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ORIGINAL ARTICLE

An L^2 to L^∞ Framework for the Landau Equation

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Abstract
Consider the Landau equation with Coulomb potential in a periodic box. We develop a new L^2 to L^∞ framework to construct global unique solutions near Maxwellian with small L^∞ norm. The first step is to establish global L^2 estimates with strong velocity weight and time decay, under the assumption of L^∞ bound, which is further controlled by such L^2 estimates via De Giorgi's method (Golse et al. in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19(1), 253–295 (2019), Imbert and Mouhot in arXiv:1505.04608 (2015)). The second step is to employ estimates in S_p spaces to control velocity derivatives to ensure uniqueness, which is based on Hölder estimates via De Giorgi's method (Golse et al. in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19(1), 253–295 (2019), Golse and Vasseur in arXiv:1506.01908 (2015), Imbert and Mouhot in arXiv:1505.04608 (2015)).

Keywords Landau equation · Weak solution · Existence and uniqueness · L^2 to L^∞ framework

Mathematics Subject Classification 35Qxx

1 Introduction

We consider the following Landau equation:

$$\begin{aligned}
&\partial_t F + v\cdot\nabla_x F = Q(F,F) = \nabla_v\cdot\left\{\int_{\mathbb{R}^3}\phi(v-v')[F(v')\nabla_v F(v) - F(v)\nabla_v F(v')]\mathrm{d}v'\right\},\\
&F(0,x,v) = F_0(x,v),
\end{aligned} \tag{1.1}$$

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where $F(t,x,v)\geq 0$ is the spatially periodic distribution function for particles at time $t\geq 0$, with spatial coordinates $x=(x_1,x_2,x_3)\in[-\pi,\pi]^3=\mathbb{T}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The non-negative matrix ϕ is

$$\phi^{ij}(v)=\left\{\delta_{i,j}-\frac{v_iv_j}{|v|^2}\right\}|v|^{-1}. \tag{1.2}$$

As in the Boltzmann equation, it is well known that Maxwellians are steady states to (1.1) [13], etc. Let μ be a normalized Maxwellian

$$\mu(v)=e^{-|v|^2}, \tag{1.3}$$

and set

$$F(t,x,v)=\mu(v)+\mu^{1/2}(v)f(t,x,v). \tag{1.4}$$

Then the standard perturbation $f(t,x,v)$ to μ satisfies

$$f_t+v\cdot\partial_x f+Lf=\Gamma(f,f), \tag{1.5}$$

$$f(0,x,v)=f_0(x,v), \tag{1.6}$$

where f_0 is the initial data satisfying the conservation laws:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)\sqrt{\mu}=\int_{\mathbb{T}^3\times\mathbb{R}^3} v_if_0(x,v)\sqrt{\mu}=\int_{\mathbb{T}^3\times\mathbb{R}^3}|v|^2f_0(x,v)\sqrt{\mu}=0. \tag{1.7}$$

The linear operator L and the nonlinear part Γ are defined as

$$L=-A-K, \tag{1.8}$$

$$Af\ :=\mu^{-1/2}\partial_i\{\mu^{1/2}\sigma^{ij}[\partial_jf+v_jf]\}=\partial_i[\sigma^{ij}\partial_jf]-\sigma^{ij}v_iv_jf+\partial_i\sigma^if, \tag{1.9}$$

$$Kf\ :=-\mu^{-1/2}\partial_i\{\mu[\phi^{ij}*\{\mu^{1/2}[\partial_jf+v_jf]\}]\}, \tag{1.10}$$

$$\begin{aligned}\Gamma[g,f]\ :=\ &\partial_i[\{\phi^{ij}*[\mu^{1/2}g]\}\partial_jf]-\{\phi^{ij}*[v_i\mu^{1/2}g]\}\partial_jf\\ &-\partial_i[\{\phi^{ij}*[\mu^{1/2}\partial_jg]\}f]+\{\phi^{ij}*[v_i\mu^{1/2}\partial_jg]\}f.\end{aligned} \tag{1.11}$$

Define the weighted norm and weighted energy associated with (1.5):

$$w\ :=(1+|v|),\quad |f|^p_{p,\vartheta}\ :=\int_{\mathbb{R}^3}w^{p\vartheta}|f|^p\mathrm{d}v,\quad \|f\|^p_{p,\vartheta}\ :=\int_{\mathbb{T}^3\times\mathbb{R}^3}w^{p\vartheta}|f|^p\mathrm{d}x\mathrm{d}v. \tag{1.12}$$

$$\|f\|^2_{\sigma,\vartheta}\ :=\iint_{\mathbb{T}^3\times\mathbb{R}^3}w^{2\vartheta}[\sigma^{ij}\partial_if\partial_jf+\sigma^{ij}v_iv_jf^2]\mathrm{d}v\mathrm{d}x. \tag{1.13}$$

$$|f|_{\infty,\vartheta} = \sup_{\mathbb{R}^3} w^\vartheta(v) f(v), \quad \| f\|_{\infty,\vartheta} = \sup_{\mathbb{T}^3\times\mathbb{R}^3} w^\vartheta(v) f(x,v). \tag{1.14}$$

$$\begin{aligned} |f|_2 &:= |f|_{2,0}, \quad \| f\|_2 := \| f\|_{2,0}, \\ |f|_\sigma &:= |f|_{\sigma,0}, \quad \| f\|_\sigma := \| f\|_{\sigma,0}, \\ |f|_\infty &:= |f|_{\infty,0}, \quad \| f\|_\infty := \| f\|_{\infty,0}, \\ \mathcal{E}_\vartheta(f(t)) &:= \frac{1}{2}\| f(t)\|^2_{2,\vartheta} + \int_0^t \| f(s)\|^2_{\sigma,\vartheta} \mathrm{d}s. \end{aligned} \tag{1.15}$$

Here we introduce the main result.

Theorem 1.1 (Main result) *There exist* ϑ' *and* $0 < \varepsilon_0 \ll 1$ *such that for some* $\vartheta \geq \vartheta'$ *if* f_0 *satisfies*

$$\|f_0\|_{\infty,\vartheta} \leq \varepsilon_0, \quad \|f_{0t}\|_{\infty,\vartheta} + \|D_v f_0\|_{\infty,\vartheta} < \infty, \tag{1.16}$$

where $f_{0t} := -v\cdot\nabla_x f_0 + \bar{A}_{f_0} f_0$, *then there exists a unique weak solution (see* Definition 4.1*)* f *of* (1.5), (1.6) *on* $(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$ *such that*

(1) *if* $F(0) := \mu(v) + \sqrt{\mu}(v) f_0 \geq 0$, *then* $F(t) := \mu(v) + \sqrt{\mu}(v) f(t) \geq 0$ *for every* $t \geq 0$.
(2) *Moreover for any* $t > 0$, $\vartheta_0 \in \mathbb{N}$, *and* $\vartheta \geq \vartheta'$, *there exist* C, $C_{\vartheta,\vartheta_0}$, $l_0(\vartheta_0)$, *and* $0 < \alpha < 1$ *such that* f *satisfies*

$$\sup_{0\leq s\leq\infty} \mathcal{E}_\vartheta(f(s)) \leq C 2^{2\vartheta} \mathcal{E}_\vartheta(0), \tag{1.17}$$

$$\| f(t)\|_{2,\vartheta} \leq C_{\vartheta,\vartheta_0} \mathcal{E}_{\vartheta+\vartheta_0/2}(0)^{1/2} \left(1 + \frac{t}{\vartheta_0}\right)^{-\vartheta_0/2}, \tag{1.18}$$

$$\| f(t)\|_{\infty,\vartheta} \leq C_{\vartheta,\vartheta_0} (1+t)^{-\vartheta_0} \|f_0\|_{\infty,\vartheta+l_0}, \tag{1.19}$$

$$\| f\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \leq C\big(\|f_{0t}\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\big), \tag{1.20}$$

and

$$\|D_v f\|_{L^\infty((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \leq C\big(\|f_{0t}\|_{\infty,\vartheta} + \|D_v f_0\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\big). \tag{1.21}$$

Motivated by the study of global well-posedness for the Landau equation in a bounded domain with physical boundary conditions, our current study is the first step to develop an L^2 to L^∞ framework with necessary analytic tools in a simpler periodic domain. There have been many results for Landau equations in either a periodic box or whole domain [2, 3, 6–8, 13, 15–17, 22, 23, 25–27], in which high-order Sobolev norms can be employed. On the other hand, in a bounded domain, even with the velocity diffusion, the solutions can not be smooth up to the grazing set [19]. New mathematical tools involving weaker norms are needed to be developed. In the case of Boltzmann equations, an L^2 to L^∞ framework has been developed to construct unique global solutions in bounded domains [14].

Our work can be viewed as a similar L^2 to L^∞ approach for the Landau equation. Our techniques are inspired by recent remarkable progresses of [10, 11, 20], in which a general machinery in the spirit of De Giorgi, has been developed for the Fokker–Planck equations, even for the Landau equation [20], to bootstrap L^∞ and Hölder space $C^{0,\alpha}$ from a L^2 weak solution. Unfortunately, to our knowledge, there is still no construction for L^∞ global weak solutions to the Landau equation.

Our paper settles the global existence and uniqueness for an L^2 weak solution with a small weighted L^∞ perturbation of a Maxwellian initially. Our method is an intricate combination of different tools. Our starting point is a design of an iterating sequence

$$\begin{aligned}(\partial_t + v\cdot\nabla_x)f^{n+1} &= -Lf^{n+1} + \Gamma(f^n, f^{n+1})\\ &\equiv A_{f^n}(f^{n+1}) + K_{f^n}(f^{n+1}),\end{aligned}$$

where all terms in $A_{f^n}(f^{n+1})$ contain at least one momentum derivative of f^n or f^{n+1}, so that f^n appears in the coefficients of the Landau operator for f^{n+1}. The crucial lemma states that if $\| f^n\|_\infty$ is sufficiently small, the main part of $A_{f^n}(f^{n+1})$ retains the same analytical properties of the linearized Landau operator A.

We first establish global energy estimates and time decay under the assumption f^n is small, in Sect. 4.

Let us consider the linearized Landau equation with a given g:

$$\partial_t f + v\cdot\nabla_x f + Lf = \Gamma(g, f). \tag{1.22}$$

Theorem 1.2 *Suppose that* $\|g\|_\infty < \varepsilon$. *Let* $\vartheta \in 2^{-1}\mathbb{N}\cup\{0\}$ *and* f *be a classical solution of* (1.6), (1.7), *and* (1.22). *Then there exist* C *and* $\varepsilon = \varepsilon(\vartheta) > 0$ *such that*

$$\sup_{0\le s<\infty} \mathcal{E}_\vartheta(f(s)) \le C2^{2\vartheta}\mathcal{E}_\vartheta(0), \tag{1.23}$$

and

$$\| f(t)\|_{2,\vartheta} \le C_{\vartheta,k}\big(\mathcal{E}_{\vartheta+k/2}(0)\big)^{1/2}\Big(1+\frac{t}{k}\Big)^{-k/2} \tag{1.24}$$

for any $t > 0$ *and* $k \in \mathbb{N}$.

It is important to note that, thanks to the nonlinearity, the velocity weight can be arbitrarily strong. The proof of this step is a combination of energy estimates with positivity estimates for $\mathbf{P}f$ [9, 13] and a timedecay estimate [25], but in the absence of high-order Sobolev regularity.

We next bootstrap such an L^2 bound to an L^∞ bound.

Theorem 1.3 *Let f be a weak solution of* (1.6), (1.7), *and* (1.22) *in a periodic box and* $\vartheta \in \mathbb{N} \cup \{0\}$, $\vartheta_0 \in \mathbb{N}$. *Then there exist* ε, $l_0(\vartheta_0) > 0$ *and* $C_{\vartheta,\vartheta_0}$ *such that if g satisfies*

$$\sup_{0 \le s \le \infty} \|g(s)\|_\infty \le \varepsilon$$

then

$$\| f(t)\|_{\infty,\vartheta+\vartheta_0} \le C_{\vartheta,\vartheta_0}(1+t)^{-\vartheta_0}\|f_0\|_{\infty,\vartheta+l_0}. \tag{1.25}$$

It is important to note that even though there is a finite loss of velocity weight, we are still able to close the estimates thanks to the strong gain of velocity weight in (1.24). The proof of such an L^∞ estimate locally in x and v is an adaptation of recent work of [11, 20]. It is well known that the Landau operator is delicate to study and estimate for large velocities. Together with the maximum principle of the Landau operator as well as strong time decay for L^2 norm in (1.24), we are able to control the 'tails' of solutions for large velocities, and obtain global (in x and v) L^∞ estimate.

Unfortunately, unlike in the Boltzmann case (see [14]), to establish the convergence of $\{f^n\}$ and more importantly, uniqueness of our solution, such an L^∞ bound is not sufficient due to the presence of velocity derivative in the nonlinear Landau equation. We need to further control $\|\nabla_v f^n\|_\infty$ as in Lemma 8.2, which follows from S_p estimates established in [5, 24]. One crucial requirement for such S_p estimates (as the classical $W^{2,p}$ estimate in the elliptic theory), is the $C^{0,\alpha}$ estimate (uniform in x and v) for the coefficients containing f^n. We establish

Theorem 1.4 *Let f be a solution of* (1.6), (1.7), *and* (1.22). *Then there exist* ε, ϑ, C, $\alpha > 0$ *such that if g satisfies*

$$\sup_{0 \le s \le \infty} \|g(s)\|_\infty \le \varepsilon,$$

then we have

$$\| f\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C(f_0),$$

where

$$C(f_0) = C\big(\|f_{0t}\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\big). \tag{1.26}$$

Again, we follow the methods in [10, 11, 20] to establish such an estimate locally in (x, v), and use a delicate change of coordinates (6.16) locally to capture precisely the isotropic behavior of the Landau operator, thanks to Lemma 2.4 and our strong weighted L^∞ estimates to obtain uniform $C^{0,\alpha}$ estimate. It is well known that the Landau equation is degenerate for $|v| \to \infty$ and our strong energy estimate provides the control of velocity (tails) of the Landau solutions. An additional regularity condition $\|f_{0t}\|_{\infty,\theta} < +\infty$ is needed for such a Hölder estimate, but no smallness is required. A further bound $\|f_{0v}\|_{\infty,\theta} < +\infty$ is needed to apply the S_p theory in a non-divergent form.

Such an L^2 to L^∞ framework is robust and is currently being applied to the study of several other problems in the kinetic theory.

2 Basic Estimates

For the reader's convenience, we summarize and modify some basic estimates. We will adapt techniques in [13].

