ISSN 0012-9593

quatrième série - tome 52 fascicule 1 janvier-février 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € ($ 863). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 52, 2019, p. 139 à 179

GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE FOCUSING, CUBIC SCHRÖDINGER EQUATION IN DIMENSION $d = 4$

BY BENJAMIN DODSON

ABSTRACT. – In this paper we prove global well-posedness and scattering for the focusing, cubic Schrödinger equation in four dimensions below the ground state. Previous work proved this in five dimensions and higher. To prove this we combine the double Duhamel method with the long time Strichartz estimates.

RÉSUMÉ. – Nous prouvons l'existence globale et la diffusion des ondes pour l'équation de Schrödinger cubique focalisante en dimension quatre. Des travaux antérieurs ont montré de tels résultats en dimension supérieure ou égale à cinq. Nous utilisons ici la méthode de Duhamel double et les estimations de Strichartz en temps long.

1. Introduction

In this paper we study the nonlinear Schrödinger initial value problem

$$iu_t + \Delta u = F(u) = -|u|^2 u, \quad u(0,x) = u_0 \in \dot{H}^1(\mathbf{R}^4), \tag{1.1}$$

which belongs to a class of problems known as the focusing, nonlinear Schrödinger initial value problems,

$$iu_t + \Delta u = F(u) = -|u|^p u, \quad u(0,x) = u_0 \in \dot{H}^1(\mathbf{R}^d), \tag{1.2}$$

In general a solution to (1.2) conserves mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{1.3}$$

and energy,

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{1}{p+2}\int |u(t,x)|^{p+2} dx = E(u(0)). \tag{1.4}$$

 doi:10.24033/asens.2385

When $p = \frac{4}{d-2}$, (1.2) is called energy-critical since a solution to (1.2) is invariant under the scaling

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x), \tag{1.5}$$

and (1.5) preserves the energy (1.4) when $p = \frac{4}{d-2}$.

The global behavior of the defocusing, energy-critical problem ($F(u) = |u|^{\frac{4}{d-2}} u$) is now completely worked out for any $d \geq 3$.

Theorem 1.1. – *The defocusing initial value problem* (1.2), $F(u) = |u|^{\frac{4}{d-2}} u$, *is globally well-posed for any* $u_0 \in \dot{H}^1(\mathbf{R}^d)$, $d \geq 3$, *and the solution scatters both forward and backward in time.*

Definition 1.1 (Scattering). – *A solution* u *to* (1.2) *with* $p = \frac{4}{d-2}$ *is said to scatter forward in time if there exists* $u_+ \in \dot{H}^1$ *such that*

$$\lim_{t \nearrow +\infty} \| u(t) - e^{it\Delta} u_+ \|_{\dot{H}^1(\mathbf{R}^d)} = 0. \tag{1.6}$$

Likewise, u *is said to scatter backward in time if there exists* $u_- \in \dot{H}^1$ *such that*

$$\lim_{t \searrow -\infty} \| u(t) - e^{it\Delta} u_- \|_{\dot{H}^1(\mathbf{R}^d)} = 0. \tag{1.7}$$

Proof. – The proof of Theorem 1.1 involved contributions from numerous authors. [10] proved Theorem 1.1 for small data for both the focusing and defocusing problem. [10] also proved that (1.2) has a local solution for any initial data $u_0 \in \dot{H}^1(\mathbf{R}^d)$, where the time of existence depends on the size and profile of u_0.

For large data, the seminal result was the work of [5] (also see [4]), proving Theorem 1.1 for radial data in dimensions $d = 3, 4$, and also that for more regular u_0, this additional smoothness is preserved. See [23] for another proof of this last fact. [42] then extended Theorem 1.1 to radial data in higher dimensions. Both [5] and [42] used the induction on energy method.

For nonradial data, the first progress came when [13] extended Theorem 1.1 to general $u_0 \in \dot{H}^1(\mathbf{R}^3)$. Subsequently, [36] extended this result dimension $d = 4$, and [49] (also see [48]) extended Theorem 1.1 to dimensions $d \geq 5$. □

Remark. – [31] and [?] have since used the long time Strichartz estimates of [14] to reprove Theorem 1.1 in dimensions three and four, respectively.

However, Theorem 1.1 does not hold for arbitrary data in the focusing case. By the virial identity (see for example [22])

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 8 \Big[\int |\nabla u(t,x)|^2 dx - \int |u(t,x)|^{\frac{2d}{d-2}} dx \Big], \tag{1.8}$$

so if $x u_0 \in L^2(\mathbf{R}^d)$ and $E(u_0) < 0$, $\int |x|^2 |u(t,x)|^2 dx$ is a function of t that is concave down and has two real roots, $t_1 < 0 < t_2$. Then the positive definiteness of $\int |x|^2 |u(t,x)|^2 dx$ implies that the solution to (1.1) with such u_0 cannot exist outside of $[t_1, t_2]$.

There also exist global solutions to (1.1) that do not scatter.

$$W(x) = \frac{1}{(1 + \frac{|x|^2}{d(d-2)})^{\frac{d-2}{2}}} \tag{1.9}$$

lies in $\dot{H}^1(\mathbf{R}^d)$ and solves the elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0, \tag{1.10}$$

so $W(x,t) = W(x)$ solves (1.1) but is clearly non scattering. Therefore, as in the mass-critical problem we conjecture that scattering holds for initial data below the threshold given by (1.9).

Conjecture 1.1. – *Let $d \geq 3$ and let $u : I \times \mathbf{R}^d \to \mathbf{C}$ be a solution to* (1.2), $p = \frac{4}{d-2}$. *If*

$$\|u_0\|_{\dot{H}^1(\mathbf{R}^d)} < \|W\|_{\dot{H}^1(\mathbf{R}^d)}, \tag{1.11}$$

and

$$E(u_0) < E(W), \tag{1.12}$$

then

$$\int_I \int |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt \leq C(\|u_0\|_{\dot{H}^1}, E(u_0)) < \infty. \tag{1.13}$$

The quantity $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)}$ is the key quantity to determining whether or not u scatters forward in time or backward in time.

Definition 1.2 (Scattering size). – *The scattering size of a solution to* (1.2) *on a time interval I is given by*

$$S_I(u) = \int_I \int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt. \tag{1.14}$$

Theorem 1.2. – *When $p = \frac{4}{d-2}$,* (1.2) *is well-posed on some open interval $I(u_0)$. Additionally, u scatters forward in time if and only if $S_{[t_1,\infty)}(u) < \infty$ for some $t_1 \in \mathbf{R}$. Likewise, u scatters backward in time if and only if $S_{(-\infty,t_1]}(u) < \infty$ for some $t_1 \in \mathbf{R}$.*

Proof. – See [10] and [11]. □

Therefore a solution may either scatter or blow-up.

Definition 1.3 (Blow up). – *A solution u to* (1.2) *blows up forward in time on I if there exists $t_1 \in I$ such that*

$$S_{[t_1,\sup(I))}(u) = \infty. \tag{1.15}$$

u blows up backward in time if there exists $t_1 \in I$ such that

$$S_{(\inf(I),t_1]}(u) = \infty. \tag{1.16}$$

[25] proved Conjecture 1.1 for radial data in dimensions $d = 3, 4, 5$. The proof uses the concentration compactness argument.

Theorem 1.3. – *Assume that $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $d = 3, 4, 5$, and u_0 is radial. Then* (1.2) *is globally well-posed and scatters forward and backward in time.*

Proof. – See [25]. □

[28] treated the nonradial case in dimensions $d \geq 5$.

Theorem 1.4. – *Assume that $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $d \geq 5$. Then* (1.2) *is globally well-posed and scatters forward and backward in time.*

Proof. – See [28]. □

REMARK. – The result of [28] was actually proved under the possibly weaker assumption

$$\|u\|_{L_t^\infty \dot{H}_x^1(I\times\mathbf{R}^4)} < \|\nabla W\|_{L^2(\mathbf{R}^4)}. \tag{1.17}$$

Now by the energy trapping lemma of [25], if $E(u_0) < E(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, then (1.17) holds.

LEMMA 1.5 (Energy trapping lemma). – *If $E(u_0) \leq (1-\delta)E(W)$ and $\|\nabla u_0\|_{L^2(\mathbf{R}^d)} < (1-\delta)\|\nabla W\|_{L^2(\mathbf{R}^d)}$ for some $\delta > 0$, then there exists $\bar{\delta}(\delta, d) > 0$ such that for all $t \in I$, where I is the maximal interval of existence of u,*

$$\|\nabla u(t)\|_{L_x^2(\mathbf{R}^d)} \leq (1-\bar{\delta})\|\nabla W\|_{L^2(\mathbf{R}^d)}. \tag{1.18}$$

Proof. – This follows from the work of [1] and [40], which proved that if C_d is the best constant in the Sobolev embedding theorem:

$$\|f\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)} \leq C_d\|\nabla f\|_{L_x^2(\mathbf{R}^d)}. \tag{1.19}$$

That is, if

$$\|u\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)} = C_d\|\nabla u\|_{L_x^2(\mathbf{R}^d)}, \tag{1.20}$$

then $u = CW_{\theta_0,x_0,\lambda_0}$ for some constant $C \in \mathbf{C}$, $\theta_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^d$, and $\lambda_0 \in (0,\infty)$, where

$$W_{\theta_0,x_0,\lambda_0} = \frac{1}{\lambda_0^{\frac{d-2}{2}}} e^{i\theta_0} W\left(\frac{x-x_0}{\lambda_0}\right), \tag{1.21}$$

and W is given by (1.9).

So when $d = 4$, (1.10) implies that

$$0 = \langle \Delta W, W\rangle + \langle W, |W|^2 W\rangle = -\int |\nabla W|^2 dx + \int |W|^4 dx. \tag{1.22}$$

Then by (1.20),

$$C_4 = \frac{1}{\|W\|_{L_x^4(\mathbf{R}^4)}}, \tag{1.23}$$

so

$$(1-\delta)E(W) \geq E(u_0) = \frac{1}{2}\int |\nabla u(t)|^2 dx \left(1 - \frac{1}{2}\frac{\|u(t)\|_{L_x^4(\mathbf{R}^4)}^2}{\|W\|_{L_x^4(\mathbf{R}^4)}^2}\right). \tag{1.24}$$

Now make a bootstrap argument. Let

$$J = \{t \in I : \|u(t)\|_{\dot{H}^1} \leq \|\nabla W\|_{L^2}\}. \tag{1.25}$$

By the Sobolev embedding theorem and the fact that C_4 is the best constant,

$$\|u(t)\|_{L^4(\mathbf{R}^4)} \leq \|W\|_{L^4(\mathbf{R}^4)} \tag{1.26}$$

for all $t \in J$. Also by the dominated convergence theorem and local well-posedness, J is closed. Also by $\|u(0)\|_{\dot{H}^1} \leq (1-\delta)\|W\|_{\dot{H}^1}$, J is not empty. Then by (1.20), (1.23), (1.24), and conservation of energy

$$(1-\delta)E(W) = (1-\delta)\frac{1}{4}\|W\|_{\dot{H}^1(\mathbf{R}^4)}^2 \geq \frac{1}{4}\|\nabla u(t)\|_{L_x^2(\mathbf{R}^4)}^2, \tag{1.27}$$

which in turn implies that for $t \in J$, $\|u(t)\|^2_{\dot{H}^1(\mathbf{R}^4)} \leq (1-\delta)\|W\|^2_{\dot{H}^1(\mathbf{R}^4)}$. Local well-posedness then implies J is also open in I, therefore $J = I$. □

Scattering results for the focusing, mass-critical problem ([15], [33], [34], [44]) assume that the initial data u_0 has mass below the mass of a ground state. However, unlike the mass (1.3), the $\dot{H}^1(\mathbf{R}^d)$ norm is not conserved. Energy is conserved, but is not positive definite (1.4), so $E(u(t)) < E(W)$ does not by itself give a bound on the size of $u(t)$, hence the two conditions in Theorem 1.3.

Remark. – The author of this paper is personally unaware of any solutions $u(t)$ to (1.2), $p = \frac{4}{d-2}$ that satisfy (1.17) but not the initial conditions of Theorem 1.3, and would be interested in more information on the matter.

In this paper we prove global well-posedness and scattering for nonradial data in dimension four.

Theorem 1.6. – *Assume that* $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, *and* $d = 4$. *Then* (1.2) *is globally well-posed and scatters forward and backward in time.*

As in [25] and [28], the proof uses the concentration compactness method. The proof may be separated into two theorems.

Theorem 1.7. – *If* (1.1) *is not globally well-posed and scattering for all data satisfying* $\|u_0\|_{\dot{H}^1(\mathbf{R}^4)} < \|W\|_{\dot{H}^1(\mathbf{R}^4)}$ *and* $E(u_0) < E(W)$, *then there exists a nonzero solution to* (1.1) *that is almost periodic for the entire time of its existence. That is, u solves* (1.1) *on I, where I is the maximal interval of its existence, and u is almost periodic for all $t \in I$.*

Definition 1.4 (Almost periodicity). – *$u(t)$ is said to be almost periodic for all $t \in I$ if there exists $N(t) : I \to (0, \infty)$ and $x(t) : I \to \mathbf{R}^4$ such that for all $t \in I$, $\frac{1}{N(t)} u(\frac{x - x(t)}{N(t)})$ lies in a compact set $K \subset \dot{H}^1(\mathbf{R}^4)$.*

Theorem 1.8. – *The only almost periodic solution to* (1.1) *on the maximal interval of its existence I, with $\|\nabla u(t)\|_{L_t^\infty L_x^2(I \times \mathbf{R}^4)} < \|\nabla W\|_{L^2}$, is $u \equiv 0$.*

Theorem 1.7 is already well-known, so its proof will merely be sketched in section three. The novel part of this paper is the proof of Theorem 1.8.

In fact, [28] proved the reduction

Theorem 1.9. – *To prove Theorem 1.8 it suffices to show that the only global, almost periodic solution to* (1.1) *on* $\mathbf{R}$ *with*

$$N(t) \geq 1, \qquad N(0) = 1, \tag{1.28}$$

is $u \equiv 0$.

Thus we shall prove that

Theorem 1.10. – *The only global, almost periodic solution to* (1.1) *on* $\mathbf{R}$ *with*

$$N(t) \geq 1, \qquad N(0) = 1, \tag{1.29}$$

is $u \equiv 0$.

[28] utilized the double Duhamel method to prove Theorem 1.10 for dimensions $d \geq 5$. The double Duhamel method was introduced in [13] for the energy-critical, defocusing, nonlinear Schrödinger equation (see also [31]). This method has proved to be extremely useful throughout dispersive partial differential equations, for Schrödinger, ([13], [28], [31], [41]), wave ([8], [7], [6], [20], [19], [30], [29]) and mKdV ([18]) problems.

The main new difficulty in dimension $d = 4$ is that the dispersive estimate is not doubly integrable, unlike in dimensions $d \geq 5$. We prove Theorem 4.3, which implies $u \in L_t^\infty L_x^3$, and thus $F(u) \in L_t^\infty L^1$, however even with this fact the double integral of (2.18) diverges logarithmically.

Remark. – This is not merely a technical obstacle, since (1.9) gives an example of an almost periodic solution that does not lie in L^2.

Despite this difficulty, we are able to combine this logarithmically divergent result with the long time Strichartz estimates to establish an interaction Morawetz estimate, proving Theorem 1.10.

Outline of Proof. – In §2, some linear estimates and harmonic analysis results will be discussed. These results will be used frequently throughout the rest of the paper. Only one of the results in this section (Theorem 2.6 when $d \geq 4$) is new.

In §3, the concentration compactness method will be discussed, sketching [25]'s (see [28] for higher dimensions) proof of Theorem 1.7. We will also discuss almost periodic solutions to (1.1) and sketch [28]'s proof of Theorem 1.9.

In §4 we prove the long time Strichartz estimate. In contrast to [?] and [29], we will consider the quantity

$$\int_I \frac{1}{N(t)^2} dt. \tag{1.30}$$

The long time Strichartz estimates allow us to easily exclude the case when $\int_{\mathbf{R}} N(t)^{-2} dt < \infty$. Finally we will bound the $L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)$ norm of a solution satisfying (1.29) and $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$.

In §5 we use the long time Strichartz estimates of section four to show that the soliton blowup solution, that is $N(t) \equiv 1$, is $u \equiv 0$. Finally, in §6 we will extend this argument to a quasi soliton solution, (1.30) $= \infty$. This completes the proof of Theorem 1.6. ☐

Acknowledgements. – During the time of researching this paper, the author was supported by NSF postdoctoral fellowship DMS-1103914. The author also performed much of the research while a guest of the Hausdorff Institute at the University of Bonn for the summer trimester program in harmonic analysis and partial differential equations. The author is now supported on NSF grant DMS-1500424. The author is grateful to Rowan Killip, Jason Murphy, and Monica Visan for several helpful discussions regarding this problem.

2. Linear Estimates and harmonic analysis

In this section we describe the tools from harmonic analysis that will be used in this paper.

Definition 2.1 (Fourier transform). – *Suppose $f \in L^1(\mathbf{R}^d)$. Let $\mathcal{F} f$ denote the Fourier transform*

$$\mathcal{F} f(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x)dx. \tag{2.1}$$

The inverse Fourier transform is given by

$$(\mathcal{F}^{-1} g)(x) = (2\pi)^{-d/2} \int e^{ix\cdot\xi} g(\xi)d\xi. \tag{2.2}$$

Plancherel's theorem proved that the Fourier transform and inverse Fourier transform provide a unitary transformation between functions in $L_x^2(\mathbf{R}^d)$ and functions in $L_\xi^2(\mathbf{R}^d)$. Because of this fact it is useful to decompose a function via a partition of unity in Fourier space, or a Littlewood-Paley decomposition.

