_0^2$. Denote by

$$\mathcal{E}(t) = \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t) \text{ for } 0 < t \le T, \text{ and } \mathcal{E}_0 = \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2).$$

From Lemma 2.13, we have that for $0 < t \le t_0$

$$\begin{aligned}\mathcal{E}(t) + \int_{Q_t} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2) &\le \psi(T)\big(\mathcal{E}_0 + C\|(\mathbf{h}, \partial_t \mathbf{h})\|^2_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big) \\ &\le C(T),\end{aligned} \tag{2.57}$$

where $C(T) > 0$ depends on T, $\mathcal{E}_0$, $\|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}$, and

$$\psi(T) \equiv \exp\Big(C\int_0^T \|\partial_t \mathbf{h}(\tau)\|_{H^{\frac{3}{2}}(\Gamma)}\, d\tau\Big).$$

Hence by Lemma 2.17 and (2.57) we have that for $0 < t \le t_0 \le r_0^2$,

$$\begin{aligned}\int_{Q_t} |\nabla \mathbf{d}|^4 &\le M_0 \sup_{(x,\tau)\in Q_t} \int_{\Omega\cap B_{r_0}(x)} |\nabla \mathbf{d}|^2(\tau)\Big(\int_{Q_t} |\Delta \mathbf{d}|^2 + \frac{1}{r_0^2}\int_{Q_t} |\nabla \mathbf{d}|^2\Big) \\ &\le 2M_0\varepsilon_1^2\Big(\int_{Q_t} |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 + \int_{Q_t} |\nabla \mathbf{d}|^4 + \frac{C(T)t}{r_0^2}\Big) \\ &\le CM_0\varepsilon_1^2\Big(C(T) + \int_{Q_t} |\nabla \mathbf{d}|^4\Big),\end{aligned}$$

which implies that

$$\int_{Q_t} |\nabla \mathbf{d}|^4 \le \frac{C(T)\varepsilon_1^2}{1 - C(T)\varepsilon_1^2} \le C(T)\varepsilon_1^2, \tag{2.58}$$

provided $0 < \varepsilon_1^2 < \frac{1}{2C(T)}$.

It follows from (2.58) and (2.57) that

$$\int_{Q_t} |\nabla^2 \mathbf{d}|^2 \le C(T), \ \forall\, t \in (0, T]. \tag{2.59}$$

On the other hand, we can estimate

$$\begin{aligned}\int_{Q_t} |\mathbf{u}|^4 &\le M_0 \sup_{(x,\tau)\in Q_t} \int_{\Omega\cap B_{r_0}(x)} |\mathbf{u}|^2(\tau)\Big(\int_{Q_t} |\nabla \mathbf{u}|^2 + \frac{1}{r_0^2}\int_{Q_t} |\mathbf{u}|^2\Big) \\ &\le 2M_0\varepsilon_1^2\Big(\int_{Q_t} |\nabla \mathbf{u}|^2 + \frac{C(T)t}{r_0^2}\Big) \le C(T)\varepsilon_1^2.\end{aligned} \tag{2.60}$$

It follows from (2.57), (2.58), (2.59), (2.60), and Lemma 2.16 that

$$\left\| P - P_\Omega \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(Q_t)} \leq C(T)\varepsilon_1^{\frac{1}{2}}, \ \forall\, t \in (0, T]. \tag{2.61}$$

From $\partial_t \mathbf{d} = -\mathbf{u} \cdot \nabla \mathbf{d} + (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d})$ and (2.57), (2.58), (2.60), we have that

$$\begin{aligned} \left\| \partial_t \mathbf{d} \right\|_{L^2(Q_t)} &\leq C\big(\|\mathbf{u}\|_{L^4(Q_t)} \|\nabla \mathbf{d}\|_{L^4(Q_t)} + \|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\|_{L^2(Q_t)}\big) \\ &\leq C(T), \ \forall\, t \in (0, T]. \end{aligned} \tag{2.62}$$

Now we are ready to refine the estimate of the quantity

$$\max_{x \in \overline{\Omega}} \int_{\Omega \cap B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(t), \ 0 \leq t \leq t_0.$$

To do it, for any $x \in \overline{\Omega}$, let $\phi \in C_0^\infty(B_{2r_0}(x))$ be a cut-off function of $B_{r_0}(x)$ such that

$$0 \leq \phi \leq 1; \ \phi = 1 \text{ in } B_{r_0}(x); \ \phi = 0 \text{ outside } B_{2r_0}(x); \ \text{ and } |\nabla \phi| \leq \frac{4}{r_0}.$$

Applying Lemma 2.14, we see that for any $B_{2r_0}(x) \subset \Omega$, it holds that

$$\begin{aligned} &\sup_{0 \leq t \leq t_0} \int_{B_{r_0}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) - \int_{B_{2r_0}(x)} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \\ &\leq \sup_{0 \leq t \leq t_0} \int_{B_{2r_0}(x)} \phi(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) - \int_{B_{2r_0}(x)} \phi(|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \\ &\leq C \int_0^{t_0} \int_{B_{2r_0}(x)} |\nabla \phi|(|\mathbf{u}|^3 + |\nabla \mathbf{u}||\mathbf{u}| + |P - P_\Omega||\mathbf{u}| \\ &\quad + |\nabla \mathbf{d}|^2 |\mathbf{u}| + |\partial_t \mathbf{d}||\nabla \mathbf{d}|) \\ &\leq C(\frac{t_0}{r_0^2})^{\frac{1}{4}} \Big[\|\mathbf{u}\|^3_{L^4(Q_{t_0})} + \|\nabla \mathbf{u}\|_{L^2(Q_{t_0})} \|\mathbf{u}\|_{L^4(\Omega_{t_0})} \\ &\quad + \|\nabla \mathbf{d}\|^2_{L^4(Q_{t_0})} \|\mathbf{u}\|_{L^4(Q_{t_0})} + \|\partial_t \mathbf{d}\|_{L^2(Q_{t_0})} \|\nabla \mathbf{d}\|_{L^4(Q_{t_0})} \\ &\quad + \left\| P - P_\Omega \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(Q_{t_0})} \|\mathbf{u}\|_{L_t^\infty L_x^2(B_{2r_0}(x) \times [0, t_0])} \Big] \\ &\leq C(\frac{t_0}{r_0^2})^{\frac{1}{4}} \varepsilon_1^{\frac{1}{2}}, \end{aligned} \tag{2.63}$$

where we have used (2.56), (2.57), (2.58), (2.60), (2.61), and (2.62) in the last step.

For $B_{2r_0}(x_0)\cap\Gamma\neq\emptyset$, we can apply (2.52) of Lemma 2.15 to get that

$$\begin{aligned}
&\sup_{0\le t\le t_0}\int_{\Omega\cap B_{r_0}(x_0)}(|\mathbf{u}|^2+|\nabla\widehat{\mathbf{d}}|^2)-\int_{\Omega\cap B_{2r_0}(x_0)}(|\mathbf{u}_0|^2+|\nabla\widehat{\mathbf{d}}_0|^2)\\
&\le\sup_{0\le t\le t_0}\int_{\Omega\cap B_{2r_0}(x_0)}\phi(|\mathbf{u}|^2+|\nabla\widehat{\mathbf{d}}|^2)-\int_{\Omega\cap B_{2r_0}(x_0)}\phi(|\mathbf{u}_0|^2+|\nabla\widehat{\mathbf{d}}_0|^2)\\
&\le\int_0^{t_0}\int_{\Omega\cap B_{2r_0}(x_0)}\phi(|\nabla\mathbf{d}|^2|\partial_t\mathbf{h}_E|+|\partial_t\mathbf{h}_E|^2)+|\nabla\phi||\partial_t\widehat{\mathbf{d}}||\nabla\widehat{\mathbf{d}}|\\
&\quad+\int_0^{t_0}\int_{\Omega\cap B_{2r_0}(x_0)}|\nabla\phi||\mathbf{u}|(|\mathbf{u}|^2+|\nabla\mathbf{u}|+|P-P_\Omega|+|\nabla\mathbf{d}|^2)\\
&=I+II+III.
\end{aligned}\tag{2.64}$$

As in (2.63), we can estimate III by

$$|III|\le C\left(\frac{t_0}{r_0^2}\right)^{\frac14}\varepsilon_1^{\frac12}.$$

From $\partial_t\mathbf{h}_E\in L_t^2H_x^2(Q_T)$ and the Sobolev embedding theorem, we have that $\partial_t\mathbf{h}_E\in L_t^2L_x^\infty(Q_T)$, and

$$\left\|\partial_t\mathbf{h}_E\right\|_{L_t^2L_x^\infty(Q_T)}\le C\left\|\partial_t\mathbf{h}_E\right\|_{L_t^2H_x^2(Q_T)}\le C\left\|\partial_t\mathbf{h}\right\|_{L_t^2H_x^{\frac32}(\Gamma_T)}.$$

Since $\mathbf{h}\in L_t^2H_x^{\frac32}(\Gamma_T)$ and $\partial_t\mathbf{h}\in L_t^2H_x^{\frac32}(\Gamma_T)$, $\mathbf{h}\in C([0,T],H^{\frac32}(\Gamma))$ and

$$\left\|\mathbf{h}\right\|_{L_t^\infty H_x^{\frac32}(\Gamma_T)}\le C\big(T,\|\mathbf{h}\|_{L_t^2H_x^{\frac32}(\Gamma_T)},\|\partial_t\mathbf{h}\|_{L_t^2H_x^{\frac32}(\Gamma_T)}\big).$$

This, combined with the fact that $\mathbf{h}_E(\cdot,t)$ is a harmonic extension of $h(\cdot,t)$ for $t\in[0,T]$, implies that $\mathbf{h}_E\in L_t^\infty H_x^2(Q_T)$ and hence by Sobolev embedding theorem we obtain that

$$\begin{aligned}
\left\|\nabla\mathbf{h}_E\right\|_{L_t^\infty L_x^4(Q_T)}\le C\left\|\mathbf{h}_E\right\|_{L_t^\infty H_x^2(Q_T)}&\le C\left\|\mathbf{h}\right\|_{L_t^\infty H_x^{\frac32}(\Gamma_T)}\\
&\le C\big(T,\|\mathbf{h}\|_{L_t^2H_x^{\frac32}(\Gamma_T)},\|\partial_t\mathbf{h}\|_{L_t^2H_x^{\frac32}(\Gamma_T)}\big).
\end{aligned}\tag{2.65}$$

We also have that

$$\|\nabla\mathbf{h}_E(t)\|_{L^2(\Omega)}\le\|\nabla\mathbf{d}(t)\|_{L^2(\Omega)},\ 0\le t\le T.$$

Hence

$$\begin{aligned}
|I|&\le C\|\partial_t\mathbf{h}_E\|_{L_t^2L_x^\infty(Q_{t_0})}\Big(\sup_{0\le t\le t_0}\int_{\Omega\cap B_{2r_0}(x_0)}|\nabla\mathbf{d}|^2\Big)t_0^{\frac12}\\
&\quad+C\|\partial_t\mathbf{h}_E\|^2_{L_t^2L_x^\infty(Q_{t_0})}r_0^2\\
&\le C\Big(\|\partial_t\mathbf{h}\|_{L_t^2H_x^{\frac32}(\Gamma_T)}\varepsilon_1^2t_0^{\frac12}+\|\partial_t\mathbf{h}\|^2_{L_t^2H_x^{\frac32}(\Gamma_T)}r_0^2\Big)\\
&\le C(\varepsilon_1^2r_0+r_0^2).
\end{aligned}$$

While II can be estimated as follows.

$$
\begin{aligned}
|II| \leq & C \int_0^{t_0} \int_{\Omega \cap B_{2r_0}(x_0)} |\nabla \phi| (|\partial_t \mathbf{d}||\nabla \mathbf{d}| + |\partial_t \mathbf{d}||\nabla \mathbf{h}_E| \\
& + |\partial_t \mathbf{h}_E||\nabla \mathbf{d}| + |\partial_t \mathbf{h}_E||\nabla \mathbf{h}_E|) \\
\leq & C \Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}} \|\partial_t \mathbf{d}\|_{L^2(Q_{t_0})} \|\nabla \mathbf{d}\|_{L^4(Q_{t_0})} \\
& + C \frac{t_0^{\frac{1}{2}}}{r_0} \|\partial_t \mathbf{d}\|_{L^2(Q_{t_0})} \|\nabla \mathbf{h}_E\|_{L_t^\infty L_x^2(Q_{t_0})} \\
& + C t_0^{\frac{1}{2}} \|\partial_t \mathbf{h}_E\|_{L_t^2 L_x^\infty(Q_{t_0})} \|\nabla \mathbf{d}\|_{L_t^\infty L_x^2(Q_{t_0})} \\
\leq & C \Big[\varepsilon_1^{\frac{1}{2}} \Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}} + \frac{t_0^{\frac{1}{2}}}{r_0} + t_0^{\frac{1}{2}}\Big] \leq C \Big[\varepsilon_1^{\frac{1}{2}} \Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}} + t_0^{\frac{1}{2}}\Big].
\end{aligned}
$$

Putting these estimates of I, II, and III into (2.64) yields that

$$
\begin{aligned}
& \sup_{0 \leq t \leq t_0} \int_{\Omega \cap B_{r_0}(x_0)} (|\mathbf{u}|^2 + |\nabla \widehat{\mathbf{d}}|^2) - \int_{\Omega \cap B_{2r_0}(x_0)} (|\mathbf{u}_0|^2 + |\nabla \widehat{\mathbf{d}}_0|^2) \\
& \quad \leq C \Big[t_0^{\frac{1}{2}} + \varepsilon_1^{\frac{1}{2}} \Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}}\Big].
\end{aligned} \tag{2.66}
$$

Applying (2.65), we can estimate

$$
\begin{aligned}
& \int_{\Omega \cap B_{r_0}(x_0)} (|\mathbf{u}|^2 + |\nabla \widehat{\mathbf{d}}|^2) \\
& \geq \frac{4}{5} \int_{\Omega \cap B_{r_0}(x_0)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) - C \int_{\Omega \cap B_{r_0}(x_0)} |\nabla \mathbf{h}_E|^2 \\
& \geq \frac{4}{5} \int_{\Omega \cap B_{r_0}(x_0)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) - C \|\mathbf{h}_E\|^2_{L_t^\infty H_x^2(Q_T)} r_0 \\
& \geq \frac{4}{5} \int_{\Omega \cap B_{r_0}(x_0)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) - C r_0, \ \forall\, t \in [0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\Omega \cap B_{2r_0}(x_0)} (|\mathbf{u}_0|^2 + |\nabla \widehat{\mathbf{d}}_0|^2) \\
& \leq \frac{5}{4} \int_{\Omega \cap B_{2r_0}(x_0)} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) + C \int_{\Omega \cap B_{2r_0}(x_0)} |\nabla \mathbf{h}_E|^2 \\
& \leq \frac{5}{4} \int_{\Omega \cap B_{2r_0}(x_0)} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) + C r_0.
\end{aligned}
$$

Therefore we obtain

$$\begin{aligned}\sup_{0\le t\le t_0}&\int_{\Omega\cap B_{r_0}(x_0)}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\\&\le\Big(\frac{5}{4}\Big)^2\int_{\Omega\cap B_{2r_0}(x_0)}(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2)+C\Big[r_0+t_0^{\frac{1}{2}}+\varepsilon_1^{\frac{1}{2}}\Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}}\Big]\\&\le\Big(\frac{5}{4}\Big)^2\varepsilon_1^2+C\Big[r_0+t_0^{\frac{1}{2}}+\varepsilon_1^{\frac{1}{2}}\Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}}\Big].\end{aligned}\tag{2.67}$$

Combining (2.63) with (2.67), we obtain that

$$\begin{aligned}2\epsilon_1^2&=\sup_{0\le t\le t_0}\max_{x_0\in\overline{\Omega}}\int_{\Omega\cap B_{r_0}(x_0)}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\\&\le\Big(\frac{5}{4}\Big)^2\varepsilon_1^2+C\Big[r_0+\varepsilon_1^{\frac{1}{2}}\Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}}\Big]\\&\le\Big(\frac{25}{16}+C\varepsilon_1^2\Big)\varepsilon_1^2+C\varepsilon_1^{\frac{1}{2}}\Big(\frac{t_0}{r_0^2}\Big)^{\frac{1}{4}}.\end{aligned}\tag{2.68}$$

Therefore if we choose $\varepsilon_0\le\frac{5}{16C}$, then $t_0\ge\theta_0 r_0^2$ with $\theta_0=\big(\frac{3\varepsilon_1^{\frac{3}{2}}}{8C}\big)^4$. This gives the desired estimates of T_0 and (2.55). The proof is now complete. □

Proof of Theorem 2.1 From $\mathbf{u}_0\in\mathbf{H}$, there exists $\{\mathbf{u}_0^k\}\subset C^{2,\alpha}(\overline{\Omega},\mathbb{R}^2)$ with $\nabla\cdot\mathbf{u}_0^k=0$ in Ω such that

$$\lim_{k\uparrow\infty}\|\mathbf{u}_0^k-\mathbf{u}_0\|_{L^2(\Omega)}=0.$$

Since dimension of $\partial_p Q_T=\Omega\cup\Gamma_T$ is 2, $\mathbf{h}\in L_t^2H_x^{\frac{3}{2}}(\Gamma_T,\mathbb{S}^2)$, $\partial_t\mathbf{h}\in L_t^2H_x^{\frac{3}{2}}(\Gamma_T)$, $\mathbf{d}_0\in H^1(\Omega,\mathbb{S}^2)$, and $\mathbf{d}_0|_\Gamma=\mathbf{h}|_{\Gamma\times\{0\}}$, there exist maps $(\mathbf{h}^k,\mathbf{d}_0^k)$ such that $\mathbf{h}^k\in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T,\mathbb{S}^2)$ and $\mathbf{d}_0^k\in C^{2,\alpha}(\overline{\Omega},\mathbb{S}^2)$ with $\mathbf{d}_0^k|_\Gamma=\mathbf{h}^k|_{\Gamma\times\{0\}}$, and

$$\lim_{k\uparrow\infty}\big\|(\mathbf{h}^k-\mathbf{h},\partial_t(\mathbf{h}^k-\mathbf{h}))\big\|_{L_t^2H_x^{\frac{3}{2}}(\Gamma_T)}=\lim_{k\uparrow\infty}\big\|\mathbf{d}_0^k-\mathbf{d}_0\big\|_{H^1(\Omega)}=0.\tag{2.69}$$

From the absolute continuity of $\int(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2)$, there exists $r_0\in(0,\varepsilon_1^2)$ such that

$$\sup_{x\in\overline{\Omega}}\int_{\Omega\cap B_{2r_0}(x)}(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2)\le\frac{\varepsilon_1^2}{2},$$

where $\varepsilon_1>0$ is the constant given by Lemma 2.18. By the strong convergence of $(\mathbf{u}_0^k,\nabla\mathbf{d}_0^k)$ to $(\mathbf{u}_0,\nabla\mathbf{d}_0)$ in $L^2(\Omega)$, we may assume that

$$\sup_{x\in\overline{\Omega}}\int_{\Omega\cap B_{2r_0}(x)}(|\mathbf{u}_0^k|^2+|\nabla\mathbf{d}_0^k|^2)\le\varepsilon_1^2\quad\text{for }k\ge1.\tag{2.70}$$

We may also assume that

$$\big\|(\mathbf{h}^k,\partial_t\mathbf{h}^k)\big\|_{L_t^2H_x^{\frac{3}{2}}(\Gamma_T)}\le C\quad\text{for }k\ge1.\tag{2.71}$$

By Lemma 2.18, there is $\theta_0 > 0$ depending on $T, \varepsilon_1, \mathcal{E}_0, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}$ and smooth solutions $(\mathbf{u}^k, \mathbf{d}^k) \in C^\infty(\overline{\Omega} \times [0, T^k], \mathbb{R}^2 \times \mathbb{S}^2)$, with $T^k \geq \theta_0 r_0^2$, to the system (1.1) under the initial and boundary condition

$$
\begin{aligned}
(\mathbf{u}^k, \mathbf{d}^k) &= (\mathbf{u}_0^k, \mathbf{d}_0^k) \quad \text{in } \Omega \times \{0\}, \\
(\mathbf{u}^k, \mathbf{d}^k) &= (0, \mathbf{h}^k) \quad \text{on } \Gamma_{T^k}.
\end{aligned}
$$

Moreover, it holds that

$$
\sup_{(x,t) \in \overline{\Omega} \times [0, T^k]} \int_{\Omega \cap B_{r_0}(x)} (|\mathbf{u}^k|^2 + |\nabla \mathbf{d}^k|^2) \leq \varepsilon_1^2, \tag{2.72}
$$

and for any $0 < t \leq T^k$,

$$
\begin{aligned}
&\sup_{0 \leq \tau \leq t} \int_\Omega (|\mathbf{u}^k|^2 + |\nabla \mathbf{d}^k|^2)(\tau) + \int_{Q_t} (|\nabla \mathbf{u}^k|^2 + |\Delta \mathbf{d}^k + |\nabla \mathbf{d}^k|^2 \mathbf{d}^k|^2) \\
&\leq \psi_k(t) \Big[\int_\Omega (|\mathbf{u}_0^k|^2 + |\nabla \mathbf{d}_0^k|^2) + C \big\| (\mathbf{h}^k, \partial_t \mathbf{h}^k) \big\|^2_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_t)} \Big] \\
&\leq C\big(T, \mathcal{E}_0, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big),
\end{aligned} \tag{2.73}
$$

where

$$
\psi_k(t) = \exp\Big(C \int_0^t \|\partial_t \mathbf{h}^k(\tau)\|_{H^{\frac{3}{2}}(\Gamma)} \, d\tau\Big) \leq C < \infty, \ \forall\, 0 \leq t \leq T.
$$

Combining (2.72), (2.73) together with Lemma 2.17, we conclude that

$$
\int_{Q_{T^k}} (|\mathbf{u}^k|^4 + |\nabla \mathbf{d}^k|^4) \leq C \varepsilon_1^2, \ \forall\, k \geq 1, \tag{2.74}
$$

and

$$
\|\partial_t \mathbf{d}^k\|^2_{L^2(Q_{T^k})} + \|\nabla \mathbf{u}^k\|^2_{L^2(Q_{T^k})} + \|\nabla^2 \mathbf{d}^k\|^2_{L^2(Q_{T^k})} \leq C, \ \forall\, k \geq 1. \tag{2.75}
$$

It follows from Lemma 2.16, (2.72)–(2.75) that

$$
\|\nabla P^k\|_{L^{\frac{4}{3}}(Q_{T^k})} \leq C \varepsilon_1^{\frac{1}{2}}, \ \forall\, k \geq 1. \tag{2.76}
$$

Furthermore, $(1.1)_1$, (2.74)–(2.76) imply that

$$
\|\partial_t \mathbf{u}^k\|_{L_t^{\frac{4}{3}} H_x^{-1}(Q_{T^k})} \leq C, \ \forall\, k \geq 1. \tag{2.77}
$$