Proposition 2.1 *There exists a uniform constant C such that for every function f and a constant $\vartheta \in \mathbb{R}$, we have*

$$\|f\|_{2,\vartheta} \le C\|f\|_{\infty,\vartheta+2}.$$

Proof

$$\begin{aligned}\|f\|_{2,\vartheta}^2 &= \iint_{\mathbb{T}^3\times\mathbb{R}^3} (w^{\vartheta}(v)f(x,v))^2 \mathrm{d}x\mathrm{d}v \\ &= \iint_{\mathbb{T}^3\times\mathbb{R}^3} w^{-4}(v)(w^{\vartheta+2}(v)f(x,v))^2 \mathrm{d}x\mathrm{d}v \\ &\le \|f\|_{\infty,\vartheta+2} \iint_{\mathbb{T}^3\times\mathbb{R}^3} w^{-4}(v)\mathrm{d}x\mathrm{d}v \\ &\le C\|f\|_{\infty,\vartheta+2},\end{aligned}$$

for some constant $C > 0$. □

Lemma 2.2 (Lemma 2 in [13]) *Let $\vartheta > -3$, $a(v) \in C^\infty(\mathbb{R}^3\backslash\{0\})$ and $b(v) \in C^\infty(\mathbb{R}^3)$. Assume for any positive multi-index β, there is $C_\beta > 0$ such that*

$$\begin{aligned}|\partial_\beta a(v)| &\le C_\beta |v|^{\vartheta-|\beta|}, \\ |\partial_\beta b(v)| &\le C_\beta e^{-\tau_\beta |v|^2},\end{aligned}$$

with some $\tau_\beta > 0$. Then there is $C_\beta^* > 0$ such that

$$|\partial_\beta [a * b](v)| \le C_\beta^* [1 + |v|]^{\vartheta-\beta}.$$

For the reader's convenience, we will use the following notation in this paper.

$$\sigma_u^{ij}(v) := \phi^{ij} * u = \int_{\mathbb{R}^3} \phi^{ij}(v-v')u(v')\mathrm{d}v', \tag{2.1}$$

$$\sigma^{ij} = \sigma_\mu^{ij}, \quad \sigma^i = \sigma^{ij}v_j. \tag{2.2}$$

For every $v, \nu \in \mathbb{R}^3$, define

$$D_u(\nu;\nu) := \nu^T \sigma_u(v)\nu \tag{2.3}$$

and P_v is the projection onto the vector v as

$$P_v g := \sum_{j=1}^{3} \langle g_j, v_j \rangle \frac{v_i}{|v|^2}, \quad 1 \le i \le 3. \tag{2.4}$$

Lemma 2.3 (Lemma 3 in [13]) *If $u = \mu$ or $\sqrt{\mu}$, then*

$$D_u(\nu;\nu) = \lambda_1(v)|P_v \nu|^2 + \lambda_2(v)|(I - P_v)\nu|^2. \tag{2.5}$$

Moreover, there exists C such that

$$\frac{1}{C}(1 + |v|)^{-3} \le \lambda_1(v) \le C(1 + |v|)^{-3},$$

and

$$\frac{1}{C}(1 + |v|)^{-1} \le \lambda_2(v) \le C(1 + |v|)^{-1}.$$

We can derive upper and lower bounds of eigenvalues for $\sigma + \sigma_{\sqrt{\mu}g}$ by adapting ideas in the proof of Theorem 3 in [13].

Lemma 2.4 *Let g be a given function in $L^\infty((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$ and $G = \mu + \sqrt{\mu}g$. Let σ_G be the matrix defined as in (2.1). Then there exists $0 < \varepsilon \ll 1$ such that if g satisfies*

$$\sup_{0 \le s \le \infty} \|g(s)\|_\infty \le \varepsilon, \tag{2.6}$$

then

$$D_G(\nu;\nu) \ge \frac{1}{2C}((1 + |v|)^{-3}|P_v \nu|^2 + (1 + |v|)^{-1}|(I - P_v)\nu|^2),$$
$$D_G(\nu;\nu) \le 2C((1 + |v|)^{-3}|P_v \nu|^2 + (1 + |v|)^{-1}|(I - P_v)\nu|^2),$$

for every $\nu \in \mathbb{R}^3$. Thus $\sigma_G(v)$ has three non-negative eigenvalues. Moreover, $\lambda(v)$, eigenvalue of $\sigma_G(v)$, has the following estimate

$$\frac{1}{C}(1 + |v|)^{-3} \le \lambda(v) \le C(1 + |v|)^{-1},$$

for some constant $C > 0$.

Proof Let $u = \sqrt{\mu}g$. Then we claim that there exists $C' > 0$ such that

$$|D_u(v;v)| \leq C'\|g\|_\infty((1+|v|)^{-3}|P_v v|^2 + (1+|v|)^{-1}|(I-P_v)v|^2). \tag{2.7}$$

Consider

$$\begin{aligned} D_u(v;v) &= \sum_{i,j}\int_{2|v'|>|v|} v_i v_j \phi^{ij}(v-v')\sqrt{\mu}(v')g(v')\mathrm{d}v' \\ &\quad + \sum_{i,j}\int_{2|v'|\leq|v|} v_i v_j \phi^{ij}(v-v')\sqrt{\mu}(v')g(v')\mathrm{d}v' \\ &= (I) + (II). \end{aligned}$$

Note that for $2|v'| > |v|$, $\sqrt{\mu}(v') \leq C'\mu(v'/4)\mu(v/4)$. Therefore,

$$\begin{aligned} |(I)| &\leq C'\mu\left(\frac{v}{4}\right)\|g\|_\infty |v|^2 \int_{\mathbb{R}^3} \phi^{ij}(v-v')\mu\left(\frac{v'}{4}\right)\mathrm{d}v' \\ &\leq C'^{-1}\mu\left(\frac{v}{4}\right)\|g\|_\infty |v|^2. \end{aligned} \tag{2.8}$$

To control (II), we expand $\phi^{ij}(v-v')$ to get

$$\phi^{ij}(v-v') = \phi^{ij}(v) - \sum_k \partial_k\phi^{ij}(v)v'_k + \frac{1}{2}\sum_{k,l}\partial_{kl}\phi^{ij}(\bar{v})v'_k v'_l,$$

for some $\bar{v}$ in a line segment of v and $v - v'$. Then we have

$$\begin{aligned} (II) &= \sum_{i,j} v_i v_j \phi^{ij}(v) \int_{2|v'|\leq|v|} \sqrt{\mu}(v')g(v')\mathrm{d}v' \\ &\quad - \sum_{i,j} v_i v_j \sum_k \partial_k\phi^{ij}(v) \int_{2|v'|\leq|v|} v'_k\sqrt{\mu}(v')g(v')\mathrm{d}v' \\ &\quad + \frac{1}{2}\sum_{i,j}\int_{2|v'|\leq|v|} v_i v_j \sum_{k,l}\partial_{kl}\phi^{ij}(\bar{v})v'_k v'_* \sqrt{\mu}(v')g(v')\mathrm{d}v' \\ &= (II)_1 + (II)_2 + (II)_3. \end{aligned}$$

Since

$$\sum_i \phi^{ij}(v)v_i = \sum_j \phi^{ij}(v)v_j = 0,$$

we have

$$
\begin{aligned}
|(II)_1| &= \big((I-P_v)v\big)^T \phi(v)\big((I-P_v)v\big)\int_{2|v'|\le|v|}\sqrt{\mu}(v')g(v')\mathrm{d}v' \\
&\le C\|g\|_\infty(1+|v|)^{-1}|(I-P_v)v|^2.
\end{aligned}
\tag{2.9}
$$

Note that

$$
\sum_{i,j}\partial_k\phi^{ij}(v)v_iv_j = 0.
$$

Therefore

$$
\begin{aligned}
|(II)_2| &\le \left|\sum_k \big((I-P_v)v\big)^T\partial_k\phi(v)\big((I-P_v)v\big)\int_{2|v'|\le|v|} v'_k\sqrt{\mu}(v')g(v')\mathrm{d}v'\right| \\
&\quad + 2\left|\sum_k (P_vv)^T\partial_k\phi(v)\big((I-P_v)v\big)\int_{2|v'|\le|v|} v'_k\sqrt{\mu}(v')g(v')\mathrm{d}v'\right| \\
&\le C\|g\|_\infty(1+|v|)^{-2}\big(|(I-P_v)v|^2 + |P_vv||(I-P_v)v|\big) \\
&\le C\|g\|_\infty\big((1+|v|)^{-3}|P_vv|^2 + (1+|v|)^{-1}|(I-P_v)v|^2\big).
\end{aligned}
\tag{2.10}
$$

Since $\bar{v}$ is in $(v, v-v')$ and $2|v'|\le|v|$, we have $|v|/2\le|\bar{v}|\le 3|v|/2$. Therefore, $\partial_{kl}\phi^{ij}(\bar{v})\le C'|v|^{-3}$. Thus we have

$$
|(II)_3| \le C'\|g\|_\infty(1+|v|)^{-3}|v|^2. \tag{2.11}
$$

Combining (2.8)–(2.11), we have (2.7).

Now we can compute $D_G(v;v)$. Since $\varepsilon>0$ is a given small enough constant, from (2.5), (2.7), we have

$$
\begin{aligned}
D_G(v;v) &\ge \frac{1}{2C}((1+|v|)^{-3}|P_vv|^2 + (1+|v|)^{-1}|(I-P_v)v|^2), \\
D_G(v;v) &\le 2C((1+|v|)^{-3}|P_vv|^2 + (1+|v|)^{-1}|(I-P_v)v|^2).
\end{aligned}
$$

Therefore,

$$
\frac{1}{2C}(1+|v|)^{-3} \le \lambda \le 2C(1+|v|)^{-1}.
$$

$\square$

Let

$$
|f|^2_{\sigma,\vartheta} := \int_{\mathbb{R}^3} w^{2\vartheta}\big[\sigma^{ij}\partial_i f\partial_j f + \sigma^{ij}v_iv_jf^2\big]\mathrm{d}v. \tag{2.12}
$$

Lemma 2.5 (Corollary 1 in [13]) *There exists* $c = c_\vartheta > 0$, *such that*

$$|g|_{\sigma,\vartheta}^2 \geq c\{|w^\vartheta[1+|v|]^{-3/2}\{P_v\partial_i g\}|_2^2 + |w^\vartheta[1+|v|]^{-1/2}\{[I-P_v]\partial_i g\}|_2^2 + |w^\vartheta[1+|v|]^{-1/2}g|_2^2\}.$$

Define

$$\langle f,g\rangle := \int_{\mathbb{R}^3} fg\mathrm{d}v, \quad (f,g) := \int_{\mathbb{T}^3\times\mathbb{R}^3} fg\mathrm{d}x\mathrm{d}v. \tag{2.13}$$

For any real-valued function $f(v)$, we define the projection onto the span$\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$ in $L^2(\mathbb{R}^3)$ as

$$\mathbf{P}f := \Big(\mathbf{a}_f(t,x) + v\cdot\mathbf{b}_f(t,x) + \Big(|v|^2 - \frac{3}{2}\Big)\mathbf{c}_f(t,x)\Big)\sqrt{\mu}, \tag{2.14}$$

where

$$\mathbf{a}_f := \frac{\langle f,\sqrt{\mu}\rangle}{|\langle\sqrt{\mu},\sqrt{\mu}\rangle|^2},$$
$$\mathbf{b}_f^i := \frac{\langle f, v_i\sqrt{\mu}\rangle}{|\langle v_i\sqrt{\mu}, v_i\sqrt{\mu}\rangle|^2},$$
$$\mathbf{c}_f := \frac{\langle f,(|v|^2-3/2)\sqrt{\mu}\rangle}{|\langle(|v|^2-3/2)\sqrt{\mu},(|v|^2-3/2)\sqrt{\mu}\rangle|^2}.$$

Lemma 2.6 (Lemma 5 in [13]) *Let L, K, and σ^i be defined as in (1.8), (2.2), and (1.10). Let $\vartheta\in\mathbb{R}$. For any $m>1$, there is $0<C(m)<\infty$, such that*

$$|\langle w^{2\vartheta}\partial_i\sigma^i g_1, g_2\rangle| + |\langle w^{2\vartheta}Kg_1,g_2\rangle| \leq \frac{C}{m}|g_1|_{\sigma,\vartheta}|g_2|_{\sigma,\vartheta} + C(m)\left\{\int_{|v|\leq C(m)}|w^\vartheta g_1|^2\mathrm{d}v\right\}^{1/2}\left\{\int_{|v|\leq C(m)}|w^\vartheta g_2|^2\mathrm{d}v\right\}^{1/2}.$$

Moreover, there is $\delta>0$, such that

$$\langle Lg,g\rangle \geq \delta|(I-\mathbf{P})g|_\sigma^2.$$

Lemma 2.7 (Lemma 6 in [13]) *Let L, A, and K be defined as in (1.8), (1.9), and (1.10). Let $\vartheta\in\mathbb{R}$ and $|\beta|\geq 0$. For small $\delta>0$, there exists $C_\delta = C_\delta(\vartheta)>0$ such that*

$$-\langle w^{2\vartheta}Ag,g\rangle \geq |g|_{\sigma,\vartheta}^2 - \delta|g|_{\sigma,\vartheta}^2 - C_\delta|\mu g|_2^2,$$
$$|\langle w^{2\vartheta}Kg_1,g_2\rangle| \leq \left\{\delta|g_1|_{\sigma,\vartheta} + C_\delta|\mu g_1|_2\right\}|g_2|_{\sigma,\vartheta}.$$

Thus we have

$$\frac{1}{2}|g|^2_{\sigma,\vartheta} - C_\vartheta |g|^2_\sigma \le \langle w^{2\vartheta} Lg, g\rangle \le \frac{3}{2}|g|^2_{\sigma,\vartheta} + C_\vartheta |g|^2_\sigma.$$

For the nonlinear estimate in Theorem 3 in [13], they estimated

$$\left(w^{2\vartheta}\Gamma[g_1, g_2], g_3\right)$$

in terms of $\|g_i\|_{2,\vartheta}$ and $\|g_i\|_{\sigma,\vartheta}$ for $i = 1, 2, 3$ and $\vartheta \ge 0$. To get such an L^2 estimate, they need a higher order regularity like $\|D^\alpha_\beta g_i\|_{2,\vartheta}$ and $\|D^\alpha_\beta g_i\|_{\sigma,\vartheta}$ for $i = 1, 2, 3$, $\vartheta \ge 0$, $|\alpha| + |\beta| \le N$ and $N \ge 8$.

The following lemma is a refinement of Theorem 3 in [13]. First, the range of ϑ is extended to $\mathbb{R}$. Second, we estimate the nonlinear term in terms of $\|\cdot\|_\infty$, $\|\cdot\|_{2,\vartheta}$, and $\|\cdot\|_{\sigma,\vartheta}$ without a higher order regularity.

Theorem 2.8 *Let* Γ *be defined as in* (1.11).