Definition 2.2 (Littlewood-Paley decomposition). – *Let $\phi \in C_0^\infty(\mathbf{R}^d)$ be a radial, decreasing function, $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x)$ is supported on $|x| \leq 2$. Then for any $j \in \mathbf{Z}$ let*

$$P_j f = (2\pi)^{-d/2} \int e^{ix\cdot\xi} [\phi(2^{-j-1}\xi) - \phi(2^{-j}\xi)] \hat{f}(\xi)d\xi, \tag{2.3}$$

$$P_{\leq j} f = (2\pi)^{-d/2} \int e^{ix\cdot\xi} \phi(2^{-j-1}\xi) \hat{f}(\xi)d\xi, \tag{2.4}$$

and

$$P_{\geq j} f = (2\pi)^{-d/2} \int e^{ix\cdot\xi} [1 - \phi(2^{-j}\xi)] \hat{f}(\xi)d\xi. \tag{2.5}$$

Remark. – It is often convenient to write P_N, which is given by the multiplier

$$[\phi(\frac{1}{N}\xi) - \phi(\frac{1}{2N}\xi)]. \tag{2.6}$$

$P_{\leq N}$ and $P_{\leq N}$ are defined in the obvious fashion. When summing over Littlewood-Paley pieces, $\sum_{M \geq N}$ denotes $\sum_{j \geq 0; M = 2^j N}$. $\sum_{M \leq N}$ is similarly defined.

Remark. – To simplify notation it is convenient to write u_k or u_N instead of $P_k u$ or $P_N u$.

Theorem 2.1 (Littlewood-Paley theorem). – *For any $1 < p < \infty$,*

$$\|(\sum_j |P_j f|^2)^{1/2}\|_{L_x^p(\mathbf{R}^d)} \sim_{p,d} \|f\|_{L^p(\mathbf{R}^d)}. \tag{2.7}$$

Proof. – This is a well-known fact from harmonic analysis. See [37], [38], [45], or many other sources. □

The proof of Theorem 2.1 utilizes the maximal function, which can be defined in any dimension. This paper will only use the maximal function in one dimension.

Definition 2.3 (Maximal function). – *For a function $f \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, let*

$$\mathcal{M}(f)(x) = \sup_{T>0} \frac{1}{T} \int_{x-T}^{x+T} |f(t)| dt. \tag{2.8}$$

Theorem 2.2 (Maximal function theorem). – *For any $1 < p \leq \infty$,*

$$\|\mathcal{M}(f)\|_{L^p(\mathbf{R})} \lesssim_p \|f\|_{L^p(\mathbf{R})}. \tag{2.9}$$

Proof. – See [37], [38], or [45]. The proof there is described in any dimension. □

Theorem 2.3 (Sobolev embedding theorem). – *For $1 \leq p \leq q \leq \infty$,*

$$\|P_j f\|_{L^q(\mathbf{R}^d)} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})} \|P_j f\|_{L^p(\mathbf{R}^d)}. \tag{2.10}$$

Proof. – See for example [47]. □

Lemma 2.4 (Bernstein's inequality). – *For any $s \in \mathbf{R}$, $j \in \mathbf{Z}$, $1 < p < \infty$,*

$$2^{js} \|P_j f\|_{L^p(\mathbf{R}^d)} \sim_{p,d} \||\nabla|^s P_j f\|_{L^p(\mathbf{R}^d)}. \tag{2.11}$$

Proof. – See [46]. □

Theorem 2.1, Theorem 2.3, and Lemma 2.4 will be used throughout this paper, frequently in combination.

The Fourier transform intertwines the multiplication and differentiation operators, so the solution to the initial value problem

$$(i\partial_t + \Delta)u = F, \qquad u(0,x) = u_0, \tag{2.12}$$

when $F = 0$ is given by

$$e^{it\Delta}u_0 = (2\pi)^{-d/2} \int e^{-it|\xi|^2} e^{ix\cdot\xi} (\mathcal{F} f)(\xi) d\xi. \tag{2.13}$$

The general strong solution to (2.12) is given by

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i \int_{t_0}^{t} e^{i(t-\tau)\Delta} F(\tau) d\tau. \tag{2.14}$$

Since $|e^{it|\xi|^2}| = 1$,

$$\|e^{it\Delta} f\|_{L^2(\mathbf{R}^d)} = \|f\|_{L^2(\mathbf{R}^d)}, \tag{2.15}$$

and in fact, for any L^2-based Sobolev space,

$$\|e^{it\Delta} f\|_{\dot{H}^s_x(\mathbf{R}^d)} = \|f\|_{\dot{H}^s(\mathbf{R}^d)}. \tag{2.16}$$

By completing the square in the exponent of (2.13) and by stationary phase computations,

$$e^{it\Delta} f(x) = \frac{1}{(4\pi t)^{d/2}} e^{-id\pi/4} \int e^{-i\frac{|x-y|^2}{4t}} f(y) dy, \tag{2.17}$$

which implies

$$\|e^{it\Delta} f\|_{L^\infty_x(\mathbf{R}^d)} \lesssim_d t^{-d/2} \|f\|_{L^1(\mathbf{R}^d)}. \tag{2.18}$$

REMARK. – More generally, if P is any Fourier multiplier, that is,

(2.19) $$Pf(x) = \mathscr{F}^{-1}(P(\xi)(\mathscr{F}f)(\xi)),$$

then

(2.20) $$Pf(x) = \int (\mathscr{F}^{-1}P)(x-y)f(y)dy.$$

Using both analysis on the Fourier side (2.13) (see [39]), and analysis on the spatial side (2.17) (see [21], [24], and [50]), the sharp result has been proved,

THEOREM 2.5 (Strichartz estimates). – *For $d \geq 3$, suppose (p_1, q_1), (p_2, q_2) are admissible pairs, that is $p_j \geq 2$ and*

(2.21) $$\frac{2}{p_j} = d(\frac{1}{2} - \frac{1}{q_j}).$$

If u solves (2.12) on I, $t_0 \in I$, and $\frac{1}{p'} = 1 - \frac{1}{p}$, then

(2.22) $$\|u\|_{L_t^{p_1} L_x^{q_1}(I \times \mathbf{R}^d)} \lesssim_d \|u(t_0)\|_{L_x^2(\mathbf{R}^d)} + \|F\|_{L_t^{p_2'} L_x^{q_2'}(I \times \mathbf{R}^d)}.$$

Proof. – See [39] for the seminal result, [21] and [50] for the non-endpoint results ($p_j > 2$), and [24] for the endpoint case. See [43] for a nice overview of this work. □

Following Theorem 2.5, it is convenient to use the Strichartz spaces of [43].

DEFINITION 2.4 (Strichartz space). – *When $d \geq 3$ let*

(2.23) $$\|u\|_{S^0(I \times \mathbf{R}^d)} = \sup_{(p,q) \text{ admissible}} \|u\|_{L_t^p L_x^q(I \times \mathbf{R}^d)}.$$

Let N^0 be the dual to S^0. Also, for any $s \in \mathbf{R}$ let

(2.24) $$\|u\|_{\dot{S}^s(I \times \mathbf{R}^d)} = \||\nabla|^s u\|_{S^0(I \times \mathbf{R}^d)} \quad \text{and} \quad \|F\|_{\dot{N}^s(I \times \mathbf{R}^d)} = \||\nabla|^s F\|_{N^0(I \times \mathbf{R}^d)}.$$

Then Theorem 2.5 implies

(2.25) $$\|u\|_{S^0(I \times \mathbf{R}^d)} \lesssim \|u(t_0)\|_{L^2(\mathbf{R}^d)} + \|F\|_{N^0(I \times \mathbf{R}^d)}.$$

The Strichartz estimates are quite important to the study of nonlinear Schrödinger initial value problem for a number of reasons. In this paper, as in [14], [?], and [31], the Strichartz estimates are the building blocks of the long time Strichartz estimates. For the three dimensional, energy-critical initial value problem, [31] made use of a maximal Strichartz estimate. Since we are analyzing a four dimensional equation, we will extend this result to dimensions $d \geq 4$.

THEOREM 2.6 (Maximal Strichartz estimate). – *Suppose $t, t_0 \in I$, and*

(2.26) $$v(t) = \int_{t_0}^{t} e^{i(t-\tau)\Delta} F(\tau) d\tau.$$

Then for any $d \geq 3$, $q > \frac{2d}{d-2}$,

(2.27) $$\|\sup_j 2^{j(\frac{d}{q}-(d-2))} \|P_j v(t)\|_{L_x^q(\mathbf{R}^d)}\|_{L_t^2(I)} \lesssim_q \|F\|_{L_t^2 L_x^1(I \times \mathbf{R}^d)}.$$

Proof. – This is proved by combining the dispersive estimate (2.18) with the Sobolev embedding theorem (Theorem 2.3). If $q > \frac{2d}{d-2}$ then $d(\frac{1}{2} - \frac{1}{q}) > 1$, so

$$(2.28) \quad 2^{j(\frac{d}{q}-(d-2))} \int_{|t-\tau|>2^{-2j}} \frac{1}{(t-\tau)^{d(\frac{1}{2}-\frac{1}{q})}} \|P_j F(u(\tau))\|_{L_x^{q'}(\mathbf{R}^d)} d\tau$$
$$\lesssim \sum_{k\geq 0} 2^{-kd(\frac{1}{2}-\frac{1}{q})} 2^{2j} \int_{|t-\tau|\sim 2^k 2^{-2j}} \|F(\tau)\|_{L_x^1(\mathbf{R}^d)} d\tau \lesssim_q \mathcal{M}(\|F(\tau)\|_{L_x^1(\mathbf{R}^d)})(t).$$

By the Sobolev embedding theorem and (2.16),

$$(2.29) \quad 2^{j(\frac{d}{q}-(d-2))} \| \int_{|t-\tau|\leq 2^{-2j}} P_j e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \|_{L_x^q(\mathbf{R}^d)}$$
$$\lesssim 2^{2j} \int_{|t-\tau|\leq 2^{-2j}} \|F(\tau)\|_{L_x^1(\mathbf{R}^d)} d\tau \lesssim \mathcal{M}(\|F(\tau)\|_{L_x^1(\mathbf{R}^d)})(t).$$

Therefore,

$$(2.30) \quad 2^{j(\frac{d}{q}-(d-2))} \|P_j v(t)\|_{L_x^q(\mathbf{R}^d)} \lesssim \mathcal{M}(\|F(\tau)\|_{L_x^1(\mathbf{R}^d)})(t),$$

so by Theorem 2.2 the proof is complete. □

3. Concentration compactness

In this section we briefly discuss the reduction to the almost periodic solution (1.29).

Sketch of the proof of Theorem 1.7. – Since this is merely a sketch, the interested reader should consult [25] or [28] for a complete treatment of the concentration compactness method. Define the increasing function

$$(3.1) \quad C(E) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)} : \|u\|_{L_t^\infty \dot{H}_x^1(\mathbf{R}\times\mathbf{R}^d)} \leq E\}.$$

To prove Theorem 1.6 it suffices to prove $C(E) < \infty$ for $E < \|\nabla W\|_{L^2(\mathbf{R}^4)}$. The small data results of [10] show that $C(E) \lesssim E$ for E small. Moreover, by a stability result in $d \geq 5$ (see [28]) and a simple calculation in dimensions $d = 3, 4$, $C(E)$ is a continuous function of E. Therefore, if Theorem 1.6 fails, then by the continuity of $C(E)$, there exists $E_* < \|\nabla W\|_{L^2}$ such that $C(E_*) = \infty$ and $C(E) < \infty$ for all $E < E_*$. E_* is called the minimal energy. We wish to show that $E_* = \|\nabla W\|_{L^2(\mathbf{R}^4)}$.

Now take a sequence $u_n(t)$ of solutions to (1.1) such that

$$(3.2) \quad \|u_n(t)\|_{L_t^\infty \dot{H}_x^1(\mathbf{R}\times\mathbf{R}^d)} \nearrow E_*,$$

and

$$(3.3) \quad S_{[0,\infty)}(u_n) = S_{(-\infty,0]}(u_n) = n.$$

Now by a straightforward application of Strichartz estimates, there exists $\delta > 0$ such that

$$(3.4) \quad \|e^{it\Delta} u_n(0)\|_{L_{t,x}^6(\mathbf{R}\times\mathbf{R}^4)} \geq \delta > 0.$$

Then [26] proved that $u_n(t_n)$ can be decomposed into asymptotically decoupling profiles, such that for any J,

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + w_n^J, \tag{3.5}$$

where g_n^j is an element of a group generated by scaling and translation symmetries, w_n^J is the error, and the group elements g_n^j asymptotically decouple. (See Definition 1.4 for more information on the group.) The asymptotic decoupling implies that if $u^j(t)$ is a global solution to (1.1) with initial data given by $e^{it_n^j \Delta}\phi^j$ and $\sup_j \|u^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)} < \infty$, the solution to (1.1) with initial data $u_n(0)$ is well approximated by

$$\sum_{j=1}^{J} u^j(t) + e^{it\Delta} w_n^J. \tag{3.6}$$

Then by the minimality of E_* and (3.3), there exists one j_0, $t_n^{j_0} \to 0$ and

$$\|u^{j_0}(t)\|_{L_t^\infty \dot{H}_x^1(I\times\mathbf{R}^d)} = E_*, \tag{3.7}$$

all other $\phi^j = 0$, and

$$\|u^{j_0}(t)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbf{R}^d)} = \infty, \tag{3.8}$$

where I is the maximal interval of existence of u^{j_0}.

Repeating the argument for $u^{j_0}(t_n)$ for any sequence $t_n \in I$ shows that $u^{j_0}(t_n)$ has a subsequence that converges in $\dot{H}^1/G$, where G is the group of symmetries g_n^j. This proves Theorem 1.7. □

By the Arzela-Ascoli theorem, if u is an almost periodic solution to (1.1), then there exists $x(t) : I \to \mathbf{R}^d$ and $N(t) : I \to (0,\infty)$, such that for any $\eta > 0$ there exists $C(\eta) < \infty$ such that

$$\int_{|x-x(t)|>\frac{C(\eta)}{N(t)}} |\nabla u(t,x)|^2 dx + \int_{|\xi|>C(\eta)N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi + \int_{|\xi|<\frac{1}{C(\eta)}N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi < \eta. \tag{3.9}$$

REMARK. – $N(t)$ is not uniquely defined. For example, modifying $C(\eta)$ by a constant, one may also modify $N(t)$ by a constant. Thus (see [32] for a proof) one can choose $N(t)$ such that

$$|N'(t)| \lesssim N(t)^3, \tag{3.10}$$

and

$$\int_I N(t)^2 dt \lesssim \int_I \int |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt \lesssim \int_I N(t)^2 dt + 1. \tag{3.11}$$

Sketch of proof of Theorem 1.9. – Suppose $u(t)$ is an almost periodic solution to (1.1). [28] showed that one can take a limit of $u(t_n)$ in $\dot{H}^1/G$ and obtain a solution to (1.1) satisfying either

$$N(t) \geq 1, \qquad t \in \mathbf{R}, \qquad N(0) = 1, \tag{3.12}$$

or than u blows up in finite time.

(3.9) and conservation of mass (1.3) show that finite time blowup cannot occur. Suppose u blows up in finite time, say at $T = 0$, and by time reversal symmetry suppose $u(t)$ blows up as $t \searrow 0$. Then by (3.10) and (3.11), $N(t) \nearrow \infty$ as $t \searrow 0$. Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be a radial function, $\psi = 1$ on $|x| \leq 1$, ψ supported on $|x| \leq 2$. By (3.9) and Hölder's inequality, for any $R > 0$,

$$\lim_{t \searrow 0} \int \psi(\frac{x}{R})^2 |u(t,x)|^2 dx = \lim_{t \searrow 0} M_R(t) = 0. \tag{3.13}$$

Moreover, integrating by parts,

$$\frac{d}{dt} M_R(t) \leq \frac{1}{R} \int \psi'(\frac{x}{R})\psi(\frac{x}{R})|\nabla u(t,x)||u(t,x)| dx \leq \frac{1}{R} M_R(t)^{1/2} \|\nabla u(t)\|_{L_x^2(\mathbf{R}^d)}. \tag{3.14}$$

Therefore, (3.13) combined with the fundamental theorem of calculus and (3.14) implies that $\int |u(t,x)|^2 dx = 0$ for any $t > 0$. However, this implies $u \equiv 0$, which contradicts u blowing up in finite time. □

Thus Theorem 1.6 has been reduced to Theorem 1.10. Since the rest of the argument will analyze some specific almost periodic solution u to (1.1) (which will turn out to be identically zero), $A \lesssim B$ will denote $A \leq C(u) B$.

4. Long Time Strichartz estimate

Thus it only remains to investigate when u is an almost periodic solution on $\mathbf{R}$ and $N(t) \geq 1$ for all $t \in \mathbf{R}$. In general for critical nonlinear Schrödinger problems one would like to prove that the minimal blowup solution has both finite mass (1.3) and finite energy (1.4). Having done so, it only remains to use a nonlinear estimate (e.g., a conservation law or Morawetz estimate) to prove that such an almost periodic solution must be identically zero. The sketch of the proof of Theorem 1.9 is a good example of this, using conservation of mass to prove $u \equiv 0$ in the case of a finite time blowup solution.

Long time Strichartz estimates provide a useful substitute to proving finite conserved quantities. Long time Strichartz estimates were introduced in [14] to study the mass-criticial nonlinear Schrödinger equation in dimensions $d \geq 3$. There, the initial data a priori lay in L^2, but rather than proving that u also had finite energy, [14] utilized the long time Strichartz estimates to bound the error of the frequency truncated interaction Morawetz estimate.