By Theorem 2.7, we conclude that for any $\alpha \in (0, 1)$ such that for any $\delta > 0$,

$$
\big\|(\mathbf{u}^k, \mathbf{d}^k)\big\|_{C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [\delta, T^k])} \leq C\big(\alpha, \delta, \mathcal{E}_0, \varepsilon_1, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_{T^k})}\big),
$$

for any compact subdomain $\omega \subset\subset \Omega$,

$$
\big\|(\mathbf{u}^k, \mathbf{d}^k)\big\|_{C^\ell(\omega \times [\delta, T^k])} \leq C(\mathrm{dist}(\omega, \partial\Omega), \delta, \ell, \mathcal{E}_0) \quad \text{for all } \ell \geq 1.
$$

There exist $T_0 \geq \theta_0 r_0^2$, $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{T_0}, \mathbb{R}^2)$, $\mathbf{d} \in L_t^2 H_x^2(Q_{T_0}, \mathbb{S}^2)$ such that after passing to a possible subsequence, $T^k \to T_0$,

$$\begin{aligned}
&\mathbf{u}^k \to \mathbf{u} \text{ weakly in } W_2^{1,0}(Q_{T_0}, \mathbb{R}^2) \text{ and strongly in } L^2(Q_{T_0}),\\
&\mathbf{d}^k \to \mathbf{d} \text{ weakly in } W_2^{2,1}(Q_{T_0}, \mathbb{R}^3) \text{ and strongly in } L_t^2 H_x^1(Q_{T_0}),\\
&\lim_{k\uparrow\infty} \left(\|\mathbf{u}^k - \mathbf{u}\|_{L^4(Q_{T_0})} + \|\nabla \mathbf{d}^k - \nabla \mathbf{d}\|_{L^4(Q_{T_0})} \right) = 0,
\end{aligned}$$

and for any $\ell \geq 2$, $\delta > 0$, and compact $\omega \subset\subset \Omega$,

$$\begin{aligned}
&\lim_{k\uparrow\infty} \|(\mathbf{u}^k, \mathbf{d}^k) - (\mathbf{u}, \mathbf{d})\|_{C^\ell(\omega\times[\delta, T_0])} = 0,\\
&\lim_{k\uparrow\infty} \|(\mathbf{u}^k, \mathbf{d}^k) - (\mathbf{u}, \mathbf{d})\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[\delta, T_0])} = 0.
\end{aligned}$$

Thus $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0, T_0], \mathbb{R}^2 \times \mathbb{S}^2) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ solves the system (1.1)–(1.3) in $\Omega \times (0, T_0]$. From (2.73), we can show that

$$(\mathbf{u}, \nabla \mathbf{d})(\cdot, t) \to (\mathbf{u}_0, \nabla \mathbf{d}_0) \text{ in } L^2(\Omega) \text{ as } t \downarrow 0.$$

Hence $(\mathbf{u}, \mathbf{d})$ satisfies the initial and boundary condition (1.2) and (1.3). Let $T_1 \in (0, T)$ be the first singular time of $(\mathbf{u}, \mathbf{d})$, that is

$$(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0, T_1), \mathbb{R}^2 \times \mathbb{S}^2) \bigcap C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0, T_1), \mathbb{R}^2 \times \mathbb{S}^2),$$

but

$$(\mathbf{u}, \mathbf{d}) \notin C^\infty(\Omega \times (0, T_1], \mathbb{R}^2 \times \mathbb{S}^2) \bigcap C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (0, T_1], \mathbb{R}^2 \times \mathbb{S}^2).$$

Thus we must have

$$\limsup_{t\uparrow T_1} \max_{x\in\overline{\Omega}} \int_{\Omega\cap B_r(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\cdot, t) \geq \varepsilon_1^2 \quad \text{for all } r > 0. \tag{2.78}$$

In what follows, we will look for an eternal extension of this weak solution beyond T_1. To do it, we need to define $(\mathbf{u}, \mathbf{d})$ at time T_1, which follows from the claim that

$$(\mathbf{u}, \mathbf{d}) \in C([0, T_1], L^2(\Omega, \mathbb{R}^2 \times \mathbb{S}^2)). \tag{2.79}$$

In fact, for any $\phi \in H_0^2(\Omega, \mathbb{R}^3)$, we can derive from $(1.1)_3$ that

$$\begin{aligned}
|\langle \partial_t \mathbf{d}, \phi \rangle| &= \Big| \int_\Omega (\langle \nabla \mathbf{d}, \nabla \phi \rangle + \langle \mathbf{u} \cdot \nabla \mathbf{d}, \phi \rangle - |\nabla \mathbf{d}|^2 \langle \mathbf{d}, \phi \rangle) \Big|\\
&\leq C\|\nabla \mathbf{d}\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + (\|\mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mathbf{d}\|_{L^2(\Omega)}) \|\nabla \mathbf{d}\|_{L^2(\Omega)} \|\phi\|_{L^\infty(\Omega)}\\
&\leq C\big[\|\nabla \mathbf{d}\|_{L^2(\Omega)} + (\|\mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mathbf{d}\|_{L^2(\Omega)}) \|\nabla \mathbf{d}\|_{L^2(\Omega)}\big] \|\phi\|_{H^2(\Omega)},
\end{aligned}$$

so that $\partial_t \mathbf{d} \in L^2([0, T_1], H^{-2}(\Omega, \mathbb{R}^3))$. This and $\mathbf{d} \in L_t^2 H_x^1(Q_T)$ imply that $\mathbf{d} \in C([0, T_1], L^2(\Omega, \mathbb{S}^2))$.

For any $\phi \in H_0^3(\Omega, \mathbb{R}^2)$, with $\nabla \cdot \phi = 0$, $(1.1)_1$ implies that

$$\begin{aligned}
|\langle \partial_t \mathbf{u}, \phi \rangle| &= \Big| \int_\Omega (\nabla \mathbf{u} \cdot \nabla \phi + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \phi - \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla \phi) \Big|\\
&\leq C\|\nabla \mathbf{u}\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}\\
&\quad + C(\|\mathbf{u}\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mathbf{d}\|_{L^2(\Omega)}^2) \|\nabla \phi\|_{L^\infty(\Omega)}\\
&\leq C(\|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mathbf{d}\|_{L^2(\Omega)}^2) \|\phi\|_{H^3(\Omega)},
\end{aligned}$$

so that $\partial_t \mathbf{u} \in L^2([0,T_1], H^{-3}(\Omega,\mathbb{R}^2))$. This and $\mathbf{u} \in L^2_t H^1_x(Q_T)$ imply $\mathbf{u} \in C([0,T_1], L^2(\Omega))$. Thus (2.79) follows.

It follows from (2.79) that

$$(\mathbf{u},\mathbf{d})(\cdot,T_1) = \lim_{t\uparrow T_1} (\mathbf{u},\mathbf{d})(\cdot,t) \text{ in } L^2(\Omega).$$

This and (2.73) imply that

$$\nabla\mathbf{d}(\cdot,t) \to \nabla\mathbf{d}(\cdot,T_1) \text{ weakly in } L^2(\Omega) \text{ as } t\uparrow T_1.$$

Thus $\mathbf{u}(\cdot,T_1)\in\mathbf{H}$ and $\mathbf{d}(\cdot,T_1)\in H^1(\Omega,\mathbb{S}^2)$. Since $H^1(\Omega)\subset L^2(\Gamma)$ is compact, we also have that

$$\mathbf{d}(\cdot,t)(=\mathbf{h}(\cdot,t)) \to \mathbf{d}(\cdot,T_1) \text{ in } L^2(\Gamma) \text{ as } t\uparrow T_1.$$

This and $\mathbf{h}\in C([0,T], H^{\frac32}(\Gamma))$ imply that $\mathbf{d}(\cdot,T_1)=\mathbf{h}(\cdot,T_1)$ on Γ.

Now, we can use $(\mathbf{u},\mathbf{d})(\cdot,T_1)$ and $(0,\mathbf{h})$ as the initial and boundary value to extend the weak solution of (1.1)–(1.3) to the time interval $[0,T_2]$ for some $T_2>T_1$. Repeating this procedure, we eventually obtain the existence of global weak solution in the time interval $[0,T)$. Next we want to show

Claim 1. There are at most finitely singular times To show it, first observe that at any singular time $T_\sharp\in(0,T)$, there is at least a loss of energy of $\frac12\varepsilon_1^2$. It follows from (2.78) that for any $r>0$, there exist $t_i\uparrow T_\sharp$ and $x_i\in\overline{\Omega}$ such that $x_i\to x_0\in\overline{\Omega}$, and

$$\int_{\Omega\cap B_r(x_i)} (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(t_i) \ge \frac12\varepsilon_1^2,$$

and hence

$$\begin{aligned}
\int_\Omega (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(T_\sharp) &= \lim_{r\downarrow 0}\int_{\Omega\setminus B_{2r}(x_0)} (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(T_\sharp)\\
&\le \lim_{r\downarrow 0}\liminf_{t_i\uparrow T_\sharp}\int_{\Omega\setminus B_{2r}(x_0)} (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(t_i)\\
&\le \lim_{r\downarrow 0}\Big[\liminf_{t_i\uparrow T_\sharp}\int_\Omega (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(t_i)\\
&\quad -\limsup_{t_i\uparrow T_\sharp}\int_{\Omega\cap B_{2r}(x_0)} (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(t_i)\Big]\\
&\le \liminf_{t_i\uparrow T_\sharp}\int_\Omega (|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(t_i) - \frac12\varepsilon_1^2,
\end{aligned} \tag{2.80}$$

We will prove *Claim* 1 by contradiction. Suppose that there were infinitely many singular times $\{T_j\}_{j=1}^\infty\subset(0,T]$, with $0<T_1<T_2<\cdots<T_j<\cdots$, and $\lim_{j\uparrow+\infty}T_j=T_*\le T$. Hence for any $\delta>0$, there exists a sufficiently large $j_0=j_0(\delta)\ge 1$ such that for $j\ge j_0$, we have

$$\exp\Big(C\int_{T_j}^{T_{j+1}} \|\partial_t\mathbf{h}(\tau)\|_{H^{\frac32}(\Gamma)}\,d\tau\Big)\le 1+\delta,$$

and

$$\big\|(\mathbf{h},\partial_t\mathbf{h})\big\|^2_{L^2_tH^{\frac32}_x(\Gamma\times[T_j,T_{j+1}])}\le\delta.$$

Then by (2.44) we have that for any $t \in [T_j, T_{j+1})$

$$
\begin{aligned}
\mathcal{E}(t) &= \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(t) \\
&\le \exp\big(C \int_{T_j}^{T_{j+1}} \|\partial_t \mathbf{h}(\tau)\|_{H^{\frac{3}{2}}(\Gamma)} \, d\tau\big)\big[\int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_j) \\
&\quad + C \big\|(\mathbf{h}, \partial_t \mathbf{h})\big\|^2_{L_t^2 H_x^{\frac{3}{2}}(\Gamma \times [T_j, T_{j+1}])}\big] \\
&\le (1+\delta)\big[\int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_j) + \delta \big] \\
&\le \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_j) + C\delta.
\end{aligned}
\tag{2.81}
$$

Putting (2.80) and (2.81) together, we obtain

$$
\begin{aligned}
&\int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_{j+1}) \le \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_j) + C\delta - \frac{1}{2}\varepsilon_1^2 \\
&\le \int_\Omega (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(T_j) - \frac{1}{4}\varepsilon_1^2,
\end{aligned}
\tag{2.82}
$$

provided $\delta \le \frac{\varepsilon_1^2}{4C}$.

Iterating the above inequality m times, we obtain that

$$
0 \le \mathcal{E}(T_{j_0+m}) \le \mathcal{E}(T_{j_0}) - \frac{m\varepsilon_1^2}{4}.
$$

This yields that

$$
m \le \big[\frac{4K_0}{\varepsilon_1^2}\big] + 1,
$$

where $K_0 = \mathcal{E}(T_{j_0})$. This proves *Claim* 1.

If $T_L < T$ is the last singular time, then we can use $(\mathbf{u}(T_L), \mathbf{d}(T_L))$ and $(0, \mathbf{h})|_{\Gamma \times (T_L, T]}$ as the initial and boundary data to construct a weak solution $(\mathbf{u}, \mathbf{d})$ to system (1.1)–(1.3) on $[T_L, T]$ as before so that we obtain a global weak solution $(\mathbf{u}, \mathbf{d})$ to (1.1)–(1.3) in the time interval $[0, T)$. This completes the proof of Theorem 2.1. □

2.5 Proof of Theorem 2.3

The proof of Theorem 2.3 is similar to [31] Theorem 1.3. For the convenience of reader, we sketch it here. Let $(\mathbf{u}_0, \mathbf{d}_0)$ and $\mathbf{h}$ satisfy the assumptions of Theorem 2.3. By Lemma 2.11, the weak solution $(\mathbf{u}, \mathbf{d})$ to (1.1)–(1.3), obtained by Theorem 2.1, satisfies

$$
\mathbf{d}(x, t) \in \mathbb{S}_+^2, \ \text{ for a.e. } (x, t) \in Q_T.
$$

Assume that $(\mathbf{u}, \mathbf{d})$ has a singular time $T_1 \in (0, T)$. Then, it follows from (2.2) that for $\mathcal{M} > 1$, to be determined later, there exist $t_m \uparrow T_1^-$ and $r_m \downarrow 0^+$ such that

$$
\frac{\varepsilon_1^2}{\mathcal{M}} = \sup_{x \in \overline{\Omega}, 0 \le t \le t_m} \int_{\Omega \cap B_{r_m}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2).
\tag{2.83}
$$

It follows from the proof of Lemma 2.18, there exist θ_0, depending only on ε_1, $\mathcal{E}_0$, and $\|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}$, and $x_m \in \overline{\Omega}$, such that

$$\begin{aligned}
&\int_{\Omega\cap B_{2r_m}(x_m)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(t_m - \theta_0 r_m^2) \\
&\geq \frac{1}{2} \sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{2r_m}(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(t_m - \theta_0 r_m^2) \geq \frac{\varepsilon_1^2}{4\mathcal{M}}.
\end{aligned} \tag{2.84}$$

By (2.44) in Lemma 2.13, (2.83) and the Ladyzhenskaya inequality, we have

$$\begin{cases}
\int_{Q_{t_m}} (|\nabla \mathbf{u}|^2 + |\nabla^2 \mathbf{d}|^2) \leq C\big(\varepsilon_1, \mathcal{E}_0, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big), \\
\int_{Q_{t_m}} (|\mathbf{u}|^4 + |\nabla \mathbf{d}|^4) \leq \frac{C\varepsilon_1^2}{\mathcal{M}}.
\end{cases} \tag{2.85}$$

Set $\Omega_m = r_m^{-1}(\Omega\backslash\{x_m\})$ and define $(\mathbf{u}_m, \mathbf{d}_m) : \Omega_m \times [-\frac{t_m}{r_m^2}, 0] \mapsto \mathbb{R}^2 \times \mathbb{S}_+^2$ by

$$(\mathbf{u}_m, \mathbf{d}_m)(x, t) = (r_m \mathbf{u}(x_m + r_m x, t_m + r_m^2 t), \mathbf{d}(x_m + r_m x, t_m + r_m^2 t)).$$

Then $(\mathbf{u}_m, \mathbf{d}_m)$ solves (1.1)–(1.3) in $\Omega_m \times [-\frac{t_m}{r_m^2}, 0]$, along with

$$(\mathbf{u}_m, \mathbf{d}_m)\left(x, -\frac{t_m}{r_m^2}\right) = (r_m \mathbf{u}(x_m + r_m x, 0), \mathbf{d}(x_m + r_m x, 0))$$

and

$$(\mathbf{u}_m, \mathbf{d}_m)(x, t) = (0, \mathbf{h}(x_m + r_m x, t_m + r_m^2 t)) \text{ on } \partial\Omega_m \times \left[-\frac{t_m}{r_m^2}, 0\right].$$

Moreover,

$$\begin{aligned}
&\int_{\Omega_m\cap B_2(0)} (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2)(-\theta_0) \geq \frac{\varepsilon_1^2}{4\mathcal{M}}, \\
&\int_{\Omega_m\cap B_1(x)} (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2)(t) \leq \frac{\varepsilon_1^2}{\mathcal{M}}, \forall x \in \Omega_m, -\frac{t_m}{r_m^2} \leq t \leq 0, \\
&\int_{\Omega_m\times[-\frac{t_m}{r_m^2}, 0]} (|\mathbf{u}_m|^4 + |\nabla \mathbf{d}_m|^4) \leq \frac{C\varepsilon_1^2}{\mathcal{M}}, \\
&\int_{\Omega_m\times[-\frac{t_m}{r_m^2}, 0]} (|\nabla \mathbf{u}_m|^2 + |\nabla^2 \mathbf{d}_m|^2) \leq C\big(\varepsilon_1, \mathcal{E}_0, \|(\mathbf{h}, \partial_t \mathbf{h})\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}\big).
\end{aligned} \tag{2.86}$$

Assume $x_m \to x_0 \in \overline{\Omega}$ and $\mathcal{M} > 0$ is chosen sufficiently large. We divide the discussion into two cases:

Case 1: $x_0 \in \Omega$. Then $r_m < \text{dist}(x_0, \Gamma)$ and $\Omega_m \to \mathbb{R}^2, -\frac{t_m}{r_m^2} \to -\infty$. By Theorem 2.4, there exists a smooth solution $(\mathbf{u}_\infty, \mathbf{d}_\infty) : \mathbb{R}^2 \times (-\infty, 0] \mapsto \mathbb{R}^2 \times \mathbb{S}_+^2$ to the system (1.1)–(1.3) such that

$$(\mathbf{u}_m, \mathbf{d}_m) \to (\mathbf{u}_\infty, \mathbf{d}_\infty) \quad \text{in } C_{loc}^2(\mathbb{R}^2 \times (-\infty, 0], \mathbb{R}^2 \times \mathbb{S}_+^2).$$

For any set $P_R = B_R \times [-R^2, 0] \subset \mathbb{R}^2 \times (-\infty, 0]$, it is easy to see that

$$\int_{P_R} |\mathbf{u}_\infty|^4 = \lim_{m\uparrow\infty} \int_{P_R} |\mathbf{u}_m|^4 = \lim_{m\uparrow\infty} \int_{B_{Rr_m}(x_m)\times[t_m-R^2r_m^2, t_m]} |\mathbf{u}|^4 = 0.$$

Thus $\mathbf{u}_\infty \equiv 0$.

It is also easy to see that for any compact $\omega \subset \mathbb{R}^2$,

$$\begin{aligned} &\int_{-1}^{0}\int_\omega |\Delta \mathbf{d}_\infty + |\nabla \mathbf{d}_\infty|^2 \mathbf{d}_\infty|^2 \\ &\le \liminf_{m\uparrow\infty} \int_{-1}^{0}\int_{\Omega_m} |\Delta \mathbf{d}_m + |\nabla \mathbf{d}_m|^2 \mathbf{d}_m|^2 \\ &\le \lim_{m\uparrow\infty} \int_{t_m-r_m^2}^{t_m}\int_{\Omega} |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 = 0, \end{aligned}$$

which, together with (1.1)$_3$, implies that

$$\partial_t \mathbf{d}_\infty + \mathbf{u}_\infty \cdot \nabla \mathbf{d}_\infty = \mathbf{0} \text{ in } \mathbb{R}^2 \times [-1, 0].$$

Hence $\partial_t \mathbf{d}_\infty \equiv 0$ and $\mathbf{d}_\infty : \mathbb{R}^2 \mapsto \mathbb{S}^2_+$ is a nontrivial smooth harmonic map with finite energy according to (2.86), which contradicts to Lemma 2.12.

Case 2. $x_0 \in \Gamma$. Then we have either

(a) $\lim_{m\uparrow\infty} \frac{|x_m - x_0|}{r_m} = \infty$. Then, as in *Case* 1, $\Omega_m \to \mathbb{R}^2$ and $(\mathbf{u}_m, \mathbf{d}_m)$ converges to $(0, \mathbf{d}_\infty)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \times [-1, 0])$, where $\mathbf{d}_\infty : \mathbb{R}^2 \mapsto \mathbb{S}^2_+$ is a nontrivial smooth harmonic map with finite energy, which contradicts to Lemma 2.12.

or

(b) $\lim_{m\uparrow\infty} \frac{|x_m - x_0|}{r_m} = a \in [0, \infty)$. Then we would have

$$(\Omega_m,\ \partial\Omega_m) \to \left(\mathbb{R}^2_{-a} = \left\{x = (x_1, x_2) \in \mathbb{R}^2 :\ x_2 \ge -a\right\},\ \partial\mathbb{R}^2_{-a}\right).$$

Observe that $\mathbf{h}_m(x,t) = \mathbf{h}(x_m + r_m x, t_m + r_m^2 t)$ is uniformly bounded in $C^{\alpha,\frac{\alpha}{2}}(\partial\Omega_m \times [-1, 0])$ for any $\alpha \in (0, 1)$. Hence we may assume that $\mathbf{h}_m \to \mathbf{p}$, in $C^0_{\mathrm{loc}}(\mathbb{R}^2_{-a} \times [-1, 0])$, for some point $\mathbf{p} \in \mathbb{S}^2$. Thus, similar to [31] Theorem 1.3, we would obtain a nontrivial harmonic map $\mathbf{d}_\infty : \mathbb{R}^2_{-a} \to \mathbb{S}^2_+$ with $\mathbf{d}_\infty = \mathbf{p}$ on $\partial\mathbb{R}^2_{-a}$, that has finite energy. This is again impossible.

From *Case* 1 and *Case* 2, we conclude that (2.83) never occurs in $[0, T]$. This completes the proof of Theorem 2.3.