(1) *For every* $\vartheta \in \mathbb{R}$, *there exists* C_ϑ *such that*

$$|\langle w^{2\vartheta}\Gamma[g_1, g_2], g_3\rangle| \le C_\vartheta |g_1|_\infty |g_2|_{\sigma,\vartheta} |g_3|_{\sigma,\vartheta}, \tag{2.15}$$

and

$$\left|\left(w^{2\vartheta}\Gamma[g_1, g_2], g_3\right)\right| \le C_\vartheta \|g_1\|_\infty \|g_2\|_{\sigma,\vartheta} \|g_3\|_{\sigma,\vartheta}. \tag{2.16}$$

(2) *There exists* $\bar{\vartheta} < 0$ *such that for any* $\vartheta \le \bar{\vartheta}$,

$$\left|\left(w^{2\vartheta}\Gamma[g_1, g_2], g_3\right)\right| \le C_\vartheta \min\{\|g_1\|_{2,\vartheta}, \|g_1\|_{\sigma,\vartheta}\}(\|g_2\|_\infty + \|D_v g_2\|_\infty)\|g_3\|_{\sigma,\vartheta}. \tag{2.17}$$

Proof

(1) By the integration by parts, we have

$$\begin{aligned}
|\langle w^{2\vartheta}\Gamma[g_1, g_2], g_3\rangle| \le\, & |\langle \partial_i w^{2\vartheta}\{\phi^{ij} * [\sqrt{\mu} g_1]\}\partial_j g_2, g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\partial_j \phi^{ij} * [\sqrt{\mu} g_1]\} g_2, \partial_i g_3\rangle| \\
& + |\langle \partial_i w^{2\vartheta}\{\partial_j \phi^{ij} * [\sqrt{\mu} g_1]\} g_2, g_3\rangle| \\
& + |\langle \partial_i w^{2\vartheta}\{\phi^{ij} * [v_j \sqrt{\mu} g_1]\} g_2, g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\partial_j \phi^{ij} * [v_i \sqrt{\mu} g_1]\} g_2, g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\phi^{ij} * [\partial_j\{v_i \sqrt{\mu}\} g_1]\} g_2, g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\phi^{ij} * [v_i \sqrt{\mu} g_1]\}\partial_j g_2, g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\phi^{ij} * [v_j \sqrt{\mu} g_1]\} g_2, \partial_i g_3\rangle| \\
& + |\langle w^{2\vartheta}\{\phi^{ij} * [\sqrt{\mu} g_1]\}\partial_j g_2, \partial_i g_3\rangle| \\
=\, & (I) + (II) + \cdots + (IX),
\end{aligned} \tag{2.18}$$

where ϕ is the matrix defined as in (1.2). Clearly, $|\partial_i w^{2\vartheta}| \leq C_\vartheta(1+|v|)^{-1}w^{2\vartheta}$ and by Lemma 2.2, we have

$$
\begin{aligned}
&|\phi^{ij} * [\sqrt{\mu} g_1]| + |\phi^{ij} * [v_i\sqrt{\mu} g_1]| + |\phi^{ij} * [v_j\sqrt{\mu} g_1]| + |\phi^{ij} * [\partial_j\{v_i\sqrt{\mu}\} g_1]| \\
&\qquad \leq C(1+|v|)^{-1}\|g_1\|_\infty,
\end{aligned}
$$

$$
|\partial_j\phi^{ij} * [\sqrt{\mu} g_1]| + |\partial_j\phi^{ij} * [v_i\sqrt{\mu} g_1]| \leq C(1+|v|)^{-2}\|g_1\|_\infty.
$$

Therefore, by Lemma 2.5 and the Hölder inequality,

$$
\begin{aligned}
(I) &\leq C_\vartheta |g_1|_\infty |\langle w^\vartheta (1+|v|)^{-3/2}\partial_j g_2, w^\vartheta(1+|v|)^{-1/2} g_3\rangle| \\
&\leq C_\vartheta |g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}, \\
(II) &\leq C |g_1|_\infty |\langle w^\vartheta (1+|v|)^{-1/2} g_2, w^\vartheta(1+|v|)^{-3/2} \partial_i g_3\rangle| \\
&\leq C |g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}, \\
(III)+(IV)+(V)+(VI) &\leq C_\vartheta |g_1|_\infty |\langle w^\vartheta (1+|v|)^{-1/2} g_2, w^\vartheta(1+|v|)^{-1/2} g_3\rangle| \\
&\leq C_\vartheta |g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}.
\end{aligned}
$$

By (2.7) and the Hölder inequality,

$$
\begin{aligned}
(VII) &\leq C|g_1|_\infty \int w^{2\vartheta}\big|(1+|v|)^{-3/2}|P_v\partial_j g_2| + (1+|v|)^{-1/2}|(I-P_v)\partial_j g_2|\big|(1+|v|)^{-1/2}|g_3|\mathrm{d}v \\
&\leq C|g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}, \\
(VIII) &\leq C|g_1|_\infty \int w^{2\vartheta}(1+|v|)^{-1/2}|g_2|\big|(1+|v|)^{-3/2}|P_v\partial_j g_3| + (1+|v|)^{-1/2}|(I-P_v)\partial_j g_3|\big|\mathrm{d}v \\
&\leq C|g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta},
\end{aligned}
$$

and

$$
\begin{aligned}
(IX) &\leq C|g_1|_\infty \int w^{2\vartheta}\big|(1+|v|)^{-3/2}|P_v\partial_j g_2| + (1+|v|)^{-1/2}|(I-P_v)\partial_j g_2|\big| \\
&\quad \times \big|(1+|v|)^{-3/2}|P_v\partial_j g_3| + (1+|v|)^{-1/2}|(I-P_v)\partial_j g_3|\big|\mathrm{d}v \\
&\leq C|g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}.
\end{aligned}
$$

Thus we obtain (2.15). By applying the Hölder inequality to (2.15),

$$
\begin{aligned}
\Big|\big(w^{2\vartheta}\Gamma[g_1,g_2],g_3\big)\Big| &= \int |\langle w^{2\vartheta}\Gamma[g_1,g_2],g_3\rangle|\mathrm{d}x \\
&\leq \int C_\vartheta |g_1|_\infty |g_2|_{\sigma,\vartheta}|g_3|_{\sigma,\vartheta}\mathrm{d}x \\
&\leq C_\vartheta \|g_1\|_\infty \|g_2\|_{\sigma,\vartheta}\|g_3\|_{\sigma,\vartheta}.
\end{aligned}
$$

Thus we have (2.16).

(2) By the integration by parts again, we have

$$
\begin{aligned}
|\langle w^{2\vartheta}\Gamma[g_1,g_2],g_3\rangle| :=& |\langle w^{2\vartheta}\{\phi^{ij}*[\mu^{1/2}g_1]\}\partial_j g_2,\partial_i g_3\rangle| \\
&+|\langle w^{2\vartheta}\{\phi^{ij}*[v_i\mu^{1/2}g_1]\}\partial_j g_2,g_3\rangle| \\
&+|\langle w^{2\vartheta}\{\phi^{ij}*[\mu^{1/2}\partial_j g_1]\}g_2,\partial_i g_3\rangle| \\
&+|\langle w^{2\vartheta}\{\phi^{ij}*[v_i\mu^{1/2}\partial_j g_1]\}g_2,g_3\rangle| \\
&+|\langle \partial_i w^{2\vartheta}\{\phi^{ij}*[\mu^{1/2}g_1]\}\partial_j g_2,g_3\rangle| \\
&+|\langle \partial_i w^{2\vartheta}\{\phi^{ij}*[\mu^{1/2}\partial_j g_1]\}g_2,g_3\rangle| \\
=& (i)+(ii)+\cdots+(vi).
\end{aligned}
\tag{2.19}
$$

By the Hölder inequality and the integration by parts, we have

$$
\begin{aligned}
&|\langle\{\phi^{ij}*[\mu^{1/2}g_1]\}\rangle|+|\langle\{\phi^{ij}*[v_i\mu^{1/2}g_1]\}\rangle|+|\langle\{\phi^{ij}*[\mu^{1/2}\partial_j g_1]\}\rangle|+|\langle\{\phi^{ij}*[v_i\mu^{1/2}\partial_j g_1]\}\rangle| \\
&\le C_\vartheta(1+|v|)^{-1}\min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}.
\end{aligned}
$$

Let $\bar\vartheta := -2$, then by applying the Hölder inequality to (i) and Lemma 2.5, we have

$$
\begin{aligned}
(i) &\le C_\vartheta \min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}\left|\int_{\mathbb{R}^3} w^{2\vartheta}(1+|v|)^{-1}\partial_j g_2(v)\partial_i g_3(v)\mathrm{d}v\right| \\
&\le C_\vartheta \min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}|\partial_j g_2|_\infty\left(\int_{\mathbb{R}^3}(1+|v|)^{2\vartheta+1}\mathrm{d}v\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}^3} w^{2\vartheta}(1+|v|)^{-3}|\partial_i g_3|^2\mathrm{d}v\right)^{1/2} \\
&\le C_\vartheta \min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}|D_v g_2|_\infty|g_3|_{\sigma,\vartheta}.
\end{aligned}
$$

Similarly,

$$
(ii)+(iii)+\cdots+(vi)\le C_\vartheta\min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}(|g_2|_\infty+|D_v g_2|_\infty)|g_3|_{\sigma,\vartheta}.
$$

Therefore, by the Hölder inequality again, we have

$$
\begin{aligned}
\left|\left(w^{2\vartheta}\Gamma[g_1,g_2],g_3\right)\right| &= \int|\langle w^{2\vartheta}\Gamma[g_1,g_2],g_3\rangle|\mathrm{d}x \\
&\le \int C_\vartheta\min\{|g_1|_{2,\vartheta},|g_1|_{\sigma,\vartheta}\}(|g_2|_\infty+|D_v g_2|_\infty)|g_3|_{\sigma,\vartheta}\mathrm{d}x \\
&\le C_\vartheta\min\{\|g_1\|_{2,\vartheta},\|g_1\|_{\sigma,\vartheta}\}(\|g_2\|_\infty+\|D_v g_2\|_\infty)\|g_3\|_{\sigma,\vartheta}.
\end{aligned}
$$

□

Lemma 2.9 *Let K be defined as in (1.10). Then there exists $C=C_\vartheta>0$ such that for every $N,M>0$,*

$$
\|w^\theta Kf\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}\le C\|f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)},
\tag{2.20}
$$

$$\|w^{\theta}K1_{|v|>M}f\|_{L^{\infty}(\mathbb{T}^3\times\mathbb{R}^3)} \le C(1+M)^{-1}\|f^{\vartheta}\|_{L^{\infty}(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{2.21}$$

and

$$\|w^{\theta}Kf\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} \le CN^2\|f^{\vartheta}\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} + \frac{C}{N}\|f^{\vartheta}\|_{L^{\infty}(\mathbb{T}^3\times\mathbb{R}^3)}. \tag{2.22}$$

Proof After the integration by parts, we have

$$\begin{aligned}
w^{\vartheta}Kf &= -w^{\vartheta}\mu^{-1/2}\partial_i\{\mu[\phi^{ij} * \{\mu^{1/2}[\partial_j f + v_j f]\}]\} \\
&= 2w^{\vartheta}v_i\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}[\partial_j f + v_j f]\}] - w^{\vartheta}\mu^{1/2}[\partial_i\phi^{ij} * \{\mu^{1/2}[\partial_j f + v_j f]\}] \\
&= 2w^{\vartheta}v_i\mu^{1/2}[\phi^{ij} * (v_j\mu^{1/2}f)] - 2w^{\vartheta}v_i\mu^{1/2}[\phi^{ij} * (\partial_j\mu^{1/2}f)] \\
&\quad + 2w^{\vartheta}v_i\mu^{1/2}[\partial_j\phi^{ij} * (\mu^{1/2}f)] - w^{\vartheta}\mu^{1/2}[\partial_i\phi^{ij} * (v_j\mu^{1/2}f)] \\
&\quad + w^{\vartheta}\mu^{1/2}[\partial_i\phi^{ij} * (\partial_j\mu^{1/2}f)] - w^{\vartheta}\mu^{1/2}[\partial_{ij}\phi^{ij} * (\mu^{1/2}f)] \\
&= 4w^{\vartheta}v_i\mu^{1/2}[\phi^{ij} * (v_j\mu^{1/2}f)] + 2w^{\vartheta}v_i\mu^{1/2}[\partial_j\phi^{ij} * (\mu^{1/2}f)] \\
&\quad - w^{\vartheta}2\mu^{1/2}[\partial_i\phi^{ij} * (v_j\mu^{1/2}f)] - w^{\vartheta}\mu^{1/2}[\partial_{ij}\phi^{ij} * (\mu^{1/2}f)] \\
&= 4w^{\vartheta}v_i\mu^{1/2}[\phi^{ij} * (v_jw^{-\vartheta}\mu^{1/2}f^{\vartheta})] + 2w^{\vartheta}v_i\mu^{1/2}[\partial_j\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^{\vartheta})] \\
&\quad - w^{\vartheta}2\mu^{1/2}[\partial_i\phi^{ij} * (v_jw^{-\vartheta}\mu^{1/2}f^{\vartheta})] - w^{\vartheta}\mu^{1/2}[\partial_{ij}\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^{\vartheta})] \\
&= (I) + (II) + (III) + (VI).
\end{aligned}$$

Applying Lemma 2.2 to $(I)+(II)+(III)$, we have $(I)+(II)+(III) \le C\|f^{\vartheta}\|_{L^{\infty}(\mathbb{T}^3\times\mathbb{R}^3)}$. Note that $\partial_{ij}\phi^{ij} * (\mu^{1/2}f) = -8\pi\mu^{1/2}f$. Thus we also have $(VI) \le C\|f^{\vartheta}\|_{L^{\infty}(\mathbb{T}^3\times\mathbb{R}^3)}$. Therefore, we have (2.20).

Since every convolution term of Kf contains $\mu^{1/2}$, we have (2.21).

