THE RANK OF G-CROSSED BRAIDED EXTENSIONS OF MODULAR TENSOR CATEGORIES

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Abstract. We give a short proof for a well-known formula for the rank of a G-crossed braided extension of a modular tensor category.

1. Introduction

G-crossed braided extensions of modular tensor categories are an important ingredient for the process of gauging in topological phases of matter in the framework of modular tensor categories [BBCW14]. Let $\mathcal{C}$ be a modular tensor category, which for the purpose of this note is a semisimple, $\mathbb{C}$-linear abelian ribbon category with simple tensor unit, such that the set of isomorphism classes of simple objects $\mathrm{Irr}(\mathcal{C})$ is finite and the braiding is non-degenerate, see e.g. [BK01]. A global symmetry [BBCW14] of $\mathcal{C}$ is a pair (G,ρ) consisting of a finite group G and homomorphism $\rho\colon G\to \mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{C})$, where $\mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{C})$ is the group of isomorphism classes of braided autoequivalences of $\mathcal{C}$. Gauging a global symmetry (G,ρ) of $\mathcal{C}$ [CGPW16] is a two step process which eventually produces a new modular tensor category and is given as follows:

(1) Construct a G-crossed braided extension $\mathcal{F}=\bigoplus_{g\in G}\mathcal{F}_g$ of $\mathcal{C}$.
(2) Consider the equivariantization $\mathcal{F}^G$ of $\mathcal{F}$ to obtain a new modular tensor category, which therefore contains $\mathrm{Rep}(G)$ as a symmetric subcategory.

A G-crossed braided fusion category is a fusion category $\mathcal{F}$ equipped with the following structure:

- $\mathcal{F}$ is faithfully G-graded, i.e. $\mathcal{F}=\bigoplus\mathcal{F}_g$ and $X_g\otimes Y_h\in\mathcal{F}_{gh}$ for every $g,h\in G$, $X_g\in\mathcal{F}_g$, and $Y_h\in\mathcal{F}_h$.
- There is an G-action $X\mapsto {}^gX=\rho(g)(X)$ given by a monoidal functor $\underline{\rho}\colon \underline{G}\to\underline{\mathrm{Aut}_\otimes}(\mathcal{F})$, where $\underline{\mathrm{Aut}_\otimes}(\mathcal{F})$ is the categorical group of monoidal autoequivalences.
- There is a natural family of isomorphisms
$$c_{X,Y}\colon X\otimes Y\to {}^gY\otimes X\,,\qquad\qquad g\in G, X\in\mathcal{F}_g, Y\in\mathcal{F}$$
with $\rho(g)(\mathcal{F}_h)\subset\mathcal{F}_{ghg^{-1}}$ fulfilling certain coeherence diagrams, see e.g. [Tur10].

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We note that $\mathcal{F}_e$ is a braided fusion category and we get a global symmetry $\rho\colon G \to \mathrm{Aut}^{\mathrm{br}}_{\otimes}(\mathcal{F}_e)$ given by restriction and truncation of $\underline{\rho}$. We are only interested in the case where $\mathcal{F}_e$ is a modular tensor category. We refer to [DGNO10, Tur10] for more details.

We call a G-crossed braided fusion category $\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g$ a G-crossed braided extension of a modular tensor category $\mathcal{C}$ if $\mathcal{F}_e$ is braided equivalent to $\mathcal{C}$. The above data of $\mathcal{F}$ gives by restriction a global symmetry (G,ρ) on $\mathcal{C} \cong \mathcal{F}_e$. We note that given a modular tensor category $\mathcal{C}$ and a global action (G,ρ) as an input one has to check that certain obstructions have to vanish in order for a G-crossed braided to exist, see [ENO10].