Here we will do something similar. As was already observed, when $d = 4$, (1.9) barely fails to lie in L^2. Since (1.9) is clearly an almost periodic solution to (1.1), this precludes the possibility of proving that an almost periodic solution lies in L^2 by purely linear arguments.

REMARK. – Of course, Theorem 1.6 implies that if u lies below the energy threshold of W, $u \equiv 0$, and therefore lies in L^2. However, any proof of a result which depends upon the size of u must contain a nonlinear argument, by definition of the term linear estimate. So linear estimates such as Duhamel's principle, Strichartz estimates, and the double Duhamel arugment are not enough to prove that u has finite mass.

[31] and [?] proved long-time Strichartz estimates to the defocusing, energy-critical nonlinear Schrödinger problem in dimensions $d = 3$ and $d = 4$ respectively. In each case the crucial quantity considered was $\int_I \frac{1}{N(t)} dt$, since it scales like the interaction Morawetz estimates of [12], [13], and [44]. Examining (1.9), one can see that proving such long time Strichartz estimates in $d = 4$ is relatively straightforward, but when $d = 3$ the quantity $\int_I \frac{1}{N(t)} dt$ is optimal in the sense that (1.9) lies in $\dot{H}^{1/2}$ when $d = 3$, and $\|u(t)\|_{\dot{H}^{1/2}}$ scales like $\frac{1}{N(t)}$. This is why [31] is technically more complicated than [?].

In this paper, since the usual interaction Morawetz estimates are not positive definite in the focusing case, and therefore will not be used, we will rely on long time Strichartz estimates based on

$$K = \int_I N(t)^{-2} dt. \tag{4.1}$$

Thus, many of the more technically difficult arguments, such as Theorem 2.6 of [31] will be adapted to this paper.

THEOREM 4.1 (Long time Strichartz estimate). – *Suppose I is an interval and K is given by* (4.1). *Then for any $j \in \mathbf{Z}$,*

$$(\sum_{k \leq j} \|u_k\|^2_{\dot{S}^1(I \times \mathbf{R}^4)})^{1/2} + 2^{2j} \| \sup_{k \geq j} 2^{-2k} \|u_k(t)\|_{L^\infty_x(\mathbf{R}^4)} \|_{L^2_t(I)} \lesssim (1 + 2^{4j} K)^{1/2}. \tag{4.2}$$

(1.5) and (1.9) imply that (4.2) cannot be improved to any $K = \int_I N(t)^{-2-\epsilon} dt$ for any $\epsilon > 0$.

Proof. – By the Sobolev embedding theorem, Bernstein's inequality, and Strichartz estimates,

$$\begin{aligned} 2^{4j} \sum_{k \geq j} 2^{-4k} \|e^{i(t-t_0)\Delta} P_k u(t_0)\|^2_{L^2_t L^\infty_x(I \times \mathbf{R}^4)} &\lesssim 2^{4j} \sum_{k \geq j} 2^{-2k} \|e^{i(t-t_0)\Delta} P_k u(t_0)\|^2_{L^2_t L^4_x(I \times \mathbf{R}^4)} \\ &\lesssim 2^{2j} \|P_{\geq j} u(t_0)\|^2_{L^2_x(\mathbf{R}^4)} \lesssim \|u(t_0)\|^2_{\dot{H}^1(\mathbf{R}^4)} \lesssim 1, \end{aligned} \tag{4.3}$$

and

$$\sum_{k \leq j} \|\nabla e^{i(t-t_0)\Delta} P_k u(t_0)\|^2_{L^2_t L^4_x(I \times \mathbf{R}^4)} \lesssim \|\nabla u(t_0)\|^2_{L^2_x(\mathbf{R}^4)} \lesssim 1. \tag{4.4}$$

Let

$$\|u\|_{Y(I\times\mathbf{R}^4)} = \sup_j 2^{2j}(1+2^{4j}K)^{-1/2}\|\sup_{k\geq j} 2^{-2k}\|u_j(t)\|_{L_x^\infty(\mathbf{R}^4)}\|_{L_t^2(I)} + \sup_j (1+2^{4j}K)^{-1/2}(\sum_{k\leq j} 2^{2k}\|u_k(t)\|^2_{L_t^2L_x^4(I\times\mathbf{R}^4)})^{1/2}. \tag{4.5}$$

The goal is to use (4.3), (4.4), and the smallness of u away from the scale $N(t)$ to prove an estimate of the form

$$\|u\|_{Y(I\times\mathbf{R}^4)} \lesssim 1 + \eta\|u\|_{Y(I\times\mathbf{R}^4)}, \tag{4.6}$$

and then proceed by the usual small data arguments. To that end decompose

$$F(u) = F(u_{\geq j}) + O(u_{\geq j}^2 u_{\leq j}) + O(u_{\geq j}u_{\leq j}^2) + F(u_{\leq j}). \tag{4.7}$$

By (3.9) it is possible to choose $c(\eta) > 0$ such that

$$\|u_{\leq c(\eta)N(t)}\|_{L_t^\infty \dot{H}^1(\mathbf{R}\times\mathbf{R}^4)} \leq \eta. \tag{4.8}$$

By Bernstein's inequality,

(4.9)

$$\begin{aligned}\|P_{\leq cN(t)}(u_{\geq j})\|^3_{L_t^6L_x^3(I\times\mathbf{R}^4)} &\lesssim \|\sum_{j\leq k_1\leq k_2\leq k_3}\|P_{\leq cN(t)}u_{k_1}\|_{L_x^\infty(\mathbf{R}^4)}\|P_{\leq cN(t)}u_{k_2}\|_{L_x^2(\mathbf{R}^4)}\|P_{\leq cN(t)}u_{k_3}\|_{L_x^2(\mathbf{R}^4)}\|_{L_t^2(I)} \\ &\lesssim \|P_{\leq cN(t)}u\|^2_{L_t^\infty\dot{H}^1(\mathbf{R}\times\mathbf{R}^4)}\|\sup_{k\geq j}2^{-2k}\|u_k(t)\|_{L_x^\infty(\mathbf{R}^4)}\|_{L_t^2(I)} \\ &\lesssim \eta^2 2^{-2j}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

Also by Bernstein's inequality,

$$\begin{aligned}\|P_{\geq cN(t)}u\|^3_{L_t^6L_x^3(I\times\mathbf{R}^4)} &\lesssim (\int_I \|u_{\geq cN(t)}(t)\|^2_{L_x^2(\mathbf{R}^4)}\|u(t)\|^4_{L_x^4(\mathbf{R}^4)}dt)^{1/2} \\ &\lesssim \|u\|^3_{L_t^\infty\dot{H}^1(\mathbf{R}\times\mathbf{R}^4)}(\int_I c^{-2}N(t)^{-2}dt)^{1/2} \lesssim c^{-1}K^{1/2}.\end{aligned} \tag{4.10}$$

Therefore,

$$\|u^3_{\geq j}\|_{L_t^2L_x^1(I\times\mathbf{R}^4)} \lesssim c^{-1}K^{1/2} + \eta^2(1+2^{4j}K)^{1/2}2^{-2j}\|u\|_{Y(I\times\mathbf{R}^4)}. \tag{4.11}$$

By Theorem 2.6 (the maximal Strichartz estimate),

$$\begin{aligned}\|\sup_{k\geq j}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}F(u_{\geq j})d\tau\|_{L_x^\infty(\mathbf{R}^4)}\|_{L_t^2(I)} &\lesssim \|(u_{\geq j})^3\|_{L_t^2L_x^1(I\times\mathbf{R}^4)} \\ &\lesssim c^{-1}K^{1/2} + \eta^2(1+2^{4j}K)^{1/2}2^{-2j}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned} \tag{4.12}$$

Also by the Sobolev embedding theorem and Strichartz estimates,

(4.13)

$$\begin{aligned}(\sum_{k\leq j}2^{2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}F(u_{\geq j})d\tau\|^2_{L_t^2L_x^4(I\times\mathbf{R}^4)})^{1/2} &\lesssim (\sum_{k\leq j}2^{4k})^{1/2}\|(u_{\geq j})^3\|_{L_t^2L_x^1(I\times\mathbf{R}^4)} \\ &\lesssim c^{-1}2^{2j}K^{1/2} + \eta^2(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

This takes care of $F(u_{\geq j})$. Next, by the Sobolev embedding theorem, the Littlewood-Paley theorem, and interpolation,

$$\begin{aligned}(4.14)\quad \|u_{\leq j}\|_{L^6_{t,x}(I\times\mathbf{R}^4)} &\lesssim \|\nabla u_{\leq j}\|_{L^6_t L^{12/5}_x(I\times\mathbf{R}^4)}\\ &\lesssim (\sum_{k\leq j} 2^{2k}\|u_k\|^2_{L^2_t L^4_x(I\times\mathbf{R}^4)})^{1/6}\|\nabla u\|^{2/3}_{L^\infty_t L^2_x(I\times\mathbf{R}^4)} \lesssim (1+2^{4j}K)^{1/6}\|u\|^{1/3}_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

By the Sobolev embedding theorem, (4.11), and (4.14),

$$\begin{aligned}(4.15)\quad &\|(u^2_{\geq j}u_{\leq j})\|_{L^2_t L^{4/3}_x(I\times\mathbf{R}^4)} \lesssim \|u_{\geq j}\|^2_{L^6_t L^3_x(I\times\mathbf{R}^4)}\|u_{\leq j}\|_{L^6_t L^{12}_x(I\times\mathbf{R}^4)}\\ &\lesssim 2^{j/3}\|u_{\geq j}\|^2_{L^6_t L^3_x(I\times\mathbf{R}^4)}\|u_{\leq j}\|_{L^6_{t,x}(I\times\mathbf{R}^4)}\\ &\lesssim 2^{j/3}(c^{-1}K^{1/2}+\eta^2 2^{-2j}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)})^{2/3}(1+2^{4j}K)^{1/6}\|u\|^{1/3}_{Y(I\times\mathbf{R}^4)}\\ &\lesssim 2^{j/3}c^{-2/3}K^{1/3}(1+2^{4j}K)^{1/6}\|u\|^{1/3}_{Y(I\times\mathbf{R}^4)} + 2^{-j}\eta^{4/3}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

By the Sobolev embedding theorem, (4.15), and Strichartz estimates,

$$\begin{aligned}(4.16)\quad &\|\sup_{k\geq j} 2^{-2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}O(u^2_{\geq j}u_{\leq j})d\tau\|_{L^\infty_x(\mathbf{R}^4)}\|_{L^2_t(I)}\\ &\lesssim 2^{-j}\|\int_{t_0}^t e^{i(t-\tau)\Delta}O(u^2_{\geq j}u_{\leq j})d\tau\|_{L^2_t L^4_x(I\times\mathbf{R}^4)} \lesssim 2^{-j}\|u^2_{\geq j}u_{\leq j}\|_{L^2_t L^{4/3}_x(I\times\mathbf{R}^4)}\\ &\lesssim 2^{-2j/3}c^{-2/3}K^{1/3}(1+2^{4j}K)^{1/6}\|u\|^{1/3}_{Y(I\times\mathbf{R}^4)} + 2^{-2j}\eta^{4/3}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)},\end{aligned}$$

and

$$\begin{aligned}(4.17)\quad &(\sum_{k\leq j}2^{2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}O(u^2_{\geq j}u_{\leq j})d\tau\|^2_{L^2_t L^4_x(I\times\mathbf{R}^4)})^{1/2} \lesssim 2^j\|u^2_{\geq j}u_{\leq j}\|_{L^2_t L^{4/3}_x(I\times\mathbf{R}^4)}\\ &\lesssim c^{-2/3}2^{4j/3}K^{1/3}(1+2^{4j}K)^{1/6}\|u\|^{1/3}_{Y(I\times\mathbf{R}^4)} + \eta^{4/3}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

This takes care of $O(u^2_{\geq j}u_{\leq j})$. Next, by the Sobolev embedding theorem and (3.9),

$$\begin{aligned}(4.18)\quad \|(P_{\leq cN(t)}u_{\leq j})^2\|_{L^2_t L^4_x(I\times\mathbf{R}^4)} &\lesssim \|\nabla u_{\leq j}\|_{L^2_t L^4_x(I\times\mathbf{R}^4)}\|u_{\leq cN(t)}\|_{L^\infty_t L^4_x(I\times\mathbf{R}^4)}\\ &\lesssim \eta(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)},\end{aligned}$$

and by Bernstein's inequality and the Sobolev embedding theorem

$$\begin{aligned}(4.19)\quad &\|(P_{\geq cN(t)}u_{\leq j})^2\|_{L^2_t L^4_x(I\times\mathbf{R}^4)}\\ &\lesssim 2^j(\int \|u_{>cN(t)}\|^2_{L^2_x(\mathbf{R}^4)}\|u_{\leq j}(t)\|^2_{L^\infty_x(\mathbf{R}^4)}dt)^{1/2} \lesssim c^{-1}K^{1/2}2^{2j}.\end{aligned}$$

(4.18) and (4.19) imply

$$\begin{aligned}(4.20)\quad &\|\nabla O(u^2_{\leq j}u_{>j})\|_{L^2_t L^{4/3}_x(I\times\mathbf{R}^4)} + \|\nabla u^3_{\leq j}\|_{L^2_t L^{4/3}_x(I\times\mathbf{R}^4)}\\ &\lesssim \|\nabla u\|_{L^\infty_t L^2_x(I\times\mathbf{R}^4)}\|u^2_{\leq j}\|_{L^2_t L^4_x(I\times\mathbf{R}^4)} \lesssim c^{-1}K^{1/2}2^{2j} + \eta(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

Again by Strichartz estimates (4.20) implies

$$(4.21)\quad (\sum_{k\leq j} 2^{2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}O(u_{\leq j}^2 u)d\tau\|^2_{L_t^2L_x^4(I\times\mathbf{R}^4)})^{1/2} \lesssim c^{-1}K^{1/2}2^{2j} + \eta(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.$$

Also, by the Sobolev embedding theorem, Bernstein's inequality, Strichartz estimates, and (4.21),

$$(4.22)\quad \begin{aligned}&\|\sup_{k\geq j} 2^{-2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}O(u_{\leq j}^2 u)d\tau\|_{L_x^\infty(\mathbf{R}^4)}\|_{L_t^2(I)}\\ &\lesssim 2^{-2j}\|\nabla\int_{t_0}^t e^{i(t-\tau)\Delta}O(u_{\leq j}^2u)d\tau\|_{L_t^2L_x^4(I\times\mathbf{R}^4)}\\ &\lesssim c^{-1}K^{1/2} + \eta 2^{-2j}(1+2^{4j}K)^{1/2}\|u\|_{Y(I\times\mathbf{R}^4)}.\end{aligned}$$

Therefore, combining (4.3), (4.4), (4.12), (4.13), (4.16), (4.17), (4.21), and (4.22),

$$(4.23)\quad \|u\|_{Y(I\times\mathbf{R}^4)} \lesssim c(\eta)^{-1} + \eta\|u\|_{Y(I\times\mathbf{R}^4)}.$$

Choosing $\eta > 0$ sufficiently small, the proof of Theorem 4.1 is complete. □

Remark. – By Theorem 2.6 combined with the above analysis, we have also proved

$$(4.24)\quad 2^{2j}\|\sup_{k\geq j} 2^{-4k/3}\|u_k(t)\|_{L_x^6(\mathbf{R}^4)}\|_{L_t^2(I)} \lesssim (1+2^{4j}K)^{1/2}.$$

This estimate will be utilized in section six.

The long time Strichartz estimates directly yield the fact that if u is an almost periodic, rapid frequency cascade solution to (1.1) then $u \equiv 0$. (1.9) does not fall into this category, and in this case it is possible to prove $u \in L_t^\infty \dot{H}^{-\epsilon}$ for some $\epsilon > 0$ using only linear estimates. $u \equiv 0$ follows directly from interpolation and (3.9).