For (2.22), clearly we have

$$\|1_{|v|\ge N}w^{\vartheta}Kf\|_2 \le \frac{C}{N}\|w^{\vartheta}f\|_{\infty}. \tag{2.23}$$

Now we will estimate $\|1_{|v|<N}w^{\vartheta}Kf\|_2$. First, consider $\|1_{|v|<N}v_iw^{\vartheta}\mu^{1/2}[\partial_j\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^{\vartheta})]\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}$.

$$\begin{aligned}
&\left\|1_{|v|<N}v_iw^{\vartheta}\mu^{1/2}\left[\partial_j\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^{\vartheta})\right]\right\|_2^2 \\
&= \int_{\mathbb{T}^3}\int_{|v|<N}\left(\int \partial_j\phi^{ij}(v-v')v_iw^{\vartheta}(v)\mu^{1/2}(v)\mu^{1/2}(v')w^{-\vartheta}(v')f^{\vartheta}(v')\mathrm{d}v'\right)^2\mathrm{d}v\mathrm{d}x \\
&\le C\Bigg(\int_{\mathbb{T}^3}\int_{|v|<N}\left(\int_{1/N<|v-v'|<2N}\mathrm{d}v'\right)^2\mathrm{d}v\mathrm{d}x + \int_{\mathbb{T}^3}\int_{|v|<N}\left(\int_{|v-v'|>2N}\mathrm{d}v'\right)^2\mathrm{d}v\mathrm{d}x \\
&\quad + \int_{\mathbb{T}^3}\int_{|v|<N}\left(\int_{|v-v'|<1/N}\mathrm{d}v'\right)^2\mathrm{d}v\mathrm{d}x\Bigg) \\
&= (i) + (ii) + (iii).
\end{aligned}$$

Since $|\partial_j \phi^{ij}(v-v')| \le C|v-v'|^{-2}$, by the Minkowski and Hölder inequalities,

$$\begin{aligned}
(i) &\le \int_{\mathbb{T}^3}\left(\int_{|v'|<3N}\left(\int_{1/N<|v-v'|<2N} |v-v'|^{-4} w^{2\vartheta}(v) v_i^2 \mu(v) w^{-2\vartheta}(v')\mu(v')(f^\vartheta)^2(v')\mathrm{d}v\right)^{1/2}\mathrm{d}v'\right)^2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3}\left(\int_{|v'|<3N} w^{-\vartheta}(v')\mu^{1/2}(v')f^\vartheta(v')\left(\int_{1/N<|v-v'|<2N}|v-v'|^{-4}w^{2\vartheta}(v)v_i^2\mu(v)\mathrm{d}v\right)^{1/2}\mathrm{d}v'\right)^2\mathrm{d}x \\
&\le C\int_{\mathbb{T}^3}\left(\int_{|v'|<3N}\mu^{1/2}(v')(f^\vartheta)^2(v')\mathrm{d}v'\right)\left(\int_{|v'|<3N}\int_{1/N<|v-v'|<2N}|v-v'|^{-4}w^{2\vartheta}(v)v_i^2\mu(v)\mathrm{d}v\mathrm{d}v'\right)\mathrm{d}x \\
&\le CN^4\|f^\vartheta\|_2^2.
\end{aligned}$$

Note that if $|v|<N$ and $|v-v'|>2N$, then $|v'|>N$. Since the integrand of (ii) contains a Maxwellian and $|v'|>N$, for every $\beta>0$ we have

$$(ii) \le \frac{C_\beta}{N^{2\beta}}\|f^\vartheta\|_\infty^2.$$

Finally,

$$(iii) \le C\|f^\vartheta\|_\infty^2 \iint_{\mathbb{T}^3\times[0,\infty)}\int_{\mathbb{R}^3}\mu(v)\left(\int_{|v-v'|<1/N}|v-v'|^{-2}\mathrm{d}v'\right)^2\mathrm{d}v\mathrm{d}x\mathrm{d}t \le C\frac{1}{N^2}\|f^\vartheta\|_\infty^2.$$

So we have

$$\|1_{|v|<N} w^\vartheta \mu^{1/2}[\partial_{ij}\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^\vartheta)]\|_2 \le CN^2\|f^\vartheta\|_2 + \frac{C}{N}\|f^\vartheta\|_\infty. \tag{2.24}$$

In a similar manner,

$$\|1_{|v|<N} w^\vartheta \mu^{1/2}[\partial_i\phi^{ij} * (v_j w^{-\vartheta}\mu^{1/2}f^\vartheta)]\|_2 \le CN^2\|f^\vartheta\|_2 + \frac{C}{N}\|f^\vartheta\|_\infty \tag{2.25}$$

and

$$\|1_{|v|<N} v_i w^\vartheta \mu^{1/2}[\phi^{ij} * (v_j w^{-\vartheta}\mu^{1/2}f^\vartheta)]\|_2 \le CN\|f^\vartheta\|_2 + \frac{C}{N^2}\|f^\vartheta\|_\infty. \tag{2.26}$$

Note that

$$\begin{aligned}
w^\vartheta\mu^{1/2}[\partial_{ij}\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^\vartheta)] &= w^\vartheta\mu^{1/2}[-8\pi w^{-\vartheta}\mu^{1/2}f^\vartheta] \\
&= -8\pi\mu f^\vartheta.
\end{aligned}$$

Thus,

$$\|1_{|v|<N} w^\vartheta \mu^{1/2}[\partial_{ij}\phi^{ij} * (w^{-\vartheta}\mu^{1/2}f^\vartheta)]\|_2 \le C\|f^\vartheta\|_2. \tag{2.27}$$

From (2.23)–(2.27), we have (2.22).

So the proof is complete. □

3 Maximum Principle

To get L^∞ estimates, we rearrange (1.5) as follows:

$$f_t + v \cdot \nabla_x f = \bar{A}_f f + \bar{K}_f f, \tag{3.1}$$

where

$$\begin{aligned} \bar{A}_g f &:= \partial_i[\{\phi^{ij} * [\mu + \mu^{1/2} g]\}\partial_j f] \\ &\quad -\{\phi^{ij} * [v_i \mu^{1/2} g]\}\partial_j f - \{\phi^{ij} * [\mu^{1/2}\partial_j g]\}\partial_i f \\ &=: \nabla_v \cdot (\sigma_G \nabla_v f) + a_g \cdot \nabla_v f, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \bar{K}_g f &:= Kf + \partial_i \sigma^i f - \sigma^{ij} v_i v_j f \\ &\quad - \partial_i\{\phi^{ij} * [\mu^{1/2}\partial_j g]\} f + \{\phi^{ij} * [v_i \mu^{1/2} \partial_j g]\} f. \end{aligned} \tag{3.3}$$

If f satisfies (1.22), then for every $\vartheta \in \mathbb{R}$, $f^\vartheta := w^\vartheta f$ satisfies

$$\begin{aligned} &\partial_t f^\vartheta + v \cdot \nabla_x f^\vartheta = \bar{A}_g^\vartheta f^\vartheta + \bar{K}_g^\vartheta f, \\ &f^\vartheta(0) = w^\vartheta f_0 =: f_0^\vartheta, \end{aligned}$$

where

$$\bar{K}_g^\vartheta f = w^\vartheta \bar{K}_g f + \left(2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) f^\vartheta, \tag{3.4}$$

$$\bar{A}_g^\vartheta := \bar{A}_g - 2\frac{\partial_i w^\vartheta}{w^\vartheta}\sigma_G^{ij}\partial_j. \tag{3.5}$$

Note that $\bar{A}_g^0 = \bar{A}_g$ and $f^0 = f$. In this section, we first define a weak solution for

$$h_t + v \cdot \nabla_x h = \bar{A}_g^\vartheta h \tag{3.6}$$

and obtain the well-posedness and the maximum principle of the weak solution for (3.6). Due to the lack of regularity, we cannot use a direct contradiction argument for the weak solution as in the case of strong solutions. Therefore, we first construct a smooth approximated solution and then pass to the limit to obtain the maximum principle for the weak solution.

Remark 3.1 If $g \in C_v^1$, $f \in C_v^2$, and $\varphi \in C_v^2$ with a compact support in $\mathbb{R}^3$, then

$$\begin{aligned} \int_{\mathbb{R}^3} \left(\bar{A}_g(f) + \bar{K}_g(f)\right)\varphi = -\langle f, \varphi\rangle_\sigma + \int_{\mathbb{R}^3} \big[&-\sigma_{\mu^{1/2}g}\partial_i f \partial_j \varphi - (a_g \cdot \nabla_v \varphi) f \\ &+ \left(K\varphi + \partial_i \sigma^i \varphi - \partial_i\left\{\phi^{ij} * [\mu^{1/2}\partial_j g]\right\}\varphi + \left\{\phi^{ij} * [v_i \mu^{1/2}\partial_j g]\right\}\varphi\right) f \big] \mathrm{d}v. \end{aligned}$$

Definition 3.2 Let $h(t,x,v) \in L^\infty((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3, w^\vartheta(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v)$ be a periodic function in $x \in \mathbb{T}^3 = [-\pi,\pi]^3$ satisfying

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\sigma^{ij}\partial_i h \partial_j h\right)(s,x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}s < \infty,$$

where σ is defined as in (2.2). We say that h is a weak solution of (3.6), with $h(0) = h_0$ on $(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$ if for all $t \in (0,\infty)$ and all $\varphi \in C^{1,1,1}_{t,x,v}\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)$ such that $\varphi(t,x,v)$ is a periodic function in $x \in \mathbb{T}^3 = [-\pi,\pi]^3$ and $\varphi(t,x,\cdot)$ is compactly supported in $\mathbb{R}^3$, it satisfies

$$\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} h(t,x,v)\varphi(t,x,v)\mathrm{d}x\mathrm{d}v - \iint_{\mathbb{T}^3\times\mathbb{R}^3} h_0(x,v)\varphi(0,x,v)\mathrm{d}x\mathrm{d}v \\
&= \iiint_{(0,t)\times\mathbb{T}^3\times\mathbb{R}^3} \Bigg[h(s,x,v)\left(\partial_s\varphi + v\cdot\nabla_x\varphi - \left(a_g + 2\frac{\nabla w^\vartheta}{w^\vartheta}\sigma_G\right)\cdot\nabla_v\varphi\right)(s,x,v) \\
&\quad - \nabla_v h(s,x,v)\cdot(\sigma_G\nabla_v\varphi)(s,x,v)\Bigg]\mathrm{d}s\mathrm{d}x\mathrm{d}v,
\end{aligned} \tag{3.7}$$

where σ_G is defined as in (2.1) with $G = \mu + \mu^{1/2}g$.

Lemma 3.3 *Assume* (2.6). *Let* σ_G *be the matrix defined as in* (2.1) *with* $G = \mu + \mu^{1/2}g$. *Let* $\vartheta \in \mathbb{N}\cup\{0\}$, $\delta \geq 0$, *and* h *be a classical solution of* (3.6). *Then there exist* $C = C(\vartheta)$, $0 < \varepsilon \ll 1$ *such that if* $\|g\|_\infty < \varepsilon$, *then*

$$\sup_{0\le s\le t}\|h(s)\|^2_{L^2} + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\sigma^{ij}\partial_i h\partial_j h\right)(s,x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}s \le C(\vartheta)\|h(0)\|^2_{L^2}. \tag{3.8}$$

Proof Multiplying (3.6) by h and integrating both sides of the resulting equation, we have

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{1}{2}\left(h^2(t,x,v) - h^2(0,x,v)\right)\mathrm{d}x\mathrm{d}v = \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} (\bar{A}^\vartheta_g h(s,x,v))h(s,x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}s. \tag{3.9}$$

By Lemma 2.4, we have

$$C^{-1}\sigma^{ij}\partial_i h\partial_j h \le \sigma^{ij}_G\partial_i h\partial_j h \le C\sigma^{ij}\partial_i h\partial_j h, \tag{3.10}$$

for some C. By Lemma 2.4 and the Young inequality, we have

$$\begin{aligned}
\left|\frac{\partial_i w^\vartheta}{w^\vartheta}\sigma^{ij}_G(\partial_j h)h\right| &\le C(1+|v|)^{-1}(\sigma^{ij}_G\partial_i h\partial_j h)^{1/2}(\sigma^{ij}_G h^2)^{1/2} \\
&\le \varepsilon\sigma^{ij}_G\partial_i h\partial_j h + C_\varepsilon(1+|v|)^{-1}\sigma^{ij}_G h^2 \\
&\le \varepsilon\sigma^{ij}\partial_i h\partial_j h + C_\varepsilon h^2.
\end{aligned} \tag{3.11}$$

In a similar manner, by (2.7) and the Young inequality, we have

$$\begin{aligned} |\{\phi * [v_i \mu^{1/2} g]\}(\partial_i h)h| &\le C\|g\|_\infty D_\mu(\nabla_v h; v)^{1/2} (\sigma^{ij} h^2)^{1/2} \\ &\le \varepsilon \sigma^{ij} \partial_i h \partial_j h + \varepsilon h^2 \end{aligned} \tag{3.12}$$

and

$$|\{\phi^{ij} * [\mu^{1/2} \partial_j g]\} \partial_i h h| \le \varepsilon \sigma^{ij} \partial_i h \partial_j h + \varepsilon h^2. \tag{3.13}$$

Thus from (3.9)–(3.13), we have

$$\begin{aligned} &\iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(t,x,v) \mathrm{d}x \mathrm{d}v + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(\sigma^{ij} \partial_i h \partial_j h\right)(s,x,v) \mathrm{d}x \mathrm{d}v \mathrm{d}s \\ &\le \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(0,x,v) \mathrm{d}x \mathrm{d}v + \varepsilon \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(\sigma^{ij} \partial_i h \partial_j h\right)(s,x,v) \mathrm{d}x \mathrm{d}v \mathrm{d}s \\ &\quad + C_\varepsilon \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(s,x,v) \mathrm{d}x \mathrm{d}v \mathrm{d}s. \end{aligned}$$

Absorbing the second term of the RHS to the LHS and applying the Gronwall inequality to the resulting equation, we have (3.8). □

Lemma 3.4 *Assume* (2.6). *Then there exists a unique weak solution to* (3.6) *which satisfies* (3.8).

Proof We approximate g by $g^\delta \in C^\infty$ and h_0 by $h_0^\delta \in C^\infty$ such that $\|g^\delta\|_\infty \le \|g\|_\infty$, $\|h_0^\delta\|_\infty \le \|h_0\|_\infty$ and

$$\lim \|g^\delta - g\|_\infty = 0, \quad \lim \|h_0^\delta - h_0\|_1 = 0.$$

Consider

$$\begin{aligned} &\partial_t h^\delta + v \cdot \nabla_x h^\delta = \bar{A}^{\vartheta}_{g^\delta} h^\delta, \\ &h^\delta(0,x,v) = h_0^\delta(x,v). \end{aligned} \tag{3.14}$$

By Lemma 2.4, $\sigma_G \ge 0$. Since $\sigma_G \ge 0$, it is rather standard (for instance, by adding regularization $\varepsilon(\nabla_{x,v})^{2m}$, for some large integer m, then letting $\varepsilon \to 0$, if necessary) that there exists a solution h^δ to the linear equation (3.14). Since g^δ and h_0^δ are smooth, we can derive a similar energy estimate for the derivatives of h^δ by taking derivatives of the above equation and multiplying by the derivatives of h^δ and integrating both sides of the resulting equation as in [13]. For more details, see [13]. Therefore, h^δ is smooth.