G-crossed braided extensions arise naturally in rational conformal field theory. Let $\mathcal{A}$ be a (completely) rational conformal net, then the category of representations $\mathrm{Rep}(\mathcal{A})$ is a unitary modular tensor category [KLM01]. Let $G \leq \mathrm{Aut}(\mathcal{A})$ be a finite group of automorphisms of the net $\mathcal{A}$. There is a category $G\text{–}\mathrm{Rep}(\mathcal{A})$ of G-twisted representations of $\mathcal{A}$, which is a G-crossed braided extension of the unitary modular tensor category $\mathrm{Rep}(\mathcal{A})$. The fixed point or "orbifold net" $\mathcal{A}^G$ is again completely rational and $\mathrm{Rep}(\mathcal{A}^G)$ is a gauging of $\mathrm{Rep}(\mathcal{A})$ by G, i.e. $\mathrm{Rep}(\mathcal{A}^G)$ is braided equivalent to the G-equivariantization $(G\text{–}\mathrm{Rep}(\mathcal{A}))^G$, see [Müg05] for the original reference and [Bis18] for a review.

Starting with a modular tensor category $\mathcal{C}$ we want to compute certain invariants of possible G-crossed braided extensions $\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g$ with $\mathcal{F}_e \cong \mathcal{C}$. The simplest invariant of $\mathcal{F}$ is its rank $\mathrm{rk}(\mathcal{F}) := |\operatorname{Irr}(\mathcal{F})|$.

We note that the global symmetry (G,ρ) associated with a G-crossed braided extension $\mathcal{F}$ of $\mathcal{C}$ equips $\operatorname{Irr}(\mathcal{C})$ with a G-action. This G-space already contains all the information about the rank of possible G-crossed braided extensions associated with (G,ρ). Namely, the following well-know formula, cf. [BBCW14, Eq. (348)] holds.

Proposition 1.1. *Let $\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g$ be a (faithful) G-crossed braided extension of a modular tensor category $\mathcal{C} \cong \mathcal{F}_e$. Then the rank of $\mathcal{F}_g$ is equal to the size of the stabilizer $\operatorname{Irr}(\mathcal{C})^g$, i.e.*

$$\mathrm{rk}(\mathcal{F}_g) = |\operatorname{Irr}(\mathcal{C})^g|\,.$$

In particular, the rank of $\mathcal{F}$ is determined via the G-action on $\operatorname{Irr}(\mathcal{C})$ by

$$\mathrm{rk}(\mathcal{F}) = \sum_{g\in G} |\operatorname{Irr}(\mathcal{C})^g| = |G||\operatorname{Irr}(\mathcal{C})/G|\,.$$

The goal of this note is to provide a short proof of this statement using modular invariants, which were used in the operator algebra literature, e.g. [BEK99, BEK00].

2. The rank of module categories and G-crossed braided extensions

Let $\mathcal{C}$ be a modular tensor category and A a simple non-degenerate algebra object in $\mathcal{C}$. Here non-degenerate means that the trace pairing $\Phi_A\colon A \to \overline{A}$

given by

$$\Phi_A = ((\mathrm{ev}_A) \circ (\mathrm{id}_{\bar A} \otimes m) \otimes \mathrm{id}_{\bar A}) \circ (\mathrm{id}_{\bar A} \otimes (\mathrm{id}_A \otimes \mathrm{coev}_A) \circ m) \circ (\mathrm{coev}_{\bar A} \otimes \mathrm{id}_A)$$

is invertible, which implies that A has the structure of a special symmetric Frobenius algebra object, see [KR08, Lemma 2.3].