Theorem 4.2. – *If u is an almost periodic solution to (1.1) on $\mathbf{R}$ satisfying $\int_{\mathbf{R}} N(t)^{-2}dt = K < \infty$, then $u \equiv 0$.*

Proof. – Again by (3.9), for any $\eta > 0$ there exists $j_0(\eta)$ such that

$$(4.25)\quad \|P_{\leq j_0}u(t)\|_{L_t^\infty \dot{H}_x^1(\mathbf{R}\times\mathbf{R}^4)} \leq \eta.$$

Let k_0 be the integer such that $2^{k_0} \leq K^{-1/4} \leq 2^{k_0+1}$. By Duhamel's formula, for $j \leq k_0$ and $t \in [-T, T]$,

$$(4.26)\quad \nabla P_{\leq j}u(t) = \nabla P_{\leq j}u(-T) - i\nabla P_{\leq j}\int_{-T}^t e^{i(t-\tau)\Delta}F(u(\tau))d\tau.$$

For $j \leq j_0(\eta)$ and k_0,

$$(4.27)\quad \|\nabla F(u_{\leq j})\|_{L_t^2L_x^{4/3}([-T,T]\times\mathbf{R}^4)} \lesssim \eta^2\|\nabla u_{\leq j}\|_{L_t^2L_x^4([-T,T]\times\mathbf{R}^4)}.$$

Next, by the Sobolev embedding theorem

$$(4.28)\quad \begin{aligned}&\|\nabla P_{\leq j}O(u_{\leq j}^2 u_{j\leq\cdot\leq k_0})\|_{L_t^2L_x^{4/3}([-T,T]\times\mathbf{R}^4)}\\ &\lesssim 2^j\|u_{j\leq\cdot\leq k_0}\|_{L_t^\infty L_x^2(\mathbf{R}\times\mathbf{R}^4)}\|\nabla u_{\leq j}\|_{L_t^2L_x^4([-T,T]\times\mathbf{R}^4)}\|\nabla u_{\leq j}\|_{L_t^\infty L_x^2(\mathbf{R}\times\mathbf{R}^4)}\\ &\lesssim \eta\|\nabla u_{\leq j}\|_{L_t^2L_x^4(I\times\mathbf{R}^4)}.\end{aligned}$$

By Bernstein's inequality and the Sobolev embedding theorem,

(4.29)
$$\|\nabla P_{\leq j} O(u^2_{j\leq\cdot\leq k_0} u)\|_{L^2_t L^{4/3}_x([-T,T]\times\mathbf{R}^4)}$$
$$\lesssim 2^{2j} \sum_{j\leq k_1\leq k_2\leq k_0} \|P_{k_1} u\|_{L^2_t L^4_x([-T,T]\times\mathbf{R}^4)} \|P_{k_2} u\|_{L^\infty_t L^2_x([-T,T]\times\mathbf{R}^4)} \|u\|_{L^\infty_t L^4_x([-T,T]\times\mathbf{R}^4)}$$
$$\lesssim \Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|u_l\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)}\Big)\Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|u_l\|_{L^\infty_t \dot{H}^1([-T,T]\times\mathbf{R}^4)}\Big).$$

Combining (3.9) with (4.27)-(4.29),

$$\|\nabla P_{\leq j} F(u_{\leq k_0})\|_{L^2_t L^{4/3}_x([-T,T]\times\mathbf{R}^4)} \lesssim \eta \|u_{\leq j}\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)} + \Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|u_l\|_{L^\infty_t \dot{H}^1([-T,T]\times\mathbf{R}^4)}\Big)\Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|u_l\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)}\Big). \tag{4.30}$$

Next, by the Sobolev embedding theorem, Strichartz estimates, Bernstein's inequality, (3.9), and Theorem 4.1,

(4.31)
$$\|\nabla P_{\leq j}[F(u) - F(u_{\leq k_0})]\|_{L^2_t L^{4/3}_x([-T,T]\times\mathbf{R}^4)}$$
$$\lesssim 2^{2j}\|(u_{\geq k_0})^3\|_{L^2_t L^1_x([-T,T]\times\mathbf{R}^4)}$$
$$+ 2^j \|u_{j\leq\cdot\leq k_0}\|^2_{L^4_t L^8_x([-T,T]\times\mathbf{R}^4)} \|u_{>k_0}\|_{L^\infty_t L^2_x([-T,T]\times\mathbf{R}^4)}$$
$$+ 2^j \|\nabla u_{\leq j}\|_{L^2_t L^4_x([-T,T]\times\mathbf{R}^4)} \|u_{\leq j}\|_{L^\infty_t \dot{H}^1([-T,T]\times\mathbf{R}^4)} \|u_{>k_0}\|_{L^\infty_t L^2_x([-T,T]\times\mathbf{R}^4)}$$
$$\lesssim 2^{2j} K^{1/2} + 2^{j-k_0}\Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|\nabla u_l\|_{L^2_t L^4_x([-T,T]\times\mathbf{R}^4)}\Big)$$
$$+ 2^{j-k_0}\eta\|\nabla u_{\leq j}\|_{L^2_t L^4_x([-T,T]\times\mathbf{R}^4)}.$$

Combining (4.26), (4.30), and (4.31),

$$\|u_{\leq j}(t)\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)} \lesssim \|\nabla P_{\leq j} u(-T)\|_{L^2_x(\mathbf{R}^4)} + \eta\|u_{\leq j}\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)} + \Big(\sum_{j\leq l\leq k_0} 2^{j-l}\|u_{\leq l}(t)\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)}\Big)\Big(\sum_{l\geq j} 2^{j-l}\|\nabla u_l\|_{L^\infty_t L^2_x([-T,T]\times\mathbf{R}^4)}\Big) + 2^{2j}K^{1/2}. \tag{4.32}$$

By (3.9),

$$\Big(\sum_{j\leq l}\|u_l\|_{L^\infty_t \dot{H}^1([-T,T]\times\mathbf{R}^4)}\Big) \lesssim \eta + 2^{j-j_0(\eta)}. \tag{4.33}$$

Since $\int_{\mathbf{R}} N(t)^{-2}dt = K < \infty$, $N(-T) \nearrow +\infty$ as $T \nearrow +\infty$, so for any j,

$$\inf_T \|\nabla P_{\leq j} u(-T)\|_{L^2_x(\mathbf{R}^4)} = 0. \tag{4.34}$$

Let $a_j = \|u_{\leq j}\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)}$. Theorem 4.1 implies

$$\|u_{\leq k_0}\|_{\dot{S}^1([-T,T]\times\mathbf{R}^4)} \lesssim 1 \tag{4.35}$$

uniformly in T, so by (4.32) and (4.34),

$$a_j \lesssim \eta \sum_{j \leq l \leq k_0} 2^{j-l} a_l + 2^{2j} K^{1/2} + 2^{j-j_0}. \tag{4.36}$$

Let $\beta_m = \sum_{m \leq j \leq k_0} 2^{\frac{3}{4}(m-j)} a_j$. Clearly $\beta_m \leq a_m$ for any m. Then (4.36) implies $\beta_m \lesssim K^{1/2} 2^{3m/4}$. Plugging $\|u_j\|_{\dot{S}^1(\mathbf{R}\times\mathbf{R}^4)} \lesssim 2^{3j/4}$ for $j \leq k_0$ back into (4.32),

$$\|u_{\leq j}(t)\|_{\dot{S}^1(\mathbf{R}\times\mathbf{R}^4)} \lesssim K 2^{3j/2}. \tag{4.37}$$

In particular this implies

$$\|u(t)\|_{H_x^{-1/4}(\mathbf{R}^4)} \lesssim K. \tag{4.38}$$

Then by Bernstein's inequality, interpolation, (3.9), and (4.38), for any $\eta > 0$,

$$\|u(t)\|_{L_x^2(\mathbf{R}^4)} \lesssim \|P_{\leq \frac{1}{C(\eta)} N(t)} u(t)\|_{H_x^{-1/4}(\mathbf{R}^4)}^{4/5} \|P_{\leq \frac{1}{C(\eta)} N(t)} u(t)\|_{\dot{H}_x^1(\mathbf{R}^4)}^{1/5} + \|P_{\geq \frac{1}{C(\eta)} N(t)} u(t)\|_{L_x^2(\mathbf{R}^4)} \lesssim K^{2/3} \eta^{1/5} + \frac{C(\eta)}{N(t)}. \tag{4.39}$$

$N(t) \nearrow +\infty$ as $t \nearrow \infty$, so there exists $\eta(t) \searrow 0$, possibly very slowly, such that (4.39) implies

$$\|u(t)\|_{L_x^2(\mathbf{R}^4)} \to 0. \tag{4.40}$$

Then conservation of mass (1.3) implies $u \equiv 0$. □

For the case $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$, $N(t) \geq 1$, we begin by proving $u \in L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)$. This is the endpoint of [35]'s proposition 3.1. Observe that this bound breaks the scaling in (1.5) since L_x^3 scales like $\dot{H}^{2/3}(\mathbf{R}^4)$. The proof utilizes Duhamel's formula, but not the double Duhamel argument. The long time Strichartz estimates are utilized to estimate error terms. The reader may find it helpful to assume $u_2 = 0$ on a first reading of Theorem 4.3, since u_2 will be treated as an error term.

THEOREM 4.3. – *If $u(t)$ is an almost periodic solution to* (1.1) *satisfying $N(t) \geq 1$ on $\mathbf{R}$, then*

$$\|u(t)\|_{L_t^\infty L_x^3(\mathbf{R}\times\mathbf{R}^4)} < \infty. \tag{4.41}$$

Proof. – Partition $\mathbf{R}$ into intervals I_l such that on each interval

$$\int_{I_l} N(t)^{-2} dt = 2^{-4 j_0(\eta)}, \tag{4.42}$$

where (since $N(t) \geq 1$) $j_0(\eta)$ satisfies

$$\int_{|x-x(t)| \geq 2^{-2j_0}} |\nabla u(t,x)|^2 dx + \int_{|x-x(t)| \geq 2^{-2j_0}} |u(t,x)|^4 dx < \eta. \tag{4.43}$$

(3.9) implies that such a $j_0(\eta) > 0$ exists. Then decompose $u = u_1 + u_2$, $u_1 = P_{\geq j_0} u$. Theorem 4.1 and (3.9) imply that for each I_l,

$$\|u_2\|_{\dot{S}^1(I_l \times \mathbf{R}^4)} \lesssim 1, \quad \text{and} \quad \|u_2\|_{L_t^\infty \dot{H}^1(I_l \times \mathbf{R}^4)} \leq \eta, \tag{4.44}$$

so by interpolation,

$$\|\nabla u_2\|_{L_t^4 L_x^{8/3}(I_l \times \mathbf{R}^4)} \lesssim \|u_2\|_{\dot{S}^1(I_l\times\mathbf{R}^4)}^{1/2} \|u_2\|_{L_t^\infty \dot{H}^1(I_l\times\mathbf{R}^4)}^{1/2} \lesssim \eta^{1/2}. \tag{4.45}$$

Also, by Bernstein's inequality and the Sobolev embedding theorem

$$\|u_1\|_{L_t^\infty L_x^3(I_l\times\mathbf{R}^4)} \lesssim \|u\|_{L_t^\infty \dot{H}^{2/3}(I_l\times\mathbf{R}^4)} \lesssim 2^{-j_0/3}. \tag{4.46}$$

Now let $I_l^{(1)}$ be a new partition of $\mathbf{R}$ such that each interval $I_l^{(1)} = [a_l, b_l]$ is the union of two adjacent intervals in the previous partition, I_l. For any $t \in [\frac{a_l+b_l}{2}, b_l]$,

$$u(t) = e^{i(t-a_l)\Delta}u(a_l) - i\int_{a_l}^{t} e^{i(t-\tau)\Delta}F(u(\tau))d\tau. \tag{4.47}$$

Decompose

$$F(u) = F(u_1) + O(u_1^2u_2) + O(u_1u_2^2) + F(u_2). \tag{4.48}$$

By direct computation and the Sobolev embedding theorem,

$$\begin{aligned}\|\nabla F(u_2)\|_{L_t^2L_x^{4/3}(I_l^{(1)}\times\mathbf{R}^4)} &\lesssim \|\nabla u_2\|^2_{L_t^4L_x^{8/3}(I_l^{(1)}\times\mathbf{R}^4)}\|\nabla u_2\|_{L_t^\infty L_x^2(I_l^{(1)}\times\mathbf{R}^4)}\\ &\lesssim \|u_2\|_{\dot{S}^1(I_l^{(1)}\times\mathbf{R}^4)}\|u_2\|^2_{L_t^\infty\dot{H}^1(I_l^{(1)}\times\mathbf{R}^4)}\\ &\lesssim \eta^2\|u_2\|_{\dot{S}^1(I_l^{(1)}\times\mathbf{R}^4)}.\end{aligned} \tag{4.49}$$

Now by (2.10) and (2.15),

$$\|P_j\int_{t-2^{-2j}}^{t} e^{i(t-\tau)\Delta}F(u_1(\tau))d\tau\|_{L_x^\infty(\mathbf{R}^4)} \lesssim 2^{2j}\|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)}\times\mathbf{R}^4)}. \tag{4.50}$$

Also by the dispersive estimate (2.18),

$$\begin{aligned}&\|\int_{a_l}^{t-2^{-2j}} e^{i(t-\tau)\Delta}P_jF(u_1(\tau))d\tau\|_{L_x^\infty(\mathbf{R}^4)}\\ &\qquad\lesssim \|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)}\times\mathbf{R}^4)}\int_{t>2^{-2j}}\frac{1}{t^2}dt \lesssim 2^{2j}\|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)}\times\mathbf{R}^4)}.\end{aligned} \tag{4.51}$$

Finally split $O(u_1^2u_2 + u_1u_2^2)$ into two pieces. Let $\chi \in C_0^\infty(\mathbf{C})$, $\chi(x) = 1$ for $|x| \leq 3$ and $\chi(x) = 0$ on $|x| > 4$. Then by (4.50), (4.51), and the fact that $|u_2| \leq 4|u_1|$ on the support of $\chi(\frac{u_2}{u_1})$,

$$\|P_j\int_{a_l}^{t} e^{i(t-\tau)\Delta}\chi(\frac{u_2}{u_1})O(u_1^2u_2+u_1u_2^2)d\tau\|_{L_x^\infty(\mathbf{R}^4)} \lesssim 2^{2j}\|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)}\times\mathbf{R}^4)}. \tag{4.52}$$

Meanwhile, on the support of $(1-\chi)(\frac{u_2}{u_1})$, $|u_1| \leq \frac{1}{3}|u_2|$, so by (4.45) and (3.9),

$$\begin{aligned}&\|\nabla(1-\chi(\frac{u_2}{u_1}))O(u_1^2u_2+u_1u_2^2)d\tau\|_{L_t^2L_x^{4/3}(I_l^{(1)}\times\mathbf{R}^4)}\\ &\qquad\lesssim (\|\nabla u\|_{L_t^\infty L_x^2(I_l^{(1)}\times\mathbf{R}^4)} + \|\nabla u_2\|_{L_t^\infty L_x^2(I_l^{(1)}\times\mathbf{R}^4)})\|u_2\|^2_{L_t^4L_x^8(I\times\mathbf{R}^4)}\\ &\qquad\lesssim \eta\|u_2\|_{\dot{S}^1(I_l^{(1)}\times\mathbf{R}^4)}.\end{aligned} \tag{4.53}$$

Now let

$$\begin{aligned}u_1^{(1)}(t) = -i\int_{a_l}^{t} e^{i(t-\tau)\Delta}F(u_1)d\tau - i\int_{a_l}^{t} e^{i(t-\tau)\Delta}\chi(\frac{u_2}{u_1})O(u_1^2u_2+u_1u_2^2)d\tau\\ + e^{i(t-a_l)\Delta}P_{\geq j_0}\chi(\frac{x-x(a_l)}{2^{-2j_0}})u(a_l),\end{aligned} \tag{4.54}$$

where $x(a_l)$ refers to the $x(t)$ in (3.9). Then by the dispersive estimate (2.18), Hölder's inequality, and the fact that $N(t) \geq 1$, which implies $|I_l^{(1)}| \geq 2^{-4j_0(\eta)+1}$,

(4.55)
$$\|e^{i(t-a_l)\Delta} P_{\geq j_0} \chi(\frac{x - x(a_l)}{2^{-2j_0}}) u(a_l)\|_{L_x^\infty(\mathbf{R}^4)} \lesssim 2^{8j_0} \|\chi(\frac{x - x(a_l)}{2^{-2j_0}}) u(a_l)\|_{L_x^1(\mathbf{R}^4)} \lesssim 2^{2j_0}.$$

For $t \in [a_l, \frac{a_l+b_l}{2}]$ replace a_l with b_l in (4.54). Therefore, by (4.50), (4.51), (4.52), and the Fourier support of (4.55), for any j,

$$\|P_j u_1^{(1)}(t)\|_{L_x^\infty(\mathbf{R}^4)} \lesssim 2^{2j}(1 + \|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)}). \tag{4.56}$$

Then

$$u_2^{(1)}(t) = -i\int_{a_l}^t e^{i(t-\tau)\Delta} F(u_2) d\tau - i\int_{a_l}^t e^{i(t-\tau)\Delta}(1 - \chi(\frac{u_2}{u_1})) O(u_1^2 u_2 + u_1 u_2^2) d\tau$$
$$e^{i(t-a_l)\Delta} u(a_l) - e^{i(t-a_l)\Delta} P_{\geq j_0}(\chi(\frac{x - x(a_l)}{2^{-j_0/2}}) u(a_l)). \tag{4.57}$$

Then by Strichartz estimates, (3.9), (4.49), and (4.53),

$$\|u_2^{(1)}\|_{\dot S^1(I_l^{(1)} \times \mathbf{R}^4)} \lesssim \eta + \eta\|u_2\|_{\dot S^1(I_l^{(1)} \times \mathbf{R}^4)} \leq \eta + 2\eta(\sup_l \|u_2\|_{\dot S^1(I_l \times \mathbf{R}^4)}), \tag{4.58}$$

which, using Theorem 4.1 in the last inequality, implies

$$(\sup_l \|u_2^{(1)}\|_{\dot S^1(I_l^{(1)} \times \mathbf{R}^4)}) \lesssim \eta + \eta(\sup_l \|u_2\|_{\dot S^1(I_l \times \mathbf{R}^4)}) \lesssim \eta. \tag{4.59}$$

Also by (3.9), $N(t) \geq 1$, (4.56), (4.58), and Bernstein's inequality,

$$\begin{aligned} &\|P_{\leq j_0(\eta)} u_1^{(1)}(t)\|^3_{L_x^3(\mathbf{R}^4)} \\ &\lesssim \sum_{k_1 \leq k_2 \leq k_3 \leq j_0(\eta)} \|P_{k_1} u_1(t)\|_{L_x^\infty(\mathbf{R}^4)} \|P_{k_2} u_1(t)\|_{L_x^2(\mathbf{R}^4)} \|P_{k_3} u_1(t)\|_{L_x^2(\mathbf{R}^4)} \\ &\lesssim (1 + \|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)}) \\ &\quad \times \sum_{k_1 \leq k_2 \leq k_3 \leq j_0(\eta)} 2^{2k_1} 2^{-k_2} 2^{-k_3} \|\nabla P_{k_2} u_1(t)\|_{L_x^2(\mathbf{R}^4)} \|\nabla P_{k_3} u_1(t)\|_{L_x^2(\mathbf{R}^4)} \\ &\lesssim \eta^2 \|u\|^2_{L_t^\infty \dot H_x^1(\mathbf{R} \times \mathbf{R}^4)} \|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)} \lesssim \eta^2 \|u_1\|^3_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)}. \end{aligned} \tag{4.60}$$