By (3.8), $\|h^\delta(s)\|_{L^2}^2$ is uniformly bounded on $0 \le s \le t$. Therefore, there exists h such that $h^\delta \to h$ weakly in L^2. Multiplying (3.14) by a test function φ, integrating both sides of the resulting equation, and taking the integration by parts, we have

$$
\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} h^\delta(t,x,v)\varphi(t,x,v)\mathrm{d}x\mathrm{d}v - \iint_{\mathbb{T}^3\times\mathbb{R}^3} h_0^\delta(x,v)\varphi(0,x,v)\mathrm{d}x\mathrm{d}v \\
&= \iiint_{(0,t)\times\mathbb{T}^3\times\mathbb{R}^3} h^\delta(s,x,v)\Big(\partial_s\varphi(s,x,v) + v\cdot\nabla_x\varphi(s,x,v) - \Big(a_{g^\delta} + 2\frac{\nabla w^\vartheta}{w^\vartheta}\sigma_{G^\delta}\Big)\cdot\nabla_v\varphi \\
&\quad + \nabla_v\cdot(\sigma_{G^\delta}\nabla_v\varphi)\Big)\mathrm{d}s\mathrm{d}x\mathrm{d}v,
\end{aligned}
\tag{3.15}
$$

where $G^\delta = \mu + \sqrt{\mu}g^\delta$. Since $h^\delta \to h$ weakly in L^2, taking $\delta \to 0$ in (3.15) we have (3.7). Therefore h is a weak solution of (3.6). The second assertion is an analogue of Lemma 3.3. Let h and $\tilde{h}$ be weak solutions to (3.6). Then $h-\tilde{h}$ is also a weak solution to (3.6) with zero initial data. Therefore, we have $\sup_{0\le s\le t}\|(h-\tilde{h})(s)\|_{L^2}^2 = 0$. Thus we obtain the uniqueness. □

Before we derive the maximum principle for weak solutions, we establish the maximum principle for strong solutions. We first derive the maximum principle for strong solutions in bounded domains. The following technique is similar to that in [18] and [19]. Let

$$
\mathcal{M}_g^\vartheta h := (\partial_t + v\cdot\nabla_x - \bar{A}_g^\vartheta)h.
$$

Lemma 3.5 *Assume* (2.6). *Let* $h\in C^{1,1,2}_{t,x,v}\big([0,T]\times\mathbb{T}^3\times B(0;M)\big)$ *be a periodic function satisfying* $\mathcal{M}_g^\vartheta h \le 0$. *Then* h *attains its maximum only at* $t=0$ *or* $|v|=M$.

Proof Let us assume that $\max_{(t,x,v)\in[0,T]\times\mathbb{T}^3\times B(0;M)} h(t,x,v) > 0$ and $\mathcal{M}_g^\vartheta h < 0$. Suppose that h attains its maximum at an interior point $(t,x,v)\in[0,T]\times\mathbb{T}^3\times B(0;M)$ or at (T,x,v) with (x,v) lying in the interior. Since $\sigma_G \ge 0$ by Lemma 2.4, we have $\partial_t h \ge 0$, $\nabla_x h = 0$, and $\nabla_v h = 0$ while $\sigma_G^{ij}\partial_{ij}h \le 0$ and $h(t,x,v) > 0$. Thus $\mathcal{M}_g^\vartheta h(t,x,v) \ge 0$ and this gives a contradiction.

In the case of $\mathcal{M}_g^\vartheta h \le 0$, define $h^k := h - kt$ for $k>0$, then $\mathcal{M}_g^\vartheta h^k < 0$. Thus we have

$$
\sup_{(t,x,v)\in[0,T]\times\mathbb{T}^3\times B(0;M)} h^k(t,x,v) = \sup_{t=0 \text{ or } |v|=M} h^k(t,x,v).
$$

Taking $k\to 0$, we complete the proof. □

Lemma 3.6 *Assume* (2.6). *There exists* $\varphi\in C^{1,1,2}\big([0,T]\times\mathbb{T}^3\times\mathbb{R}^3\big)$ *with* $\varphi\ge 0$, *which satisfies* $\mathcal{M}_g^\vartheta\varphi \ge 0$ *and* $\varphi\to\infty$ *as* $|v|\to\infty$ *uniformly in* $t\in[0,T]$.

Proof Define

$$
\varphi(t,x,v) := \varphi(t,v) = \alpha_1(t) + \alpha_2(t)|v|^2. \tag{3.16}
$$

Then

$$\begin{aligned}\mathcal{M}_g^{\vartheta}\varphi &= \alpha_1'(t) + \alpha_2'(t)|v|^2 - 2\alpha_2(t)\nabla_v \cdot (\sigma_G v)\\ &\quad - 2\alpha_2(t)a_g \cdot v - 2\alpha_2(t)\frac{\partial_i w^{\vartheta}}{w^{\vartheta}}\sigma^{ij}v_j\\ &= \alpha_1'(t) + \alpha_2'(t)|v|^2 - 2\alpha_2(t)\partial_i\sigma_G^{ij}v_j - 2\alpha_2(t)\sigma_G^{ii}\\ &\quad - 2\alpha_2(t)a_g \cdot v - 2\alpha_2(t)\frac{\partial_i w^{\vartheta}}{w^{\vartheta}}\sigma^{ij}v_j.\end{aligned}$$

Note that

$$\begin{aligned}&|\partial_i\sigma_G^{ij}| \le C\|g\|_\infty(1+|v|)^{-2}, \quad |\sigma_G^{ii}| \le C\|g\|_\infty(1+|v|)^{-1},\\ &|a_g| \le C\|g\|_\infty(1+|v|)^{-1}, \text{ and } \frac{\partial_i w^{\vartheta}}{w^{\vartheta}}\sigma^{ij}v_j \le C\|g\|_\infty(1+|v|)^{-3}.\end{aligned}$$

Choose $\alpha_1(t) = \alpha_2(t) := \exp(kt)$. Then $\varphi \ge 0$. Moreover,

$$\mathcal{M}_g^{\vartheta}\varphi \ge e^{kt}\big(k(1+|v|)^2 - C(1+|v|)^{-1} - C(1+|v|)^{-1}(1+|v|)\big) \ge 0$$

for a sufficiently large k. □

Lemma 3.7 *Assume* (2.6). *Let $h \in C^{1,1,2}_{t,x,v}\big([0,T]\times\mathbb{T}^3\times\mathbb{R}^3\big)$ be a periodic function satisfying $\mathcal{M}_g^{\vartheta}h \le 0$. Then h attains its maximum only at $t=0$.*

Proof Fix $\lambda > 0$. Let φ be a barrier function obtained in Lemma 3.6. Define $\eta^\lambda(t,x,v) := h(t,x,v) - \lambda\varphi(t,x,v)$, then $\mathcal{M}_g^{\vartheta}\eta^\lambda \le 0$. Thus we can apply Lemma 3.5 on the domain $[0,T]\times\mathbb{T}^3\times B(0;M)$. Then we have

$$\eta^\lambda(t,x,v) \le \sup_{t=0 \text{ or } |v|=M}\eta^\lambda(t,x,v), \quad \text{for } (t,x,v) \in [0,T]\times\mathbb{T}^3\times B(0;M).$$

Note that

$$\eta^\lambda(0,x,v) = h(0,x,v) - \lambda\varphi(0,x,v) \le h(0,x,v) \le \sup_{x,v} h(0,x,v).$$

For a sufficiently large M, we have

$$\eta^\lambda(t,x,v) = h(t,x,v) - \lambda\varphi(t,x,v) = h(t,x,v) - \lambda(\alpha_1(t) + \alpha_2(t)M^2) \le \sup_{x,v} h(0,x,v)$$

for $|v| = M$. Thus

$$\eta^\lambda(t,x,v) \le \sup_{x,v} h(0,x,v), \quad \text{for } (t,x,v) \in [0,T]\times\mathbb{T}^3\times B(0;M).$$

Since M is an arbitrary large enough constant, we can take $M \to \infty$. Then we have

$$\eta^\lambda(t,x,v) \le \sup_{x,v} h(0,x,v), \quad \text{for } (t,x,v) \in [0,T]\times\mathbb{T}^3\times\mathbb{R}^3.$$

Taking $\lambda \to 0$, we have

$$h(t,x,v) \leq \sup_{x,v} h(0,x,v).$$

Thus we complete the proof. □

Now we will derive the maximum principle for weak solutions.

Lemma 3.8 *Assume* (2.6) *and* $g \in C^0$ *and* $\|h_0\|_\infty < \infty$. *Then the weak solution to* (3.6) *satisfies*

$$\sup_t \|h(t)\|_\infty \leq \|h_0\|_\infty. \tag{3.17}$$

Proof Approximating g by $g^\delta \in C^\infty$ and h_0 by $h_0^\delta \in C^\infty$ as in Lemma 3.4, we can obtain a smooth solution h^δ to (3.6). Thus by Lemma 3.7, we have

$$\sup_t \|h^\delta(t)\|_\infty \leq \|h_0^\delta\|_\infty \leq \|h_0\|_\infty.$$

In a similar manner to Lemma 3.3 we can derive an energy estimate for $h^\delta - h^{\delta'}$ and we can show that h^δ is a Cauchy sequence in L^2. Therefore there exists h such that $\|h^\delta - h\|_{L^2} \to 0$. In a similar manner to Lemma 3.4, we can show that h is a weak solution. Since $\sup_t \|h^\delta(t)\|_\infty \leq \|h_0\|_\infty$ and $\|h^\delta - h\|_{L^2} \to 0$, we can obtain (3.17). □

4 L^2 Decay

In this section, we will establish a weighted L^2 estimate for (1.22). We will adapt techniques in [9, 13, 25].

As a starting point, we prove that (1.22) has a unique weak solution globally in time. Define the inner product associated with weighted norm (2.12) and (1.13):

$$\langle f, g\rangle_\sigma := \int_{\mathbb{R}^3} [\sigma^{ij}\partial_i f \partial_j g + \sigma^{ij} v_i v_j f g]\mathrm{d}v, \tag{4.1}$$

$$(f, g)_\sigma := \iint_{\mathbb{T}^3\times\mathbb{R}^3} [\sigma^{ij}\partial_i f \partial_j g + \sigma^{ij} v_i v_j f g]\mathrm{d}v\mathrm{d}x. \tag{4.2}$$

Definition 4.1 Let $f(t,x,v) \in L^\infty((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3, w^\vartheta(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v)$ be a periodic function in $x \in \mathbb{T}^3 = [-\pi,\pi]^3$ satisfying

$$\int_0^t \|f(s)\|_{\sigma,\vartheta}^2 \mathrm{d}s < \infty. \tag{4.3}$$

We say that f is a weak solution of the Landau Eqs. (1.6), (1.22) on $(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$ if for all $t \in (0,\infty)$ and all $\varphi \in C^{1,1,1}_{t,x,v}\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)$ such that $\varphi(t,x,v)$ is a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ and $\varphi(t, x, \cdot)$ is compactly supported in $\mathbb{R}^3$, it satisfies

$$\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\varphi(t,x,v)\mathrm{d}x\mathrm{d}v - \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)\varphi(0,x,v)\mathrm{d}x\mathrm{d}v \\
&\quad = -(f,\varphi)_\sigma + \iiint_{(0,t)\times\mathbb{T}^3\times\mathbb{R}^3} [f(s,x,v)(\partial_s\varphi(s,x,v) + v\cdot\nabla_x\varphi(s,x,v) - a_g(s,x,v)\cdot\nabla_v\varphi(s,x,v) \\
&\qquad + K\varphi(s,x,v) + \partial_i\sigma^i(s,x,v)\varphi(s,x,v) - \partial_i\{\phi^{ij} * [\mu^{1/2}\partial_j g]\}(s,x,v)\varphi(s,x,v) \\
&\qquad + \{\phi^{ij} * [v_i\mu^{1/2}\partial_j g]\}(s,x,v)\varphi(s,x,v)) - \sigma_{\mu^{1/2}g}(s,x,v)\partial_i f(s,x,v)\partial_j\varphi(s,x,v)]\mathrm{d}s\mathrm{d}x\mathrm{d}v.
\end{aligned} \tag{4.4}$$

Moreover if $g = f$, then we say that f is a weak solution of the Landau equation (1.5), (1.6) on $(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$.

Let $f(t) = U(t,s)f_0$ be a solution of the following equation

$$\begin{aligned}
&f_t + v\cdot\partial_x f = \bar{A}_g f, \\
&f(s) = U(s,s)f_0 = f_0,
\end{aligned} \tag{4.5}$$

where $\bar{A}_g$ is defined as in (3.2). Then by the Duhamel principle, the solution of (1.22) is

$$f(t) = U(t,0)f_0 + \int_0^t U(t,\tau)\bar{K}_g f(\tau)\mathrm{d}\tau. \tag{4.6}$$

Lemma 4.2 *Assume* (2.6). *Then there exists a unique weak solution f to* (1.22) *in the sense of* Definition 4.1 *with* $f(0) = f_0$, *which satisfies*

$$\sup_{0\le s\le t} \| f(s)\|_\infty \le C(t)\|f_0\|_\infty.$$

Sketch of proof It is clear from (4.6) and by the Gronwall inequality. □

Let $\mathbf{P}$ be the projection onto the span$\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$ in $L^2(\mathbb{R}^3)$ and $\mathbf{a}_f$, $\mathbf{b}_f$, and $\mathbf{c}_f$ are coefficient functions as we defined in (2.14). We will prove the positivity of L. By Lemma 2.6, L is only semi-positive;

$$(Lf, f) \ge C\|(I-\mathbf{P})f\|_\sigma^2.$$

Now we will estimate $\mathbf{P}f$ in terms of $(I-\mathbf{P})f$. The following lemma is an adaptation of Lemma 6.1 in [9].

Lemma 4.3 (Lemma 6.1 in [9]) *Assume* (2.6). *Let f be a weak solution of* (1.6), (1.7), (1.22). *Then there exist C and a function* $\eta(t) \le C\| f(t)\|_2^2$, *such that*

$$\int_s^t \|\mathbf{P}f(\tau)\|_\sigma^2 \le \eta(t) - \eta(s) + C\int_s^t \|(I-\mathbf{P})f(\tau)\|_\sigma^2.$$

Proof For every periodic test function ψ, f satisfies

$$\begin{aligned}
&-\int_s^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v\cdot\nabla_x\psi f-\int_s^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \partial_t\psi f\\
&\quad=-\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(t)+\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(s)+\int_s^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\big[-\psi L(I-\mathbf{P})f+\psi\Gamma(g,f)\big].
\end{aligned}\tag{4.7}$$

By convention, we denote $a(t,x)=\mathbf{a}_f(t,x)$, $b(t,x)=\mathbf{b}_f(t,x)$, and $c(t,x)=\mathbf{c}_f(t,x)$, where $\mathbf{a}_f$, $\mathbf{b}_f$, and $\mathbf{c}_f$ are defined as in (2.14). We note that, with such choices $\eta(t)=-\int\int_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(t)\mathrm{d}x\mathrm{d}v$, and $\psi=p(v)\phi(t,x)$ for some $|p(v)|\le\exp(-|v|^2/4)$ and $\|\phi(t)\|_2\le C(\|a(t)\|_2+\|b(t)\|_2+\|c(t)\|_2)$. Thus,

$$\begin{aligned}
|\eta(t)|&\le\|f(t)\|_2\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}p(v)|\phi(t,x)|^2\mathrm{d}x\mathrm{d}v\right)^{1/2}\\
&=C\|f(t)\|_2\left(\int_{\mathbb{T}^3}|\phi(t,x)|^2\mathrm{d}x\right)^{1/2}\\
&\le C\|f(t)\|_2\|\phi(t)\|_2\\
&\le C\|f(t)\|_2^2.
\end{aligned}$$

Without loss of generality, we can take $s=0$.