We denote the category with the opposite braiding by $\overline{\mathcal{C}}$. The full center $Z(A)$ is a Lagrangian algebra object in $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \overline{\mathcal{C}}$ [FFRS06, KR08, DMNO13]. We note that the forgetful functor $F\colon \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{C}$ maps $X \boxtimes Y \mapsto X \otimes Y$ and has an adjoint $I\colon \mathcal{C} \to \mathcal{C} \boxtimes \overline{\mathcal{C}}$. The image $I(\mathbb{1})$ has the canonical structure of a Lagrangian algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. If $X \in \mathcal{C}$ we denote by $\bar X \in \mathcal{C}$ a dual object. The full center $Z(A)$ can be realized as the left center of the braided product of $(A \boxtimes \mathbb{1}) \otimes^+ I(\mathbb{1})$ [FFRS06, KR08]. Given a simple non-degenerate algebra A there are two functors $\alpha^\pm$ from $\mathcal{C}$ to the category ${}_A\mathcal{C}_A$ of A-bimodules in $\mathcal{C}$. They are given by equipping the image of the free right module functor $-\otimes A$ with a left action using the braiding or opposite braiding, respectively, see [FFRS06].

The following proposition is well-known to experts.

Proposition 2.1. *Let $\mathcal{C}$ be a modular tensor category, A a simple non-degenerate algebra object in $\mathcal{C}$, and Z the associated modular invariant matrix, i.e. $Z = (Z_{X,Y})_{X,Y\in\mathrm{Irr}(\mathcal{C})}$ is the square matrix given by*

$$Z_{X,Y} = \dim_{\mathbb{C}} \mathrm{Hom}_{{}_A\mathcal{C}_A}(\alpha^+(X), \alpha^-(\bar Y)) .$$

Then

(1) *The full center $Z(A)$ of A as an object in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ is given by*

$$Z(A) \cong \bigoplus_{X,Y\in\mathrm{Irr}(\mathcal{C})} Z_{X,Y} \cdot X \boxtimes \bar Y .$$

(2) *The rank $\mathrm{rk}(\mathcal{C}_A)$ of the category of right A-modules $\mathcal{C}_A$ is given by the trace of the matrix Z:*

$$\mathrm{rk}(\mathcal{C}_A) = \mathrm{tr}(Z) = \sum_{X\in\mathrm{Irr}(\mathcal{C})} Z_{X,X} .$$

Proof. The first statement is [FFRS06, Remark 3.7]. The second statement is [BEK99, Corollary 6.1] in the case that A is a Q-system and $\mathcal{C}$ is a unitary modular tensor category realized as endomorphisms of a type III factor. In the more general setting, it follows from [KR08, Eq. (4.4) and Prop. 4.3]. Namely, $F(Z(A)) \cong \bigoplus_{X,Y\in\mathrm{Irr}(\mathcal{C})} Z_{X,Y} X \otimes \bar Y$ thus $\dim\mathrm{Hom}(\mathbb{1}, F(Z(A)) = \mathrm{tr}(Z)$ equals the number of isomorphism classes of simple modules in $\mathcal{C}_A$, since $F(Z(A))$ is equivalent to $\bigoplus_{M\in\mathrm{Irr}(\mathcal{C}_A)} M \otimes_A \bar M$ and since $M \otimes_A \bar M$ is a connected algebra by [KR08, Proof of Prop. 4.10]. □

For $\phi \in \mathrm{Aut}^{\mathrm{br}}_{\otimes}(\mathcal{C})$ there is a canonical Lagrangian algebra $L_\phi = (\mathrm{id} \boxtimes \phi) I(1)$ in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$, see e.g. [DNO13, Sec 3.2-3.3]. The algebra L_ϕ corresponds to the full center of a $\mathcal{C}$-module category, which we denote by $\mathcal{C}_\phi$, cf. [KR08, DMNO13]. This module category can be obtained, using e.g. [KR08], by taking any connected summand A of the algebra $F(L_\phi)$ (which always fulfills $Z(A) = L_\phi$) and considering the $\mathcal{C}$-module category $\mathcal{C}_\phi := \mathcal{C}_A$, see [KR08], and also [ENO10, Lemma 5.1]. Since $L_\phi \cong \bigoplus_{X\in\operatorname{Irr}(\mathcal{C})} X \boxtimes \phi(\bar{X})$, i.e. $Z_{X,Y} = \delta_{\phi(X),Y}$, Proposition 2.1 immediately gives the following statement.