Therefore by (4.46), (4.54), and (4.60),

$$\|u_1^{(1)}(t)\|_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)} \lesssim 2^{-j_0(\eta)/3} + \eta^{2/3} \|u_1(t)\|_{L_t^\infty L_x^3(I_l^{(1)} \times \mathbf{R}^4)} \lesssim 2^{-j_0(\eta)/3}. \tag{4.61}$$

Now for any partition $I_l^{(n)}$ of $\mathbf{R}$ let $I_l^{(n+1)} = [a_l^{(n+1)}, b_l^{(n+1)}]$ be the union of two adjacent intervals in $I_l^{(n)}$. By induction,

$$\int_{I_l^{(n+1)}} N(t)^{-2} dt = 2^{-4j_{n+1}} = 2^{n+1} 2^{-4j_0}, \tag{4.62}$$

so since $N(t) \geq 1$, $|I_l^{(n+1)}| \geq 2^{-4j_{n+1}}$. By induction suppose that for each n there exists some $\eta_n \leq \eta$ such that

$$\sup_l \|u_1^{(n)}\|_{L_t^\infty L_x^3(I_l^{(n)}\times\mathbf{R}^4)} \lesssim 2^{-j_0(\eta)/3},$$
$$\sup_l \|u_2^{(n)}\|_{\dot S^1(I_l^{(n)}\times\mathbf{R}^4)} \lesssim \eta_n. \tag{4.63}$$

For $t \in [\frac{a_l^{(n+1)}+b_l^{(n+1)}}{2}, b_l^{(n+1)}]$, let

$$\begin{aligned} u_1^{(n+1)}(t) = &-i\int_{a_l^{(n+1)}}^t e^{i(t-\tau)\Delta}F(u_1^{(n)})d\tau \\ &- i\int_{a_l^{(n+1)}}^t e^{i(t-\tau)\Delta}\chi(\frac{u_2^{(n)}}{u_1^{(n)}})O((u_1^{(n)})^2(u_2^{(n)}) + (u_1^{(n)})(u_2^{(n)})^2)d\tau \\ &+ e^{i(t-a_l^{(n+1)})\Delta}P_{\geq j_{n+1}}(\chi(\frac{x-x(a_l^{(n+1)})}{2^{-j_{n+1}/4}})u(a_l^{(n+1)})), \end{aligned} \tag{4.64}$$

and

$$\begin{aligned} u_2^{(n+1)}(t) = &-i\int_{a_l^{(n+1)}}^t e^{i(t-\tau)\Delta}F(u_2^{(n)})d\tau \\ &- i\int_{a_l^{(n+1)}}^t e^{i(t-\tau)\Delta}(1-\chi(\frac{u_2^{(n)}}{u_1^{(n)}}))O((u_1^{(n)})^2(u_2^{(n)}) + (u_1^{(n)})(u_2^{(n)})^2)d\tau \\ &+ e^{i(t-a_l^{(n+1)})\Delta}(u(a_l^{(n+1)}) - P_{\geq j_{n+1}}(\chi(\frac{x-x(a_l^{(n+1)})}{2^{-j_{n+1}/2}})u(a_l^{(n+1)}))). \end{aligned} \tag{4.65}$$

For $t \in [a_l^{(n+1)}, \frac{a_l^{(n+1)}+b_l^{(n+1)}}{2}]$ replace $a_l^{(n+1)}$ with $b_l^{(n+1)}$ in (4.64) and (4.65). Then by (4.61) and (4.63),

$$\sup_l \|u_1^{(n+1)}\|_{L_t^\infty L_x^3(I_l^{(n+1)}\times\mathbf{R}^4)} \lesssim 2^{-j_0/3}. \tag{4.66}$$

Next, by (3.9) and Hölder's inequality, there exists $\eta_n \leq \eta$, $\eta_n \searrow 0$ as $n \nearrow \infty$, such that

$$\|u(a_l^{(n+1)}) - P_{\geq j_n}(\chi(\frac{x-x(a_l^{(n+1)})}{2^{-j_{n+1}/2}})u(a_l^{(n+1)}))\|_{\dot H^1(\mathbf{R}^4)} \leq \eta_n. \tag{4.67}$$

Then by (4.49), (4.53), (4.58), and (4.67),

$$(\sup_l \|u_2^{(n+1)}\|_{\dot S^1(I_l^{(n+1)}\times\mathbf{R}^4)}) \lesssim \eta_n + \eta(\sup_l \|u_2^{(n)}\|_{\dot S^1(I_l^{(n)}\times\mathbf{R}^4)}), \tag{4.68}$$

so by induction, $\|u_2^{(n)}\|_{L_t^\infty \dot H_x^1(\mathbf{R}\times\mathbf{R}^4)} \searrow 0$ as $n \nearrow \infty$. Therefore $u_1^{(n)}(t)$ converges to $u(t)$ uniformly in $\dot H^1(\mathbf{R}^4)$, so (4.66) implies a uniform bound on $\|u(t)\|_{L_x^3(\mathbf{R}^4)}$. □

5. Modified Soliton

Next, prove that in an average sense, the L^2 norm of an almost periodic solution satisfying $\int_{\mathbf{R}} N(t)^{-2}dt = \infty$ diverges logarithmically.

Lemma 5.1. – *Suppose $\psi \in C_0^\infty(\mathbf{R}^4)$, is a positive, radial, decreasing function,*

$$\psi(x) = \begin{cases} 1 \ \textit{if} \ |x| \le 1 \\ 0 \ \textit{if} \ |x| \ge 2. \end{cases} \tag{5.1}$$

Suppose $\int_I N(t)^{-2} dt = K$. Then for any $1 \le R \le K^{1/5}$,

$$\int_I \iint |u(t,y)|^2 \psi(\frac{N(t)(x-y)}{R})[|\nabla u(t,x)|^2 + |u(t,x)|^4] dx dy dt \lesssim K(1 + \ln(R)). \tag{5.2}$$

Remark. – By (1.9) this estimate is sharp for almost periodic solutions.

Proof. – The proof uses the double Duhamel method. This method was introduced in [13] to study the defocusing, energy-critical Schrödinger initial value problem when $d = 3$. There, as here, it was used in conjunction with a frequency localized interaction Morawetz estimate. See also [31].

By simple linear algebra, if $A + B = A' + B'$,

$$\langle A + B, A' + B' \rangle \lesssim |A|^2 + |A'|^2 + \langle B, B' \rangle. \tag{5.3}$$

Suppose $I = [t_-, t_+]$ and u solves the equation

$$(i\partial_t + \Delta) u = G(t) + F(t). \tag{5.4}$$

Then by (2.14), for any $t \in I$,

$$\begin{aligned} u(t) &= e^{i(t-t_-)\Delta} u(t_-) - i \int_{t_-}^t e^{i(t-s_-)\Delta} G(s_-) ds_- - i \int_{t_-}^t e^{i(t-s_-)\Delta} F(s_-) ds_- \\ &= e^{i(t-t_+)\Delta} u(t_+) - i \int_{t_+}^t e^{i(t-s_+)\Delta} G(s_+) ds_+ - i \int_{t_+}^t e^{i(t-s_+)\Delta} F(s_+) ds_+. \end{aligned} \tag{5.5}$$

Then if X is some Hilbert space, such as $L^2(\mathbf{R}^4)$ or a weighted $L^2(\mathbf{R}^4)$ space, (5.3) implies

$$\begin{aligned} \|u(t)\|_X^2 &\lesssim \|e^{i(t-t_-)\Delta} u(t_-)\|_X^2 + \|e^{i(t-t_+)\Delta} u(t_+)\|_X^2 + \|\int_{t_-}^t e^{i(t-s_-)\Delta} G(s_-) ds_-\|_X^2 \\ &+ \|\int_{t_+}^t e^{i(t-s_+)\Delta} G(s_+) ds_+\|_X^2 + |\int_{t_-}^t \int_t^{t_+} \langle e^{i(t-s_-)\Delta} F(s_-), e^{i(t-s_+)\Delta} F(s_+) \rangle_X ds_- ds_+|. \end{aligned} \tag{5.6}$$

Let $P_h = P_{\ge K^{-1/4}}$ and let $P_l = 1 - P_h$. Also, for a fixed $x \in \mathbf{R}^4$ define the inner product

$$\langle f, g \rangle_x = \int \psi(\frac{x-y}{R}) f(y) \overline{g(y)} dy. \tag{5.7}$$

Let $1_A(\tau)$ be the indicator function of a set $A \subset \mathbf{R}$. For fixed $t \in [t_-, t_+]$ let

$$\begin{aligned} G(\tau) = P_h O(u_l u^2)(\tau) &+ 1_{[t_-, t - \frac{R^2}{N(t)^2}]}(\tau) P_h F(u_h)(\tau) + 1_{[t - \frac{1}{N(t)^2}, t]}(\tau) P_h F(u_h)(\tau) \\ &+ 1_{[t + \frac{R^2}{N(t)^2}, t_+]}(\tau) P_h F(u_h)(\tau) + 1_{[t, t + \frac{1}{N(t)^2}]}(\tau) P_h F(u_h)(\tau), \end{aligned} \tag{5.8}$$

and let

$$F(\tau) = 1_{[t - \frac{R^2}{N(t)^2}, t - \frac{1}{N(t)^2}]}(\tau) P_h F(u_h)(\tau) + 1_{[t + \frac{1}{N(t)^2}, t + \frac{R^2}{N(t)^2}]}(\tau) P_h F(u_h)(\tau). \tag{5.9}$$

By Hölder's inequality, $N(t) \geq 1$, Strichartz estimates, (4.15), (4.20), Bernstein's inequality, and $R \leq K^{1/5}$,

(5.10)
$$\int_I \sup_{x \in \mathbf{R}^4} \int \psi(\frac{N(t)(x-y)}{R}) \left[|e^{i(t-t_-)\Delta} u_h(t_-)(y)|^2 + \left| \int_{t_-}^t e^{i(t-s_-)\Delta} P_h O(u_l u^2)(y) ds_- \right|^2 \right] dy dt$$
$$\lesssim R^2 \| e^{i(t-t_-)\Delta} u_h(t_-) \|^2_{L_t^2 L_x^4(I \times \mathbf{R}^4)} + R^2 \| \int_{t_-}^t e^{i(t-s_-)\Delta} P_h O(u_l u^2)(y) ds_- \|^2_{L_t^2 L_x^4(I \times \mathbf{R}^4)}$$
$$\lesssim R^2 \| u_h(t_-) \|^2_{L_x^2(\mathbf{R}^4)} + R^2 \| P_h O(u_l u^2) \|^2_{L_t^2 L_x^{4/3}(I \times \mathbf{R}^4)} \lesssim R^2 K^{1/2} \lesssim K^{9/10}.$$

Identically,
(5.11)
$$\int_I \sup_{x \in \mathbf{R}^4} \int \psi(\frac{N(t)(x-y)}{R}) [|e^{i(t-t_+)\Delta} u_h(t_+)(y)|^2 + | \int_{t_+}^t e^{i(t-s_+)\Delta} P_h O(u_l u^2)(y) ds_+ |^2] dy dt \lesssim R^2 K^{1/2} \lesssim K^{9/10}.$$

Next, by the dispersive estimate (2.18) and Hölder's inequality,

(5.12)
$$\sup_{x \in \mathbf{R}^4} \int \psi(\frac{N(t)(x-y)}{R}) | \int_{t_-}^{t - \frac{R^2}{N(t)^2}} e^{i(t-s_-)\Delta} P_h F(u_h)(s_-) ds_- |^2 dy$$
$$\lesssim \frac{R^4}{N(t)^4} (\int_{t-s_- > \frac{R^2}{N(t)^2}} \frac{1}{(t-s_-)^2} \| F(u_h)(s_-) \|_{L_x^1} ds_-)^2$$
$$\lesssim (\sum_{j \geq 0} \frac{N(t)^2}{2^{2j} R^2} \int_{|t-s_-| \sim 2^j \frac{R^2}{N(t)^2}} \| F(u_h)(s_-) \|_{L_x^1(\mathbf{R}^4)} ds_-)^2 \lesssim (\mathcal{M}(\| F(u_h) \|_{L_x^1(\mathbf{R}^4)})(t))^2.$$

Once again, an identical calculation implies
(5.13)
$$\sup_{x \in \mathbf{R}^4} \int \psi(\frac{N(t)(x-y)}{R}) | \int_{t + \frac{R^2}{N(t)^2}}^{t_+} e^{i(t-s_+)\Delta} P_h F(u_h)(s_+) ds_+ |^2 dy \lesssim (\mathcal{M}(\| F(u_h) \|_{L_x^1(\mathbf{R}^4)}))(t)^2.$$

Recalling L^p estimates for maximal functions (Theorem 2.2), (5.12) and (5.13) and (4.11),

(5.14)
$$\int_I (\mathcal{M}(\| F(u_h) \|_{L_x^1(\mathbf{R}^4)})(t))^2 dt \lesssim \int_I \| u_h(t) \|^6_{L_x^3(\mathbf{R}^4)} dt \lesssim K.$$

Finally, by Strichartz estimates and the Sobolev embedding theorem,

(5.15)
$$\| \int_{t + \frac{1}{N(t)^2}}^t e^{i(t-s_+)\Delta} P_h F(u_h)(s_+) ds_+ \|^2_{L_x^2(\mathbf{R}^4)} + \| \int_{t - \frac{1}{N(t)^2}}^t e^{i(t-s_-)\Delta} P_h F(u_h)(s_-) ds_- \|^2_{L_x^2(\mathbf{R}^4)} \lesssim \frac{1}{N(t)^2}.$$

Since $\int_I \frac{1}{N(t)^2} dt = K$, by (5.6) it only remains to estimate

(5.16)
$$\int_I \sup_{x \in \mathbf{R}^4} \int_{t - \frac{R^2}{N(t)^2}}^{t - \frac{1}{N(t)^2}} \int_{t + \frac{1}{N(t)^2}}^{t + \frac{R^2}{N(t)^2}} \langle e^{i(t-s_-)\Delta} P_h F(u_h)(s_-), e^{i(t-s_+)\Delta} P_h F(u_h)(s_+)\rangle_x ds_- ds_+ dt.$$

Since $F(u_h) \in L_t^2 L_x^1$ it suffices to compute the kernel of $e^{i(t-s_-)\Delta}\psi(\frac{x-y}{R})e^{i(s_+-t)\Delta}$ for a fixed x. Since t is also fixed, to simplify notation let $x = 0$, $s = s_+ - t$ and $t = t - s_-$. The kernel of $e^{it\Delta}\psi(\frac{y}{R})e^{is\Delta}$ is given by

$$K(s, t; y, z) = \frac{C}{s^2 t^2} \int e^{-i\frac{|w-y|^2}{4t}} \psi(\frac{w}{R}) e^{-i\frac{|w-z|^2}{4s}} dw. \tag{5.17}$$

Now let $q(s, t, y, z) = \frac{sy+zt}{s+t} \cdot \frac{(s+t)^{1/2}}{(ts)^{1/2}}$. After making a change of variables in w,

$$|K(s, t; y, z)| = \frac{C}{(s+t)^2} \int e^{-i|w - q(s,t,y,z)|^2} \psi(\frac{w}{R} \cdot \frac{(st)^{1/2}}{(s+t)^{1/2}}) dy. \tag{5.18}$$

When $R \cdot \frac{(s+t)^{1/2}}{(st)^{1/2}} \leq 1$, Hölder's inequality implies that $|K(s, t; y, z)| \lesssim \frac{1}{(t+s)^2}$. For $R_0 = R \cdot \frac{(s+t)^{1/2}}{(st)^{1/2}} > 1$, stationary phase calculations imply that for $\chi \in C_0^\infty$, $\chi = 1$ on $|x| \leq 1$, for any N,

(5.19)
$$\int e^{-i|w-q|^2}(1-\chi)(w-q)\psi(\frac{w}{R_0})dw = \int ((\frac{i(w-q)\cdot\nabla}{|w-q|^2})^N e^{-i|w-q|^2})(1-\chi)(w-q)\psi(\frac{w}{R_0})dw.$$

Taking $N = 5$ and integrating by parts,

$$|K(s, t; x, z)| \lesssim \frac{1}{(t+s)^2} \int_{|y|>1} \frac{1}{|y|^5} dy \lesssim \frac{1}{(t+s)^2}. \tag{5.20}$$

Therefore,

(5.21)
$$\begin{aligned}
&\int_{\frac{1}{N(t)^2} < t - s_- < \frac{R^2}{N(t)^2}} \int_{\frac{1}{N(t)^2} < s_+ - t < \frac{R^2}{N(t)^2}} \langle e^{i(t-s_-)\Delta} P_h F(u_h)(s_-), e^{i(t-s_+)\Delta} P_h F(u_h)(s_+)\rangle_x ds_- ds_+ \\
&\lesssim \int_{\frac{1}{N(t)^2} < t - s_- < \frac{R^2}{N(t)^2}} \int_{\frac{1}{N(t)^2} < s_+ - t < \frac{R^2}{N(t)^2}} \frac{1}{(s_+ - s_-)^2} \|F(s_-)\|_{L_x^1} \|F(s_+)\|_{L_x^1} ds_- ds_+ \\
&\lesssim \sum_{0 \leq j \leq k \leq \ln_2(R^2)} 2^{-2k} (\int_{t - s_+ \sim \frac{2^k}{N(t)^2}} \|F(u_h)(s_+)\|_{L_x^1(\mathbf{R}^4)} ds_+) \\
&\quad \times (\int_{t - s_- \sim \frac{2^j}{N(t)^2}} \|F(u_h)(s_-)\|_{L_x^1(\mathbf{R}^4)} ds_-) \lesssim \ln(R) \mathscr{M}(\|F(u_h)\|_{L_x^1(\mathbf{R}^4)}(t))^2.
\end{aligned}$$

Therefore, (5.10), (5.11), (5.12), (5.13), (5.14), (5.15), and (5.21) imply

$$\int_I \sup_x \int \psi(\frac{N(t)(x-y)}{R})|u_h(t, y)|^2 dy dt \lesssim (1 + \ln(R))K. \tag{5.22}$$

Now since $\int [|\nabla u(t, x)|^2 + |u(t, x)|^4 dx] < \|W\|_{\dot{H}^1}^2 + \|W\|_{L_x^4}^4$,

$$\int_I \iint \psi(\frac{N(t)(x-y)}{R})|u_h(t, y)|^2 [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx dy dt \lesssim (\ln(R) + 1)K. \tag{5.23}$$

Now notice that by the Sobolev embedding theorem and Theorem 4.1,

(5.24)
$$\|u_l^2\|_{L_t^{3/2}L_x^6(I\times\mathbf{R}^4)} \lesssim \|\nabla u_l^2\|_{L_t^{3/2}L_x^{12/5}(I\times\mathbf{R}^4)} \lesssim \|\nabla u_l\|^{4/3}_{L_t^2L_x^4(I\times\mathbf{R}^4)}\|u_l\|^{2/3}_{L_t^\infty L_x^4(I\times\mathbf{R}^4)} \lesssim 1,$$

so by Hölder's inequality and $N(t)\geq 1$,

(5.25)
$$\int_I\iint |u_l(t,y)|^2\psi(\frac{N(t)(x-y)}{R})[|\nabla u(t,x)|^2+|u(t,x)|^4]dxdy$$
$$\lesssim \|u_l\|^2_{L_t^3L_x^{12}(I\times\mathbf{R}^4)}[\|\nabla u\|^2_{L_t^\infty L_x^2(I\times\mathbf{R}^4)}+\|u\|^4_{L_t^\infty L_x^4(I\times\mathbf{R}^4)}](\int_I\frac{R^{10}}{N(t)^{10}}dt)^{1/3}\lesssim R^{10/3}K^{1/3}.$$

When $R\leq K^{1/5}$, (5.2) holds and the proof of Lemma 5.1 is complete. □

Now consider the case when $N(t)\equiv 1$.