Step 1. Estimate of $\nabla_x\Delta^{-1}\partial_t a=\nabla_x\partial_t\phi_a$. Choosing a test function $\psi=\phi\sqrt{\mu}$ with ϕ dependent only on x, we have (note that $\int\sqrt{\mu}Lf=\int\sqrt{\mu}\Gamma(g,f)=0$)

$$\sqrt{2\pi}\int_{\mathbb{T}^3}[a(t+\varepsilon)-a(t)]\phi(x)=2\pi\sqrt{2\pi}\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}(b\cdot\nabla_x)\phi(x).$$

Therefore,

$$\int_{\mathbb{T}^3}\phi\partial_t a=\sqrt{2\pi}\int_{\mathbb{T}^3}(b\cdot\nabla_x)\phi.$$

First, take $\phi=1$. Then, we have $\int_{\mathbb{T}^3}\partial_t a(t)\mathrm{d}x=0$ for all $t>0$. On the other hand, for all $\phi(x)\in H^1(\mathbb{T}^3)$, we have

$$\left|\int_{\mathbb{T}^3}\phi(x)\partial_t a\mathrm{d}x\right|\lesssim\|b\|_2\|\phi\|_{H^1}.$$

Therefore, for all $t>0$, $\|\partial_t a(t)\|_{(H^1)^*}\lesssim\|b(t)\|_2$. Since $\int_{\mathbb{T}^3}\partial_t a\mathrm{d}x=0$ for all $t>0$, we can find a solution of the Poisson equation with the periodic boundary condition $-\Delta\Phi_a=\partial_t a(t)$. Let ϕ_a be a solution of the Poisson equation with the periodic boundary condition $-\Delta\phi_a=a(t)$. Then $\Phi_a=\partial_t\phi_a$. Moreover, we have

$$\|\nabla_x\partial_t\phi_a\|_2=\|\Phi_a\|_{H^1}\lesssim\|\partial_t a(t)\|_{(H^1)^*}\lesssim\|b(t)\|_2.\tag{4.8}$$

Step 2. Estimate of $\nabla_x\Delta^{-1}\partial_t b^j=\nabla_x\partial_t\phi_b^j$. Choosing a test function $\psi=\phi(x)v_i\sqrt{\mu}$, we have

$$
\begin{aligned}
&\frac{3}{2}\pi\sqrt{\pi}\int_{\mathbb{T}^3}[b_i(t+\varepsilon)-b_i(t)]\phi \\
&\quad=\frac{3}{2}\pi\sqrt{\pi}\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\partial_i\phi[a+c]+\int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\sum_{j=1}^d v_jv_i\sqrt{\mu}\partial_j\phi(I-\mathbf{P})f \\
&\qquad+\int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi v_i\Gamma(g,f)\sqrt{\mu}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{T}^3}\partial_t b_i(t)\phi=\int_{\mathbb{T}^3}\partial_i\phi[a(t)+c(t)]+\frac{2}{3\pi\sqrt{\pi}}\Bigg\{&\iint_{\mathbb{T}^3\times\mathbb{R}^3}\sum_{j=1}^d v_iv_j\sqrt{\mu}\partial_j\phi(I-\mathbf{P})f(t) \\
&+\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi v_i\Gamma(f,f)(t)\sqrt{\mu}\Bigg\}.
\end{aligned}
$$

By the Hölder inequality and Theorem 2.8,

$$
\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{R}^3}\sum_{j=1}^d v_iv_j\sqrt{\mu}\partial_j\phi(I-\mathbf{P})f(t) \\
&\quad\le C\|(1+|v|)^{-1/2}(I-\mathbf{P})f(t)\|_2^2\left\|(1+|v|)^{-\frac{-1}{2}}\sum_{j=1}^d v_iv_j\sqrt{\mu}\partial_j\phi\right\|_2 \\
&\quad\le C\|(I-\mathbf{P})f(t)\|_\sigma\|\nabla_x\phi\|_2
\end{aligned}
$$

and

$$
\left\|\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi v_i\Gamma(g,f)(t)\sqrt{\mu}\right\|\le C\|g\|_\infty\|f\|_\sigma\|\phi v_i\mu\|_\sigma\le\|g\|_\infty\|f\|_\sigma\|\phi\|_2.
$$

For fixed $t>0$, we choose $\phi=\Phi_b^i$, where Φ_b^i is a solution of the Poisson equation with the periodic boundary condition $-\Delta\Phi_b^i=\partial_t b_i(t)$. Let ϕ_b^i be a solution of the Poisson equation with the periodic boundary condition $-\Delta\phi_b^i=b_i(t)$. Then $\Phi_b^i=\partial_t\phi_b^i$. By the Poincaré inequality,

$$
\begin{aligned}
\int_{\mathbb{T}^3}|\nabla_x\partial_t\phi_b^i(t)|^2\mathrm{d}x=\int_{\mathbb{T}^3}|\nabla_x\Phi_b^i|^2\mathrm{d}x&=-\int_{T^3}\Delta_x\Phi_b^i\Phi_b^i\mathrm{d}x \\
&\le\varepsilon\{\|\nabla_x\Phi_b^i\|_2^2+\|\Phi_b^i\|_2^2\} \\
&\quad+C_\varepsilon\{\|a(t)\|_2^2+\|c(t)\|_2^2+\|(I-\mathbf{P})f(t)\|_\sigma^2+\|g(t)\|_\infty^2\|f(t)\|_\sigma^2\} \\
&\le C\varepsilon\|\nabla_x\Phi_b^i\|_2^2+C_\varepsilon\{\|a(t)\|_2^2+\|c(t)\|_2^2 \\
&\quad+\|(I-\mathbf{P})f(t)\|_\sigma^2+\|g(t)\|_\infty^2\|f(t)\|_\sigma^2\},
\end{aligned}
$$

for every $\varepsilon>0$. Now, we choose small ε, such that $C\varepsilon\le 1/4$. Then we can absorb the first term in RHS to the LHS. Then we have for all $t>0$,

$$\|\nabla_x\partial_t\phi_b^i(t)\|_2 \le C_\varepsilon\{\|a(t)\|_2 + \|c(t)\|_2 + \|(I-\mathbf{P})f(t)\|_\sigma + \|g(t)\|_\infty\|f(t)\|_\sigma\}. \tag{4.9}$$

Step 3. Estimate of $\nabla_x\Delta^{-1}\partial_t\phi_c = \nabla_x\partial_t\phi_c$. Choosing a test function $\psi = \phi(x)\left(|v|^2 - \frac{3}{2}\right)\sqrt{\mu}$, we have

$$\begin{aligned}
&\frac{3}{2}\pi\sqrt{\pi}\int_{\mathbb{T}^3}\phi(x)[c(t+\varepsilon)-c(t)]\\
&\quad= \frac{3}{2}\pi\sqrt{\pi}\int_t^{t+\varepsilon}\int_{\mathbb{T}^3} b\cdot\nabla_x\phi - \int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}(I-\mathbf{P})f\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}(v\cdot\nabla_x)\phi\\
&\qquad+\int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi\Gamma(g,f)\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{T}^3}\phi(x)\partial_t c(t)\\
&\quad= \int_{\mathbb{T}^3} b(t)\cdot\nabla_x\phi + \frac{2}{3\pi\sqrt{\pi}}\iint_{\mathbb{T}^3\times\mathbb{R}^3}(I-\mathbf{P})f(t)\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}(v\cdot\nabla_x)\phi\\
&\qquad+\frac{2}{3\pi\sqrt{\pi}}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi\Gamma(g,f)(t)\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}.
\end{aligned}$$

Similar to Step 2,

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3}(I-\mathbf{P})f(t)\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}(v\cdot\nabla_x)\phi \le C\|(I-\mathbf{P})f(t)\|_\sigma\|\nabla_x\phi\|_2$$

and

$$\left\|\iint_{\mathbb{T}^3\times\mathbb{R}^3}\phi\Gamma(g,f)(t)\left(|v|^2-\frac{3}{2}\right)\sqrt{\mu}\right\| \le \|g\|_\infty\|f\|_\sigma\|\phi\|_2.$$

For fixed $t>0$, we choose $\phi = \Phi_c$, where Φ_c is a solution of the Poisson equation with the periodic boundary condition $-\Delta\Phi_c = \partial_t c(t)$. Let ϕ_c be a solution of the Poisson equation with the periodic boundary condition $-\Delta\phi_c = c(t)$. Then $\Phi_c = \partial_t\phi_c$. By the Poincaré inequality,

$$\begin{aligned}
\int_{\mathbb{T}^3}|\nabla_x\partial_t\phi_c(t)|^2\mathrm{d}x &= \int_{\mathbb{T}^3}|\nabla_x\Phi_c|^2\mathrm{d}x = -\int_{\mathbb{T}^3}\Delta_x\Phi_c\Phi_c\mathrm{d}x\\
&\le \varepsilon\{\|\nabla_x\Phi_c\|_2^2 + \|\Phi_c\|_2^2\}\\
&\quad + C_\varepsilon\{\|b(t)\|_2^2 + \|(I-\mathbf{P})f(t)\|_\sigma^2 + \|g(t)\|_\infty^2\|f(t)\|_\sigma^2\}\\
&\le C\varepsilon\|\nabla_x\Phi_c\|_2^2 + C_\varepsilon\{\|b(t)\|_2^2 + \|(I-\mathbf{P})f(t)\|_\sigma^2 + \|g(t)\|_\infty^2\|f(t)\|_\sigma^2\}.
\end{aligned}$$

Therefore, for all $t>0$,

$$\|\nabla_x\partial_t\phi_c(t)\|_2 \le C_\varepsilon\{\|b(t)\|_2 + \|(I-\mathbf{P})f(t)\|_\sigma + \|g(t)\|_\infty\|f(t)\|_\sigma\}. \tag{4.10}$$

Step 4. Estimate of c. Choosing a test function $\psi = \left(|v|^2 - \frac{5}{2}\right)\sqrt{\mu}v \cdot \nabla_x \phi_c$, we have

$$
\begin{aligned}
-\frac{45}{2}\pi\sqrt{\pi}\int_0^t \int_{\mathbb{T}^3} \Delta_x \phi_c c = &- \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(0) \\
&+ \sum_{i=1}^{d} \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(|v|^2 - \frac{5}{2}\right) v_i \sqrt{\mu}\partial_t \partial_i \phi_c f \\
&+ \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v \cdot \nabla_x \psi (I-\mathbf{P})f \\
&- \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi \Gamma(g,f). \\
= &- \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(0) \\
&+ \sum_{i,j=1}^{d} \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(|v|^2 - \frac{5}{2}\right) v_i v_j \mu \partial_t \partial_i \phi_c b_j \\
&+ \sum_{i=1}^{d} \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(|v|^2 - \frac{5}{2}\right) v_i \sqrt{\mu}\partial_t \partial_i \phi_c (I-\mathbf{P})f \\
&+ \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v \cdot \nabla_x \psi (I-\mathbf{P})f \\
&- \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi \Gamma(g,f).
\end{aligned}
$$

Note that $\int \left(|v|^2 - \frac{5}{2}\right)|v|^2\mu = 0$. Therefore, the third term of RHS is zero. Moreover,

$$
\begin{aligned}
&\sum_{i=1}^{d} \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(|v|^2 - \frac{5}{2}\right) v_i \sqrt{\mu}\partial_t \partial_i \phi_c (I-\mathbf{P})f \\
&\quad \le C\|\nabla_x \partial_t \phi_c\|_2 \|(I-\mathbf{P})f\|_\sigma \\
&\quad \le C(C_\varepsilon\{\|b\|_2 + \|(I-\mathbf{P})f\|_\sigma + \|g\|_\infty \|f\|_\sigma\})\|(I-\mathbf{P})f\|_\sigma \\
&\quad \le \varepsilon\|b\|_2^2 + C_\varepsilon\{\|(I-\mathbf{P})f\|_\sigma^2 + \|g\|_\infty^2 \|f\|_\sigma^2\},
\end{aligned}
$$

$$
\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v \cdot \nabla_x \psi (I-\mathbf{P})f \le C\|c\|_2 \|(I-\mathbf{P})f\|_\sigma \le \varepsilon\|c\|_2^2 + C_\varepsilon \|(I-\mathbf{P})\|_\sigma^2,
$$

$$
\begin{aligned}
\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f &= \iint_{\mathbb{T}^3\times\mathbb{R}^3} L\psi (I-\mathbf{P})f \\
&\le C\|\nabla_x \phi_c\|_2 \|(I-\mathbf{P})f\|_\sigma \\
&\le C\|c\|_2 \|(I-\mathbf{P})f\|_\sigma \\
&\le \varepsilon\|c\|_2^2 + C_\varepsilon \|(I-\mathbf{P})\|_\sigma^2
\end{aligned}
$$

and

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi\Gamma(g,f) \le C\|g\|_\infty\|f\|_\sigma\|c\|_2 \le \varepsilon\|c\|_2^2 + C_\varepsilon\|g\|_\infty^2\|f\|_\sigma^2.$$

For a small $\varepsilon > 0$, we can absorb $\|c\|_2^2$ on the RHS to the LHS. By (4.10), we have

$$\int_0^t \|c(s)\|^2 \mathrm{d}s \le C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon\left\{\|(I-\mathbf{P})f\|_\sigma^2 + \|g\|_\infty^2\|f\|_\sigma^2\right\} + \varepsilon\|b\|_2^2 \mathrm{d}s. \tag{4.11}$$

Step 5. Estimate of b. We will estimate $(\partial_{ij}\phi_b^j)b_i$ for all $i,j = 1,\dots,d$, and $(\partial_{jj}\phi_b^i)b_i$ for $i \ne j$.