□

3 Global strong solution

In this section, we will show the existence of unique, global strong solutions to the system (1.1)–(1.3). For this purpose, we will assume that the initial data

$$(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+),$$

and the boundary data

$$\mathbf{h} \in H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T, \mathbb{S}^2_+) \text{ and } \partial_t \mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T). \tag{3.1}$$

More precisely, we will prove

Theorem 3.1 *Let* $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$*, and* $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2_+$ *satisfy* (3.1) *and the compatibility condition* (1.5)*. Let* $(\mathbf{u}, \mathbf{d}) : \Omega \times [0, T] \mapsto \mathbb{R}^2 \times \mathbb{S}^2_+$ *be a weak solution to the system* (1.1)–(1.3)*, with initial value* $(\mathbf{u}_0, \mathbf{d}_0)$ *and boundary value* $(0, \mathbf{h})$*, obtained by Theorem* 2.3*. Then* $(\mathbf{u}, \mathbf{d})$ *is a unique global strong solution to the system* (1.1)–(1.3)*, that satisfies*

$$\mathbf{u} \in L^\infty([0, T], \mathbf{V}) \cap L^2_t H^2_x(Q_T, \mathbb{R}^2),$$
$$\mathbf{d} \in L^\infty_t H^2_x(Q_T, \mathbb{S}^2_+) \cap L^2_t H^3_x(Q_T, \mathbb{S}^2_+).$$

Moreover, the following estimate holds

$$\|\mathbf{u}(t)\|^2_{L^\infty_t H^1_x(Q_T)} + \|\mathbf{d}(t)\|^2_{L^\infty_t H^2_x(Q_T)} + \int_0^T \left(\|\mathbf{u}(\tau)\|^2_{H^2(\Omega)} + \|\mathbf{d}(\tau)\|^2_{H^3(\Omega)}\right) \leq C_T, \tag{3.2}$$

where $C_T > 0$ *depends on* $\|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}$, $\|\mathbf{h}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}$, $\|\partial_t \mathbf{h}\|_{L^2_t H_x^{\frac{3}{2}}(\Gamma_T)}$, ε_1, T, *and* Ω.

Remark 3.2 Employing $(1.1)_1$ and $(1.1)_3$ and the estimate (3.2), we can verify that the global strong solution $(\mathbf{u}, \mathbf{d})$ obtained by Theorem 3.1 satisfies

$$\partial_t \mathbf{u} \in L^2([0, T], \mathbf{H}) \text{ and } \partial_t \mathbf{d} \in L^2_t H^1_x(Q_T),$$

which, combined with the Aubin-Lions Lemma, implies that

$$\mathbf{u} \in C([0, T], \mathbf{V}) \text{ and } \mathbf{d} \in C([0, T], H^2(\Omega)).$$

Proof of Theorem 3.1 Since the uniqueness part of strong solutions follows immediately from the continuous dependence Theorem 3.3 below, we will focus on the proof of the existence of a global strong solution $(\mathbf{u}, \mathbf{d})$ that satisfies the estimate (3.2).

For $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$ and $\mathbf{h}$ satisfies (3.1), recall that Theorem 2.3 implies that there exists a global weak solution $(\mathbf{u}, \mathbf{d})$ to the system (1.1)–(1.3), with initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and boundary condition $(0, \mathbf{h})$, which satisfies

$$\mathbf{u} \in L^\infty([0, T], \mathbf{H}) \cap L^2([0, T], \mathbf{V}), \quad \mathbf{d} \in L^\infty_t H^1_x(Q_T) \cap L^2_t H^2_x(Q_T).$$

In order to prove that this global weak solution $(\mathbf{u}, \mathbf{d})$ is the desired strong solution satisfying (3.2), we need to show that the sequence of smooth solutions $(\mathbf{u}^k, \mathbf{d}^k) : Q_T \mapsto \mathbb{R}^2 \times \mathbb{S}^2_+$ of the system (1.1), under the initial and boundary conditions $(\mathbf{u}^k_0, \mathbf{d}^k_0)$ and $(0, \mathbf{h}^k)$, from Theorem 2.3 satisfy (3.2) with a constant C_T that is independent of k.

In fact, it follows from the proof of Theorem 2.3 that

(i)
$$\begin{cases} (\nabla \mathbf{u}^k, \nabla^2 \mathbf{d}^k) \to (\nabla \mathbf{u}, \nabla^2 \mathbf{d}) \text{ weakly in } L^2(Q_T), \\ (\mathbf{u}^k, \mathbf{d}^k) \to (\mathbf{u}, \mathbf{d}) \text{ in } C([0, T], L^2(\Omega) \times H^1(\Omega)). \end{cases} \tag{3.3}$$

(ii) there exists ε_1 and $r_0 > 0$ such that

$$\sup_{(x,t) \in \overline{Q}_T} \int_{\Omega \cap B_{r_0}(x)} (|\mathbf{u}^k|^2 + |\nabla \mathbf{d}^k|^2)(\cdot, t) \leq \varepsilon_1^2, \forall k \geq 1, \tag{3.4}$$

$$\|(\mathbf{u}^k, \nabla \mathbf{d}^k)\|_{L^\infty_t L^2_x(Q_T)} + \|(\nabla \mathbf{u}^k, \nabla^2 \mathbf{d}^k)\|_{L^2(Q_T)} \leq K_{T,\varepsilon_1}, \forall k \geq 1, \tag{3.5}$$

where $K_{T,\varepsilon_1} > 0$ depends on $\|(\mathbf{u}_0, \mathbf{d}_0)\|_{\mathbf{H} \times H^1(\Omega)}$, $\|(\mathbf{h}, \partial_t \mathbf{h})\|_{L^2_t H_x^{\frac{3}{2}}(\Gamma_T)}$, ε_1, T, and Ω.

Now we claim that

$$\lim_{k\to\infty}\int_{Q_T}(|\nabla \mathbf{u}^k|^2+|\Delta \mathbf{d}^k+|\nabla \mathbf{d}^k|^2\mathbf{d}^k|^2)$$
$$=\int_{Q_T}(|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2). \tag{3.6}$$

To show (3.6), first observe from the proof of Lemma 2.13 that $(\mathbf{u}^k,\mathbf{d}^k)$ satisfies the following energy equality:

$$\int_\Omega(|\mathbf{u}^k|^2+|\nabla \mathbf{d}^k|^2)(T)+2\int_{Q_T}(|\nabla \mathbf{u}^k|^2+|\Delta \mathbf{d}^k+|\nabla \mathbf{d}^k|^2\mathbf{d}^k|^2)$$
$$=\int_\Omega(|\mathbf{u}_0^k|^2+|\nabla \mathbf{d}_0^k|^2)+2\int_{Q_T}\langle\Delta \mathbf{d}^k,\partial_t \mathbf{h}_E^k\rangle+2\int_{\Gamma_T}\left\langle\frac{\partial \mathbf{h}_E^k}{\partial\nu},\ \partial_t \mathbf{h}^k\right\rangle, \tag{3.7}$$

where $\mathbf{h}_E^k(\cdot,t)$ is the harmonic extension of $\mathbf{h}^k(\cdot,t)$.

Since $\mathbf{d}\in L_t^2H_x^2(Q_T)$, an argument similar to Lemma 2.13 also yields that $(\mathbf{u},\mathbf{d})$ satisfies the same energy equality as (3.7):

$$\int_\Omega(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2)(T)+2\int_{Q_T}(|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2)$$
$$=\int_\Omega(|\mathbf{u}_0|^2+|\nabla \mathbf{d}_0|^2)+2\int_{Q_T}\langle\Delta \mathbf{d},\partial_t \mathbf{h}_E\rangle+2\int_{\Gamma_T}\langle\frac{\partial \mathbf{h}_E}{\partial\nu},\ \partial_t \mathbf{h}\rangle, \tag{3.8}$$

Since $(\mathbf{h}^k,\partial_t\mathbf{h}^k)\to(\mathbf{h},\partial_t\mathbf{h})$ in $L_t^2H_x^{\frac{3}{2}}(\Gamma_T)$, it follows the standard estimate on harmonic functions that

$$\partial_t \mathbf{h}_E^k\to\partial_t \mathbf{h}_E \text{ in } L_t^2H_x^2(Q_T),$$
$$\frac{\partial \mathbf{h}_E^k}{\partial\nu}\to\frac{\partial \mathbf{h}_E}{\partial\nu} \text{ in } L_t^2H_x^{\frac{1}{2}}(\Gamma_T).$$

Therefore, after sending $k\to\infty$ in (3.7) and comparing the resulting equality with (3.8), we see that (3.6) holds true.

On the other hand, (3.3) and the Ladyzhenskaya's inequality imply that

$$(\mathbf{u}^k,\nabla \mathbf{d}^k)\to(\mathbf{u},\nabla \mathbf{d}) \text{ in } L^4(Q_T). \tag{3.9}$$

Now it is easy to see from (3.6) and (3.9) that

$$\lim_{k\to\infty}\int_{Q_T}(|\nabla \mathbf{u}^k|^2+|\Delta \mathbf{d}^k|^2)=\int_{Q_T}(|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}|^2). \tag{3.10}$$

In particular, we conclude that

$$(\mathbf{u}^k,\mathbf{d}^k)\to(\mathbf{u},\mathbf{d}) \text{ in } L_t^2H_x^1(Q_T)\times L_t^2H_x^2(Q_T). \tag{3.11}$$

It is clear that (3.11) yields the following uniform absolute continuity: for any $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ such that

$$\|\mathbf{u}^k\|^2_{L^2([s_1,s_2],H^1(\Omega))}+\|\mathbf{d}^k\|^2_{L^2([s_1,s_2],H^2(\Omega))}\le\epsilon^2, \tag{3.12}$$

provided $0\le s_1<s_2\le T$ satisfies $|s_2-s_1|\le\delta$.

Now we can show that $(\mathbf{u}^k, \mathbf{d}^k)$ satisfies the estimate (3.2) with a constant C_T, that is independent of k, as follows. For simplicity, we drop the upper index k and write $(\mathbf{u}, \mathbf{d}, P)$ for $(\mathbf{u}^k, \mathbf{d}^k, P^k)$.

By employing the $W_2^{2,1}$-regularity theory on the non-stationary Stokes system, we have that $\partial_t \mathbf{u}, \nabla^2 \mathbf{u}, \nabla P \in L^2(Q_T)$, and for all $0 < t \leq T$

$$
\begin{aligned}
&\|\partial_t \mathbf{u}\|_{L^2(Q_t)} + \|\nabla^2 \mathbf{u}\|_{L^2(Q_t)} \\
&\quad \leq C\big[\|\nabla \mathbf{u}_0\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2(Q_t)} + \|\nabla(\nabla \mathbf{d} \odot \nabla \mathbf{d})\|_{L^2(Q_t)}\big] \\
&\quad \leq C\big[\|\nabla \mathbf{u}_0\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^4(Q_t)}\|\nabla \mathbf{u}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}\|_{L^4(Q_t)}\big].
\end{aligned}
$$

On the other hand, it follows from the trace theorem $W_2^{2,1}(Q_T) \hookrightarrow H^1(\Omega \times \{t\}), \forall t \in [0, T]$, that

$$
\|\nabla \mathbf{u}\|_{L_t^\infty L_x^2(Q_t)} \leq C\|\mathbf{u}\|_{W_2^{2,1}(Q_t)} \leq C\big[\|\partial_t \mathbf{u}\|_{L^2(Q_t)} + \|(\nabla \mathbf{u}, \nabla^2 \mathbf{u})\|_{L^2(Q_t)}\big].
$$

Putting these two estimates together, we obtain that

$$
\begin{aligned}
&\|\nabla \mathbf{u}\|_{L_t^\infty L_x^2(Q_t)} + \|\partial_t \mathbf{u}\|_{L^2(Q_t)} + \|\nabla^2 \mathbf{u}\|_{L^2(Q_t)} \\
&\quad \leq C\big[\|\nabla \mathbf{u}_0\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^4(Q_t)}\|\nabla \mathbf{u}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}\|_{L^4(Q_t)}\big].
\end{aligned}
\tag{3.13}
$$

To estimate $\mathbf{d}$, let $\mathbf{h}_P$ be the parabolic lifting function of $\mathbf{h}$, i.e.,

$$
\begin{cases}
\partial_t \mathbf{h}_P - \Delta \mathbf{h}_P = 0 & \text{in} \quad Q_T, \\
\mathbf{h}_P = \mathbf{h} & \text{on} \quad \Gamma_T, \\
\mathbf{h}_P = \mathbf{d}_0 & \text{in} \quad \Omega \times \{0\}.
\end{cases}
$$

It follows from (3.1) and the regularity theory of parabolic equations (cf. [38]) that

$$
\begin{aligned}
&\|\mathbf{h}_P\|_{L_t^\infty H_x^2(Q_T)} + \|\mathbf{h}_P\|_{H^{3,\frac{3}{2}}(Q_T)} \\
&\quad \leq C\big[\|\mathbf{h}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\mathbf{d}_0\|_{H^2(\Omega)}\big].
\end{aligned}
\tag{3.14}
$$

Set $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{h}_P$. Then we have

$$
\begin{cases}
\partial_t \widetilde{\mathbf{d}} - \Delta \widetilde{\mathbf{d}} = -\mathbf{u} \cdot \nabla \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} & \text{in} \quad Q_T, \\
\widetilde{\mathbf{d}} = 0 & \text{on} \quad \partial_p Q_T.
\end{cases}
\tag{3.15}
$$

It follows from the regularity theory of parabolic equations, the trace theorem, and the estimate of $\mathbf{h}_P$ that for $0 < t \leq T$, it holds

$$
\begin{aligned}
&\|\mathbf{d}\|_{L_t^\infty H_x^2(Q_t)} + \|\mathbf{d}\|_{H^{3,\frac{3}{2}}(Q_t)} \\
&\quad \leq C\big[\|\mathbf{d}_0\|_{H^2(\Omega)} + \|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\mathbf{h}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} \\
&\qquad + \|\mathbf{u} \cdot \nabla \mathbf{d}\|_{H^{1,\frac{1}{2}}(Q_t)} + \||\nabla \mathbf{d}|^2 \mathbf{d}\|_{H^{1,\frac{1}{2}}(Q_t)}\big].
\end{aligned}
\tag{3.16}
$$

From the interpolation inequality $W^{0,0}_{\frac{8}{3}}(Q_t) \cap W^{0,1}_{\frac{8}{5}}(Q_t) \hookrightarrow W^{0,\frac{1}{2}}_{2}(Q_t)$ (cf. [39] Lemma 7), we can estimate the last two terms in the right hand side of (3.16) by

$$\begin{aligned}
&\|\mathbf{u}\cdot\nabla\mathbf{d}\|_{H^{1,\frac{1}{2}}(Q_t)}\\
&\quad\le C\Big[\||\mathbf{u}||\nabla\mathbf{d}|\|_{L^2(Q_t)} + \||\nabla\mathbf{u}||\nabla\mathbf{d}|\|_{L^2(Q_t)} + \||\mathbf{u}||\nabla^2\mathbf{d}|\|_{L^2(Q_t)}\\
&\qquad + \|\mathbf{u}\cdot\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^{\frac{8}{3}}(Q_t)}\|\mathbf{u}\cdot\nabla\mathbf{d}\|^{\frac{1}{2}}_{W^{0,1}_{\frac{8}{5}}(Q_t)}\Big]\\
&\quad\le C[(\|\mathbf{u}\|_{L^4(Q_t)} + \|\nabla\mathbf{u}\|_{L^4(Q_t)})\|\nabla\mathbf{d}\|_{L^4(Q_t)} + \|\mathbf{u}\|_{L^4(Q_t)}\|\nabla^2\mathbf{d}\|_{L^4(Q_t)}\\
&\qquad + \|\mathbf{u}\|^{\frac{1}{2}}_{L^4(Q_t)}\|\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^8(Q_t)}\\
&\quad\cdot(\|\partial_t\mathbf{u}\|^{\frac{1}{2}}_{L^2(Q_t)}\|\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^8(Q_t)} + \|\partial_t\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^2(Q_t)}\|\mathbf{u}\|^{\frac{1}{2}}_{L^8(Q_t)})],
\end{aligned}$$

and

$$\begin{aligned}
&\||\nabla\mathbf{d}|^2\mathbf{d}\|_{H^{1,\frac{1}{2}}(Q_t)}\\
&\quad\le C\Big[\|\nabla\mathbf{d}\|^2_{L^4(Q_t)} + \||\nabla\mathbf{d}|^3\|_{L^2(Q_t)} + \||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2(Q_t)}\\
&\qquad + \||\nabla\mathbf{d}|^2\mathbf{d}\|^{\frac{1}{2}}_{L^{\frac{8}{3}}(Q_t)}\|\partial_t(|\nabla\mathbf{d}|^2\mathbf{d})\|^{\frac{1}{2}}_{L^{\frac{8}{5}}(Q_t)}\Big]\\
&\quad\le C\Big[\|\nabla\mathbf{d}\|^2_{L^4(Q_t)} + \|\nabla\mathbf{d}\|^3_{L^6(Q_t)} + \|\nabla\mathbf{d}\|_{L^4(Q_t)}\|\nabla^2\mathbf{d}\|_{L^4(Q_t)}\\
&\qquad + \|\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^4(Q_t)}\|\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^8(Q_t)}\\
&\quad\cdot(\|\partial_t\mathbf{d}\|^{\frac{1}{2}}_{L^2(Q_t)}\|\nabla\mathbf{d}\|_{L^{16}(Q_t)} + \|\partial_t\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^2(Q_t)}\|\nabla\mathbf{d}\|^{\frac{1}{2}}_{L^8(Q_t)})\Big].
\end{aligned}$$

Substituting these two estimates into (3.13) and (3.16), we obtain that

$$\begin{aligned}
&\big(\|\mathbf{u}\|^2_{L^\infty_t H^1_x(Q_t)} + \|\mathbf{d}\|^2_{L^\infty_t H^2_x(Q_t)}\big) + \int_0^t (\|\mathbf{u}\|^2_{H^2(\Omega)} + \|\mathbf{d}\|^2_{H^3(\Omega)})\,dt\\
&\quad\le C\Big[\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)} + \|\mathbf{d}_0\|^2_{H^2(\Omega)} + \|\mathbf{h}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t\mathbf{h}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}\Big]\\
&\qquad + C\Big[\|\nabla\mathbf{d}\|^4_{L^4(Q_t)} + \|\nabla\mathbf{d}\|^6_{L^6(Q_t)} + \|\partial_t\mathbf{u}\|_{L^2(Q_t)}\|\mathbf{u}\|_{L^4(Q_t)}\|\nabla\mathbf{d}\|^2_{L^8(Q_t)}\\
&\qquad + (\|\mathbf{u}\|^2_{L^4(Q_t)} + \|\nabla\mathbf{d}\|^2_{L^4(Q_t)})\cdot(\|\nabla\mathbf{u}\|^2_{L^4(Q_t)} + \|\nabla^2\mathbf{d}\|^2_{L^4(Q_t)})\\
&\qquad + \|\mathbf{u}\|^2_{L^4(Q_t)}\|\nabla\mathbf{d}\|^2_{L^4(Q_t)} + \|\partial_t\nabla\mathbf{d}\|_{L^2(Q_t)}\|\nabla\mathbf{d}\|_{L^8(Q_t)}\\
&\quad\cdot(\|\mathbf{u}\|_{L^4(Q_t)}\|\mathbf{u}\|_{L^8(Q_t)} + \|\nabla\mathbf{d}\|_{L^4(Q_t)}\|\nabla\mathbf{d}\|_{L^8(Q_t)})\\
&\qquad + \|\partial_t\mathbf{d}\|_{L^2(Q_t)}\|\nabla\mathbf{d}\|_{L^4(Q_t)}\|\nabla\mathbf{d}\|_{L^8(Q_t)}\|\nabla\mathbf{d}\|^2_{L^{16}(Q_t)}\Big]
\end{aligned}\tag{3.17}$$

for any $0 < t \le T$, where $Q_t = \Omega \times [0, t]$.

To simplify the presentation, we set two auxiliary functions

$$\Phi(t) = \|\mathbf{u}\|^2_{L^\infty_t H^1_x(Q_t)} + \|\mathbf{d}\|^2_{L^\infty_t H^2_x(Q_t)} + \int_0^t (\|\mathbf{u}\|^2_{H^2(\Omega)} + \|\mathbf{d}\|^2_{H^3(\Omega)})\,dt,$$

and

$$\eta(t) = \int_0^t (\|\mathbf{u}\|^2_{H^1(\Omega)} + \|\mathbf{d}\|^2_{H^2(\Omega)})\,dt$$

for $t \in [0, T]$.