THEOREM 5.2 (No nonzero modified solitons). – *Suppose u is an almost periodic solution to* (1.1) *with $N(t)\equiv 1$ on $\mathbf{R}$ and $\|u\|_{L_t^\infty\dot H_x^1(\mathbf{R}\times\mathbf{R}^4)}<\|\nabla W\|_{L_x^2(\mathbf{R}^4)}$. Then $u\equiv 0$.*

Proof. – The first step is to prove that any almost periodic solution to (1.1) lying below the ground state with $N(t)\equiv 1$ lies in $L^2(\mathbf{R}^4)$. To prove such a fact necessarily demands a nonlinear estimate, in this case an interaction Morawetz estimate. This interaction Morawetz estimate is in the same vein as the Morawetz estimate in [15] for the focusing, mass-critical problem, and provides a logarithmic improvement over (5.2).

Let $\psi\in C_0^\infty(\mathbf{R})$ be a radial function satisfying (5.1) and let J be a large number such that $e^J\leq K^{1/10}$. Then let

(5.26)
$$\begin{aligned}\phi(x-y)&=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int_{\mathbf{R}^4}\psi^2(\frac{x}{R}-s)\psi^2(\frac{y}{R}-s)dsdR\\&=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int_{\mathbf{R}^4}\psi^2(\frac{x-y}{R}-s)\psi^2(s)dsdR.\end{aligned}$$

Notice that $\psi(s)=0$ for $|s|\geq 2$ implies $\phi(x-y)$ is supported on $|x-y|\leq 4e^J$ and that $\|\phi\|_{L^\infty}$ is uniformly bounded. Next, estimate the derivatives of ϕ.

LEMMA 5.3. – *For $k=1,2,3$,*

(5.27)
$$|\nabla^k\phi(x)|\lesssim\frac{1}{J}\frac{1}{|x|^k}.$$

Proof of the lemma. – The proof follows by direct computation. Since $\psi(s)=0$ when $|s|\geq 2$,

(5.28)
$$\begin{aligned}\nabla\phi(x)&=\frac{2}{J}\iint_1^{e^J}\frac{1}{R^2}\psi(\frac{x}{R}-s)\psi'(\frac{x}{R}-s)\psi^2(s)\frac{(\frac{x}{R}-s)}{|\frac{x}{R}-s|}dsdR\\&=\frac{2}{J}\int_{\frac{|x|}{4}}^{e^J}\int\frac{1}{R^2}\psi(\frac{x}{R}-s)\psi'(\frac{x}{R}-s)\psi^2(s)\frac{(\frac{x}{R}-s)}{|\frac{x}{R}-s|}dsdR\lesssim\frac{1}{J}\frac{1}{|x|}.\end{aligned}$$

Similar computations prove (5.27) for $k=2,3$. □

Now let $M(t)$ be the interaction Morawetz potential

$$M(t) = \int |u(t,y)|^2 \phi(x-y)(x-y) \cdot \mathrm{Im}[\bar{u}\nabla u](t,x)dxdy. \tag{5.29}$$

By Hölder's inequality, the Sobolev embedding theorem, and Young's inequality,

$$\sup_{t\in I} |M(t)| \lesssim \|u\|^3_{L_t^\infty L_x^4(I\times\mathbf{R}^4)} \|\nabla u\|_{L_t^\infty L_x^2(I\times\mathbf{R}^4)} \|\phi(x-y)(x-y)\|_{L^{4/3}(\mathbf{R}^4)} \lesssim e^{4J}. \tag{5.30}$$

Integrating by parts,

$$\frac{d}{dt}M(t) = 2\int |u(t,y)|^2 \phi(x-y)[|\nabla u(t,x)|^2 - |u(t,x)|^4]dxdy \tag{5.31}$$

$$-2\int \mathrm{Im}[\bar{u}\partial_j u](t,y)\phi(x-y)\mathrm{Im}[\bar{u}\partial_j u](t,x)dxdy \tag{5.32}$$

$$+2\int |u(t,y)|^2 (\partial_k \phi(x-y))(x-y)_j [\mathrm{Re}(\partial_j \bar{u}\partial_k u)(t,x) - \frac{1}{4}\delta_{jk}|u(t,x)|^4]dxdy \tag{5.33}$$

$$-2\int \mathrm{Im}[\bar{u}\partial_k u](t,y)(\partial_k \phi(x-y))(x-y)_j \mathrm{Im}[\bar{u}\partial_j u](t,x)dxdy \tag{5.34}$$

$$-\frac{1}{2}\int |u(t,y)|^2 (\partial_j \Delta\phi(x-y)(x-y)_j)|u(t,x)|^2 dxdy. \tag{5.35}$$

(5.33) and (5.34) involves integrating an energy and a mass term. A term of this type may be estimated by (5.2) and (5.27).

$$\begin{aligned} &\int_I (5.33) + (5.34)dt \\ &\lesssim \frac{1}{J}\int_I \int_{|x-y|\le 4e^J} \int_{|x-x(t)|\ge C(\eta)} |u(t,y)|^2 [|\nabla u(t,x)|^2 + |u(t,x)|^4]dxdydt \\ &\quad + \frac{1}{J}\int_I \int_{|x-y|\le 4e^J} \int_{|x-x(t)|\le C(\eta)} |u(t,y)|^2 [|\nabla u(t,x)|^2 + |u(t,x)|^4]dxdydt \end{aligned} \tag{5.36}$$

By (3.9) the energy of u is small outside of $|x-x(t)| \ge C(\eta)$, so

$$\begin{aligned} &\lesssim \frac{\eta}{J}(\int_I \sup_{y\in\mathbf{R}^4} \int_{|x-y|\le 4e^J} |u(t,y)|^2 dydt) + \frac{1}{J}\int_I \int_{|x-x(t)|\le 8e^J} |u(t,x)|^2 dxdt \\ &\lesssim \eta K + \frac{1}{J}\int_I \int_{|x-x(t)|\le 8e^J} |u(t,x)|^2 dxdt. \end{aligned} \tag{5.37}$$

This action fixes the mass of the main part of u in place, in preparation for a bootstrap type argument to prove $u \equiv 0$.

Observe that if (5.37) provides a bound on $\int_I \int_{|x-x(t)|\le 4e^J} |u(t,x)|^2 dxdt$ then by (5.2),

$$\int_I \int_{|x-x(t)|\le e^{J/2}} |u(t,x)|^2 dxdt \lesssim K = \int_I \frac{1}{N(t)^2}dt. \tag{5.38}$$

Feeding (5.38) back into (5.37) then implies that the left hand side of (5.38) $\ll K$, which by (3.9 and Bernstein's inequality forces $u \equiv 0$. Of course, such an argument necessitates estimating (5.31), (5.32), and (5.35).

To estimate (5.35), (5.27) implies

$$\int_I (5.35)\,dt \lesssim \frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |u(t,x)|^2 \frac{1}{|x-y|^2}|u(t,y)|^2 dxdydt. \tag{5.39}$$

$$\begin{aligned}&\lesssim \frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h u_{\geq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt\\ &+ \frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h u_{\leq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt\\ &+ \frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |u_l(t,x)|^2 \frac{1}{|x-y|^2}|u_l(t,y)|^2 dxdydt.\end{aligned} \tag{5.40}$$

Once again u is split into a high frequency piece lying in L^2, and by (3.9) a piece with small energy. Now by Bernstein's inequality and Hardy's inequality, since $N(t) \equiv 1$,

$$\begin{aligned}&\frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h u_{\geq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt\\ &\lesssim \frac{1}{J}\int_I \|u_{\geq c(\eta)}(t)\|_{L_x^2(\mathbf{R}^4)}^2 (\sup_{x\in\mathbf{R}^4}\int \frac{1}{|x-y|^2}|u_h(t,y)|^2 dy)dt\\ &\lesssim \frac{1}{J}\frac{1}{c(\eta)^2}\|\nabla u\|_{L_t^\infty L_x^2(I\times\mathbf{R}^4)}^2 \cdot \int_I \frac{1}{N(t)^2}dt = \frac{K}{J}\frac{1}{c(\eta)^2}.\end{aligned} \tag{5.41}$$

Remark. – The calculation in (5.41) includes $N(t)$ in the second to last step, in preparation for the next section, which considers a variable $N(t)$.

Next, by Hölder's inequality and Young's inequality,

(5.42)

$$\begin{aligned}&\frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h u_{\leq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt\\ &\lesssim \frac{1}{J}\int_I \sum_{2^j\leq 4e^J} 2^{-2j}\int_{2^j\leq|x-y|\leq 2^{j+1}} |P_{>-j}P_h u_{\leq c(\eta)}(t,x)|^2|u_h(t,y)|^2 dxdydt\\ &+ \frac{1}{J}\int_I \sum_{2^j\leq 4e^J} 2^{-2j}\int_{2^j\leq|x-y|\leq 2^{j+1}} |P_{\leq -j}P_h u_{\leq c(\eta)}(t,x)|^2|u_h(t,y)|^2 dxdydt\\ &\lesssim \frac{1}{J}\int_I (\sum_j 2^{-2j}\|P_{>-j}P_h u_{\leq c(\eta)}(t)\|_{L_x^2(\mathbf{R}^4)}^2)(\sup_y \int_{|x-y|\leq 4e^J} |u_h(t,x)|^2 dx)dt\\ &+ \frac{1}{J}\sum_{1\leq 2^j\leq 4e^J} 2^{-2j}2^{10j/3}\|P_{\leq -j}P_h u_{\leq c(\eta)}\|_{L_t^2 L_x^4(I\times\mathbf{R}^4)}^2 \|u_h\|_{L_t^\infty L_x^3(I\times\mathbf{R}^4)}^2\\ &+ \frac{1}{J}\sum_{j\leq 0} 2^{2j}\|P_{\leq c(\eta)}u_h\|_{L_t^2 L_x^4(I\times\mathbf{R}^4)}^2 \|u_h\|_{L_t^\infty L_x^4(I\times\mathbf{R}^4)}^2.\end{aligned}$$

Now for any fixed t, by Bernstein's inequality and rearranging the order of summation,

$$\begin{aligned}(5.43)\quad &\sum_j 2^{-2j}\|P_{>-j}P_{\leq c(\eta)}u(t)\|_{L^2}^2 \\ &\lesssim \sum_j \sum_{-j<k_1\leq k_2} 2^{-2j}\|P_{k_1}P_{\leq c(\eta)}u(t)\|_{L^2}\|P_{k_2}P_{\leq c(\eta)}u(t)\|_{L^2} \\ &\lesssim \sum_j \sum_{-j<k_1\leq k_2} 2^{-2j-k_1-k_2}(2^{k_1}2^{k_2}\|P_{k_1}P_{\leq c(\eta)}u(t)\|_{L^2}\|P_{k_2}P_{\leq c(\eta)}u(t)\|_{L^2}) \\ &\lesssim \sum_{k_1\leq k_2} 2^{k_1-k_2}\|\nabla P_{k_1}P_{\leq c(\eta)}u(t)\|_{L^2}\|\nabla P_{k_2}P_{\leq c(\eta)}u(t)\|_{L^2}\lesssim \eta^2.\end{aligned}$$

Therefore, by (5.2),

$$(5.44)\quad \frac{1}{J}\int_I(\sum_j 2^{-2j}\|P_{>-j}u_{\leq c(\eta)}(t)\|^2_{L^2_x(\mathbf{R}^4)})(\sup_y\int_{|x-y|\leq 4e^J}|u_h(t,x)|^2dx)dt\lesssim \eta^2 K.$$

By (4.41), Bernstein's inequality, (5.2), the long time Strichartz estimates of Theorem 4.1, $P_h = P_{\geq K^{-1/4}}$, Hölder's inequality, and Young's inequality,

$$\begin{aligned}(5.45)\quad &\frac{1}{J}\sum_{1\leq 2^j\leq 4e^J} 2^{-2j}2^{10j/3}\|P_{\leq -j}P_h u_{\leq c(\eta)}\|^2_{L^2_tL^4_x(I\times\mathbf{R}^4)}\|u_h\|^2_{L^\infty_tL^3_x(I\times\mathbf{R}^4)} \\ &\qquad\lesssim \frac{1}{J}\sum_{1\leq 2^j\leq 4e^J}2^{-2j/3}K\lesssim\frac{K}{J}.\end{aligned}$$

Also since $N(t)\equiv 1$ and $u\in L^\infty_tL^4_x$,

$$(5.46)\quad \frac{1}{J}\sum_{j\leq 0}2^{2j}\|P_{\leq c(\eta)}u_h\|^2_{L^2_tL^4_x(I\times\mathbf{R}^4)}\|u_h\|^2_{L^\infty_tL^4_x(I\times\mathbf{R}^4)}\lesssim\frac{K}{J}.$$

Finally, by (4.41), Theorem 4.1, and the Sobolev embedding theorem,

$$(5.47)\quad \|u_l^2\|_{L^2_tL^3_x(I\times\mathbf{R}^4)}\lesssim\|\nabla u_l\|_{L^2_tL^4_x(I\times\mathbf{R}^4)}\|u_l\|_{L^\infty_tL^3_x(I\times\mathbf{R}^4)}\lesssim 1,$$

so by Hölder's inequality in space and time

$$\begin{aligned}(5.48)\quad &\int_I\iint_{|x-y|\leq 4e^J}|u_l(t,y)|^2\frac{1}{|x-y|^2}|u_l(t,x)|^2dxdydt \\ &\qquad\lesssim K^{1/2}e^{2J}\|u_l^2\|_{L^2_tL^3_x(I\times\mathbf{R}^4)}\|u\|^2_{L^\infty_tL^3_x(I\times\mathbf{R}^4)}\lesssim e^{2J}K^{1/2}.\end{aligned}$$

Therefore, by (5.41), (5.44), (5.45), (5.46), and (5.48),

$$(5.49)\quad \int_I(5.35)dt\lesssim\frac{K}{J}\frac{1}{c(\eta)^2}+\eta^2K+e^{2J}K^{1/2}.$$

Now take (5.32), which is removed with a Galilean transformation. Decompose

$$(5.50)\quad \phi(x-y)=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int\psi^2(\frac{x}{R}-s)\psi^2(\frac{y}{R}-s)dsdR.$$

For each R, s, t there exists a $\xi(R,s,t)$ such that

$$\int \psi^2(\frac{x}{R} - s)\mathrm{Im}[\overline{e^{ix\cdot\xi(R,s,t)}u}\nabla e^{ix\cdot\xi(R,s,t)}u](t,x)dx = \int \xi(R,s,t)|\psi(\frac{x}{R} - s)u(t,x)|^2 dx + \int \mathrm{Im}[\bar{u}\nabla u](t,x)dx = 0. \tag{5.51}$$

Moreover, for any fixed s, t, the quantity

$$\int \psi^2(\frac{x}{R} - s)\psi^2(\frac{y}{R} - s)[|\nabla u(t,x)|^2|u(t,y)|^2 - \mathrm{Im}[\bar{u}\nabla u](t,x)\mathrm{Im}[\bar{u}\nabla u](t,y)dxdy \tag{5.52}$$

is invariant under the Galilean transformation $u \mapsto e^{-ix\cdot\xi(R,s,t)}u$. Therefore, for each R, s, t it is possible to choose $\xi(R,s,t)$ that removes the momentum squared term. Then, integrating by parts,

$$\begin{aligned}
&\int \psi^2(\frac{x}{R} - s)[|\nabla(e^{-ix\cdot\xi(R,s,t)}u(t,x))|^2 - |u(t,x)|^4]dx \\
&= \int |\nabla(\psi(\frac{x}{R} - s)e^{-ix\cdot\xi(R,s,t)}u(t,x))|^2 dx - |\psi(\frac{x}{R} - s)u(t,x)|^2|u(t,x)|^2 dx \\
&+ \int |u(t,x)|^2(\psi(\frac{x}{R} - s)\Delta\psi(\frac{x}{R} - s))dx
\end{aligned} \tag{5.53}$$