Lemma 2.2. *The rank of $\mathcal{C}_\phi$ equals $|\operatorname{Irr}(\mathcal{C})^\phi|$, i.e. the number of isomorphism classes of simple objects in $\mathcal{C}$ which are fixed by ϕ.*

Proof of Proposition 1.1. Let $\mathcal{F} = \bigoplus_{g\in G}\mathcal{F}_g$ be a G-crossed braided extension of a modular tensor category $\mathcal{C}$. By [ENO10, Sec. 5.4] the category $\mathcal{F}_g$ is equivalent as a left $\mathcal{C}$-module category to the category $\mathcal{C}_g$ in Lemma 2.2 and the statement follows. □

2.1. **The rank of permutation extensions.** Let $\mathcal{C}$ be a modular tensor category. Gannon and Jones announced [GJ18] that certain cohomological obstructions vanish and thus that there is always an S_n-crossed braided extension of $\mathcal{C}^{\boxtimes n}$ which we denote by $\mathcal{C} \wr S_n$, where the action of S_n is given by permuations of objects. More general, for any subgroup $G \leq S_n$ there is a G-crossed braided extension $\mathcal{C} \wr G$.

For the cyclic subgroup $\mathbb{Z}_n \cong \langle (12\cdots n)\rangle \subset S_n$ with n prime we obtain

$$\operatorname{rk}(\mathcal{C}\wr\mathbb{Z}_n) = \operatorname{rk}(\mathcal{C})^n + (n-1)\operatorname{rk}(\mathcal{C})\,,$$

which in the case $n = 2$ is also true by construction for the examples considered in [EMJP18].

In general, for the full symmetric group, we get that $\operatorname{rk}((\mathcal{C}\wr S_n)_g)$ depends only on the conjugacy class C_g of $g \in S_n$. Namely, $\operatorname{rk}(\mathcal{C}\wr S_n)_g) = \operatorname{rk}(\mathcal{C}))^{|a|}$ where $a = (a_1,\ldots,a_n)$ and a_j is the number of j-cycles (counting 1-cycles) in the cycle decomposition of g and $|a| = \sum_j a_j$. Thus summing over all elements of the $p(n)$ conjugacy classes we get

$$\operatorname{rk}(\mathcal{C}\wr S_n) = \sum_{a=(a_1,\ldots,a_n)} c_a \operatorname{rk}(\mathcal{C})^{|a|}\,, \qquad c_a = \frac{n!}{\prod_j (j)^{a_j}(a_j!)}\,,$$

where $c_a = |C_g|$ the size of the conjugacy class C_g of an element $g \in S_n$ with cycle type $a = (a_1,\ldots,a_n)$ and the sum runs over all unordered partitions of $\{1,\ldots,n\}$ with a_j the number of partitions of length j. Explicitly,

$$\begin{aligned}
\operatorname{rk}(\mathcal{C}\wr S_3) &= \operatorname{rk}(\mathcal{C})^3 + 3\operatorname{rk}(\mathcal{C})^2 + 2\operatorname{rk}(\mathcal{C})\\
\operatorname{rk}(\mathcal{C}\wr S_4) &= \operatorname{rk}(\mathcal{C})^4 + 6\operatorname{rk}(\mathcal{C})^3 + 11\operatorname{rk}(\mathcal{C})^2 + 6\operatorname{rk}(\mathcal{C})\\
\operatorname{rk}(\mathcal{C}\wr S_5) &= \operatorname{rk}(\mathcal{C})^5 + 10\operatorname{rk}(\mathcal{C})^4 + 35\operatorname{rk}(\mathcal{C})^3 + 50\operatorname{rk}(\mathcal{C})^2 + 24\operatorname{rk}(\mathcal{C})\\
&\cdots
\end{aligned}$$

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