From (3.5) and Sobolev's inequality, it is readily seen that

$$\begin{cases} \|\nabla \mathbf{d}\|_{L^4(Q_t)}^4 \leq C\|\nabla \mathbf{d}\|_{L_t^\infty L_x^2(Q_t)}^2 \|\mathbf{d}\|_{L_t^2 H_x^2(Q_t)}^2 \leq C\eta(t), \\ \|\nabla^2 \mathbf{d}\|_{L^4(Q_t)}^4 \leq C\|\mathbf{d}\|_{L_t^\infty H_x^2(Q_t)}^2 \|\mathbf{d}\|_{L_t^2 H_x^3(Q_t)}^2 \leq C\Phi^2(t), \\ \|\nabla \mathbf{d}\|_{L^6(Q_t)}^6 \leq C\|\nabla \mathbf{d}\|_{L_t^\infty L_x^2(Q_t)}^3 \|\mathbf{d}\|_{L_t^2 H_x^3(Q_t)} \|\mathbf{d}\|_{L_t^\infty H_x^2(Q_t)} \|\mathbf{d}\|_{L_t^2 H_x^2(Q_t)} \\ \qquad \leq C\eta^{\frac{1}{2}}(t)\Phi(t), \\ \|\nabla \mathbf{d}\|_{L^8(Q_t)}^8 \leq C\|\nabla \mathbf{d}\|_{L_t^\infty L_x^2(Q_t)}^3 \|\mathbf{d}\|_{L_t^2 H_x^3(Q_t)} \|\mathbf{d}\|_{L_t^\infty H_x^2(Q_t)}^3 \|\mathbf{d}\|_{L_t^2 H_x^2(Q_t)} \\ \qquad \leq C\eta^{\frac{1}{2}}(t)\Phi^2(t), \\ \|\nabla \mathbf{d}\|_{L^{16}(Q_t)}^{16} \leq C\|\nabla \mathbf{d}\|_{L_t^\infty L_x^2(Q_t)}^3 \|\mathbf{d}\|_{L_t^2 H_x^3(Q_t)} \|\mathbf{d}\|_{L_t^\infty H_x^2(Q_t)}^{11} \|\mathbf{d}\|_{L_t^2 H_x^2(Q_t)} \\ \qquad \leq C\eta^{\frac{1}{2}}(t)\Phi^6(t), \end{cases} \tag{3.18}$$

where C is the positive constant depending only on Ω and K_{T,ε_1}. Similarly, we also have that

$$\begin{cases} \|\mathbf{u}\|_{L^4(Q_t)}^4 \leq C\|\mathbf{u}\|_{L_t^\infty L_x^2(Q_t)}^2 \|\mathbf{u}\|_{L_t^2 H_x^1(Q_t)}^2 \leq C\eta(t), \\ \|\nabla \mathbf{u}\|_{L^4(Q_t)}^4 \leq C\|\mathbf{u}\|_{L_t^\infty H_x^1(Q_t)}^2 \|\mathbf{u}\|_{L_t^2 H_x^2(Q_t)}^2 \leq C\Phi^2(t), \\ \|\mathbf{u}\|_{L^8(Q_t)}^8 \leq C\|\mathbf{u}\|_{L_t^\infty L_x^2(Q_t)}^3 \|\mathbf{u}\|_{L_t^2 H_x^2(Q_t)} \|\mathbf{u}\|_{L_t^\infty H_x^1(Q_t)}^3 \|\mathbf{u}\|_{L_t^2 H_x^1(Q_t)} \\ \qquad \leq C\eta^{\frac{1}{2}}(t)\Phi^2(t). \end{cases} \tag{3.19}$$

It follows from (3.13), (3.18), and (3.19) that

$$\|\partial_t \mathbf{u}\|_{L^2(Q_t)} \leq C\left[\|\nabla \mathbf{u}_0\|_{L^2(\Omega)} + \eta^{\frac{1}{4}}\Phi^{\frac{1}{2}}(t)\right], \tag{3.20}$$

and it follows from (3.15) and (3.14) that

$$\begin{aligned} \|\partial_t \mathbf{d}\|_{L^2(Q_t)} &\leq C\Big[\|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\mathbf{d}_0\|_{H^2(\Omega)} \\ &\quad + \|\mathbf{u}\|_{L^4(Q_t)}\|\nabla \mathbf{d}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}\|_{L^4(Q_t)}^2\Big] \\ &\leq C\Big[\|\partial_t \mathbf{h}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\mathbf{d}_0\|_{H^2(\Omega)} + \eta^{\frac{1}{2}}(t)\Big]. \end{aligned} \tag{3.21}$$

From (3.18), (3.19), and the equation

$$\partial_t \nabla \mathbf{d} = \nabla \Delta \mathbf{d} - \nabla(\mathbf{u} \cdot \nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}),$$

we have that

$$\begin{aligned} &\|\partial_t \nabla \mathbf{d}\|_{L^2(Q_T)} \\ &\leq (\|\nabla \Delta \mathbf{d}\|_{L^2(Q_T)} + \|\nabla(\mathbf{u} \cdot \nabla \mathbf{d})\|_{L^2(Q_T)} + \|\nabla(|\nabla \mathbf{d}|^2 \mathbf{d})\|_{L^2(Q_T)}) \\ &\leq C\Big[\Phi^{\frac{1}{2}}(t) + \|\nabla \mathbf{u}\|_{L^4(Q_t)}\|\nabla \mathbf{d}\|_{L^4(Q_t)} + \|\mathbf{u}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}\|_{L^4(Q_t)} \\ &\quad + \|\nabla \mathbf{d}\|_{L^6(Q_t)}^3 + \|\nabla \mathbf{d}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}\|_{L^4(Q_t)}\Big] \\ &\leq C\Big[\Phi^{\frac{1}{2}}(t) + \eta^{\frac{1}{4}}(t)\Phi^{\frac{1}{2}}(t)\Big]. \end{aligned} \tag{3.22}$$

Putting all these estimates into (3.17), we obtain that

$$\Phi(t) \le C\big[\|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\mathbf{d}_0\|^2_{H^2(\Omega)} + \|\mathbf{h}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}\big]$$
$$+ C\big[\eta(t) + (\eta^{\frac{3}{8}}(t) + \eta^{\frac{4}{8}}(t) + \eta^{\frac{5}{8}}(t) + \eta^{\frac{7}{8}}(t))\Phi(t)\big]. \tag{3.23}$$

From the uniform absolute continuity property (3.12), there exists $0 < t_0 \le T$ such that

$$C\big[\eta^{\frac{3}{8}}(t_0) + \eta^{\frac{4}{8}}(t_0) + \eta^{\frac{5}{8}}(t_0) + \eta^{\frac{7}{8}}(t_0)\big] \le \frac{1}{2},$$

and hence we arrive at

$$\Phi(t_0) \le C\big[1 + \|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\mathbf{d}_0\|^2_{H^2(\Omega)} + \|\mathbf{h}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}\big]$$
$$+ \frac{1}{2}\Phi(t_0).$$

This further yields

$$\Phi(t_0) \le C\big[1 + \|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\mathbf{d}_0\|^2_{H^2(\Omega)} + \|\mathbf{h}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t \mathbf{h}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}\big].$$

Hence (3.2) holds with T replaced by t_0. Next we can repeat the same argument as above to show that (3.2) also holds with T replaced by $2t_0$. After iterating finitely many times, we can see that (3.2) holds with T. This completes the proof of Theorem 3.1. □

Next we will establish the continuous dependence of the global strong solution to the system (1.1)–(1.3) for initial data in $\mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$ and boundary data in $H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)$, which is crucial to the Fréchet differentiability of the control to state operator $\mathcal{S}$.

Theorem 3.3 *Under the same assumptions of Theorem 3.1, let* $(\mathbf{u}^{(i)}, \mathbf{d}^{(i)})$, $i = 1, 2$, *be the global strong solution corresponding to the initial data* $(\mathbf{u}_0^{(i)}, \mathbf{d}_0^{(i)})$ *and the boundary data* $(0, \mathbf{h}^{(i)})$. *Define* $\overline{\mathbf{u}} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$, $\overline{\mathbf{d}} = \mathbf{d}^{(1)} - \mathbf{d}^{(2)}$, $\overline{\mathbf{u}}_0 = \mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)}$, $\overline{\mathbf{d}}_0 = \mathbf{d}_0^{(1)} - \mathbf{d}_0^{(2)}$ *and* $\overline{\mathbf{h}} = \mathbf{h}^{(1)} - \mathbf{h}^{(2)}$. *Then it holds that*

$$\|\overline{\mathbf{u}}(t)\|^2_{H^1(\Omega)} + \|\overline{\mathbf{d}}(t)\|^2_{H^2(\Omega)} + \int_0^t (\|\overline{\mathbf{u}}(\tau)\|^2_{H^2(\Omega)} + \|\overline{\mathbf{d}}(\tau)\|^2_{H^3(\Omega)})\, d\tau$$
$$\le C_T\big(\|\overline{\mathbf{u}}_0\|^2_{H^1(\Omega)} + \|\overline{\mathbf{d}}_0\|^2_{H^2(\Omega)} + \|\overline{\mathbf{h}}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_t)}\big) \quad \forall t \in [0, T], \tag{3.24}$$

where $C_T > 0$, *depending on* $\|\mathbf{u}_0^{(i)}\|_{H^1(\Omega)}$, $\|\mathbf{d}_0^{(i)}\|_{H^2(\Omega)}$, $\|\mathbf{h}^{(i)}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}$, *and* $\|\partial_t \mathbf{h}^{(i)}\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}$ *for* $i = 1, 2$, ε_1, Ω, *and* T.

Proof For $0 < t \le T$, define

$$\Phi(t) = \sum_{i=1}^{2} (\|\mathbf{u}^{(i)}\|^2_{L^\infty_t H^1_x(Q_t)} + \|\mathbf{d}^{(i)}\|^2_{L^\infty_t H^2_x(Q_t)}),$$

and

$$\Psi(t) = \int_0^t \sum_{i=1}^{2} \big(\|\mathbf{u}^{(i)}(\tau)\|^2_{H^2(\Omega)} + \|\mathbf{d}^{(i)}(\tau)\|^2_{H^3(\Omega)}\big)\, d\tau.$$

By Theorem 3.1, there exists $C_T > 0$, depending on $\|\mathbf{u}_0^{(i)}\|_{H^1(\Omega)}$, $\|\mathbf{d}_0^{(i)}\|_{H^2(\Omega)}$, $\|\mathbf{h}^{(i)}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}$, and $\|\partial_t \mathbf{h}^{(i)}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)}$ for $i = 1, 2$, ε_1, Ω and T, such that

$$\Phi(T) + \Psi(T) \leq C_T^2. \tag{3.25}$$

Moreover, for any $0 < \delta < 1$, there exists $t_\delta \in (0, \delta^6]$ such that

$$\Psi(t) \leq \delta^2 \quad \text{for all } 0 < t \leq t_\delta. \tag{3.26}$$

Observe that $(\overline{\mathbf{u}}, \overline{\mathbf{d}})$ satisfies the system

$$\begin{cases} \partial_t \overline{\mathbf{u}} - \Delta \overline{\mathbf{u}} + \mathbf{u}^{(1)} \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} + \nabla \overline{P} & \\ = -\nabla \cdot (\nabla \overline{\mathbf{d}} \odot \nabla \mathbf{d}^{(1)} + \nabla \mathbf{d}^{(2)} \odot \nabla \overline{\mathbf{d}}) & \text{in} \quad Q_T, \\ \nabla \cdot \overline{\mathbf{u}} = 0 & \text{in} \quad Q_T, \\ \partial_t \overline{\mathbf{d}} - \Delta \overline{\mathbf{d}} + \mathbf{u}^{(1)} \cdot \nabla \overline{\mathbf{d}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{d}^{(2)} & \\ = |\nabla \mathbf{d}^{(1)}|^2 \overline{\mathbf{d}} + \langle \nabla (\mathbf{d}^{(1)} + \mathbf{d}^{(2)}), \nabla \overline{\mathbf{d}} \rangle \mathbf{d}^{(2)} & \text{in} \quad Q_T, \\ \left(\overline{\mathbf{u}}, \overline{\mathbf{d}}\right) = \left(0, \overline{\mathbf{h}}\right) & \text{on} \quad \Gamma_T, \\ \left(\overline{\mathbf{u}}, \overline{\mathbf{d}}\right)\big|_{t=0} = \left(\overline{\mathbf{u}}_0, \overline{\mathbf{d}}_0\right) & \text{in} \quad \Omega. \end{cases} \tag{3.27}$$

Let $\overline{\mathbf{h}}_P$ be the parabolic lifting function of $\overline{\mathbf{h}}$:

$$\begin{cases} \partial_t \overline{\mathbf{h}}_P - \Delta \overline{\mathbf{h}}_P = 0, & \text{in} \quad Q_T, \\ \overline{\mathbf{h}}_P = \overline{\mathbf{h}}, & \text{on} \quad \Gamma_T, \\ \overline{\mathbf{h}}_P|_{t=0} = \overline{\mathbf{d}}_0, & \text{in} \quad \Omega. \end{cases} \tag{3.28}$$

By the regularity theory of parabolic equations, we have that

$$\begin{aligned} &\|\overline{\mathbf{h}}_P\|_{L_t^\infty H_x^2(Q_T)} + \|\overline{\mathbf{h}}_P\|_{H^{3,\frac{3}{2}}(Q_T)} \\ &\quad \leq C\Big[\|\partial_t \overline{\mathbf{h}}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} \|\overline{\mathbf{h}}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\overline{\mathbf{d}}_0\|_{H^2(\Omega)}\Big]. \end{aligned} \tag{3.29}$$

Set $\widehat{\mathbf{d}} = \overline{\mathbf{d}} - \overline{\mathbf{h}}_P$. Then $\widehat{\mathbf{d}}$ solves

$$\begin{cases} \partial_t \widehat{\mathbf{d}} - \Delta \widehat{\mathbf{d}} = -\mathbf{u}^{(1)} \cdot \nabla \overline{\mathbf{d}} - \overline{\mathbf{u}} \cdot \nabla \mathbf{d}^{(2)} + |\nabla \mathbf{d}^{(1)}|^2 \overline{\mathbf{d}} & \\ \qquad\qquad +2\langle \nabla (\mathbf{d}^{(1)} + \mathbf{d}^{(2)}), \nabla \overline{\mathbf{d}} \rangle \mathbf{d}^{(2)} & \text{in} \quad Q_T, \\ \widehat{\mathbf{d}} = 0 & \text{on} \quad \partial_p Q_T. \end{cases}$$

It follows from the regularity theory of parabolic equations and (3.29) that for $0 < t \leq T$,

$$\begin{aligned} &\|\overline{\mathbf{d}}\|_{L_t^\infty H_x^2(Q_t)} + \|\overline{\mathbf{d}}\|_{H^{3,\frac{3}{2}}(Q_t)} \\ &\quad \leq C\Big[\|\overline{\mathbf{h}}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)} + \|\partial_t \overline{\mathbf{h}}\|_{L_t^2 H_x^{\frac{3}{2}}(\Gamma_T)} + \|\overline{\mathbf{d}}_0\|_{H^2(\Omega)} + \|\mathbf{u}^{(1)} \cdot \nabla \overline{\mathbf{d}}\|_{H^{1,\frac{1}{2}}(Q_t)} \\ &\qquad + \|\overline{\mathbf{u}} \cdot \nabla \mathbf{d}^{(2)}\|_{H^{1,\frac{1}{2}}(Q_t)} + \||\nabla \mathbf{d}^{(1)}|^2 \overline{\mathbf{d}}\|_{H^{1,\frac{1}{2}}(Q_t)} \\ &\qquad + \|\langle \nabla (\mathbf{d}^{(1)} + \mathbf{d}^{(2)}), \nabla \overline{\mathbf{d}} \rangle \mathbf{d}^{(2)}\|_{H^{1,\frac{1}{2}}(Q_t)}\Big]. \end{aligned} \tag{3.30}$$

Since the last four terms of the right hand side of (3.30) can be estimated in a way similar to that of the proof of Theorem 3.1, we will only sketch below the estimate of $\||\nabla \mathbf{d}^{(1)}|^2 \overline{\mathbf{d}}\|_{H^{1,\frac{1}{2}}(Q_t)}$.

As in Theorem 3.1 , we first have

$$\begin{aligned}\||\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}}\|_{H^{1,\frac{1}{2}}(Q_t)} \leq C\big[&\||\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}}\|_{L^2(Q_t)} + \|\nabla(|\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}})\|_{L^2(Q_t)} \\ &+ \|\langle\partial_t(|\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}})\|_{L^{\frac{3}{2}}(Q_t)}\big].\end{aligned} \tag{3.31}$$

Here we have used Sobolev's embedding: $W^{1,1}_{\frac{3}{2}}(Q_T) \hookrightarrow H^{\frac{1}{2},\frac{1}{2}}(Q_T)$.

Applying Hölder's inequality and the Sobolev inequality, we obtain that for $0 < t \leq t_\delta$, the following estimates hold:

$$\begin{aligned}\||\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}}\|_{L^2(Q_t)} \leq &\||\nabla \mathbf{d}^{(1)}|^2\|_{L^2(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty(Q_t)} \\ \leq &\|\nabla \mathbf{d}^{(1)}\|_{L^\infty_t L^2_x(Q_t)}\|\mathbf{d}^{(1)}\|_{L^2_t H^2_x(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} \\ \leq &C_T\delta\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla(|\nabla \mathbf{d}^{(1)}|^2\overline{\mathbf{d}})\|_{L^2(Q_t)} &\leq C\big[\|\nabla \mathbf{d}^{(1)}\|^2_{L^4_t L^8_x(Q_t)}\|\nabla\overline{\mathbf{d}}\|_{L^\infty_t L^4_x(Q_t)} \\ &\quad + \|\nabla \mathbf{d}^{(1)}\|_{L^4(Q_t)}\|\nabla^2 \mathbf{d}^{(1)}\|_{L^4(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty(Q_t)}\big] \\ &\leq C\|\mathbf{d}^{(1)}\|^2_{L^4_t H^2_x(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} \\ &\leq C\|\mathbf{d}^{(1)}\|_{L^\infty_t H^2_x(Q_t)}\|\mathbf{d}^{(1)}\|_{L^2_t H^2_x(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} \\ &\leq C_T\delta\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}.\end{aligned}$$

Applying the equation of $\mathbf{d}^{(1)}$, Hölder's inequality, and the interpolation inequality, we can estimate

$$\begin{aligned}&\|\partial_t\nabla \mathbf{d}^{(1)}\|_{L^2(Q_t)} \\ &\leq \|\nabla\Delta \mathbf{d}^{(1)}\|_{L^2(Q_t)} + \|\nabla(\mathbf{u}^{(1)}\cdot\nabla \mathbf{d}^{(1)})\|_{L^2(Q_t)} + \|\nabla(|\nabla \mathbf{d}^{(1)}|^2\mathbf{d}^{(1)})\|_{L^2(Q_t)} \\ &\leq \|\mathbf{d}^{(1)}\|_{L^2_t H^3_x(Q_t)} + \big(\|\mathbf{u}^{(1)}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}^{(1)}\|_{L^4(Q_t)}\big)\|\nabla^2 \mathbf{d}^{(1)}\|_{L^4(Q_t)} \\ &\quad + \|\nabla \mathbf{u}^{(1)}\|_{L^4(Q_t)}\|\nabla \mathbf{d}^{(1)}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}^{(1)}\|^3_{L^6(Q_t)} \leq C_T.\end{aligned}$$

Applying (3.27)$_3$ and Hölder's inequality, we have that

$$\begin{aligned}&\|\partial_t\overline{\mathbf{d}}\|_{L^2(Q_t)} \\ &\leq \|\Delta\overline{\mathbf{d}}\|_{L^2(Q_t)} + \|\mathbf{u}^{(1)}\cdot\nabla\overline{\mathbf{d}}\|_{L^2(Q_t)} + \|\overline{\mathbf{u}}\cdot\nabla \mathbf{d}^{(2)}\|_{L^2(Q_t)} \\ &\quad + \||\nabla \mathbf{d}^{(2)}|^2\overline{\mathbf{d}}\|_{L^2(Q_t)} + 2\|\langle\nabla(\mathbf{d}^{(1)} + \mathbf{d}^{(2)}), \nabla\overline{\mathbf{d}}\rangle \mathbf{d}^{(2)}\|_{L^2(Q_t)} \\ &\leq Ct^{\frac{1}{2}}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\mathbf{u}^{(1)}\|_{L^4(Q_t)}\|\nabla\overline{\mathbf{d}}\|_{L^4(Q_t)} \\ &\quad + \|\overline{\mathbf{u}}\|_{L^4(Q_t)}\|\nabla\overline{\mathbf{d}}^{(2)}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}^{(2)}\|^2_{L^4(Q_t)}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} \\ &\quad + 2(\|\nabla \mathbf{d}^{(1)}\|_{L^4(Q_t)} + \|\nabla \mathbf{d}^{(2)}\|_{L^4(Q_t)})\|\nabla\overline{\mathbf{d}}\|_{L^4(Q_t)} \\ &\leq t^{\frac{1}{2}}\|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + C\Psi^{\frac{1}{2}}(t)\big(\|\overline{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\big) \\ &\leq C\delta\big(\|\overline{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\overline{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\big).\end{aligned}$$

Hence

$$
\begin{aligned}
&\|\partial_t(|\nabla \mathbf{d}^{(1)}|^2\bar{\mathbf{d}})\|_{L^{\frac{3}{2}}(Q_t)}\\
&\quad\le \|\nabla \mathbf{d}^{(1)}\|^2_{L^{12}(Q_t)}\|\partial_t\bar{\mathbf{d}}\|_{L^2(Q_t)} + \|\partial_t\nabla \mathbf{d}^{(1)}\|_{L^2(Q_t)}\|\nabla \mathbf{d}^{(1)}\|_{L^6(Q_t)}\|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\\
&\quad\le Ct^{\frac{1}{6}}\|\mathbf{d}^{(1)}\|^2_{L^\infty_t H^2_x(Q_t)}\|\partial_t\bar{\mathbf{d}}\|_{L^2(Q_t)}\\
&\qquad + Ct^{\frac{1}{6}}\|\partial_t\nabla \mathbf{d}^{(1)}\|_{L^2(Q_t)}\|\mathbf{d}^{(1)}\|_{L^\infty_t H^2_x(Q_t)}\|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\\
&\quad\le (1+C_T^2)t^{\frac{1}{6}}\big(\|\partial_t\bar{\mathbf{d}}\|_{L^2(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\big)\\
&\quad\le (1+C_T^2)t^{\frac{1}{6}}\big(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}\big),
\end{aligned}
$$

where we have used the Sobolev inequalities

$$
\|\nabla \mathbf{d}^{(1)}\|_{L^6(Q_t)} \le Ct^{\frac{1}{6}}\|\mathbf{d}^{(1)}\|_{L^\infty_t H^2_x(Q_t)},\ \|\nabla \mathbf{d}^{(1)}\|_{L^{12}(Q_t)} \le Ct^{\frac{1}{12}}\|\mathbf{d}^{(1)}\|_{L^\infty_t H^2_x(Q_t)}.
$$

Putting all these estimates into (3.31), we obtain that for $0 \le t \le t_\delta$,

$$
\||\nabla \mathbf{d}^{(1)}|^2\bar{\mathbf{d}}\|_{H^{1,\frac{1}{2}}(Q_t)} \le C_T\delta(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)}).
$$

Similarly, we can estimate

$$
\begin{aligned}
\big\|\mathbf{u}^{(1)}\cdot\nabla\bar{\mathbf{d}}\big\|_{H^{1,\frac{1}{2}}(Q_t)} &\le C_T\delta\big(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^2_t H^3_x(Q_t)}\big),\\
\big\|\bar{\mathbf{u}}\cdot\nabla \mathbf{d}^{(2)}\big\|_{H^{1,\frac{1}{2}}(Q_t)} &\le C_T\delta\big(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\bar{\mathbf{u}}\|_{L^2_t H^2_x(Q_t)}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\langle\nabla(\mathbf{d}^{(1)}+\mathbf{d}^{(2)}),\nabla\bar{\mathbf{d}}\rangle\mathbf{d}^{(2)}\big\|_{H^{1,\frac{1}{2}}(Q_t)}\\
&\quad\le C_T\delta\big(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^2_t H^3_x(Q_t)}\big).
\end{aligned}
$$

Therefore we obtain that for $0 \le t \le t_\delta$,

$$
\begin{aligned}
&\|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^2_t H^3_x(Q_t)}\\
&\quad\le C\big[\|\bar{\mathbf{h}}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}\|\partial_t\bar{\mathbf{h}}\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)} + \|\bar{\mathbf{d}}_0\|_{H^2(\Omega)}\\
&\qquad + \delta(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)} + \|\bar{\mathbf{u}}\|_{L^2_t H^2_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^2_t H^3_x(Q_t)})\big].
\end{aligned}
\tag{3.32}
$$

We can apply the $W_2^{2,1}$-regularity theory of $(3.27)_1$ to estimate $\bar{\mathbf{u}}$ as follows. For $0 < t \le t_\delta$, it holds that

$$
\begin{aligned}
&\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{u}}\|_{L^2_t H^2_x(Q_t)}\\
&\quad\le C\big[\|\bar{\mathbf{u}}_0\|_{H^1(\Omega)} + \|\mathbf{u}^{(1)}\cdot\nabla\bar{\mathbf{u}}\|_{L^2(Q_t)} + \|\bar{\mathbf{u}}\cdot\nabla\mathbf{u}^{(2)}\|_{L^2(Q_t)}\\
&\qquad + \|\nabla\cdot(\nabla\bar{\mathbf{d}}\odot\nabla\mathbf{d}^{(1)} + \nabla\mathbf{d}^{(2)}\odot\nabla\bar{\mathbf{d}})\|_{L^2(Q_t)}\big]\\
&\quad\le C\big[\|\bar{\mathbf{u}}_0\|_{H^1(\Omega)} + C_T\delta(\|\bar{\mathbf{u}}\|_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|_{L^\infty_t H^2_x(Q_t)})\big].
\end{aligned}
\tag{3.33}
$$

For $0 \le t \le T$, set

$$
\overline{\Phi}(t) = (\|\bar{\mathbf{u}}\|^2_{L^\infty_t H^1_x(Q_t)} + \|\bar{\mathbf{d}}\|^2_{L^\infty_t H^2_x(Q_t)}) + (\|\bar{\mathbf{u}}\|^2_{L^2_t H^2_x(Q_t)} + \|\bar{\mathbf{d}}\|^2_{L^2_t H^3_x(Q_t)}).
$$

Then (3.32) and (3.33) imply that for $0 \le t \le t_\delta$,

$$\begin{aligned}\overline{\Phi}(t) &\le C_T \delta \overline{\Phi}(t)\\ &\quad + C\Big[\|\overline{\mathbf{u}}_0\|^2_{H^1(\Omega)} + \|\overline{\mathbf{d}}_0\|^2_{H^2(\Omega)} + \|\overline{\mathbf{h}}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_t)} + \|\partial_t \overline{\mathbf{h}}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}\Big].\end{aligned}$$

If we choose $\delta > 0$ such that $C_T\delta \le \frac{1}{2}$, then we conclude that (3.24) holds for all $t \in [0, t_\delta]$. By repeating the same argument for $t \in [it_\delta, (i+1)t_\delta]$ for $i = 1, \ldots, [\frac{T}{t_\delta}] + 1$, we see that (3.24) holds for all $t \in [0, T]$. This completes the proof of Theorem 3.3. $\square$

4 Optimal boundary control

The second main purpose of this paper is to consider the optimal boundary control problem (1.4) for the nematic liquid crystal flow (1.1)–(1.3). This part is an extension of Sects. 4 and 5 of [4]. However, due to the nonlinear constraint $|\mathbf{d}| = 1$, we need to overcome several new difficulties.