By (1.19), (1.20), and $\|u\|_{L_t^\infty \dot{H}^1} \leq (1-\bar{\delta})\|W\|_{\dot{H}^1}$,

$$\|u\|_{L^4(\mathbf{R}^4)} \leq (1-\bar{\delta})\|W\|_{L^4(\mathbf{R}^4)}, \tag{5.54}$$

and by (1.23),

$$\begin{aligned}
&\int |\nabla(\psi(\frac{x}{R} - s)e^{-ix\cdot\xi(R,s,t)}u(t,x))|^2 dx - |\psi(\frac{x}{R} - s)e^{-ix\cdot\xi(R,s,t)}u(t,x)|^2|u(t,x)|^2 dx \\
&\geq \|\nabla(\psi(\frac{x}{R} - s)e^{-ix\cdot\xi(R,s,t)}u)\|_{L^2(\mathbf{R}^4)}^2 - (1+\frac{\bar{\delta}}{2})\|\psi(\frac{x}{R} - s)u\|_{L^4(\mathbf{R}^4)}^2\|u\|_{L^4(\mathbf{R}^4)}^2 \\
&+ \frac{\bar{\delta}}{2}\|\psi(\frac{x}{R} - s)u\|_{L_x^4(\mathbf{R}^4)}^4 \geq \frac{\bar{\delta}}{2}\|\psi(\frac{x}{R} - s)u\|_{L_x^4(\mathbf{R}^4)}^4 \\
&+ \frac{\bar{\delta}}{2}\|\nabla(\psi(\frac{x}{R} - s)e^{-ix\cdot\xi(R,s,t)}u)\|_{L^2(\mathbf{R}^4)}^2.
\end{aligned} \tag{5.55}$$

Finally, if $|\frac{x}{R} - s| \leq 2$ and $|\frac{y}{R} - s| \leq 2$, $|\frac{x-y}{R}| \leq 4$, so

$$\int |\psi(\frac{x}{R} - s)||\Delta\psi(\frac{x}{R} - s)||\psi(\frac{y}{R} - s)|^2 ds \lesssim \frac{1}{R^2}\psi(\frac{x-y}{4R}). \tag{5.56}$$

Therefore, by (5.49),

$$\begin{aligned}
&\int_1^{e^J} \frac{1}{R^3}\iint \psi(\frac{x-y}{R})|u(t,x)|^2|u(t,y)|^2 dxdydtdR \\
&\lesssim \int_I \int_{|x-y|\leq 8e^J} |u(t,x)|^2 \frac{1}{|x-y|^2}|u(t,y)|^2 dxdydt \lesssim \frac{K}{J}\frac{1}{c(\eta)^2} + \eta^2 K + e^{2J}K^{1/2}.
\end{aligned} \tag{5.57}$$

Now by (5.1), for $|x-y| \leq \frac{R}{2}$,

$$\int \psi(\frac{x}{R} - s)^4\psi(\frac{y}{R} - s)^2 ds \gtrsim 1, \tag{5.58}$$

so

$$\int \psi(\frac{x}{R} - s)^4 \psi(\frac{y}{R} - s)^2 ds \gtrsim \psi(\frac{4(x-y)}{R}), \tag{5.59}$$

and

$$\frac{1}{J} \int_1^{e^J} \frac{1}{R} \psi(\frac{4(x-y)}{R}) dR \gtrsim \psi(\frac{x-y}{e^{J/2}}). \tag{5.60}$$

In fact for any $c > 0$,

$$\frac{1}{J} \int_1^{e^J} \frac{1}{R} \psi(\frac{4(x-y)}{R}) dR \gtrsim_c \psi(\frac{x-y}{e^{J(1-c)}}). \tag{5.61}$$

Therefore, combining the fundamental theorem of calculus with (5.30), (5.37), (5.49), (5.57), and (5.60),

$$\begin{aligned} e^{4J} \gtrsim \int_I \frac{d}{dt} M(t) dt \gtrsim \bar{\delta} \int_I \int_{|x-y| \leq e^{J/2}} |u(t,x)|^4 |u(t,y)|^2 dx dy dt \\ - \frac{K}{J} \frac{1}{c(\eta)^2} - \eta^2 K - e^{2J} K^{1/2} - \frac{1}{J} \int_I \int_{|x-x(t)| \leq 8e^J} |u(t,x)|^2 dx dt. \end{aligned} \tag{5.62}$$

Now by concentration compactness (see (3.9)), if u is a nonzero almost periodic solution to (1.1), then $\|u(t)\|_{L^4}$ is uniformly bounded below for all $t \in I$, as is $\int_{|x-x(t)| \leq C(\eta)} |u(t,x)|^4 dx$. Therefore, for J large,

$$\begin{aligned} & \int_{|x-y| \leq e^{J/2}} |u(t,x)|^2 |u(t,y)|^4 dx dy \\ \geq & \int_{|x-x(t)| \leq \frac{1}{2} e^{J/2}} \int_{|y-x(t)| \leq \frac{1}{2} e^{J/2}} |u(t,x)|^2 |u(t,y)|^4 dx dy \gtrsim \int_{|x-x(t)| \leq \frac{1}{2} e^{J/2}} |u(t,x)|^2 dx. \end{aligned} \tag{5.63}$$

Plugging this into (5.62),

$$\begin{aligned} \bar{\delta} \int_I \int_{|x-x(t)| \leq \frac{1}{2} e^{J/2}} |u(t,x)|^2 dx dt \\ \lesssim e^{4J} + \frac{K}{J} \frac{1}{c(\eta)^2} + \eta^2 K + e^{2J} K^{1/2} + \frac{1}{J} \int_I \int_{|x-x(t)| \leq 8e^J} |u(t,x)|^2 dx dt. \end{aligned} \tag{5.64}$$

Repeating this argument,

$$\begin{aligned} \bar{\delta}^2 \int_I \int_{|x-x(t)| \leq \frac{1}{2} e^{J/2}} |u(t,x)|^2 dx dt \\ \lesssim e^{8J} + \frac{K}{J} \frac{1}{c(\eta)^2} + \eta^2 K + e^{4J} K^{1/2} + \frac{1}{J^2} \int_I \int_{|x-x(t)| \leq 128 e^{2J}} |u(t,x)|^2 dx dt. \end{aligned} \tag{5.65}$$

Taking $e^J = K^{1/10}$, by (5.2),

$$\int_I \int_{|x-x(t)| \leq \frac{1}{2} e^{J/2}} |u(t,x)|^2 dx dt \lesssim \eta^2 K + \frac{K}{\ln(K)} \frac{1}{c(\eta)^2}. \tag{5.66}$$

Since $\eta > 0$ is arbitrary (5.66) implies that there exists a sequence $t_n \in \mathbf{R}$ such that $R_{0,n} \nearrow \infty$ and

$$\int_{|x-x(t_n)| \leq R_{0,n}^{1/4}} |u(t_n,x)|^2 dx \to 0. \tag{5.67}$$

Combining (3.9) with (5.67) implies that $u \equiv 0$. □

6. Variable $N(t)$

The arguments used in the case that $N(t) \equiv 1$ may be generalized to any $N(t)$ satisfying $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$, $N(t) \geq 1$.

Since it is possible to modify the $C(\eta)$ in (3.9) by a constant, to complete the proof of Theorem 1.10 it is enough to consider the case where $N(t) \geq 1$ and

$$\limsup_{t \to \pm\infty} N(t) = \infty. \tag{6.1}$$

Otherwise, if $N(t) \sim 1$ for all $t \in \mathbf{R}$, or even for all $t \in [0, \infty)$ or $(-\infty, 0]$, we could modify $C(\eta)$ by a constant to return to the case $N(t) \equiv 1$.

For general $N(t)$ we could naively take

$$M(t) = \int |u(t,y)|^2 \phi(N(t)(x-y))(x-y)_j \operatorname{Im}[\bar{u}\partial_j u](t,x) dx dy. \tag{6.2}$$

Then since $N(t)$ is not constant, $\frac{d}{dt} M(t)$ would have the additional term

$$\int |u(t,y)|^2 \phi(\frac{(x-y)N(t)}{R}) \frac{|x-y|(x-y)_j}{R} N'(t) \operatorname{Im}[\bar{u}\partial_j u](t,x) dx dy. \tag{6.3}$$

By Hölder's inequality, $\|u\|_{L_t^\infty L_x^4(I \times \mathbf{R}^4)} \lesssim 1$, and Young's inequality,

$$(6.3) \lesssim R^4 \frac{N'(t)}{N(t)^5}. \tag{6.4}$$

When $N(t) \equiv 1$ this approach certainly worked since then $(6.4) = 0$, but naively taking $N(t)$ would also work when $N(t)$ is monotone increasing when $t > 0$ and monotone decreasing when $t < 0$, since then by the fundamental theorem of calculus

$$\int_0^\infty \frac{N'(t)}{N(t)^5} dt = -\frac{1}{4} \int_0^\infty \frac{d}{dt}(\frac{1}{N(t)^4}) dt = \frac{1}{4}. \tag{6.5}$$

However, for general $N(t)$, the most that (3.10) implies is that $(6.4) \lesssim \frac{R^4}{N(t)^2}$, whose integral is $R^4 K$, and therefore cannot be absorbed into the left hand side, which is bounded below by some δK.

Instead, $N(t)$ is replaced with a $\tilde{N}(t)$ that satisfies the following conditions:

1. $\tilde{N}(t) \gtrsim 1$.
2. $|\tilde{N}'(t)| \lesssim \tilde{N}(t)^3$.
3. $$\int_I \frac{1}{\tilde{N}(t)^2} dt \lesssim K, \tag{6.6}$$
   and
4. $$\int_I \frac{|\tilde{N}'(t)|}{\tilde{N}(t)^5} dt \ll K. \tag{6.7}$$

This $\tilde{N}(t)$ is defined inductively, using a procedure very similar to the smoothing algorithm in [15]. The idea is that when $N(t)$ is increasing for a long period of time, then (6.5) implies that $N(t)$ should be left alone, and one should simply take $\tilde{N}(t) = N(t)$. On the other hand, when $N(t)$ is oscillating rapidly, let's say $N(t) = 1 + \frac{1}{2}\cos(t)$ for example, then simply taking $\tilde{N}(t) = \frac{3}{2}$ is well worth the cost of modifying $C(\eta)$ in (3.9) by a constant, as was briefly discussed at the beginning of the section.

The algorithm terminates after $n(\|u\|_{L_t^\infty \dot{H}^1(\mathbf{R}\times\mathbf{R}^4)})$ steps. To simplify notation let $N_m(t)$ denote $\tilde{N}_m(t)$.

Definition 6.1 (Smoothing algorithm). – *Let*

(6.8)
$$\frac{1}{N_0(t)} = \|u_h(t)\|^3_{L_x^3(\mathbf{R}^4)}.$$

$N_0(t)$ satisfies conditions one, two, and three.

Lemma 6.1. – *Possibly after modifying $N_0(t)$ by some function $\alpha(t)$, $N_0(t) \mapsto \alpha(t)N_0(t)$,*

(6.9)
$$\epsilon < \alpha(t) < \frac{1}{\epsilon},$$

1. $N_0(t) \gtrsim 1$.
2. $|N_0'(t)| \lesssim N_0(t)^3$, *and*
3.

(6.10)
$$\int_I \frac{1}{N_0(t)^2}\, dt \lesssim K.$$

Proof. – $N_0(t) \gtrsim 1$ follows directly from Theorem 4.3. (6.10) holds by (4.11).

To prove $|N_0'(t)| \lesssim N_0(t)^3$, observe that for $t_0 \in \mathbf{R}$, for any $\eta > 0$, Bernstein's inequality implies

(6.11)
$$\|P_{>\eta^{-1}N(t_0)}u(t)\|_{L_t^\infty L_x^3(\mathbf{R}^4)} \lesssim \eta^{1/3}N(t_0)^{-1}.$$

Next, by the Sobolev embedding theorem and integrating by parts,

(6.12)
$$\begin{aligned}
&\frac{d}{dt}\Big(\int |P_{\leq\eta^{-1}N(t_0)}u_h(t,x)|^3 dx\Big)\\
&\quad= \Big(\int |P_{\leq\eta^{-1}N(t_0)}u_h(t,x)|\mathrm{Re}((i\Delta P_{\leq\eta^{-1}N(t_0)}u_h + iP_{\leq\eta^{-1}N(t_0)}P_hF(u))P_{\leq N(t_0)}\bar{u}_h dx\Big)\\
&\quad\lesssim \Big(\int |\nabla P_{\leq\eta^{-1}N(t_0)}u_h(t,x)|^2|P_{\leq\eta^{-1}N(t_0)}u_h(t,x)|dx\\
&\qquad+ \int |P_{\leq\eta^{-1}N(t_0)}u_h(t,x)|^2|P_{\leq\eta^{-1}N(t_0)}P_hF(u)(t,x)|dx\Big)\\
&\quad\lesssim \|P_{\leq\eta^{-1}N(t_0)}u_h\|_{L^\infty_{t,x}(\mathbf{R}\times\mathbf{R}^4)}(\|\nabla u\|^2_{L_t^\infty L_x^2(\mathbf{R}\times\mathbf{R}^4)} + \|u\|^4_{L_t^\infty L_x^4(\mathbf{R}\times\mathbf{R}^4)}) \lesssim \eta^{-1}N(t_0).
\end{aligned}$$

Then for $c > 0$ sufficiently small, for $|t - t_0| \leq c\eta N(t_0)^{-2}$, by (6.11) and (6.12),

(6.13)
$$\|u_h(t)\|_{L_x^3(\mathbf{R}^4)} \sim N(t_0)^{-1},$$

so for $|t_0 - t_1| \leq c\eta N(t_0)^{-2}$,

$$N_0(t_0) \sim N_0(t_1), \tag{6.14}$$

and thus $|N_0'(t)| \lesssim N_0(t)^3$, possibly after modifying $N_0(t)$ by $\alpha(t)$ satisfying (6.9). □

Now inductively define $N_m(t)$ to become progressively smoother than $N_0(t)$ until (6.7) is satisfied. Partition I into subintervals J_k such that $\int_{J_k} N_0(t)^2 dt = c$ for some $c \ll 1$. Call these the small intervals. Then let

$$N_0(J_k) = \sup\{2^j : j \in \mathbf{Z}, \quad 2^j \leq N_0(t) \quad \forall t \in J_k\}. \tag{6.15}$$

For c sufficiently small, if J_k and J_{k+1} are adjacent intervals then $|N_0'(t)| \lesssim N_0^3(t)$ implies $|J_k| \sim N_0(J_k)^{-2}$, and

$$\frac{N_0(J_k)}{N_0(J_{k+1})} = 1, \quad \frac{1}{2}, \quad \text{or} \quad 2. \tag{6.16}$$

If t_k is the midpoint of some J_k, let $N_1(t_k) = N_0(J_k)$, and otherwise let $N_1(t)$ be the linear interpolation between these midpoints. Then $N_1(t) \sim N_0(t)$ and $|N_1'(t)| \lesssim N_0(t)^3$, so

$$\int_I \frac{|N_1'(t)|}{N_1(t)^5} dt \lesssim \int_I \frac{1}{N_1(t)^2} dt \sim \int_I \|u_h(t)\|_{L^3}^6 dt \lesssim K. \tag{6.17}$$

Definition 6.2. – *We call the interval* J_k *upward sloping if* $\frac{N_1(J_k)}{N_1(J_{k+1})} = \frac{1}{2}$, *downward sloping if* $\frac{N_1(J_k)}{N_1(J_{k+1})} = 2$, *and flat if* $\frac{N_1(J_k)}{N_1(J_{k+1})} = 1$.

Now if J *is a union of small intervals,* $J = J_l \cup J_{l+1} \cup \cdots \cup J_{l+m}$, J *is called a valley if* J_l *is downward sloping,* J_{l+m} *is upward sloping, and* $J_{l+1}, \dots, J_{l+m-1}$ *are constant intervals.* J *is called a peak if* $J = J_l \cup J_{l+1} \cup \cdots \cup J_{l+m}$, J_l *is upward sloping,* J_{l+m} *is downward sloping, and* $J_{l+1}, \dots, J_{l+m-1}$ *are constant intervals.*

Remark. – $N_1(t)$ is monotone in between consecutive peaks and valleys. By construction, it is impossible to have two peaks without a valley in between, or two valleys without a peak in between.

Now apply the smoothing algorithm. If

$$J = J_l \cup \cdots \cup J_{l+m}, \tag{6.18}$$

is a valley let $N_2(t) = N_1(t_l) = N_1(t_{l+m})$ *for all* $t_l < t < t_{l+m}$. *If* t *does not lie in* (t_l, t_{l+m}) *for some valley* $J = J_l \cup \cdots \cup J_{l+m}$ *let* $N_2(t) = N_1(t)$.