We first estimate $(\partial_{ij}\phi_b^j)b_i$. Choosing a test function $\psi = \left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_j\phi_b^j$, we have

$$\begin{aligned}
&-\sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l\left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_{lj}\phi_b^j f - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_t\partial_j\phi_b^j f \\
&\quad = -\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi f(0) \\
&\qquad - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi\Gamma(g,f).
\end{aligned} \tag{4.12}$$

Note that for $i \ne k$

$$\int \left[(v_i)^2 - \frac{1}{2}\right]\mu = \int \left[(v_i)^2 - \frac{1}{2}\right](v_k)^2\mu = 0,$$

and

$$\int \left[(v_i)^2 - \frac{1}{2}\right](v_i)^2\mu = \frac{1}{2}\pi\sqrt{\pi}.$$

Therefore,

$$\begin{aligned}
&\sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l\left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_{lj}\phi_b^j f \\
&\quad = \sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} (v_l)^2\left[(v_i)^2 - \frac{1}{2}\right]\mu\partial_{lj}\phi_b^j b_l \\
&\qquad + \sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l\left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_{lj}\phi_b^j(I-\mathbf{P})f \\
&\quad = \frac{1}{2}\pi\sqrt{\pi}\int_{\mathbb{T}^3} \partial_{ij}\phi_b^j b_i + \sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l\left[(v_i)^2 - \frac{1}{2}\right]\sqrt{\mu}\partial_{lj}\phi_b^j(I-\mathbf{P})f,
\end{aligned} \tag{4.13}$$

and

$$\left|\sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l\Big[(v_i)^2 - \frac{1}{2}\Big]\sqrt{\mu}\partial_{lj}\phi_b^j(I-\mathbf{P})f\right| \le C\int_0^t \|b\|_2\|(I-\mathbf{P})f\|_\sigma$$
$$\le \varepsilon\|b\|_2^2 + C_\varepsilon\|(I-\mathbf{P})f\|_\sigma^2. \tag{4.14}$$

Moreover

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3}\Big[(v_i)^2-\frac{1}{2}\Big]\sqrt{\mu}\partial_t\partial_j\phi_b^j f = \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\Big[(v_i)^2-\frac{1}{2}\Big]\mu\partial_t\partial_j\phi_b^j\Big(|v|^2-\frac{3}{2}\Big)c$$
$$+\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\Big[(v_i)^2-\frac{1}{2}\Big]\sqrt{\mu}\partial_t\partial_j\phi_b^j(I-\mathbf{P})f.$$

By (4.9),

$$\left|\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\Big[(v_i)^2-\frac{1}{2}\Big]\sqrt{\mu}\partial_t\partial_j\phi_b^j f\right| \le \int_0^t C_\varepsilon\big\{\|a\|_2+\|c\|_2+\|(I-\mathbf{P})f\|_\sigma+\|f\|_\sigma\|f\|_\sigma\big\}$$
$$\times\big\{\|c\|_2+C_\vartheta\|(I-\mathbf{P})f\|_\sigma\big\}$$
$$\le\int_0^t[C_\varepsilon\big\{\|(I-\mathbf{P})f\|_\sigma^2+\|g\|_\infty^2\|f\|_\sigma^2+\|c\|_2^2\big\}+\varepsilon\|a\|_2^2]. \tag{4.15}$$

In a similar way to Step 4,

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi L(I-\mathbf{P})f \le \varepsilon\|b\|_2^2 + C_\varepsilon\|(I-\mathbf{P})\|_\sigma^2 \tag{4.16}$$

and

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi\Gamma(g,f)\le\varepsilon\|b\|_2^2+C_\varepsilon\|g\|_\infty^2\|f\|_\sigma^2. \tag{4.17}$$

Combining (4.12)–(4.17),

$$\int\partial_{ij}\phi_b^j b_i$$
$$\le C(\eta(t)-\eta(0))+\int_0^t[C_\varepsilon\big\{\|(I-\mathbf{P})f\|_\sigma^2+\|g\|_\infty^2\|f\|_\sigma^2+\|c\|_2^2\big\}+\varepsilon\{\|a\|_2^2+\|b\|_2^2\}]. \tag{4.18}$$

Now we estimate $(\partial_{jj}\phi_b^i)b_i$. Choose test function $\psi=|v|^2v_iv_j\sqrt{\mu}\partial_j\phi_b^i$ for $i\neq j$. Then

$$-\sum_l\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}v_l|v|^2v_iv_j\sqrt{\mu}\partial_{lj}\phi_b^i f-\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}|v|^2v_iv_j\sqrt{\mu}\partial_t\partial_j\phi_b^i f$$
$$=-\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(t)+\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(0)-\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi L(I-\mathbf{P})f+\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi\Gamma(g,f). \tag{4.19}$$

Note that

$$
\begin{aligned}
&\sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l|v|^2 v_i v_j \sqrt{\mu}\partial_{lj}\phi_b^i f \\
&\quad = \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2(v_i)^2(v_j)^2\sqrt{\mu}[\partial_{ij}\phi_b^i b_j + \partial_{jj}\phi_b^i b_i] \\
&\qquad + \sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l|v|^2 v_i v_j \sqrt{\mu}\partial_{lj}\phi_b^i (I-\mathbf{P})f.
\end{aligned}
\tag{4.20}
$$

From (4.18),

$$
\begin{aligned}
&\left| \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2(v_i)^2(v_j)^2\sqrt{\mu}\partial_{ij}\phi_b^i b_j \right| \\
&\quad \le C(\eta(t)-\eta(0)) + \int_0^t [C_\varepsilon\{\|(I-\mathbf{P})f\|_{\sigma,\vartheta}^2 + \|g\|_\infty^2\|f\|_{\sigma,\vartheta}^2 + \|c\|_2^2\} + \varepsilon\{\|a\|_2^2 + \|b\|_2^2\}],
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
\left| \sum_l \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} v_l|v|^2 v_i v_j \sqrt{\mu}\partial_{lj}\phi_b^i (I-\mathbf{P})f \right| &\le \int_0^t C\|b\|_2\|(I-\mathbf{P})f\|_\sigma \\
&\le \int_0^t [\varepsilon\|b\|_2^2 + C_\varepsilon\|(I-\mathbf{P})f\|_\sigma^2].
\end{aligned}
\tag{4.22}
$$

Moreover, by (4.9)

$$
\begin{aligned}
&\left| \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu}\partial_t\partial_j\phi_b^i f \right| = \left| \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu}\partial_t\partial_j\phi_b^i (I-\mathbf{P})f \right| \\
&\quad \le \int_0^t C_\varepsilon\left\{\|a\|_2 + \|c\|_2 + \|(I-\mathbf{P})f\|_\sigma + \|g\|_\infty\|f\|_\sigma\right\}\|(I-\mathbf{P})f\|_\sigma \\
&\quad \le \int_0^t [C_\varepsilon\left\{\|(I-\mathbf{P})f\|_\sigma^2 + \|g\|_\infty^2\|f\|_\sigma^2\right\} + \varepsilon\left\{\|a\|_2^2 + \|c\|_2^2\right\}].
\end{aligned}
\tag{4.23}
$$

Similar to (4.16) and (4.17),

$$
\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f \le \varepsilon\|b\|_2^2 + C_\varepsilon\|(I-\mathbf{P})\|_\sigma^2 \tag{4.24}
$$

and

$$
\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi\Gamma(g,f) \le \varepsilon\|b\|_2^2 + C_\varepsilon\|g\|_\infty^2\|f\|_\sigma^2. \tag{4.25}
$$

Combining (4.19)–(4.25) yields

$$\int \partial_{jj}\phi_b^i b_i \\ \leq C(\eta(t)-\eta(0)) + \int_0^t [C_\varepsilon\left\{\|(I-\mathbf{P})f\|_\sigma^2 + \|g\|_\infty^2\|f\|_\sigma^2 + \|c\|_2^2\right\} + \varepsilon\{\|a\|_2^2 + \|b\|_2^2\}]. \tag{4.26}$$

From (4.18) to (4.26) for small ε, we can absorb $\|b\|_2^2$ term on RHS to the LHS. Then we can conclude that

$$\int_0^t \|b(s)\|^2 \mathrm{d}s \leq C(\eta(t)-\eta(0)) + \int_0^t [C_\varepsilon\left\{\|(I-\mathbf{P})f\|_\sigma^2 + \|g\|_\infty^2\|f\|_\sigma^2 + \|c\|_2^2\right\} + \varepsilon\|a\|_2^2]\mathrm{d}s. \tag{4.27}$$

Step 6. Estimate of a. Choosing a test function

$$\psi = (|v|^2-5)v\cdot\nabla_x\phi_a\sqrt{\mu},$$

we have

$$-\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)v_iv_j\partial_{ij}\phi_a\sqrt{\mu}f - \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)v_i\partial_t\partial_i\phi_a\sqrt{\mu}f \\ = -\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(t) + \iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi f(0) - \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi L(I-\mathbf{P})f + \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}\psi\Gamma(g,f). \tag{4.28}$$

Note that

$$\int (|v|^2-5)\Big(|v|^2-\frac{3}{2}\Big)(v_i)^2\mu = 0.$$

Therefore,

$$\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)v_iv_j\partial_{ij}\phi_a\sqrt{\mu}f \\ = \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)(v_i)^2\mu\partial_{ii}\phi_a a + \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)v_iv_j\partial_{ij}\phi_a\sqrt{\mu}(I-\mathbf{P})f \tag{4.29}$$

and

$$\left|\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3}(|v|^2-5)v_iv_j\partial_{ij}\phi_a\sqrt{\mu}(I-\mathbf{P})f\right| \leq \int_0^t C\|a\|_2\|(I-\mathbf{P})f\|_\sigma \\ \leq \int_0^t[\varepsilon\|a\|_2^2 + C_\varepsilon\|(I-\mathbf{P})f\|_\sigma^2]. \tag{4.30}$$

Moreover, by (4.8)

$$
\begin{aligned}
\left|\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} (|v|^2-5)v_i\partial_t\partial_i\phi_a\sqrt{\mu}f\right| &\le \left|\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} (|v|^2-5)(v_i)^2\mu\partial_t\partial_i\phi_a b_i\right| \\
&\quad + \left|\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} (|v|^2-5)v_i\partial_t\partial_i\phi_a\sqrt{\mu}(I-\mathbf{P})f\right| \\
&\le \int_0^t C\|b\|_2\big\{\|b\|_2 + C\|(I-\mathbf{P})f\|_\sigma\big\} \\
&\le \int_0^t C\{\|b\|_2^2 + \|(I-\mathbf{P})f\|_\sigma^2\}.
\end{aligned}
\tag{4.31}
$$

Similar to Steps 4 and 5, we have

$$
\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi L(I-\mathbf{P})f \le \varepsilon\|a\|_2^2 + C_\varepsilon\|(I-\mathbf{P})\|_\sigma^2 \tag{4.32}
$$

and

$$
\iint_{\mathbb{T}^3\times\mathbb{R}^3} \psi\Gamma(g,f) \le \varepsilon\|a\|_2^2 + C_\varepsilon\|g\|_\infty^2\|f\|_\sigma^2. \tag{4.33}
$$

Similarly, from (4.28) to (4.33) for a small ε, we can absorb $\|a\|_2^2$ on the RHS to the LHS. Then we have

$$
\int_0^t \|a(s)\|^2 \mathrm{d}s \le C(\eta(t)-\eta(0)) + \int_0^t C_\varepsilon\big\{\|(I-\mathbf{P})f(s)\|_\sigma^2 + \|g(s)\|_\infty^2\|f(s)\|_\sigma^2 + \|b(s)\|_2^2\big\}\mathrm{d}s. \tag{4.34}
$$

Combining (4.11), (4.27), and (4.34), we have

$$
\begin{aligned}
\int_0^t \|\mathbf{P}f\|_\sigma^2 &\le C(\eta(t)-\eta(0)) + \int_0^t C_\varepsilon\big\{\|(I-\mathbf{P})f(s)\|_\sigma^2 + \|g(s)\|_\infty^2\|f(s)\|_\sigma^2\big\}\mathrm{d}s \\
&\quad + \int_0^t \varepsilon\|\mathbf{P}f(s)\|_\sigma^2\mathrm{d}s \\
&\le C(\eta(t)-\eta(0)) + \int_0^t C_\varepsilon\big\{\|(I-\mathbf{P})f(s)\|_\sigma^2 + \|g(s)\|_\infty^2\|(I-\mathbf{P})f(s)\|_\sigma^2\big\}\mathrm{d}s \\
&\quad + \int_0^t (C_\varepsilon\|g(s)\|_\infty^2 + \varepsilon)\|\mathbf{P}f(s)\|_\sigma^2\mathrm{d}s.
\end{aligned}
$$

Note that $C_\varepsilon = C\varepsilon^{-1}$. Choosing $\varepsilon_0 = \varepsilon$, we have $C_\varepsilon\|g(s)\|_\infty^2 \le C_\varepsilon\varepsilon^2 = C\varepsilon \le 1/4$ so that $\|\mathbf{P}f\|_\sigma^2$ term on the RHS can be absorbed to the LHS. Thus we complete the proof. □

Corollary 4.4 *Assume* (2.6). *Let* $f(t,x,v)$ *be a weak solution of* (1.6), (1.7), *and* (1.22) *in the sense of* Definition 4.1. *Then there exist a constant* $0<\delta'\le 1/4$ *and a function* $0\le\eta(t)\le C\|f(t)\|_2^2$, *such that*

$$\int_s^t (L[f(\tau)],f(\tau))\mathrm{d}\tau \geq \delta'\left(\int_s^t \| f(\tau)\|_\sigma^2 \mathrm{d}\tau - \{\eta(t)-\eta(s)\}\right). \tag{4.35}$$

Proof By Lemma 2.6 and Lemma 4.3,

$$\begin{aligned}
\int_s^t (L[f(\tau)],f(\tau))\mathrm{d}\tau &\geq \delta \int_s^t \|(I-\mathbf{P})f(\tau)\|_\sigma^2 \mathrm{d}\tau \\
&\geq \delta\frac{C}{1+C}\int_s^t \|(I-\mathbf{P})f(\tau)\|_\sigma^2\mathrm{d}\tau + \delta\frac{1}{1+C}\int_s^t \|(I-\mathbf{P})f(\tau)\|_\sigma^2\mathrm{d}\tau \\
&\geq \delta\frac{C}{1+C}\int_s^t \|(I-\mathbf{P})f(\tau)\|_\sigma^2\mathrm{d}\tau + \delta\frac{1}{1+C}C\left(\int_s^t \|\mathbf{P}f(\tau)\|_\sigma^2\mathrm{d}\tau - \{\eta(t)-\eta(s)\}\right) \\
&= \frac{C\delta}{1+C}\left(\int_0^t \| f(\tau)\|_\sigma^2\mathrm{d}\tau - \{\eta(t)-\eta(s)\}\right).
\end{aligned}$$

□

Remark 4.5 Note that in Lemma 2.6, we can take $\delta > 0$ sufficiently small. Therefore we can also take δ' small enough.

Now we will prove Theorem 1.2. The proof is a modification of Theorem 5.1 in [25].

Proof of Theorem 1.2 We will prove

$$\begin{aligned}
&\sum_{0\leq\bar{\vartheta}\leq 2\vartheta}\left(\frac{C_{\bar{\vartheta}}^*}{2}\{\| f(t)\|_{2,\bar{\vartheta}/2}^2 - \| f(s)\|_{2,\bar{\vartheta}/2}^2\} + \delta_{\bar{\vartheta},2\vartheta}\int_s^t \| f(\tau)\|_{\sigma,\bar{\vartheta}/2}^2\mathrm{d}\tau\right) - \delta'\{\eta(t)-\eta(s)\} \\
&\quad\leq C_\vartheta \int_s^t \|g(\tau)\|_\infty \| f(\tau)\|_{\sigma,\vartheta}^2 \mathrm{d}\tau
\end{aligned} \tag{4.36}$$

by the induction on ϑ.