For a given $0 < T < \infty$, we make the following assumptions:

(A1) $\beta_i \ge 0$ $(i = 1, 2, 3, 4, 5)$ are constants that do not all vanish.
(A2) The vector-valued functions

$$\mathbf{u}_{Q_T} \in L^2([0,T], \mathbf{H}),\ \mathbf{d}_{Q_T} \in L^2(Q_T, \mathbb{S}^2),\ \mathbf{u}_\Omega \in \mathbf{H},\ \mathbf{d}_\Omega \in L^2(\Omega, \mathbb{S}^2)$$

are given target maps.

The optimal boundary control problem (1.4) seeks a boundary data $\mathbf{h}$ in a suitable function space that minimizes the cost functional:

$$\begin{aligned}2\mathcal{C}((\mathbf{u},\mathbf{d}),\mathbf{h}) := {}& \beta_1\|\mathbf{u} - \mathbf{u}_{Q_T}\|^2_{L^2(Q_T)} + \beta_2\|\mathbf{d} - \mathbf{d}_{Q_T}\|^2_{L^2(Q_T)}\\ &+ \beta_3\|\mathbf{u}(T) - \mathbf{u}_\Omega\|^2_{L^2(\Omega)} + \beta_4\|\mathbf{d}(T) - \mathbf{d}_\Omega\|^2_{L^2(\Omega)}\\ &+ \beta_5\|\mathbf{h} - \mathbf{e}_3\|^2_{L^2(\Gamma_T)},\end{aligned} \tag{4.1}$$

where $(\mathbf{u}, \mathbf{d})$ is the unique strong solution of (1.1)–(1.3) under the boundary condition $(0, \mathbf{h})$ and the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$.

4.1 Fréchet differentiability of the control to state map

In this subsection, we will study the control to state map $\mathcal{S}$ and establish its Fréchet differentiability over suitable function spaces.

4.1.1 Function space of admissible boundary control data

The natural function space for the boundary control data $\mathbf{h}$, that guarantees the existence of unique strong solutions to the system (1.1)–(1.3), is

$$\mathcal{U} \equiv \Big\{\mathbf{h} \mid \mathbf{h} \in H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T, \mathbb{S}^2_+) \text{ and } \partial_t\mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T)\Big\}, \tag{4.2}$$

which is equipped with the norm

$$\|\mathbf{h}\|_{\mathcal{U}} := \|\mathbf{h}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T))} + \|\partial_t\mathbf{h}\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)},\quad \mathbf{h} \in \mathcal{U}.$$

Given an initial data $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$, the *function space for boundary control functions* $\widetilde{\mathcal{U}}$, associated with $(\mathbf{u}_0, \mathbf{d}_0)$, is defined by

$$\widetilde{\mathcal{U}} := \Big\{\mathbf{h} \mid \mathbf{h} \in \mathcal{U}, \text{ with } \mathbf{h}(x, 0) = \mathbf{d}_0(x) \text{ on } \Gamma\Big\}. \tag{4.3}$$

Remark 4.1 By the Aubin–Lions Lemma and the Sobolev embedding Theorem, we have that

$$\mathcal{U} \hookrightarrow C([0, T], H^{\frac{3}{2}}(\Gamma, \mathbb{S}^2_+)) \hookrightarrow C(\Gamma_T, \mathbb{S}^2_+),$$

and hence any $\mathbf{h} \in \widetilde{\mathcal{U}}$ is continuous on Γ_T and the compatibility condition $\mathbf{h}(x, 0) = \mathbf{d}_0(x)$ holds for $x \in \Gamma$ in the classical sense.

The minimization problem is taken place in a bounded, *intrinsically* convex closed set in $\widetilde{\mathcal{U}}$ that will be specified below.

Let $\Pi : \mathbb{S}^2 \setminus \{-\mathbf{e}_3\} \to \mathbb{R}^2$ be the stereographic projection from the south pole $-\mathbf{e}_3$, and Π^{-1} be its inverse map. Then $\Pi : \mathbb{S}^2_+ \mapsto B^2_1 = \big\{y \in \mathbb{R}^2 : |y| \le 1\big\}$ is a smooth differeomorphism. It is clear that any map $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2_+$ belongs to $\widetilde{\mathcal{U}}$ if and only if $\Pi(\mathbf{h}) : \Gamma_T \mapsto B^2_1$ satisfies

$$\Pi(\mathbf{h}) \in H^{\frac{5}{2}, \frac{5}{4}}(\Gamma_T, B^2_1) \text{ and } \partial_t(\Pi(\mathbf{h})) \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T),$$

and $\Pi(\mathbf{h})(x, 0) = \Pi(\mathbf{d}_0)(x)$ for $x \in \Gamma$.

We also equip $\Pi(\mathbf{h})$ with the norm

$$\big\|\Pi(\mathbf{h})\big\|_{\mathcal{U}} := \big\|\Pi(\mathbf{h})\big\|_{H^{\frac{5}{2}, \frac{5}{4}}(\Gamma_T)} + \big\|\partial_t(\Pi(\mathbf{h}))\big\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}.$$

Definition 4.1 For $M > 0$, we define the intrinsic ball in $\widetilde{\mathcal{U}}$ with center 0 and radius M, denoted as $\widetilde{\mathcal{U}}_M$, by

$$\widetilde{\mathcal{U}}_M = \Big\{\mathbf{h} \in \widetilde{\mathcal{U}} \mid \|\Pi(\mathbf{h})\|_{\mathcal{U}} \le M\Big\}. \tag{4.4}$$

It is not hard to see that for sufficiently large $M > 0$, $\widetilde{\mathcal{U}}_M \neq \emptyset$. In fact, there exists $C > 1$ such that if $M \ge C\|\mathbf{d}_0\|_{H^3(\Omega)}$ then we can construct $\mathbf{h} \in \widetilde{\mathcal{U}}$ such that $\|\Pi(\mathbf{h})\|_{\mathcal{U}} \le M$ and $\mathbf{h}(\cdot, 0) = \mathbf{d}_0(\cdot)$ on Γ. For example, let $\mathbf{h} : \Gamma_T \mapsto \mathbb{S}^2$ be the solution to the heat flow of harmonic map from Γ to $\mathbb{S}^2$:

$$\begin{cases} \partial_t \mathbf{h} - \Delta_\Gamma \mathbf{h} = |\nabla_\Gamma \mathbf{h}|^2 \mathbf{h} & \text{in } \Gamma_T, \\ \mathbf{h}(\cdot, 0) = \mathbf{d}_0(\cdot) & \text{on } \Gamma. \end{cases}$$

Here ∇_Γ and Δ_Γ denote the gradient and Laplace operator on Γ. Since Γ is a 1-dimensional smooth closed curve and $\mathbf{d}_0 \in H^{\frac{5}{2}}(\Gamma, \mathbb{S}^2_+)$, it follows from the standard theory of heat flow of harmonic maps in dimensions one that there exists a unique solution $\mathbf{h} \in H^{\frac{5}{2}, \frac{5}{4}}(\Gamma_T, \mathbb{S}^2)$, with $\partial_t \mathbf{h} \in L^2_t H^{\frac{3}{2}}_x(\Gamma_T)$, such that

$$\|\Pi(\mathbf{h})\|_{\mathcal{U}} \le C\|\mathbf{d_0}\|_{H^3(\Omega)} \le M.$$

Moreover, it follows from $\mathbf{d}^3_0 \ge 0$ on Γ that $\mathbf{h}^3 \ge 0$. Therefore, $\widetilde{\mathcal{U}}_M$ is non-empty.

Remark 4.2 It is clear that $\widetilde{\mathcal{U}}_M$ is convex in the following sense: if $\mathbf{h}_1, \mathbf{h}_2 \in \widetilde{\mathcal{U}}_M$, then $\Pi^{-1}(s\Pi(\mathbf{h}_1) + (1 - s)\Pi(\mathbf{h}_2)) \in \widetilde{\mathcal{U}}_M$ for all $s \in [0, 1]$.

In fact, it follows from the definition of $\widetilde{\mathcal{U}}_M$ that $\Pi(\mathbf{h}_i) : \Gamma_T \mapsto B_1^2$ for $i = 1, 2$, this implies that $s\Pi(\mathbf{h}_1) + (1-s)\Pi(\mathbf{h}_2) : \Gamma_T \mapsto B_1^2$ for $s \in [0, 1]$. Also note that

$$\begin{aligned}\left\| s\Pi(\mathbf{h}_1) + (1-s)\Pi(\mathbf{h}_2)\right\|_{\mathcal{U}} &\leq s\left\|\Pi(\mathbf{h}_1)\right\|_{\mathcal{U}} + (1-s)\left\|\Pi(\mathbf{h}_2)\right\|_{\mathcal{U}} \\ &\leq sM + (1-s)M = M.\end{aligned}$$

Thus $\mathbf{h}(s) = \Pi^{-1}(s\Pi(\mathbf{h}_1) + (1-s)\Pi(\mathbf{h}_2)) \in C^1([0, 1], \widetilde{\mathcal{U}}_M)$ is a path joining $\mathbf{h}(0) = \mathbf{h}_1$ and $\mathbf{h}(1) = \mathbf{h}_2$.

4.1.2 The control-to-state operator $\mathcal{S}$

To define $\mathcal{S}$, we first need to introduce the function space for global strong solutions to the system (1.1)–(1.3):

$$\mathcal{H} = C([0, T], \mathbf{V}) \cap L_t^2 H_x^2(Q_T) \times C([0, T], H^2(\Omega, \mathbb{S}_+^2)) \cap L_t^2 H_x^3(Q_T), \tag{4.5}$$

which is equipped with the norm

$$\left\|(\mathbf{u}, \mathbf{d})\right\|_{\mathcal{H}} = \left\|\mathbf{u}\right\|_{L_t^\infty H_x^1(Q_T)} + \left\|\mathbf{u}\right\|_{L_t^2 H_x^2(Q_T)} + \left\|\mathbf{d}\right\|_{L_t^\infty H_x^2(Q_T)} + \left\|\mathbf{d}\right\|_{L_t^2 H_x^3(Q_T)}.$$

We also introduce the function space for the Fréchet derivative of $\mathcal{S}$:

$$\mathcal{W} = C([0, T], \mathbf{H}) \cap L_t^2 H_x^1(Q_T) \times C([0, T], H^1(\Omega, \mathbb{R}^3)) \cap L_t^2 H_x^2(Q_T), \tag{4.6}$$

which is equipped with the norm

$$\begin{aligned}&\left\|(\boldsymbol{\omega}, \boldsymbol{\phi})\right\|_{\mathcal{W}} \\ &\quad= \left\|\boldsymbol{\omega}\right\|_{L_t^\infty L_x^2(Q_T)} + \left\|\boldsymbol{\omega}\right\|_{L_t^2 H_x^1(Q_T)} + \left\|\boldsymbol{\phi}\right\|_{L_t^\infty H_x^1(Q_T)} + \left\|\boldsymbol{\phi}\right\|_{L_t^2 H_x^2(Q_T)}.\end{aligned}$$

Note that $\mathcal{H}$ is a subset of $\mathcal{W}$. Now we define the *control-to-state* map $\mathcal{S}$ as follows.

Definition 4.2 Given an initial data $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}_+^2)$, the control-to-state mapping $\mathcal{S} : \widetilde{\mathcal{U}} \mapsto \mathcal{H}$, associated with $(\mathbf{u}_0, \mathbf{d}_0)$, is defined by letting

$$\mathbf{h} \in \widetilde{\mathcal{U}} \mapsto \mathcal{S}(\mathbf{h}) = (\mathbf{u}, \mathbf{d}) \in \mathcal{H} \tag{4.7}$$

to be the unique global strong solution to the system (1.1)–(1.3) on $[0, T]$, with the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(0, \mathbf{h})$.

It follows directly from Theorems 3.1, Remark 3.2, and Theorem 3.3 that the map $\mathcal{S}$ is Lipschitz continuous. More precisely, we have

Proposition 4.3 *For $n = 2$, $T \in (0, +\infty)$, and $M > 0$, under the same assumptions of Theorem 3.1, if $\widetilde{\mathcal{U}}_M \neq \emptyset$, then the control-to-state map $\mathcal{S}$ is Lipschitz continuous from $\widetilde{\mathcal{U}}$ to $\mathcal{H}$, i.e.,*

$$\left\|\mathcal{S}(\mathbf{h}_1) - \mathcal{S}(\mathbf{h}_2)\right\|_{\mathcal{H}} \leq C_M \left\|\mathbf{h}_1 - \mathbf{h}_2\right\|_{\mathcal{U}}, \ \forall\, \mathbf{h}_1, \mathbf{h}_2 \in \widetilde{\mathcal{U}}_M,$$

where $C_M > 0$ depends only on M, Ω, $\|\mathbf{u}_0\|_{\mathbf{H}}$, and $\left\|\mathbf{d}_0\right\|_{H^1(\Omega)}$.

4.1.3 Differentiability of the control-to-state operator $\mathcal{S}$

We will establish the differentiability of the control-to-state operator $\mathcal{S} : \widetilde{\mathcal{U}} \mapsto \mathcal{H}$.

First, we will define the Fréchet differentialiblity of $\mathcal{S}$. To do it, we need to introduce tangential spaces of $\widetilde{\mathcal{U}}$ and $\mathcal{H}$. Given an element $\mathbf{h} \in \widetilde{\mathcal{U}}$, the pullback bundle of the tangent bundle $T\widetilde{\mathcal{U}}$ by $\mathbf{h}$, $\mathbf{h}^*T\widetilde{\mathcal{U}}$, is defined by

$$\mathbf{h}^*T\widetilde{\mathcal{U}} = \Big\{\boldsymbol{\xi} \in H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T, \mathbb{R}^3) \,\Big|\, \partial_t\boldsymbol{\xi} \in L_t^2 H_x^{\frac{3}{2}}(\Gamma_T),\ \boldsymbol{\xi}(x,0) = 0 \text{ for } x \in \Gamma,$$
$$\langle \boldsymbol{\xi}, \mathbf{h}\rangle(x,t) = 0 \text{ for } (x,t) \in \Gamma_T\Big\},$$

which is equipped with the same norm $\|\cdot\|_{\mathcal{U}}$ as that on $\widetilde{\mathcal{U}}$.

For a fixed $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}_+^2)$ and an element $(\mathbf{u}, \mathbf{d}) \in \mathcal{H}$, with $(\mathbf{u}, \mathbf{d}) = (\mathbf{u}_0, \mathbf{d}_0)$ at $t = 0$, the pullback bundle of the tangent bundle $T\mathcal{H}$ by $(\mathbf{u}, \mathbf{d})$, $(\mathbf{u}, \mathbf{d})^*T\mathcal{H}$, is defined by

$$(\mathbf{u}, \mathbf{d})^*T\mathcal{H} = \Big\{(\boldsymbol{\omega}, \boldsymbol{\phi})\Big|\ \boldsymbol{\omega} \in C([0,T], \mathbf{H}) \cap L_t^2 H_x^1(Q_T),$$
$$\boldsymbol{\phi} \in C([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L_t^2 H_x^2(Q_T),$$
$$\langle \boldsymbol{\phi}, \mathbf{d}\rangle = 0 \text{ a.e. in } Q_T,\ (\boldsymbol{\omega}, \boldsymbol{\phi})\big|_{t=0} = (0,0)\Big\},$$

which is equipped with the norm $\|\cdot\|_{\mathcal{W}}$.

Definition 4.3 Given a $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}_+^2)$. For any $\mathbf{h} \in \widetilde{\mathcal{U}}$, let $(\mathbf{u}, \mathbf{d}) \in \mathcal{H}$ be the unique strong solution of (1.1)–(1.3) under the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(0, \mathbf{h})$, we say that the control to state map $\mathcal{S} : \widetilde{\mathcal{U}} \mapsto \mathcal{H}$ is Fréchet differentiable at $\mathbf{h}$, if there exists a linear map $\mathcal{S}'(h) : \mathbf{h}^*T\widetilde{\mathcal{U}} \mapsto \mathcal{S}(\mathbf{h})^*T\mathcal{H}$, called the Fréchet derivative of $\mathcal{S}$ at $\mathbf{h}$, such that for any $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\big\|\mathcal{S}(\exp_{\mathbf{h}} \boldsymbol{\xi}) - \mathcal{S}(\mathbf{h}) - \mathcal{S}'(\mathbf{h})(\boldsymbol{\xi})\big\|_{\mathcal{W}} \le \epsilon \|\boldsymbol{\xi}\|_{\mathcal{U}}, \tag{4.8}$$

whenever $\boldsymbol{\xi}$ is any section of $\mathbf{h}^*T\widetilde{\mathcal{U}}$ satisfying both $\|\boldsymbol{\xi}\|_{\mathcal{U}} \le \delta$ and $\exp_{\mathbf{h}}(\boldsymbol{\xi}) \in \widetilde{\mathcal{U}}$. Here $\exp_{\mathbf{h}}(\boldsymbol{\xi})(x,t)$ is the exponential map on $\mathbb{S}^2$ from $\mathbf{h}(x,t)$ and in the direction $\boldsymbol{\xi}(x,t)$ for any $(x,t) \in Q_T$.

Let us make two comments on Definition 4.3.

Remark 4.4 If we denote the strict upper half space by

$$\mathbb{S}_+^{2,\circ} = \mathbb{S}_+^2 \setminus \partial\mathbb{S}_+^2 = \big\{y \in \mathbb{S}^2 :\ y^3 > 0\big\}.$$

Then for any function $\mathbf{h} \in \widetilde{\mathcal{U}}$ satisfying

$$\mathbf{h}(x,t) \in \mathbb{S}_+^{2,\circ},\ \forall (x,t) \in \Gamma_T,$$

there exists $\delta = \delta(\mathbf{h}) > 0$ such that if $\boldsymbol{\xi}$ is a section $\mathbf{h}^*T\widetilde{\mathcal{U}}$ such that

$$\|\boldsymbol{\xi}\|_{\mathcal{U}} \le \delta,$$

then the exponential map $(\exp_{\mathbf{h}} \boldsymbol{\xi})(x,t) = \exp_{\mathbf{h}(x,t)} \boldsymbol{\xi}(x,t) : Q_T \mapsto \mathbb{S}^2$ has the same regularity as $\mathbf{h}$ and has its third component $(\exp_{\mathbf{h}} \boldsymbol{\xi})^3 > 0$ on Γ_T. Hence $(\exp_{\mathbf{h}} \boldsymbol{\xi})(x,t) \in \mathbb{S}_+^{2,\circ}$, for $(x,t) \in \Gamma_T$, so that $\exp_{\mathbf{h}} \boldsymbol{\xi} \in \widetilde{\mathcal{U}}$.

Remark 4.5 For $\mathbf{d}_0 \in H^2(\Omega, \mathbb{S}^{2,\circ}_+)$ and $\mathbf{h} \in \widetilde{\mathcal{U}}$ with $\mathbf{h}(\Gamma_T) \subset \mathbb{S}^{2,\circ}_+$, there exist $\delta_1 > 0$, δ_2, and $\delta_3 > 0$ depending on $\|\mathbf{d}_0\|_{H^2(\Omega)}$ and $\|\mathbf{h}\|_{\mathcal{U}}$ such that

$$\mathbf{d}_0^3(x) \geq \delta_1 \quad \forall x \in \Omega; \qquad \mathbf{h}^3(y,t) \geq \delta_1 \quad \forall (y,t) \in \Gamma_T.$$

Hence $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h}) \in \mathcal{H}$ enjoys the property that $\mathbf{d} \in C^0(Q_T)$ and

$$\mathbf{d}^3(x,t) \geq \delta_2, \ \forall (x,t) \in Q_T.$$

Therefore, for any section $\boldsymbol{\xi}$ of $\mathbf{h}^* T\widetilde{\mathcal{U}}$, if $\|\boldsymbol{\xi}\|_{\mathcal{U}} \leq \delta_3$ then $\exp_{\mathbf{h}} \boldsymbol{\xi}$ maps Γ_T to $\mathbb{S}^2_+$. In particular, $\exp_{\mathbf{h}} \boldsymbol{\xi} \in \widetilde{\mathcal{U}}$ and $\mathcal{S}(\exp_{\mathbf{h}} \boldsymbol{\xi}) \in \mathcal{H}$ is well-defined in (4.8).

Now we want to study the linearized equation of the system of (1.1)–(1.3).

4.1.4 The linearized system

For a fixed $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$, let $\mathbf{h} \in \widetilde{\mathcal{U}}$ be given and $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h})$ be the unique global strong solution to the system (1.1)–(1.3), with the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(0, \mathbf{h})$, given by Theorem 3.1.