Likewise construct $N_{m+1}(t)$ using the above algorithm with $N_1(t)$ replaced by $N_m(t)$. By the fundamental theorem of calculus, if $N_m(t)$ is monotone on an interval J,

$$\int_J \frac{|N_m'(t)|}{N_m(t)^5} dt \lesssim (\inf_{t \in J} N_m(t))^{-4}. \tag{6.19}$$

Next observe from the smoothing algorithm that if $J = J_l \cup \cdots \cup J_{l+m}$ is a valley for $N_1(t)$, (6.16) implies $N_2(t) = 2N_1(t)$ for $t \in J_{l+1} \cup \cdots \cup J_{l+m-1}$. Also observe from the smoothing algorithm that $N_2'(t) \neq 0$ implies $N_1(t) = N_2(t)$, so if J is a valley for $N_2(t)$, there must exist some $J_1 \subset J$ that was a valley for $N_1(t)$. Therefore, by (6.19) and the fact that peaks and valleys must alternate,

$$\int_I \frac{|N_{m+1}'(t)|}{N_{m+1}^5(t)} dt \leq 2^{-4} \int_I \frac{|N_m'(t)|}{N_m(t)^5} dt + 2, \tag{6.20}$$

and therefore

$$\int_I \frac{|N_{m+1}'(t)|}{N_{m+1}^5(t)} dt \leq 2^{-4m} \int_I \frac{|N_1'(t)|}{N_1^5(t)} dt + 4. \tag{6.21}$$

Also observe that by induction

$$N_1(t) \leq N_{m+1}(t) \leq 2^m N_1(t). \tag{6.22}$$

Now let

$$M(t) = \iint \phi((x-y)N_m(t))(x-y)_j |u(t,y)|^2 \mathrm{Im}[\bar{u}\partial_j u](t,x)\,dx\,dy. \tag{6.23}$$

Since $N_m(t) \gtrsim 1$, by Hölder's inequality, and Young's inequality, $|M(t)| \lesssim \frac{e^{4J}}{N_m(t)^4} \lesssim e^{4J}$. Combining (6.3) with (5.31)-(5.35),

$$\frac{d}{dt} M(t) = 2 \iint \phi((x-y)N_m(t))|u(t,y)|^2 [|\nabla u(t,x)|^2 - |u(t,x)|^4]\,dx\,dy \tag{6.24}$$

$$- 2 \iint \phi((x-y)N_m(t)) \mathrm{Im}[\bar{u}\partial_j u](t,x) \mathrm{Im}[\bar{u}\partial_j u](t,y)\,dx\,dy \tag{6.25}$$

$$+ 2 \iint (\partial_k \phi((x-y)N_m(t)))(x-y)_j |u(t,y)|^2 [\mathrm{Re}(\partial_j \bar{u}\partial_k u)(t,x) - \frac{\delta_{jk}}{4}|u(t,x)|^4]\,dx\,dy \tag{6.26}$$

$$- 2 \iint \mathrm{Im}[\bar{u}\partial_k u](t,y)(\partial_k \phi((x-y)N_m(t)))(x-y)_j \mathrm{Im}[\bar{u}\partial_j u](t,x)\,dx\,dy \tag{6.27}$$

$$+ \frac{1}{2} \iint |u(t,y)|^2 \partial_j \Delta(\phi((x-y)N_m(t))(x-y)_j)|u(t,x)|^2\,dx\,dy \tag{6.28}$$

$$+ \iint \phi'((x-y)N_m(t))(x-y)_j |x-y| N_m'(t)|u(t,y)|^2 \mathrm{Im}[\bar{u}\partial_j u](t,x)\,dx\,dy. \tag{6.29}$$

Observe that by (3.10), Theorem 4.1, (5.26), (6.8), (6.21), Hölder's inequality, Young's inequality, and $N_m(t) \gtrsim 1$,

(6.30)

$$\begin{aligned}
(6.29) &\lesssim \frac{1}{J} \int_I \int_{|x-y| \leq \frac{4e^J}{N_m(t)}} |u(t,y)|^2 |x-y| \frac{|N_m'(t)|}{N_m(t)} |\nabla u(t,x)||u(t,x)|\,dx\,dy\,dt \\
&\lesssim \frac{e^{3J}}{J} \int_I \frac{|N_m'(t)|}{N_m(t)^4} \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^3 \|\nabla u(t)\|_{L_x^2(\mathbf{R}^4)} dt \\
&\quad + \frac{e^{5J}}{J} \int_I \frac{|N_m'(t)|}{N_m(t)^6} \|u_l(t)\|_{L_x^6(\mathbf{R}^4)}^3 \|\nabla u(t)\|_{L_x^2(\mathbf{R}^4)} dt \\
&\lesssim \frac{e^{3J}}{J} \int_I \frac{|N_m'(t)|}{N_m(t)^5} dt + \frac{e^{5J}}{J} \|u_l\|_{L_{t,x}^6(I \times \mathbf{R}^4)}^3 \|\nabla u\|_{L_t^\infty L_x^2(I \times \mathbf{R}^4)} (\int_I \frac{1}{N_m(t)^6} dt)^{1/2}
\end{aligned}$$

$$\lesssim 2^{-4m+4} K \frac{e^{3J}}{J} + 4\frac{e^{3J}}{J} + K^{1/2} \frac{e^{5J}}{J}.$$

The other terms may be estimated in the same manner as their counterparts in the previous section. As in (5.50))-(5.61,

$$(6.31)\quad (6.24) + (6.25) \gtrsim \frac{\bar{\delta}}{2} \int \psi\left(\frac{4(x-y)N_m(t)}{e^{11J/12}}\right)|u(t,x)|^2|u(t,y)|^4 dxdy - \frac{1}{J}\int_{|x-y|\leq \frac{4e^J}{N_m(t)}} \frac{1}{|x-y|^2}|u(t,y)|^2|u(t,x)|^2 dxdy.$$

(6.32) and (6.33) also involve a mass and an energy integral, so by (5.2), (6.8), (6.22), and the fact that $\frac{e^J}{2^m} >> 1$ (the relationship between J and m will be described in more detail later),

$$\int_I (6.26) + (6.27) dt$$

$$(6.32)\quad \lesssim \frac{1}{J}\int_I \int_{|x-y|\leq \frac{4e^J}{N_m(t)}} \int_{|x-x(t)|\geq \frac{C(\eta)}{N(t)}} |u(t,y)|^2[|\nabla u(t,x)|^2 + |u(t,x)|^4] dxdydt + \frac{1}{J}\int_I \int_{|x-y|\leq \frac{4e^J}{N_m(t)}} \int_{|x-x(t)|\leq \frac{C(\eta)}{N(t)}} |u(t,y)|^2[|\nabla u(t,x)|^2 + |u(t,x)|^4] dxdydt$$

$$(6.33)\quad \lesssim \frac{\eta}{J}\left(\int_I \sup_{y\in\mathbf{R}^4} \int_{|x-y|\leq \frac{4e^J}{N_m(t)}} |u(t,y)|^2 dydt\right) + \frac{1}{J}\int_I \int_{|x-x(t)|\leq \frac{8e^J}{N_m(t)}} |u(t,x)|^2 dxdt \lesssim \eta K + \frac{1}{J}\int_I \int_{|x-x(t)|\leq \frac{8e^J}{N_m(t)}} |u(t,x)|^2 dxdt.$$

Now, following the analysis in (5.39)-(5.49),

$$(6.34)\quad \frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |u(t,x)|^2 \frac{1}{|x-y|^2}|u(t,y)|^2 dxdydt$$
$$\lesssim \frac{1}{J}\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |P_h u_{\geq c(\eta)N(t)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt$$
$$+ \frac{1}{J}\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |P_h u_{\leq c(\eta)N(t)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt$$
$$+ \frac{1}{J}\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |u_l(t,x)|^2 \frac{1}{|x-y|^2}|u_l(t,y)|^2 dxdydt.$$

By Bernstein's inequality and Hardy's inequality,

$$(6.35)\quad \frac{1}{J}\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |P_h u_{\geq c(\eta)N(t)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dxdydt$$
$$\lesssim \frac{1}{J}\int_I \|u_{\geq c(\eta)N(t)}(t)\|^2_{L^2_x(\mathbf{R}^4)}\left(\sup_{x\in\mathbf{R}^4}\int \frac{1}{|x-y|^2}|u_h(t,y)|^2 dy\right)dt$$
$$\lesssim \frac{1}{J}\frac{1}{c(\eta)^2}\int_I \frac{1}{N(t)^2} dt = \frac{K}{J}\frac{1}{c(\eta)^2}.$$

Also by Hölder's inequality and Young's inequality,

(6.36)
$$\begin{aligned}
&\frac{1}{J}\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |P_h u_{\leq c(\eta)N(t)}(t,x)|^2 \frac{1}{|x-y|^2}|u_h(t,y)|^2 dx dy dt \\
&\lesssim \frac{1}{J}\int_I \sum_{2^j \leq \frac{4e^J}{N_m(t)}} 2^{-2j} \int_{2^j \leq |x-y| \leq 2^{j+1}} |P_{>-j} P_h u_{\leq c(\eta)N(t)}(t,x)|^2 |u_h(t,y)|^2 dx dy dt \\
&\quad + \frac{1}{J}\int_I \sum_{2^j \leq \frac{4e^J}{N_m(t)}} 2^{-2j} \int_{2^j \leq |x-y| \leq 2^{j+1}} |P_{\leq -j} P_h u_{\leq c(\eta)N(t)}(t,x)|^2 |u_h(t,y)|^2 dx dy dt \\
&\lesssim \frac{1}{J}\int_I (\sum_j 2^{-2j} \|P_{>-j} u_{\leq c(\eta)N(t)}(t)\|_{L_x^2(\mathbf{R}^4)}^2)(\sup_y \int_{|x-y|\leq 4e^J} |u_h(t,x)|^2 dx) dt \\
&\quad + \frac{1}{J}\int_I \sum_{\frac{1}{N_0(t)} \leq 2^j \leq \frac{4e^J}{N_m(t)}} 2^{2j} \|P_{\leq -j} P_h u_{\leq c(\eta)N(t)}(t)\|_{L_x^6(\mathbf{R}^4)}^2 \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^2 \\
&\quad + \frac{1}{J}\int_I \sum_{2^j \leq \frac{1}{N_0(t)}} 2^{2j} \|P_{\leq c(\eta)N(t)} u_h(t)\|_{L_x^4(\mathbf{R}^4)}^2 \|u_h(t)\|_{L_x^4(\mathbf{R}^4)}^2 dt.
\end{aligned}$$

As in (5.43),

$$\sum_j 2^{-2j} \|P_{>-j} P_{\leq c(\eta)N(t)} u(t)\|_{L^2}^2 \lesssim \eta^2, \tag{6.37}$$

so again by (5.2),

(6.38)
$$\frac{1}{J}\int_I (\sum_j 2^{-2j} \|P_{>-j} P_{\leq c(\eta)N(t)} u(t)\|_{L_x^2(\mathbf{R}^4)}^2)(\sup_y \int_{|x-y|\leq 4e^J} |u_h(t,x)|^2 dx) dt \lesssim \eta^2 K.$$

Next, since $N(t)$ is variable, using (4.26) and (6.8) along with Hölder's inequality,

(6.39)
$$\begin{aligned}
&\frac{1}{J}\int_I \sum_{\frac{1}{N_0(t)} \leq 2^j \leq \frac{4e^J}{N_m(t)}} 2^{2j} \|P_{\leq -j} P_h P_{\leq c(\eta)N(t)} u(t)\|_{L_x^6(\mathbf{R}^4)}^2 \|u_h(t)\|_{L_x^3(I\times\mathbf{R}^4)}^2 dt \\
&\lesssim \frac{1}{J}\int_I \sum_{\frac{1}{N_0(t)} \leq 2^j \leq \frac{4e^J}{N_m(t)}} 2^{-2j/3} (2^{4j/3} \|P_{\leq -j} P_h P_{\leq c(\eta)N(t)} u(t)\|_{L_x^6(\mathbf{R}^4)})^2 \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^2 dt \\
&\lesssim (\int_I (\sup_{2^j \geq K^{-1/4}} 2^{4j/3} \|P_j u(t)\|_{L_x^6(\mathbf{R}^4)})^2 dt)(\sup_{t\in I} \sum_{2^j \geq \frac{1}{N_0(t)}} 2^{-2j/3} \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^2) \lesssim \frac{K}{J}.
\end{aligned}$$

By (3.9), (6.10), and $u \in L_t^\infty L_x^4$,

(6.40)
$$\frac{1}{J}\sum_{2^j \leq \frac{1}{N_0(t)}} \int_I 2^{2j} \|P_{\leq c(\eta)N(t)} u_h(t)\|_{L_x^4(\mathbf{R}^4)}^2 \|u_h(t)\|_{L_x^4(\mathbf{R}^4)}^2 dt \lesssim \frac{\eta}{J}\int_I \frac{1}{N_0(t)^2} dt \lesssim \eta \frac{K}{J}.$$

Finally, by (5.47), (5.48), and the fact that $N_m(t) \gtrsim 1$,

$$\int_I \iint_{|x-y| \leq \frac{4e^J}{N_m(t)}} |u_l(t,y)|^2 \frac{1}{|x-y|^2} |u_l(t,x)|^2 dx dy dt \lesssim e^{2J} K^{1/2}. \tag{6.41}$$

Therefore,

$$\int_I (6.34) dt \lesssim \frac{K}{J} \frac{1}{c(\eta)^2} + \eta^2 K + e^{2J} K^{1/2}. \tag{6.42}$$

REMARK. – By (5.27), (6.28) is also bounded by a term of this form.

Choose m so that $2^{4m} = e^{10J/3}$. In this case, by (3.9), (6.22) and analysis similar to (5.63), since e^J is large and $N(t) \gtrsim N_0(t) \sim N_1(t)$,

$$\begin{aligned} &\frac{\bar{\delta}}{2} \int_I \iint \psi\Big(\frac{4(x-y)N_m(t)}{e^{11J/12}}\Big) |u(t,y)|^2 |u(t,x)|^4 dx dy dt \\ &\qquad \gtrsim \frac{\bar{\delta}}{2} \int_I \int_{|x-x(t)| \geq \frac{e^{11J/12}}{8N_m(t)}} |u(t,x)|^2 dx dt. \end{aligned} \tag{6.43}$$

Combining (6.43) with (6.30), (6.31), (6.33), (6.42), $\sup_{t \in I} |M(t)| \lesssim e^{4J}$, and the fundamental theorem of calculus,

$$\begin{aligned} \bar{\delta} \int_I \int_{|x-x(t)| \leq \frac{e^{11J/12}}{8N_m(t)}} |u(t,x)|^2 dx dt &\lesssim \frac{1}{J} \int_I \int_{|x-x(t)| \leq \frac{8e^J}{N_m(t)}} |u(t,x)|^2 dx dt \\ &+ \eta K + 2^{-4m} K \frac{e^{3J}}{J} + \frac{e^{3J}}{J} + K^{1/2} \frac{e^{5J}}{J} + e^{4J} + \frac{K}{J} \frac{1}{c(\eta)^2}, \end{aligned} \tag{6.44}$$

and therefore

$$\begin{aligned} &\bar{\delta}^2 \int_I \int_{|x-x(t)| \leq \frac{e^{11J/12}}{8N_m(t)}} |u(t,x)|^2 dx dt \\ &\lesssim \frac{\bar{\delta}}{J} \int_I \int_{|x-x(t)| \leq \frac{8e^J}{N_m(t)}} |u(t,x)|^2 dx dt \\ &+ \bar{\delta}\Big(\eta K + 2^{-4m} \frac{e^{3J}}{J} K + \frac{e^{3J}}{J} + K^{1/2} \frac{e^{5J}}{J} + e^{4J} + \frac{K}{J} \frac{1}{c(\eta)^2}\Big) \\ &\lesssim \frac{1}{J^2} \int_I \int_{|x-x(t)| \leq \frac{512 e^{12J/11}}{N_m(t)}} |u(t,x)|^2 dx dt \\ &+ \eta K + 2^{-4m} K \frac{e^{36J/11}}{J} + \frac{e^{36J/11}}{J} + K^{1/2} \frac{e^{60J/11}}{J} + e^{48J/11} + \frac{K}{J} \frac{1}{c(\eta)^2}. \end{aligned} \tag{6.45}$$

Choosing J and m such that $2^{4m} = e^{10J/3}$ and $e^{12J} = K$,

$$\begin{aligned} &\bar{\delta}^2 \int_I \int_{|x-x(t)| \leq \frac{e^{11J/12}}{8N_m(t)}} |u(t,x)|^2 dx dt \\ &\lesssim \frac{1}{J^2} \int_I \int_{|x-x(t)| \leq \frac{512 e^{12J/11}}{N_m(t)}} |u(t,x)|^2 dx dt + \eta K + \frac{e^{-2J/33}}{J} K + K^{21/22} + \frac{K}{J} \frac{1}{c(\eta)^2}. \end{aligned} \tag{6.46}$$

Now we are ready to complete the proof of Theorem 1.10.

THEOREM 6.2. – *If u is an almost periodic solution to* (1.1) *with* $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$*, then* $u \equiv 0$.

Proof. – Suppose u is a nonzero, almost periodic solution to (1.1). Let I be an interval satisfying

$$\int_I N(t)^{-2} dt = K. \tag{6.47}$$

(6.46) combined with Lemma 5.1 implies that

$$\bar{\delta}^2 \int_I \int_{|x-x(t)| \leq \frac{e^{11J/12}}{8N_m(t)}} |u(t,x)|^2 dx dt \lesssim \frac{K}{\ln(K)} \frac{1}{c(\eta)^2} + \eta K. \tag{6.48}$$

Since $\eta > 0$ is arbitrary and $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$, choosing an increasing sequence of intervals I whose union makes up $\mathbf{R}$, together with $N_m(t) \lesssim 2^m N(t)$ and $2^{4m} = e^{10J/3}$, and $e^{12J} = K$, there exists a sequence $t_n \in \mathbf{R}$ and a sequence $R_n \nearrow \infty$ such that

$$N(t_n)^2 \int_{|x-x(t)| \leq \frac{R_n}{N(t_n)}} |u(t_n,x)|^2 dx \to 0. \tag{6.49}$$

However, by (3.9) this implies that $\|u(t_n)\|_{\dot{H}^1} \to 0$, and thus by conservation of energy $u \equiv 0$. □

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(Manuscrit reçu le 12 août 2016 ;
accepté, après révision, le 23 août 2017.)

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