The linearized system of (1.1)–(1.3) near $((\mathbf{u}, \mathbf{d}), \mathbf{h})$, along a section $\boldsymbol{\xi}$ of $\mathbf{h}^* T\widetilde{\mathcal{U}}$, seeks a section $(\boldsymbol{\omega}, \boldsymbol{\phi})$ of $(\mathbf{u}, \mathbf{d})^* T\mathcal{H}$ that solves

$$\begin{cases} \partial_t \boldsymbol{\omega} - \Delta \boldsymbol{\omega} + \nabla \mathcal{P} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \\ = -\nabla \cdot (\nabla \boldsymbol{\phi} \odot \nabla \mathbf{d}) - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \boldsymbol{\phi}), \\ \nabla \cdot \boldsymbol{\omega} = 0, & \text{in } Q_T \\ \partial_t \boldsymbol{\phi} - \Delta \boldsymbol{\phi} + (\mathbf{u} \cdot \nabla)\boldsymbol{\phi} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{d} \\ = |\nabla \mathbf{d}|^2 \boldsymbol{\phi} + 2\langle \nabla \mathbf{d}, \nabla \boldsymbol{\phi} \rangle \mathbf{d}, \end{cases} \tag{4.9}$$

under the boundary and initial condition

$$\begin{cases} (\boldsymbol{\omega}, \ \boldsymbol{\phi}) = (0, \boldsymbol{\xi}), & \text{on} \quad \Gamma_T, \\ (\boldsymbol{\omega}, \ \boldsymbol{\phi}) = (0, 0), & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{4.10}$$

We have the following theorem.

Theorem 4.6 *For any section $\boldsymbol{\xi}$ of $\mathbf{h}^* T\widetilde{\mathcal{U}}$, the system (4.9) and (4.10) admits a unique strong solution $(\boldsymbol{\omega}, \boldsymbol{\phi})$, which is a section of $(\mathbf{u}, \mathbf{d})^* T\mathcal{H}$, that satisfies the following estimate:*

$$\begin{aligned} &\|(\boldsymbol{\omega}, \nabla \boldsymbol{\phi})\|^2_{L^\infty_t L^2_x(Q_T)} + \int_0^T (\|\nabla \boldsymbol{\omega}(\tau)\|^2_{L^2(\Omega)} + \|\nabla^2 \boldsymbol{\phi}(\tau)\|^2_{L^2(\Omega)})\, d\tau \\ &\leq C_T \big(\|\partial_t \boldsymbol{\xi}\|^2_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)} + \|\boldsymbol{\xi}\|^2_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}\big) \leq C_T \|\boldsymbol{\xi}\|^2_{\mathcal{U}}, \end{aligned} \tag{4.11}$$

where $C_T > 0$ depends only on $\|\mathbf{u}_0\|_{H^1(\Omega)}$, $\|\mathbf{d}_0\|_{H^2(\Omega)}$, $\|\mathbf{h}\|_{\mathcal{U}}$, Ω and T.

Proof Since the existence of a strong solution $(\boldsymbol{\omega}, \boldsymbol{\phi})$ can be shown by the standard Galerkin method (see [5] Proposition 4.1) and the uniqueness of $(\boldsymbol{\omega}, \boldsymbol{\phi})$ follows from the estimate (4.11). We will only prove (4.11). First let $\boldsymbol{\xi}_P$ be the parabolic lift function of $\boldsymbol{\xi}$, i.e.,

$$\begin{cases} \partial_t \boldsymbol{\xi}_P - \Delta \boldsymbol{\xi}_P = \mathbf{0} & \text{in} \quad Q_T, \\ \boldsymbol{\xi}_P = \boldsymbol{\xi} & \text{on} \quad \Gamma_T, \\ \boldsymbol{\xi}_P = 0 & \text{in} \quad \Omega \times \{0\}. \end{cases}$$

Then we have

$$\|\nabla \boldsymbol{\xi}_P\|_{L_t^\infty L_x^2(Q_T)} + \|\nabla^2 \boldsymbol{\xi}_P\|_{L^2(Q_T)} + \|\boldsymbol{\xi}_P\|_{H^{3,\frac{3}{2}}(Q_T)} \leq C\|\boldsymbol{\xi}\|_{\mathcal{U}}. \tag{4.12}$$

Multiplying (4.9)$_1$ by $\boldsymbol{\omega}$ and (4.9)$_3$ by $\Delta\widetilde{\boldsymbol{\phi}}$, where $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi} - \boldsymbol{\xi}_P$, and adding the two resulting equations, and using Hölder's inequality, the Sobolev embedding Theorem and Poincaré inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2) + (\|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2)\\
&= \int_\Omega [-(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\omega} + (\nabla\widetilde{\boldsymbol{\phi}} \odot \nabla\mathbf{d} + \nabla\mathbf{d} \odot \nabla\widetilde{\boldsymbol{\phi}}) : \nabla\boldsymbol{\omega}]\\
&\quad + \int_\Omega (\mathbf{u}\cdot\nabla\widetilde{\boldsymbol{\phi}} + \boldsymbol{\omega}\cdot\nabla\mathbf{d} - |\nabla\mathbf{d}|^2\widetilde{\boldsymbol{\phi}} - 2\langle\nabla\mathbf{d}, \nabla\widetilde{\boldsymbol{\phi}}\rangle\mathbf{d})\cdot\Delta\widetilde{\boldsymbol{\phi}}\\
&\quad + \int_\Omega (\nabla\boldsymbol{\xi}_P \odot \nabla\mathbf{d} + \nabla\mathbf{d} \odot \nabla\boldsymbol{\xi}_P) : \nabla\boldsymbol{\omega}\\
&\quad + \int_\Omega (\mathbf{u}\cdot\nabla\boldsymbol{\xi}_P - |\nabla\mathbf{d}|^2\boldsymbol{\xi}_P - 2\langle\nabla\mathbf{d}, \nabla\boldsymbol{\xi}_P\rangle\mathbf{d})\cdot\Delta\widetilde{\boldsymbol{\phi}}\\
&\leq C\Big[\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\boldsymbol{\omega}\|_{L^2(\Omega)}\|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}\\
&\quad + \|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}\|\nabla\mathbf{d}\|_{L^4(\Omega)}\|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{1}{2}}\\
&\quad + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{3}{2}}\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{1}{2}}\\
&\quad + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}\|\boldsymbol{\omega}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\mathbf{d}\|_{L^4(\Omega)}\\
&\quad + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}\|\nabla\mathbf{d}\|_{L^8(\Omega)}^2\|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}\\
&\quad + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{3}{2}}\|\nabla\mathbf{d}\|_{L^4(\Omega)}\|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^{\frac{1}{2}}\\
&\quad + \|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}\|\nabla\boldsymbol{\xi}_P\|_{L^4(\Omega)}\|\nabla\mathbf{d}\|_{L^4(\Omega)}\\
&\quad + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}\big(\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\boldsymbol{\xi}_P\|_{L^4(\Omega)}\\
&\quad + \|\nabla\mathbf{d}\|_{L^8(\Omega)}^2\|\boldsymbol{\xi}_P\|_{L^4(\Omega)} + \|\nabla\mathbf{d}\|_{L^4(\Omega)}\|\nabla\boldsymbol{\xi}_P\|_{L^4(\Omega)}\big)\Big]\\
&\leq \frac{1}{2}(\|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2)\\
&\quad + C(\|\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2)\cdot\Big(\|\mathbf{u}\|_{L^4(\Omega)}^4 + \|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla\mathbf{d}\|_{L^8(\Omega)}^4\Big)\\
&\quad + C\Big[\|\nabla\boldsymbol{\xi}_P\|_{L^4(\Omega)}^2(\|\nabla\mathbf{d}\|_{L^4(\Omega)}^2 + \|\mathbf{u}\|_{L^4(\Omega)}^2) + \|\nabla\mathbf{d}\|_{L^8(\Omega)}^4\|\boldsymbol{\xi}_P\|_{L^4(\Omega)}^2\Big].
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{d}{dt}(\|\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2) + (\|\nabla\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2)\\
&\quad \leq C\big(\|\mathbf{u}\|_{L^4(\Omega)}^4 + \|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla\mathbf{d}\|_{L^8(\Omega)}^4\big)(\|\boldsymbol{\omega}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\boldsymbol{\phi}}\|_{L^2(\Omega)}^2)\\
&\qquad + C\Big[\|\boldsymbol{\xi}_P\|_{H^2(\Omega)}^2(\|\mathbf{d}\|_{H^2(\Omega)}^2 + \|\mathbf{u}\|_{H^1(\Omega)}^2) + \|\boldsymbol{\xi}_P\|_{H^1(\Omega)}^2\|\mathbf{d}\|_{H^2(\Omega)}^4\Big].
\end{aligned}$$

Since $(\mathbf{u}, \mathbf{d})$ is a strong solution obtained by Theorem 3.1, it follows from (3.2) that

$$\|\mathbf{u}\|_{L_t^\infty H_x^1(Q_T)} + \|\mathbf{d}\|_{L_t^\infty H_x^2(Q_T)} \leq C_T,$$

where $C_T > 0$ depends on T, Ω, $\|\mathbf{u}_0\|_{H^1(\Omega)}$, $\|\mathbf{d}_0\|_{H^2(\Omega)}$, $\|\mathbf{h}\|_{H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T)}$, and $\|\partial_t \mathbf{h}\|_{L^2_t H^{\frac{3}{2}}_x(\Gamma_T)}$. Hence by Sobolev's embedding theorem we have that

$$\|\mathbf{u}\|_{L^\infty_t L^4_x(Q_T)} + \|\nabla \mathbf{d}\|_{L^\infty_t L^8_x(Q_T)} \le C_T.$$

Thus we obtain that

$$\int_0^T \left(\|\mathbf{u}\|^4_{L^4(\Omega)} + \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla \mathbf{d}\|^4_{L^8(\Omega)}\right) dt \le C_T.$$

Since $(\boldsymbol{\omega}, \boldsymbol{\phi})|_{t=0} = (0, 0)$, by applying Gronwall's inequality we obtain that

$$\begin{aligned}
&\sup_{0\le t\le T} \|(\boldsymbol{\omega}, \nabla\widetilde{\boldsymbol{\phi}})(t)\|^2_{L^2(\Omega)} + \int_0^T (\|\nabla\boldsymbol{\omega}(\tau)\|^2_{L^2(\Omega)} + \|\nabla^2\widetilde{\boldsymbol{\phi}}(\tau)\|^2_{L^2(\Omega)})\, d\tau \\
&\quad \le C \exp\left\{C\int_0^T \left(\|\mathbf{u}\|^4_{L^4(\Omega)} + \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla \mathbf{d}\|^4_{L^8(\Omega)}\right)(\tau)\, d\tau\right\} \\
&\quad \cdot \int_0^T \left[\|\boldsymbol{\xi}_P\|^2_{H^2(\Omega)}(\|\mathbf{d}\|^2_{H^2(\Omega)} + \|\mathbf{u}\|^2_{H^1(\Omega)}) + \|\boldsymbol{\xi}_P\|^2_{H^1(\Omega)}\|\mathbf{d}\|^4_{H^2(\Omega)}\right] d\tau \\
&\quad \le C_T \|\boldsymbol{\xi}\|^2_{\mathcal{U}}.
\end{aligned} \tag{4.13}$$

Thus (4.11) was proven.

To show that $(\boldsymbol{\omega}, \boldsymbol{\phi})$ is a section of $(\mathbf{u}, \mathbf{d})^* T\mathcal{H}$, we need to verify that

$$\langle \boldsymbol{\phi}, \mathbf{d}\rangle(x, t) = 0, \ \text{ for } (x, t) \in Q_T.$$

To see this, observe that $\langle \boldsymbol{\phi}, \mathbf{d}\rangle$ satisfies

$$\begin{aligned}
&\partial_t \langle \boldsymbol{\phi}, \mathbf{d}\rangle + \mathbf{u}\cdot\nabla\langle \boldsymbol{\phi}, \mathbf{d}\rangle - \Delta\langle \boldsymbol{\phi}, \mathbf{d}\rangle \\
&\quad = \langle \partial_t\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\boldsymbol{\phi} - \Delta\boldsymbol{\phi}, \mathbf{d}\rangle + \langle \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} - \Delta\mathbf{d}, \boldsymbol{\phi}\rangle - 2\langle\nabla\boldsymbol{\phi}, \nabla\mathbf{d}\rangle \\
&\quad = 2|\nabla\mathbf{d}|^2\langle \boldsymbol{\phi}, \mathbf{d}\rangle,
\end{aligned} \tag{4.14}$$

and

$$\langle \boldsymbol{\phi}, \mathbf{d}\rangle = 0 \ \text{ on } \ \partial_p Q_T.$$

Hence, by the parabolic maximum principle, we conclude that

$$\langle \boldsymbol{\phi}, \mathbf{d}\rangle \equiv 0 \ \text{ in } \ Q_T.$$

This completes the proof of Theorem 4.6. □

To facilitate the discussion, we also introduce a linear map associated with an element $\mathbf{h} \in \widetilde{\mathcal{U}}$, $\mathcal{L}_{\mathbf{h}} : \mathbf{h}^* T\widetilde{\mathcal{U}} \mapsto (\mathcal{S}(\mathbf{h}))^* T\mathcal{H}$ that is defined by

$$\mathcal{L}_{\mathbf{h}}(\boldsymbol{\xi}) = (\boldsymbol{\omega}, \boldsymbol{\phi}), \tag{4.15}$$

where $(\boldsymbol{\omega}, \boldsymbol{\phi})$ is the unique global strong solution to the linearized system (4.9) and (4.10) on Q_T, with $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h})$, obtained by Theorem 4.6.

It follows directly from the estimate (4.11) that

Corollary 4.7 *For any $\mathbf{h} \in \widetilde{\mathcal{U}}$, the linear map $\mathcal{L}_{\mathbf{h}} : \mathbf{h}^* T\widetilde{\mathcal{U}} \mapsto (\mathcal{S}(\mathbf{h}))^* T\mathcal{H}$ is Lipschitz continuous.*

4.1.5 Differentiability of $\mathcal{S}$

In this subsection, we will prove the Fréchet differentiability of $\mathcal{S}$. More precisely we have

Theorem 4.8 *Given* $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$, *if* $\mathbf{h} \in \widetilde{\mathcal{U}}$ *then the control to state map* $\mathcal{S}$ *is Fréchet differentiable at* $\mathbf{h}$ *in the sense of* (4.8). *Moreover, the Fréchet derivative* $\mathcal{S}'(\mathbf{h})$ *is given by*

$$\mathcal{S}'(\mathbf{h})(\boldsymbol{\xi}) = \mathcal{L}_{\mathbf{h}}(\boldsymbol{\xi}), \text{ for any section } \boldsymbol{\xi} \text{ of } \mathbf{h}^* T\widetilde{\mathcal{U}} \text{ with } \exp_{\mathbf{h}} \boldsymbol{\xi} \in \widetilde{\mathcal{U}}. \tag{4.16}$$

Proof Let $(\mathbf{u}, \mathbf{d})$ be the unique global strong solution to the system (1.1)–(1.3), obtained by Theorem 3.1, with the initial data $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary data $(0, \mathbf{h})$, namely,

$$(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h}).$$

If $\boldsymbol{\xi}$ is a section of $\mathbf{h}^* T\widetilde{\mathcal{U}}$ such that $\exp_{\mathbf{h}} \boldsymbol{\xi} \in \widetilde{\mathcal{U}}$, then we can define a new boundary data $\widehat{\mathbf{h}} = \exp_{\mathbf{h}} \boldsymbol{\xi}$, which satisfies $\widehat{\mathbf{h}}(\Gamma_T) \subset \mathbb{S}^2_+$ and belongs to $\widetilde{\mathcal{U}}$. Let $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in \mathcal{H}$ be the unique global strong solution to the problem (1.1)–(1.3) under the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(0, \widehat{\mathbf{h}})$, i.e. $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) = \mathcal{S}(\widehat{\mathbf{h}})$.

Let $(\boldsymbol{\omega}, \boldsymbol{\phi}) = \mathcal{L}_{\mathbf{h}}(\boldsymbol{\xi}) \in \mathcal{S}(\mathbf{h})^* \mathbf{T}\mathcal{H}$, which is the unique solution to problem (4.9) and (4.10) obtained by Theorem 4.6, under the initial condition $(0, 0)$ and the boundary condition $(0, \boldsymbol{\xi})$.

By Theorem 3.1 and Theorem 4.6, we have the following estimates:

$$\begin{cases} \left\|(\mathbf{u}, \mathbf{d})\right\|_{\mathcal{H}} \le C\big(T, \|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}, \|\mathbf{h}\|_{\mathcal{U}}\big), \\ \left\|(\widehat{\mathbf{u}}, \widehat{\mathbf{d}})\right\|_{\mathcal{H}} \le C\big(T, \|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}, \|\mathbf{h}\|_{\mathcal{U}}\big), \\ \left\|(\boldsymbol{\omega}, \boldsymbol{\phi})\right\|_{\mathcal{W}} \le C\big(T, \|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}, \|\mathbf{h}\|_{\mathcal{U}}\big) \|\boldsymbol{\xi}\|_{\mathcal{U}}. \end{cases} \tag{4.17}$$

Moreover, we can infer from Theorem 3.3 that

$$\begin{aligned} &\left\|\mathbf{u} - \widehat{\mathbf{u}}\right\|^2_{L^\infty_t H^1_x(Q_T)} + \left\|\mathbf{d} - \widehat{\mathbf{d}}\right\|^2_{L^\infty_t H^2_x(Q_T)} \\ &\quad + \left\|\mathbf{u} - \widehat{\mathbf{u}}\right\|^2_{L^2_t H^2_x(Q_T)} + \left\|\mathbf{d} - \widehat{\mathbf{d}}\right\|^2_{L^2_t H^3_x(Q_T)} \\ &\le C\big(T, \|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}, \|\mathbf{h}\|_{\mathcal{U}}\big) \left\|\mathbf{h} - \widehat{\mathbf{h}}\right\|^2_{\mathcal{U}} \\ &\le C\big(T, \|(\mathbf{u}_0, \nabla \mathbf{d}_0)\|_{H^1(\Omega)}, \|\mathbf{h}\|_{\mathcal{U}}\big) \|\boldsymbol{\xi}\|^2_{\mathcal{U}}. \end{aligned} \tag{4.18}$$

Now we set

$$\mathbf{w} = \widehat{\mathbf{u}} - \mathbf{u} - \boldsymbol{\omega} \text{ and } \mathbf{e} = \widehat{\mathbf{d}} - \mathbf{d} - \boldsymbol{\phi}.$$

By direct calculations, $(\mathbf{w}, \mathbf{e})$ solves, in Q_T,

$$\begin{cases} \partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla \widetilde{P} + (\widehat{\mathbf{u}} - \mathbf{u}) \cdot \nabla(\widehat{\mathbf{u}} - \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u} \\ \qquad = -\nabla \cdot [\nabla(\widehat{\mathbf{d}} - \mathbf{d}) \odot \nabla(\widehat{\mathbf{d}} - \mathbf{d}) + \nabla \mathbf{d} \odot \nabla \mathbf{e} + \nabla \mathbf{e} \odot \nabla \mathbf{d}], \\ \nabla \cdot \mathbf{w} = 0, \\ \partial_t \mathbf{e} - \Delta \mathbf{e} + (\widehat{\mathbf{u}} - \mathbf{u}) \cdot \nabla(\widehat{\mathbf{d}} - \mathbf{d}) + \mathbf{u} \cdot \nabla \mathbf{e} + \mathbf{w} \cdot \nabla \mathbf{d} \\ = |\nabla \mathbf{d}|^2 \mathbf{e} + |\nabla(\widehat{\mathbf{d}} - \mathbf{d})|^2 \widehat{\mathbf{d}} + 2\langle \nabla \mathbf{d}, \nabla \mathbf{e} \rangle \widehat{\mathbf{d}} + 2\langle \nabla \mathbf{d}, \nabla \boldsymbol{\phi} \rangle (\widehat{\mathbf{d}} - \mathbf{d}), \end{cases} \tag{4.19}$$

with the boundary and initial condition

$$\begin{cases} (\mathbf{w}, \mathbf{e}) = (0, \exp_{\mathbf{h}} \boldsymbol{\xi} - \mathbf{h} - \boldsymbol{\xi}) & \text{on} \quad \Gamma_T \\ (\mathbf{w}, \mathbf{e}) = (0, 0) & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{4.20}$$

Define the parabolic lifting function $\chi : Q_T \mapsto \mathbb{R}^3$ by

$$\begin{cases} \partial_t \chi - \Delta \chi = \mathbf{0} & \text{in} \quad Q_T, \\ \chi = \exp_{\mathbf{h}} \xi - \mathbf{h} - \xi & \text{on} \quad \Gamma_T, \\ \chi = \mathbf{0} & \text{in} \quad \Omega \times \{0\}. \end{cases}$$

By direct calculations, we find that

$$\| \exp_{\mathbf{h}} \xi - \mathbf{h} - \xi \|_{\mathcal{U}} \leq C \| \xi \|_{\mathcal{U}}^2$$

and hence

$$\| \nabla \chi \|_{L_t^\infty L_x^2(Q_T)} + \| \nabla^2 \chi \|_{L^2(Q_T)} + \| \chi \|_{H^{3,\frac{3}{2}}(Q_T)} \leq C \| \xi \|_{\mathcal{U}}^2. \tag{4.21}$$

Next we define $\widetilde{\mathbf{e}} = \mathbf{e} - \chi$. Then $(\mathbf{w}, \widetilde{\mathbf{e}})$ satisfies in Q_T:

$$\begin{cases} \partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla \widetilde{P} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u} \\ = -\nabla \cdot [\nabla \overline{\mathbf{d}} \odot \nabla \overline{\mathbf{d}} + \nabla \mathbf{d} \odot \nabla \mathbf{e} + \nabla \mathbf{e} \odot \nabla \mathbf{d}], \\ \nabla \cdot \mathbf{w} = 0, \\ \partial_t \widetilde{\mathbf{e}} - \Delta \widetilde{\mathbf{e}} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{d}} + \mathbf{u} \cdot \nabla \mathbf{e} + \mathbf{w} \cdot \nabla \mathbf{d} \\ = |\nabla \mathbf{d}|^2 \mathbf{e} + |\nabla \overline{\mathbf{d}}|^2 \widehat{\mathbf{d}} + 2\langle \nabla \mathbf{d}, \nabla \mathbf{e} \rangle \mathbf{d} + 2\langle \nabla \mathbf{d}, \nabla \phi \rangle \overline{\mathbf{d}}, \end{cases} \tag{4.22}$$

with the boundary and initial condition

$$\begin{cases} (\mathbf{w}, \widetilde{\mathbf{e}}) = (0, 0) & \text{on} \quad \Gamma_T \\ (\mathbf{w}, \widetilde{\mathbf{e}}) = (0, 0) & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{4.23}$$

Here $\overline{\mathbf{u}} = \widehat{\mathbf{u}} - \mathbf{u}$ and $\overline{\mathbf{d}} = \widehat{\mathbf{d}} - \mathbf{d}$.

Multiplying $(4.22)_1$ by $\mathbf{w}$, and $(4.22)_3$ by $-\Delta \widetilde{\mathbf{e}}$, integrating over Ω, and adding the two resulting equations, we deduce

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\| \mathbf{w} \|_{L^2(\Omega)}^2 + \| \nabla \widetilde{\mathbf{e}} \|_{L^2(\Omega)}^2) + (\| \nabla \mathbf{w} \|_{L^2(\Omega)}^2 + \| \Delta \widetilde{\mathbf{e}} \|_{L^2(\Omega)}^2) \\ &\quad = \Big[\int_\Omega (\overline{\mathbf{u}} \cdot \nabla \mathbf{w} \cdot \overline{\mathbf{u}} + \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u}) \\ &\qquad + \int_\Omega [\nabla \overline{\mathbf{d}} \odot \nabla \overline{\mathbf{d}} + \nabla \mathbf{d} \odot \nabla \widetilde{\mathbf{e}} + \widetilde{\nabla \mathbf{e}} \odot \nabla \mathbf{d}] : \nabla \mathbf{w} \\ &\qquad + \int_\Omega [\nabla \mathbf{d} \odot \nabla \chi + \nabla \chi \odot \nabla \mathbf{d}] : \nabla \mathbf{w} \\ &\qquad + \int_\Omega (\overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{d}} + \mathbf{u} \cdot \nabla \widetilde{\mathbf{e}} + \mathbf{w} \cdot \nabla \mathbf{d}) \cdot \Delta \widetilde{\mathbf{e}} \\ &\qquad - \int_\Omega (|\nabla \mathbf{d}|^2 \widetilde{\mathbf{e}} + |\nabla \overline{\mathbf{d}}|^2 \widehat{\mathbf{d}} + 2\langle \nabla \mathbf{d}, \nabla \widetilde{\mathbf{e}} \rangle \mathbf{d} + 2\langle \nabla \mathbf{d}, \nabla \phi \rangle \overline{\mathbf{d}}) \cdot \Delta \widetilde{\mathbf{e}} \\ &\qquad - \int_\Omega (|\nabla \mathbf{d}|^2 \chi + 2\langle \nabla \mathbf{d}, \nabla \chi \rangle \mathbf{d} - \mathbf{u} \cdot \nabla \chi) \Delta \widetilde{e} \Big] \\ &\quad = I + II + III + IV + V + VI. \end{aligned} \tag{4.24}$$

$I, \ldots, VI$ can be estimated as follows.

$$
\begin{aligned}
|I| &\le \frac{1}{12}\|\nabla \mathbf{w}\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{L^4(\Omega)}^4\|\mathbf{w}\|_{L^2(\Omega)}^2 + C\|\overline{\mathbf{u}}\|_{L^4(\Omega)}^4,\\
|II| &\le \frac{1}{12}(\|\nabla \mathbf{w}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2)\\
&\quad + C\|\nabla \mathbf{d}\|_{L^4(\Omega)}^4\|\nabla\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2 + C\|\nabla\overline{\mathbf{d}}\|_{L^4(\Omega)}^4,\\
|III| &\le \frac{1}{2}\|\nabla \mathbf{w}\|_{L^2(\Omega)}^2 + C\|\nabla \mathbf{d}\|_{L^4(\Omega)}^2\|\nabla\boldsymbol{\chi}\|_{L^4(\Omega)}^2,\\
|IV| &\le \frac{1}{12}(\|\nabla \mathbf{w}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2)\\
&\quad + C(\|\overline{\mathbf{u}}\|_{L^4(\Omega)}^4 + \|\nabla\overline{\mathbf{d}}\|_{L^4(\Omega)}^4)\\
&\quad + C(\|\nabla \mathbf{d}\|_{L^4(\Omega)}^4\|\mathbf{w}\|_{L^2(\Omega)}^2 + \|\mathbf{u}\|_{L^4(\Omega)}^4\|\nabla\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2),\\
|V| &\le \frac{1}{12}\|\Delta\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2 + C\|\nabla \mathbf{d}\|_{L^8(\Omega)}^4\|\nabla\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2\\
&\quad + C\|\nabla\overline{\mathbf{d}}\|_{L^4(\Omega)}^4 + C\|\nabla \mathbf{d}\|_{L^8(\Omega)}^2\|\nabla\boldsymbol{\phi}\|_{L^4(\Omega)}^2\|\overline{\mathbf{d}}\|_{L^8(\Omega)}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|VI| \le{}& \frac{1}{2}\|\Delta\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2 + C\|\nabla \mathbf{d}\|_{L^8(\Omega)}^4\|\boldsymbol{\chi}\|_{L^4(\Omega)}^2\\
&+ C(\|\nabla \mathbf{d}\|_{L^4(\Omega)}^2 + \|\mathbf{u}\|_{L^4(\Omega)}^2)\|\nabla\boldsymbol{\chi}\|_{L^4(\Omega)}^2.
\end{aligned}
$$

Substituting these estimates into (4.24), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\mathbf{w}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2) + (\|\nabla \mathbf{w}\|_{L^2(\Omega)}^2 + \|\Delta\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2)\\
&\quad \le C(\|\mathbf{u}\|_{L^4(\Omega)}^4 + \|\nabla \mathbf{d}\|_{L^8(\Omega)}^4)(\|\mathbf{w}\|_{L^2(\Omega)}^2 + \|\nabla\widetilde{\mathbf{e}}\|_{L^2(\Omega)}^2)\\
&\qquad + C(\|\overline{\mathbf{u}}\|_{L^4(\Omega)}^4 + \|\nabla\overline{\mathbf{d}}\|_{L^4(\Omega)}^4) + \|\nabla \mathbf{d}\|_{L^8(\Omega)}^2\|\nabla\boldsymbol{\phi}\|_{L^4(\Omega)}^2\|\overline{\mathbf{d}}\|_{L^8(\Omega)}^2)\\
&\qquad + C\big[\|\nabla \mathbf{d}\|_{L^8(\Omega)}^4\|\boldsymbol{\chi}\|_{L^4(\Omega)}^2 + (\|\nabla \mathbf{d}\|_{L^4(\Omega)}^2 + \|\mathbf{u}\|_{L^4(\Omega)}^2)\|\nabla\boldsymbol{\chi}\|_{L^4(\Omega)}^2\big].
\end{aligned}
$$

From (4.17), (4.18) and (4.21), it is not hard to show that

$$
\begin{aligned}
&\int_0^T \|\nabla \mathbf{d}\|_{L^8(\Omega)}^4\,dt + \int_0^T \|\mathbf{u}\|_{L^4(\Omega)}^4\,dt \le C_T,\\
&\int_0^T (\|\overline{\mathbf{u}}\|_{L^4(\Omega)}^4 + \|\nabla\overline{\mathbf{d}}\|_{L^4(\Omega)}^4)\,dt \le C_T\|\boldsymbol{\xi}\|_{\mathcal{U}}^4,\\
&\int_0^T \|\nabla \mathbf{d}\|_{L^8(\Omega)}^2\|\nabla\boldsymbol{\phi}\|_{L^4(\Omega)}^2\|\overline{\mathbf{d}}\|_{L^8(\Omega)}^2\,dt \le C_T\|\boldsymbol{\xi}\|_{\mathcal{U}}^4,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \big[\|\nabla \mathbf{d}\|_{L^8(\Omega)}^4\|\boldsymbol{\chi}\|_{L^4(\Omega)}^2 + (\|\nabla \mathbf{d}\|_{L^4(\Omega)}^2 + \|\mathbf{u}\|_{L^4(\Omega)}^2)\|\nabla\boldsymbol{\chi}\|_{L^4(\Omega)}^2\big]\,dt\\
&\quad \le C_T\|\boldsymbol{\xi}\|_{\mathcal{U}}^4.
\end{aligned}
$$

These estimates, combined with the fact that $(\mathbf{w}, \widetilde{\mathbf{e}}) = 0$ at $t = 0$ and Gronwall's inequality, imply that

$$\sup_{0\le t\le T} (\|\mathbf{w}\|^2_{L^2(\Omega)} + \|\nabla\widetilde{\mathbf{e}}\|^2_{L^2(\Omega)}) + \int_0^T (\|\nabla\mathbf{w}\|^2_{L^2(\Omega)} + \|\Delta\widetilde{\mathbf{e}}\|^2_{L^2(\Omega)})\, dt$$
$$\le C_T \|\boldsymbol{\xi}\|^4_{\mathcal{U}}.$$

This, with the help of (4.21), yields that

$$\|(\mathbf{w}, \mathbf{e})\|_{\mathcal{W}} \le C\|\boldsymbol{\xi}\|^2_{\mathcal{U}}.$$

Hence $\mathcal{S}$ is differentiable at $\mathbf{h}$, and its Fréchet derivative $\mathcal{S}'(\mathbf{h})(\boldsymbol{\xi}) = \mathcal{L}_{\mathbf{h}}(\boldsymbol{\xi})$ whenever $\boldsymbol{\xi} \in \mathbf{h}^* T\widetilde{\mathcal{U}}$ is such that $\exp_{\mathbf{h}} \boldsymbol{\xi} \in \widetilde{\mathcal{U}}$. This completes the proof. □

4.2 Existence and necessary condition of boundary optimal control

Here we will consider both the existence and a necessary condition of an optimal boundary control for the problem (4.1).

4.2.1 The existence of an optimal boundary control

We will establish the existence of an optimal boundary control for the problem (4.1).

Theorem 4.9 *Under the conditions (A1) and (A2), let* $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$ *be given. For* $M > 0$, *if* $\widetilde{\mathcal{U}}_M \neq \emptyset$, *then (4.1) admits a solution* $((\mathbf{u}, \mathbf{d}), \mathbf{h})$, *where* $\mathbf{h} \in \widetilde{\mathcal{U}}_M$ *and* $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h})$ *is the unique strong solution to (1.1)–(1.3) with the initial condition* $(\mathbf{u}_0, \mathbf{d}_0)$ *and the boundary condition* $(0, \mathbf{h})$.

Proof Let $\{((\mathbf{u}^i, \mathbf{d}^i), \mathbf{h}^i)\}_{i=1}^\infty$ be a minimizing sequence of the cost functional $\mathcal{C}$ in (4.1) over $\widetilde{\mathcal{U}}_M$, i.e.,

$$\lim_{i\to\infty} \mathcal{C}((\mathbf{u}^i, \mathbf{d}^i), \mathbf{h}^i) = \inf_{\mathbf{h}\in\widetilde{\mathcal{U}}_M, (\mathbf{u},\mathbf{d})=\mathcal{S}(\mathbf{h})} \mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}), \tag{4.25}$$

where $\mathbf{h}^i \in \widetilde{\mathcal{U}}_M$ and $(\mathbf{u}^i, \mathbf{d}^i) = \mathcal{S}(\mathbf{h}^i) \in \mathcal{H}$ is the unique strong solution to the initial boundary value problem of (1.1)–(1.3) with the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(\mathbf{0}, \mathbf{h}^i)$. Then we have

$$\left\|\Pi(\mathbf{h}^i)\right\|_{\mathcal{U}} \le M, \text{ and } \left\|(\mathbf{u}^i, \mathbf{d}^i)\right\|_{\mathcal{H}} \le CM, \ \forall\, i \ge 1.$$

From the weak compactness of $\widetilde{\mathcal{U}}_M \hookrightarrow \mathcal{U}$, we may assume, after passing to subsequences, that there exist $\mathbf{h}^* \in \widetilde{\mathcal{U}}_M$ and $(\mathbf{u}^*, \mathbf{d}^*) \in \mathcal{H}$ such that

$$\mathbf{h}^i \rightharpoonup \mathbf{h}^* \quad \text{in} \quad H^{\frac{5}{2},\frac{5}{4}}(\Gamma_T), \quad \partial_t \mathbf{h}^i \rightharpoonup \partial_t \mathbf{h}^* \quad \text{in} \quad L^2_t H^{\frac{3}{2}}_x(\Gamma_T),$$

and

$$\mathbf{u}^i \overset{*}{\rightharpoonup} \mathbf{u}^* \quad \text{in} \quad L^\infty([0, T], \mathbf{V}), \ \mathbf{u}^i \rightharpoonup \mathbf{u}^* \quad \text{in} \quad L^2_t H^2_x(Q_T),$$
$$\mathbf{d}^i \overset{*}{\rightharpoonup} \mathbf{d}^* \quad \text{in} \quad L^\infty_t H^2_x(Q_T), \ \mathbf{d}^i \rightharpoonup \mathbf{d}^* \quad \text{in} \quad H^{3,\frac{3}{2}}(Q_T).$$

Observe also that by the Aubin–Lions Lemma,

$$\mathbf{u}^i \to \mathbf{u}^* \quad \text{in} \quad C([0, T], L^2(\Omega)), \ \ \mathbf{d}^i \to \mathbf{d}^* \quad \text{in} \quad C([0, T], H^1(\Omega)).$$

It is not hard to see that $(\mathbf{u}^*, \mathbf{d}^*) \in \mathcal{H}$ is a strong solution of the system (1.1)–(1.3), with the initial condition $(\mathbf{u}_0, \mathbf{d}_0)$ and the boundary condition $(0, \mathbf{h}^*)$. By the uniqueness theorem of strong solutions, we conclude that $(\mathbf{u}^*, \mathbf{d}^*) = \mathcal{S}(\mathbf{h}^*)$.

Since the cost functional $\mathcal{C}$ is weakly lower semi-continuous in $((\mathbf{u}, \mathbf{d}), \mathbf{h}) \in \mathcal{H} \times \mathcal{U}$, we have

$$\liminf_{i\to\infty} \mathcal{C}((\mathbf{u}^i, \mathbf{d}^i), \mathbf{h}^i) \geq \mathcal{C}((\mathbf{u}^*, \mathbf{d}^*), \mathbf{h}^*). \tag{4.26}$$

On the other hand, since $\mathbf{h}^* \in \widetilde{\mathcal{U}}_M$ and $(\mathbf{u}^*, \mathbf{d}^*) = \mathcal{S}(\mathbf{h}^*)$, we also have

$$\inf_{\mathbf{h}\in\widetilde{\mathcal{U}}_M, (\mathbf{u},\mathbf{d})=\mathcal{S}(\mathbf{h})} \mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}) \leq \mathcal{C}((\mathbf{u}^*, \mathbf{d}^*), \mathbf{h}^*). \tag{4.27}$$

It follows directly from (4.25)–(4.27) that $((\mathbf{u}^*, \mathbf{d}^*), \mathbf{h}^*)$ achieves

$$\inf_{\mathbf{h}\in\widetilde{\mathcal{U}}_M, (\mathbf{u},\mathbf{d})=\mathcal{S}(\mathbf{h})} \mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}).$$

This completes the proof. □

4.2.2 The first-order necessary optimality condition

In this subsection, we will derive the first-order necessary condition for the optimal control problem (4.1) based on the Fréchet differentiability of the control-to-state operator $\mathcal{S}$ established in the previous section.

Now we are ready to prove the following theorem that gives a necessary condition of boundary optimal control.

Theorem 4.10 *Assume both (A1) and (A2). For $M > 0$, let $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{V} \times H^2(\Omega, \mathbb{S}^2_+)$ be given. If $\widetilde{\mathcal{U}}_M \neq \emptyset$ and $\mathbf{h} \in \widetilde{\mathcal{U}}_M$ is a minimizer of the optimal control for problem (4.1) over the admissible set $\widetilde{\mathcal{U}}_M$, with the associated state map $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h}) \in \mathcal{H}$. Then for any boundary data $\mathbf{h}_* \in \widetilde{\mathcal{U}}_M$, let $\boldsymbol{\xi} = \boldsymbol{\xi}_{\mathbf{h},\mathbf{h}_*}$ be the section of $\mathbf{h}^* T\widetilde{\mathcal{U}}$ given by*

$$\boldsymbol{\xi} = \frac{d}{ds}\Big|_{s=0} \Pi^{-1}(s\Pi(\mathbf{h}) + (1-s)\Pi(\mathbf{h}^*)), \tag{4.28}$$

and $(\boldsymbol{\omega}, \boldsymbol{\phi}) = \mathcal{S}'(\mathbf{h})(\boldsymbol{\xi})$, i.e., the unique global strong solution to the linearized problem (4.9) and (4.10) associated with $\boldsymbol{\xi}$, the following variational inequality holds:

$$\begin{aligned}
&\int_{Q_T} \big(\beta_1\langle \mathbf{u} - \mathbf{u}_{Q_T}, \boldsymbol{\omega}\rangle + \beta_2\langle \mathbf{d} - \mathbf{d}_{Q_T}, \boldsymbol{\phi}\rangle\big) \\
&\quad + \int_{\Omega} \big(\beta_3\langle \mathbf{u}(T) - \mathbf{u}_{\Omega}, \boldsymbol{\omega}(T)\rangle + \beta_4\langle \mathbf{d}(T) - \mathbf{d}_{\Omega}, \boldsymbol{\phi}(T)\rangle\big) \\
&\quad + \int_{\Gamma_T} \beta_5\langle \mathbf{h} - \mathbf{e}_3, \boldsymbol{\xi}\rangle \geq 0.
\end{aligned} \tag{4.29}$$

Proof Note that $((\mathbf{u}, \mathbf{d}), \mathbf{h}) = (\mathcal{S}(\mathbf{h}), \mathbf{h})$ is a minimizer of $\mathcal{C}$ over $\widetilde{\mathcal{U}}_M$. For any $\mathbf{h}_* \in \widetilde{\mathcal{U}}_M$, let $\boldsymbol{\xi}$ be the section of $\mathbf{h}^* T\widetilde{\mathcal{U}}$ given by (4.28). Then

$$\mathbf{h}(s) = \Pi^{-1}(s\Pi(\mathbf{h}) + (1-s)\Pi(\mathbf{h}^*)) \in C^1([0,1], \widetilde{\mathcal{U}})$$

is a C^1-family of maps from Γ_T to $\widetilde{\mathcal{U}}_M$ joining $\mathbf{h}$ to $\mathbf{h}^*$. If we let $(\mathbf{u}(s), \mathbf{d}(s)) = \mathcal{S}(\mathbf{h}(s))$ for $s \in [0, 1]$. Then it is not hard to verify that $(\mathbf{u}(s), \mathbf{d}(s)) \in C^1([0, 1], \mathcal{H})$ and

$$\frac{d}{ds}\Big|_{s=0}(\mathbf{u}(s), \mathbf{d}(s)) = \mathcal{S}'(\mathbf{h})(\boldsymbol{\xi}) = (\boldsymbol{\omega}, \boldsymbol{\phi}) \text{ in } Q_T.$$

Since

$$\mathcal{C}((\mathbf{u}(s), \mathbf{d}(s)), \mathbf{h}(s)) \geq \mathcal{C}((\mathbf{u}(0), \mathbf{d}(0)), \mathbf{h}(0)) = \mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}), \ \forall s \in [0, 1],$$

we conclude that

$$\frac{d}{ds}\Big|_{s=0} \mathcal{C}((\mathbf{u}(s), \mathbf{d}(s)), \mathbf{h}(s)) \geq 0.$$

It follows directly from the chain rule that

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \int_{Q_T} (\beta_1 |\mathbf{u}(s) - \mathbf{u}_{Q_T}|^2 + \beta_2 |\mathbf{d}(s) - \mathbf{d}_{Q_T}|^2) \\
&= 2 \int_{Q_T} (\beta_1 \langle \mathbf{u} - \mathbf{u}_{Q_T}, \boldsymbol{\omega} \rangle + \beta_2 \langle \mathbf{d} - \mathbf{d}_{Q_T}, \boldsymbol{\phi} \rangle), \\
&\frac{d}{ds}\Big|_{s=0} \int_{\Omega} (\beta_3 |\mathbf{u}(T) - \mathbf{u}_{\Omega}|^2 + \beta_4 |\mathbf{d}(T) - \mathbf{d}_{\Omega}|^2) \\
&= 2 \int_{\Omega} (\beta_3 \langle \mathbf{u}(T) - \mathbf{u}_{\Omega}, \boldsymbol{\omega}(T) \rangle + \beta_4 \langle \mathbf{d}(T) - \mathbf{d}_{\Omega}, \boldsymbol{\phi}(T) \rangle),
\end{aligned}$$

and

$$\frac{d}{ds}\Big|_{s=0} \int_{\Gamma_T} \beta_5 |\mathbf{h}(s) - \mathbf{e}_3|^2 = 2 \int_{\Gamma_T} \beta_5 \langle \mathbf{h}(0) - \mathbf{e}_3, \boldsymbol{\xi} \rangle.$$

Putting these together yields (4.29). This completes the proof. □

4.3 First-order necessary condition via adjoint states

In this subsection, we will eliminate the pair $(\boldsymbol{\omega}, \boldsymbol{\phi})$ from the variational inequality (4.29) and derive a first-order necessary condition in terms of the optimal solution together with its adjoint states. For this purpose, we will first derive the corresponding adjoint system of the control problem (4.1). Since this section is similar to section 6 of [5], we will only sketch it here.

4.3.1 Formal derivation of the adjoint system

The Lagrange functional $\mathcal{G}$ for the control problem (4.1), with Lagrange multipliers $\mathbf{p}_1$, $\mathbf{p}_2$, $\boldsymbol{\pi}$, $\mathbf{q}_1$ and $\mathbf{q}_2$, is given by

$$\begin{aligned}
&\mathcal{G}((\mathbf{u}, \mathbf{d}), \mathbf{h}, \mathbf{p}_1, \mathbf{p}_2, \boldsymbol{\pi}, \mathbf{q}_1, \mathbf{q}_2) = \\
&\mathcal{C}((\mathbf{u}, \mathbf{d}), \mathbf{h}) - \int_{Q_T} \langle \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P + \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \mathbf{p}_1 \rangle \\
&\quad - \int_{Q_T} (\nabla \cdot \mathbf{u}) \boldsymbol{\pi} - \int_{Q_T} \langle \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - \Delta \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbf{d}, \mathbf{p}_2 \rangle \\
&\quad - \int_{\Gamma_T} \langle \mathbf{u}, \mathbf{q}_1 \rangle - \int_{\Gamma_T} \langle \mathbf{d} - \mathbf{h}, \mathbf{q}_2 \rangle,
\end{aligned} \tag{4.30}$$

for any $\mathbf{h} \in \widetilde{\mathcal{U}}_M$ and $(\mathbf{u}, \mathbf{d}) \in \mathcal{H}$. Here, we will eliminate the five constraints due to the state problem (1.1)–(1.3) by five corresponding Lagrange multipliers $\mathbf{p}_1, \mathbf{p}_2, \boldsymbol{\pi}, \mathbf{q}_1, \mathbf{q}_2$.

For $M > 0$, let $((\mathbf{u}, \mathbf{d}), \mathbf{h})$ be a minimizer of the optimal control problem (4.1) such that $\mathbf{h} \in \widetilde{\mathcal{U}}_M$ and $(\mathbf{u}, \mathbf{d}) = \mathcal{S}(\mathbf{h}) \in \mathcal{H}$. Then we expect that $(\mathbf{u}, \mathbf{d})$ and $\mathbf{h}$ together with the corresponding Lagrange multipliers $\mathbf{p}_1$, $\mathbf{p}_2$, $\boldsymbol{\pi}$, $\mathbf{q}_1$, $\mathbf{q}_2$ satisfy the optimality conditions associated with the minimization problem for the Lagrange functional $\mathcal{G}$, i.e.,

$$\min \mathcal{G}((\mathbf{u}, \mathbf{d}), \mathbf{h}, (\mathbf{p}_1, \mathbf{p}_2, \boldsymbol{\pi}, \mathbf{q}_1, \mathbf{q}_2)), \text{ with } (\mathbf{u}, \mathbf{d}) \textit{ unconstrained} \text{ and } \mathbf{h} \in \widetilde{\mathcal{U}}_M. \tag{4.31}$$

Then we have that

$$\mathcal{G}'_{(\mathbf{u},\mathbf{d})}((\mathbf{u}, \mathbf{d}), \mathbf{h}, (\mathbf{p}_1, \mathbf{p}_2, \boldsymbol{\pi}, \mathbf{q}_1, \mathbf{q}_2))(\boldsymbol{\omega}, \boldsymbol{\phi}) = 0, \tag{4.32}$$

for all smooth functions $(\boldsymbol{\omega}, \boldsymbol{\phi})$ satisfying

$$\boldsymbol{\omega}|_{t=0} = \mathbf{0}, \quad \boldsymbol{\phi}|_{t=0} = \mathbf{0}, \quad \text{in } \Omega. \tag{4.33}$$

Here (4.33) follows from the fact the initial data $(\mathbf{u}_0, \mathbf{d}_0)$ of (4.1) is fixed.

Similar to the derivation of (4.29), it follows from (4.32) that

$$\begin{aligned}
0 = {} & \beta_1 \int_{Q_T} \langle \mathbf{u} - \mathbf{u}_{Q_T}, \boldsymbol{\omega}\rangle + \beta_2 \int_{Q_T} \langle \mathbf{d} - \mathbf{d}_{Q_T}, \boldsymbol{\phi}\rangle \\
& + \beta_3 \int_{\Omega} \langle \mathbf{u}(T) - \mathbf{u}_{\Omega}, \boldsymbol{\omega}(T)\rangle + \beta_4 \int_{\Omega} \langle \mathbf{d}(T) - \mathbf{d}_{\Omega}, \boldsymbol{\phi}(T)\rangle \\
& - \int_{Q_T} \langle \partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \Delta \boldsymbol{\omega} + \nabla \widetilde{P} + \boldsymbol{\omega} \cdot \nabla \mathbf{u} \\
& - \nabla \cdot (\nabla \boldsymbol{\phi} \odot \nabla \mathbf{d} + \nabla \mathbf{d} \odot \nabla \boldsymbol{\phi}), \mathbf{p}_1\rangle \\
& - \int_{Q_T} (\nabla \cdot \boldsymbol{\omega})\boldsymbol{\pi} - \int_{Q_T} \langle \partial_t \boldsymbol{\phi} - \Delta \boldsymbol{\phi} + \mathbf{u} \cdot \nabla \boldsymbol{\phi} + \boldsymbol{\omega} \cdot \nabla \mathbf{d} \\
& - |\nabla \mathbf{d}|^2 \boldsymbol{\phi} - 2\langle \nabla \mathbf{d}, \nabla \boldsymbol{\phi}\rangle \mathbf{d}, \mathbf{p}_2\rangle \\
& - \int_{\Gamma_T} \langle \boldsymbol{\omega}, \mathbf{q}_1\rangle - \int_{\Gamma_T} \langle \boldsymbol{\phi}, \mathbf{q}_2\rangle.
\end{aligned} \tag{4.34}$$

Performing integration by parts, using the condition (4.33), and regrouping the relevant terms in the same way as [5] page 1065, we can obtain the adjoint system for $\mathbf{p}_1$, $\mathbf{p}_2$, $\boldsymbol{\pi}$, $\mathbf{q}_1$, and $\mathbf{q}_2$ in Q_T:

$$\begin{cases}
\partial_t \mathbf{p}_1 + \Delta \mathbf{p}_1 + \nabla \text{ß} + \mathbf{u} \cdot \nabla \mathbf{p}_1 - (\nabla \mathbf{u})\mathbf{p}_1 - (\nabla \mathbf{d})\mathbf{p}_2 \\
= -\beta_1(\mathbf{u} - \mathbf{u}_{Q_T}), \\
\nabla \cdot \mathbf{p}_1 = 0, \\
\partial_t \mathbf{p}_2 + \Delta \mathbf{p}_2 + \mathbf{u} \cdot \nabla \mathbf{p}_2 - \partial_i(\partial_j \mathbf{d} \partial_j \mathbf{p}_1^i) - \partial_j(\partial_i \mathbf{d} \partial_j \mathbf{p}_1^i) \\
= -|\nabla \mathbf{d}|^2 \mathbf{p}_2 + 2\nabla \cdot (\nabla \mathbf{d}(\mathbf{d} \cdot \mathbf{p}_2)) - \beta_2(\mathbf{d} - \mathbf{d}_{Q_T}),
\end{cases} \tag{4.35}$$

with the following boundary and terminal conditions

$$\begin{cases}
\mathbf{p}_1 = 0, \quad \mathbf{p}_2 = 0 & \text{on} \quad \Gamma_T, \\
\mathbf{p}_1|_{t=T} = \beta_3(\mathbf{u}(T) - \mathbf{u}_{\Omega}), \quad \mathbf{p}_2|_{t=T} = \beta_4(\mathbf{d}(T) - \mathbf{d}_{\Omega}) & \text{in} \quad \Omega.
\end{cases} \tag{4.36}$$

Furthermore, the Lagrange multipliers $(\mathbf{q}_1, \mathbf{q}_2)$ can be uniquely determined by $(\mathbf{p}_1, \text{ß}, \mathbf{p}_2)$ through

$$\begin{cases}
\mathbf{q}_1 + \partial_{\boldsymbol{\nu}} \mathbf{p}_1 + \text{ß}\boldsymbol{\nu} = 0 & \text{on} \quad \Gamma_T, \\
\mathbf{q}_2^k + (\partial_{\boldsymbol{\nu}} \mathbf{p}_2)^k = \partial_j \mathbf{d}^k \partial_j \mathbf{p}_1^i \boldsymbol{\nu}^i + \partial_i \mathbf{d}^k \partial_j \mathbf{p}_1^i \boldsymbol{\nu}^j, \ (k = 1, 2, 3) & \text{on} \quad \Gamma_T.
\end{cases} \tag{4.37}$$

4.3.2 Solvability of the adjoint system

In this part, we will show the existence of a unique solution of (4.35) and (4.36). To do it, set

$$\widetilde{\mathbf{p}}_1(t)=\mathbf{p}_1(T-t),\quad \widetilde{\mathbf{p}}_2=\mathbf{p}_2(T-t),\quad \text{and}\quad \widetilde{\text{ß}}(t)=\text{ß}(T-t). \tag{4.38}$$

Then (4.35) and (4.36) becomes

$$\begin{cases}\partial_t\widetilde{\mathbf{p}}_1-\Delta\widetilde{\mathbf{p}}_1-\nabla\widetilde{\text{ß}}-\mathbf{u}(T-t)\cdot\nabla\widetilde{\mathbf{p}}_1\\ =-\nabla\mathbf{u}(T-t)\widetilde{\mathbf{p}}_1-\nabla\mathbf{d}(T-t)\widetilde{\mathbf{p}}_2+\beta_1(\mathbf{u}-\mathbf{u}_{Q_T})(T-t),\\ \nabla\cdot\widetilde{\mathbf{p}}_1=0,\\ \partial_t\widetilde{\mathbf{p}}_2-\Delta\widetilde{\mathbf{p}}_2-\mathbf{u}(T-t)\cdot\nabla\widetilde{\mathbf{p}}_2\\ +\partial_i(\partial_j\mathbf{d}(T-t)\partial_j\widetilde{\mathbf{p}}_1^i)+\partial_j(\partial_i\mathbf{d}(T-t)\partial_j\widetilde{\mathbf{p}}_1^i)\\ =|\nabla\mathbf{d}|^2(T-t)\widetilde{\mathbf{p}}_2-2\nabla\cdot((\nabla\mathbf{d}\mathbf{d})(T-t)\cdot\widetilde{\mathbf{p}}_2))+\beta_2(\mathbf{d}-\mathbf{d}_{Q_T})(T-t),\end{cases} \tag{4.39}$$

in Q_T, under the boundary and initial condition:

$$\begin{cases}(\widetilde{\mathbf{p}}_1,\widetilde{\mathbf{p}}_2)=(0,0) & \text{on}\quad \Gamma_T,\\ (\widetilde{\mathbf{p}}_1,\widetilde{\mathbf{p}}_2)=\big(\beta_3(\mathbf{u}(T)-\mathbf{u}_\Omega),\beta_4(\mathbf{d}(T)-\mathbf{d}_\Omega)\big) & \text{in}\quad \Omega\times\{0\}.\end{cases} \tag{4.40}$$

We have the following existence result to (4.39) and (4.40).

Theorem 4.11 *Assume (A1) and (A2) hold, let* $(\mathbf{u},\mathbf{d})\in\mathcal{H}$ *and* $(\mathbf{u}_\Omega,\mathbf{d}_\Omega)$ *satisfy*

$$\begin{cases}\mathbf{u}_\Omega\in\mathbf{V}, & \text{if } \beta_3>0,\\ \mathbf{d}_\Omega\in H^1(\Omega,\mathbb{S}^2_+)\ \text{with}\ (\mathbf{d}(T)-\mathbf{d}_\Omega)|_\Gamma=0, & \text{if } \beta_4>0.\end{cases} \tag{4.41}$$

Then the system (4.39) *and* (4.40) *admits a unique weak solution* $(\widetilde{\mathbf{p}}_1,\widetilde{\text{ß}},\widetilde{\mathbf{p}}_2)$ *such that*

$$\begin{aligned}&\widetilde{\mathbf{p}}_1\in C([0,T],\mathbf{V})\cap L^2_tH^2_x(Q_T),\\ &\widetilde{\text{ß}}\in L^2_tH^1_x(Q_T)\ \text{with}\ \int_\Omega\widetilde{\text{ß}}(x,t)\,dx=0,\\ &\widetilde{\mathbf{p}}_2\in C([0,T],L^2(\Omega,\mathbb{R}^3))\cap L^2_tH^1_0(Q_T).\end{aligned}$$

Moreover, it holds that

$$\begin{aligned}&\|\widetilde{\mathbf{p}}_1(t)\|_{L^\infty_tH^1_x(Q_T)}+\|\widetilde{\mathbf{p}}_2(t)\|_{L^\infty_tL^2_x(Q_T)}\\ &+\int_0^T(\|\widetilde{\mathbf{p}}_1(\tau)\|^2_{H^2(\Omega)}+\|\widetilde{\mathbf{p}}_2(\tau)\|^2_{H^1(\Omega)})\le C_T,\end{aligned} \tag{4.42}$$

where $C_T>0$ *is a constant depending on* $\|(\mathbf{u},\mathbf{d})\|_{\mathcal{H}}$, $\beta_1\|\mathbf{u}-\mathbf{u}_{Q_T}\|_{L^2(Q_T)}$, $\beta_2\|\mathbf{d}-\mathbf{d}_{Q_T}\|_{L^2(Q_T)}$, $\beta_3\|\mathbf{u}_\Omega\|_{H^1(\Omega)}$, $\beta_4\|\mathbf{d}_\Omega\|_{H^1(\Omega)}$, Ω, *and* T. *For any* $s\in(0,2)$, *it also holds that*

$$\partial_t\widetilde{\mathbf{p}}_2,\ \nabla^2\widetilde{\mathbf{p}}_2\in L^{2-s}(Q_T). \tag{4.43}$$

Proof The existence of weak solutions follows from the Faedo–Galerkin method, similar to [5] Proposition 4.1, which is left for the readers. Here we sketch the proof of a priori estimates.

Multiplying (4.39) by $\Delta\widetilde{\mathbf{p}}_1$, (4.40) by $\widetilde{\mathbf{p}}_2$, and adding the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)}) + (\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\nabla\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)})\\
&\quad = -\int_\Omega \langle \mathbf{u}(T-t)\cdot\nabla\widetilde{\mathbf{p}}_1, \Delta\widetilde{\mathbf{p}}_1\rangle + \int_\Omega \langle (\nabla\mathbf{u}(T-t))\widetilde{\mathbf{p}}_1, \Delta\widetilde{\mathbf{p}}_1\rangle\\
&\qquad + \int_\Omega \langle (\nabla\mathbf{d}(T-t))\widetilde{\mathbf{p}}_2, \Delta\widetilde{\mathbf{p}}_1\rangle - \int_\Omega \beta_1\langle (\mathbf{u}-\mathbf{u}_{Q_T})(T-t), \Delta\widetilde{\mathbf{p}}_1\rangle\\
&\qquad - \int_\Omega \langle \partial_i(\partial_j\mathbf{d}(T-t)\partial_j\widetilde{\mathbf{p}}_1^i) + \partial_j(\partial_i\mathbf{d}(T-t)\partial_j\widetilde{\mathbf{p}}_1^i), \widetilde{\mathbf{p}}_2\rangle\\
&\qquad + \int_\Omega \langle |\nabla\mathbf{d}|^2(T-t)\widetilde{\mathbf{p}}_2, \widetilde{\mathbf{p}}_2\rangle - 2\int_\Omega \langle \nabla\cdot(\nabla\mathbf{d}\mathbf{d}(T-t)\cdot\widetilde{\mathbf{p}}_2), \widetilde{\mathbf{p}}_2\rangle\\
&\qquad + \beta_2\int_\Omega \langle (\mathbf{d}-\mathbf{d}_{Q_T})(T-t), \widetilde{\mathbf{p}}_2\rangle = \sum_{i=1}^{8} I_i.
\end{aligned}
\tag{4.44}
$$

We can estimate I_i $(1 \le i \le 8)$ as follows.

$$
\begin{aligned}
|I_1| &\le \frac{1}{16}\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + C\|\mathbf{u}(T-t)\|^4_{H^1(\Omega)}\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)},\\
|I_2| &\le \frac{1}{16}\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + C\|\mathbf{u}(T-t)\|^2_{H^2(\Omega)}\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)},\\
|I_3| &\le \frac{1}{16}\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + C\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)}\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)},\\
|I_4| &\le \frac{1}{16}\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + C\beta_1^2\|(\mathbf{u}-\mathbf{u}_{Q_T})(T-t)\|^2_{L^2(\Omega)},\\
|I_5| &\le \frac{1}{16}\big(\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\nabla\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)}\big)\\
&\quad + C(1+\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)})\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)}\\
&\quad \cdot(\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)}),\\
|I_6| &\le \frac{1}{16}\|\nabla\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)} + C\|\mathbf{d}(T-t)\|^4_{H^2(\Omega)}\|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)},\\
|I_7| &\le \frac{1}{16}\|\nabla\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)} + C\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)}\|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)}\\
&\quad + C\|\mathbf{d}(T-t)\|^4_{H^2(\Omega)}\|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)},\\
|I_8| &\le \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)} + C\beta_2^2\|(\mathbf{d}-\mathbf{d}_{Q_T})(T-t)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Putting these estimates into (4.44), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)}) + (\|\Delta\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\nabla\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)})\\
&\quad \le C(1+\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)})\|\mathbf{d}(T-t)\|^2_{H^2(\Omega)}(\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)})\\
&\qquad + C(\|\mathbf{u}(T-t)\|^4_{H^1(\Omega)} + \|\mathbf{u}(T-t)\|^2_{H^2(\Omega)})(\|\nabla\widetilde{\mathbf{p}}_1\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{p}}_2\|^2_{L^2(\Omega)})\\
&\qquad + C\big(\beta_1^2\|(\mathbf{u}-\mathbf{u}_{Q_T})(T-t)\|^2_{L^2(\Omega)} + \beta_2^2\|(\mathbf{d}-\mathbf{d}_{Q_T})(T-t)\|^2_{L^2(\Omega)}\big).
\end{aligned}
$$

Since $(\mathbf{u},\mathbf{d}) \in \mathcal{H}$, we have that

$$\int_0^T (\|\mathbf{u}(T-t)\|^4_{H^1(\Omega)} + \|\mathbf{d}(T-t)\|^4_{H^2(\Omega)})\,dt \le C\|(\mathbf{u},\mathbf{d})\|^4_{\mathcal{H}},$$
$$\int_0^T (\|\mathbf{u}(T-t)\|^2_{H^2(\Omega)} + \|\mathbf{d}(T-t)\|^2_{H^2(\Omega)})\,dt \le C\|(\mathbf{u},\mathbf{d})\|^2_{\mathcal{H}}.$$

Hence we can apply Gronwall's inequality to show (4.42). Note that (4.42), together with (4.39) and (4.40), implies that $\partial_t\widetilde{\mathbf{p}}_1 \in L^2([0,T],\mathbf{H})$, $\partial_t\widetilde{\mathbf{p}}_2 \in L^2_t H^{-1}_x(Q_T,\mathbb{R}^3)$, and $\nabla\widetilde{\mathbf{\ss}} \in L^2(Q_T)$.

Observe that

$$\partial_t\widetilde{\mathbf{p}}_2 - \Delta\widetilde{\mathbf{p}}_2 = \mathbf{F},$$

where

$$\begin{aligned}|\mathbf{F}| \le C\big[&(|\mathbf{u}| + |\nabla\mathbf{d}|)(T-t)|\nabla\widetilde{\mathbf{p}}_2| + |\nabla^2\mathbf{d}|(T-t)|\nabla\widetilde{\mathbf{p}}_1| \\ &+ |\nabla\mathbf{d}|(T-t)|\nabla^2\widetilde{\mathbf{p}}_1| + |\nabla^2\mathbf{d}|(T-t)|\widetilde{\mathbf{p}}_2| + |(\mathbf{d}-\mathbf{d}_{Q_T})|(T-t)\big].\end{aligned}$$

It follows easily from (4.11) and the fact that $(\mathbf{u},\mathbf{d}) \in \mathcal{H}$ that

$$|\mathbf{F}| \in L^{2-s}(Q_T),\ \forall 0 < s < 2.$$

Hence we can apply the standard L^{2-s}-theory of parabolic equations to deduce $\partial_t\widetilde{\mathbf{p}}_2, \nabla^2\widetilde{\mathbf{p}}_2 \in L^{2-s}(Q_T)$. This completes the proof. □

From the relations (4.38) and (4.37), we have

Corollary 4.12 *Under the same assumptions of Theorem 4.11, the adjoint system (4.35) and (4.36) admits a unique weak solution $(\mathbf{p}_1, \mathbf{\ss}, \mathbf{p}_2)$, satisfying the same properties as for the weak solution $(\widetilde{\mathbf{p}}_1, \widetilde{\mathbf{\ss}}, \widetilde{\mathbf{p}}_2)$ to the system (4.39) and (4.40) stated in Theorem 4.11. Moreover, the Lagrange multipliers $(\mathbf{q}_1, \mathbf{q}_2)$ are uniquely determined by (4.37) such that*

$$\mathbf{q}_1 \in L^2_t H^{\frac{1}{2}}_x(\Gamma_T,\mathbb{R}^2), \quad \mathbf{q}_2 \in L^1_t H^{\frac{1}{2}}_x(\Gamma_T,\mathbb{R}^3). \tag{4.45}$$

Proof The proof is similar to [4] Corollary 6.1. We omit the detail. □

4.3.3 The first-order necessary condition via adjoint systems

With the help of previous subsections, we are able to formulate another necessary condition for optimal boundary control in terms of adjoint systems. More precisely, we have the following theorem.

Theorem 4.13 *Assume (A1) and (A2). For $M > 0$, let $(\mathbf{u}_0,\mathbf{d}_0) \in \mathbf{V}\times\mathbf{H}^2(\Omega,\mathbb{S}^2_+)$ and*

$$\begin{aligned}&\mathbf{u}_\Omega \in \mathbf{V}, \quad \text{if } \beta_3 > 0,\\ &\mathbf{d}_\Omega \in H^1(\Omega,\mathbb{S}^2_+) \ \ \textit{with}\ (\mathbf{d}(T)-\mathbf{d}_\Omega)|_\Gamma = \mathbf{0}, \quad \text{if } \beta_4 > 0.\end{aligned}$$

Let $\mathbf{h}$ be an optimal boundary control for (4.1) in $\widetilde{\mathcal{U}}_M$, with the associate state $(\mathbf{u},\mathbf{d}) = \mathcal{S}(\mathbf{h}) \in \mathcal{H}$ and the adjoint state $(\mathbf{p}_1,\mathbf{p}_2)$ given by (4.35) and (4.36). For any $\widehat{\mathbf{h}} \in \widetilde{\mathcal{U}}_M$, if $\boldsymbol{\xi}$ is the section of $\mathbf{h}^*T\widetilde{\mathcal{U}}$ given by

$$\boldsymbol{\xi} = \frac{d}{ds}\Big|_{s=0}\Pi^{-1}(s\Pi(\mathbf{h}) + (1-s)\Pi(\widehat{\mathbf{h}})),$$

then the following variational inequality holds:

$$\beta_5 \int_{\Gamma_T} \langle \mathbf{h} - \mathbf{e}_3, \boldsymbol{\xi} \rangle + \int_{\Gamma_T} \langle \partial_j \mathbf{d} \partial_j \mathbf{p}_1^i \boldsymbol{\nu}_i + \partial_i \mathbf{d} \partial_j \mathbf{p}_1^i \boldsymbol{\nu}_j - \partial_{\boldsymbol{\nu}} \mathbf{p}_2, \boldsymbol{\xi} \rangle \geq 0. \tag{4.46}$$

Proof Set

$$\mathbf{h}(s) = \Pi^{-1}(s\Pi(\mathbf{h}) + (1-s)\Pi(\widehat{\mathbf{h}})) \in C^1([0,1], \widetilde{\mathcal{U}}_M),$$

and

$$(\mathbf{u}(s), \mathbf{d}(s)) = \mathcal{S}(\mathbf{h}(s)) \text{ for } s \in [0,1].$$

Then $\mathbf{h}(0) = \mathbf{h}$, $(\mathbf{u}, \mathbf{d}) = (\mathbf{u}(0), \mathbf{d}(0))$, and $\mathbf{h}(1) = \widehat{\mathbf{h}}$. Then (4.46) follows from the minimality of $\mathcal{G}$ at $\mathbf{h}$ and

$$\frac{d}{ds}|_{s=0} \mathcal{G}((\mathbf{u}, \mathbf{d}), \mathbf{h}(s), \mathbf{p}_1, \mathbf{p}_2, \pi, \mathbf{q}_1, \mathbf{q}_2) \geq 0.$$

□

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