Basis Step ($\vartheta = 0$). Multiplying (1.22) by f, integrating both sides of the resulting equation, by Theorem 2.8 and Corollary 4.4, we have

$$\frac{1}{2}\{\| f(t)\|_2^2 - \| f(s)\|_2^2\} + \delta'\left(\int_s^t \| f(\tau)\|_\sigma^2\mathrm{d}\tau - \{\eta(t)-\eta(s)\}\right) \leq C\int_s^t \|g(\tau)\|_\infty \| f(\tau)\|_\sigma^2\mathrm{d}\tau.$$

Inductive Step. Suppose that (4.36) holds for $\vartheta - 1/2$. Multiplying (1.22) by $w^{2\vartheta}f$, integrating both sides of the resulting equation, by Lemma 2.7 and Theorem 2.8, we have

$$\frac{1}{2}\{\| f(t)\|_{2,\vartheta}^2 - \| f(s)\|_{2,\vartheta}^2\} + \int_s^t \left(\frac{1}{2}\| f(\tau)\|_{\sigma,\vartheta}^2 - C_\vartheta\| f(\tau)\|_\sigma^2\mathrm{d}\tau\right) \leq C_\vartheta\int_s^t \|g(\tau)\|_\infty \| f(\tau)\|_{\sigma,\vartheta}^2\mathrm{d}\tau. \tag{4.37}$$

Multiply (4.37) by $\frac{\delta_{0,2\vartheta-1}}{2C_\vartheta}$ and add it to (4.36). Then we have

$$\sum_{0\le\bar\vartheta\le 2\vartheta-1}\left(\frac{C^*_{\bar\vartheta}}{2}\{\|f(t)\|^2_{2,\bar\vartheta/2}-\|f(s)\|^2_{2,\bar\vartheta/2}\}+\delta_{\bar\vartheta,2\vartheta-1}\int_s^t\|f(\tau)\|^2_{\sigma,\bar\vartheta/2}\mathrm{d}\tau\right)-\delta'\{\eta(t)-\eta(s)\}$$
$$+\frac{\delta_{0,2\vartheta-1}}{2C_\vartheta}\left[\frac{1}{2}\{\|f(t)\|^2_{2,\vartheta}-\|f(s)\|^2_{2,\vartheta}\}+\int_s^t\left(\frac{1}{2}\|f(\tau)\|^2_{\sigma,\vartheta}-C_\vartheta\|f(\tau)\|^2_\sigma\mathrm{d}\tau\right)\right]$$
$$\le C_{\vartheta-1/2}\int_s^t\|g(\tau)\|_\infty\|f(\tau)\|^2_{\sigma,\vartheta-1/2}\mathrm{d}\tau+\frac{\delta_{0,2\vartheta-1}}{2}\int_s^t\|g(\tau)\|_\infty\|f(\tau)\|^2_{\sigma,\vartheta}\mathrm{d}\tau.$$

Note that $\|\cdot\|_{2,\vartheta-1/2}\le\|\cdot\|_{2,\vartheta}$, $\|\cdot\|_{\sigma,\vartheta-1/2}\le\|\cdot\|_{\sigma,\vartheta}$. Choosing sequences of $C^*_{2\vartheta}$, $\delta_{\bar\vartheta,2\vartheta}$, and C_ϑ such that

$$C^*_0=1,\quad \delta_{0,0}=\delta',\quad C_0=C,$$
$$C^*_\vartheta=\frac{\delta_{0,2\vartheta-1}}{2C_\vartheta},\tag{4.38}$$

$$\delta_{\bar\vartheta,2\vartheta}=\begin{cases}\frac{\delta_{0,2\vartheta-1}}{2}, & \text{if } \bar\vartheta=0\\ \delta_{\bar\vartheta,2\vartheta-1}, & \text{if } \bar{2}\vartheta=1,\dots,\vartheta-1,\\ \frac{\delta_{0,2\vartheta-1}}{4C_\vartheta}, & \text{if } \bar\vartheta=2\vartheta,\end{cases}\tag{4.39}$$

and

$$C_\vartheta=C_{\vartheta-1/2}+\frac{\delta_{0,2\vartheta-1}}{2},\tag{4.40}$$

we have (4.36) for all ϑ.

Note that from (4.38)–(4.40), we have

$$\delta_{0,k}=\frac{\delta'}{2^k},\ \text{for } k=1,2,\dots,2\vartheta.$$
$$C<C_\vartheta=C+\sum_{0\le\bar\vartheta\le 2\vartheta-1}\frac{\delta_{0,\bar\vartheta}}{2}<C+\delta'<C+1,$$
$$\frac{\delta'}{2^{2\vartheta}(C+1)}<C^*_\vartheta<\frac{\delta'}{2^{2\vartheta}C},$$

and

$$\delta_{2\vartheta,2\vartheta}=\frac{\delta_{0,2\vartheta-1}}{4C_\vartheta}=\frac{C^*_\vartheta}{2}.$$

Let $\varepsilon=\frac{\delta_{2\vartheta,2\vartheta}}{2C_\vartheta}$. By Remark 4.5, we can choose δ' small enough such that

$$\delta'\eta(t)\le\frac{C^*_0}{4}\|f(t)\|^2_2=\frac{1}{4}\|f(t)\|^2_2.\tag{4.41}$$

From (4.36), we have

$$
\begin{aligned}
\frac{C^*_\vartheta}{2}\|f(t)\|^2_{2,\vartheta}+\frac{\delta_{2\vartheta,2\vartheta}}{2}\int_s^t\|f(\tau)\|^2_{\sigma,\vartheta}\mathrm{d}\tau &\le \frac{1}{4}\|f(s)\|^2_2+\sum_{1\le\bar\vartheta\le2\vartheta}\frac{C^*_{\bar\vartheta}}{2}\|f(s)\|^2_{2,\bar\vartheta/2}\\
&\le\left(\frac{1}{4}+\frac{\delta'}{C}\right)\|f(s)\|^2_{2,\vartheta}\\
&\le\frac{1}{2}\|f(s)\|^2_{2,\vartheta}.
\end{aligned}
\tag{4.42}
$$

Taking $s=0$ and dividing by $\frac{\delta_{2\vartheta,2\vartheta}}{2}$ both sides of (4.42), we have

$$
2\|f(t)\|^2_{2,\vartheta}+\int_0^t\|f(\tau)\|^2_{\sigma,\vartheta}\mathrm{d}\tau\le\frac{2}{C^*_\vartheta}\|f(0)\|^2_{2,\vartheta}\le C2^{2\vartheta}\|f(0)\|^2_{2,\vartheta}.
$$

Therefore, we have (1.23).

Fix $\vartheta,k\ge0$, by the Hölder inequality and (1.23),

$$
\begin{aligned}
\|f\|^2_{2,\vartheta}&=\int w^{2\vartheta}f^2\\
&=\int\left(w^{2\left(\vartheta-\frac{1}{2}\right)}f^2\right)^{\frac{k}{k+1}}\left(w^{2\left(\vartheta+\frac{k}{2}\right)}f^2\right)^{\frac{1}{k+1}}\\
&\le\left(\int w^{2\left(\vartheta-\frac{1}{2}\right)}f^2\right)^{\frac{k}{k+1}}\left(\int w^{2\left(\vartheta+\frac{k}{2}\right)}f^2\right)^{\frac{1}{k+1}}\\
&\le\|f\|^{2\frac{k}{k+1}}_{2,\vartheta-1/2}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{\frac{1}{k+1}}.
\end{aligned}
\tag{4.43}
$$

By Lemma 2.5,

$$
\|f\|_{\sigma,\vartheta}\ge\|(1+|v|)^{-1/2}f\|_{2,\vartheta}=\|f\|_{2,\vartheta-1/2}.
\tag{4.44}
$$

Combining (4.36), (4.43), and (4.44), we have

$$
\begin{aligned}
&\sum_{1\le\bar\vartheta\le2\vartheta}\frac{C^*_{\bar\vartheta}}{2}\left(\|f(t)\|^2_{2,\bar\vartheta/2}-\|f(s)\|^2_{2,\bar\vartheta/2}\right)+\left\{\frac{1}{2}\|f(t)\|^2_2-\delta'\eta(t)\right\}-\left\{\frac{1}{2}\|f(s)\|^2_2-\delta'\eta(s)\right\}\\
&\le-\frac{\delta_{2\vartheta,2\vartheta}}{2}\int_s^t\|f(\tau)\|^2_{\sigma,\vartheta}\mathrm{d}\tau\\
&\le-\frac{\delta_{2\vartheta,2\vartheta}}{2}\int_s^t\|f(\tau)\|^2_{2,\vartheta-1/2}\mathrm{d}\tau\\
&\le-\frac{\delta_{2\vartheta,2\vartheta}}{2}\int_s^t\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}\|f(\tau)\|^{2\frac{k+1}{k}}_{2,\vartheta}\mathrm{d}\tau.
\end{aligned}
\tag{4.45}
$$

Let

$$
y(t):=\left\{\frac{1}{2}\|f(t)\|^2_2-\delta'\eta(t)\right\}+\sum_{1\le\bar\vartheta\le2\vartheta}\frac{C^*_{\bar\vartheta}}{2}\|f(t)\|^2_{2,\bar\vartheta/2}.
$$

Then

$$\frac{C^*_\vartheta}{2}\| f(t)\|^2_{2,\vartheta} \le y(t) \le \left(\frac{1}{2}+\sum_{\bar\vartheta=1}^{2\vartheta}\frac{C^*_{\bar\vartheta}}{2}\right)\| f(t)\|^2_{2,\vartheta} \le \| f(t)\|^2_{2,\vartheta}. \tag{4.46}$$

Combining (4.45), (4.46), we have

$$y(t)-y(s) \le -\frac{\delta_{2\vartheta,2\vartheta}}{2}\int_s^t \left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}(y(\tau))^{\frac{k+1}{k}}\,\mathrm{d}\tau.$$

Therefore, we have

$$y'(t) \le -\frac{1}{2}\delta_{2\vartheta,2\vartheta}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}y(t)^{\frac{k+1}{k}} \le -\frac{1}{2^{2\vartheta}C}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}y(t)^{\frac{k+1}{k}}. \tag{4.47}$$

Multiplying (4.47) by $-\frac{1}{k}y^{-\frac{k+1}{k}}$, we have

$$\partial_t\left(y(t)^{-\frac{1}{k}}\right) \ge \frac{1}{2^{2\vartheta}Ck}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}.$$

Integrating above over $[0,t]$ yields

$$\begin{aligned} y(t)^{-\frac{1}{k}} &\ge \frac{t}{2^{2\vartheta}Ck}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}} + y(0)^{-\frac{1}{k}} \\ &\ge \frac{t}{2^{2\vartheta}Ck}\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}} + \left(\| f(0)\|^2_{2,\vartheta}\right)^{-\frac{1}{k}} \\ &\ge \frac{\left(C2^{2\vartheta+k}\mathcal{E}_{\vartheta+k/2}(0)\right)^{-\frac{1}{k}}}{2^{2\vartheta}C}\left(\frac{t}{k}+1\right). \end{aligned}$$

Therefore,

$$\| f(t)\|^2_{2,\vartheta} \le \frac{2}{C^*_\vartheta}y(t) \le C_{\vartheta,k}\mathcal{E}_{\vartheta+k/2}(0)\left(1+\frac{t}{k}\right)^{-k},$$

where we use (4.46) in the first inequality. Thus we complete the proof. □

Theorem 4.6 *Assume* (2.6). *Let* $\vartheta \in 2^{-1}\mathbb{N}\cup\{0\}$ *and* f *be a classical solution of* (1.7), (4.5). *Then there exist* $C, \varepsilon(\vartheta) > 0$ *such that if* $\|g\|_\infty < \varepsilon$, *then*

$$\sup_{0\le s<\infty}\mathcal{E}_\vartheta(f(s)) \le C2^{2\vartheta}\mathcal{E}_\vartheta(0), \tag{4.48}$$

and for any $t > 0$, $k \in \mathbb{N}$,

$$\| f(t)\|_{2,\vartheta} \le C_{\vartheta,k}\mathcal{E}_{\vartheta+k/2}(0)\left(1+\frac{t}{k}\right)^{-k/2}. \tag{4.49}$$

Sketch of proof The proof can be done by choosing $\Gamma = 0$ in Theorem 1.2. □

5 $L^2 - L^\infty$ Estimate

5.1 Local $L^2 - L^\infty$ Estimate

In this section we will derive a local L^∞ estimate for h.

$$\mathcal{M}_g^\vartheta h := (\partial_t + v \cdot \nabla_x - \bar{A}_g^\vartheta)h, \tag{5.1}$$

where $\bar{A}_g^\vartheta$ is defined as in (3.5).

Here we will refine the results about the $L^2 - L^\infty$ estimate in [20]. Comparing with [20], we have an additional term $a_g \cdot \nabla_v f - 2\frac{\partial_i w^\vartheta}{w^\vartheta}\sigma_G^{ij}\partial_j f$ and a diffusion matrix of $\mathcal{M}_g^\vartheta$ which is not uniformly elliptic. Moreover, to get an $L^2 - L^\infty$ estimate for $\mathbb{T}^3 \times \mathbb{R}^3$, we need to know the local $L^2 - L^\infty$ estimate more explicitly.

Define $Q_n := [-t_n, 0] \times \mathbb{T}^3 \times B(0;R_n)$, for $t_n \geq t_{n+1}$ and $R_n \geq R_{n+1}$. The following estimates are refinements of Lemmas 4–6 and Theorem 2 and Theorem 7 in [20].

Lemma 5.1 (Lemma 4 in [20]) *Assume* (2.6). *Let h be a non-negative periodic function in x satisfying $\mathcal{M}_g^\vartheta h \leq 0$. Then h satisfies*

$$\int_{Q_1} |\nabla_v h|^2 \leq C \int_{Q_0} h^2, \tag{5.2}$$

$$\|h\|^2_{L_t^2 L_x^2 L_v^q(Q_1)} \leq C \int_{Q_0} h^2, \tag{5.3}$$

$$\|h\|^2_{L_t^\infty L_x^2 L_v^2(Q_1)} \leq C \int_{Q_0} h^2 \tag{5.4}$$

for some $q > 2$ and $C = \bar{C}(R_0)\left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2}\right)$, $\bar{C}(R_0) = C'(1 + R_0)^3$.

Proof Consider a test function $\Phi \in C^\infty(\mathbb{R} \times \mathbb{T}^3 \times \mathbb{R}^3)$, periodic with respect to x and $\Phi(t, x, v) = 0$ for $|v| > R_0$. Multiplying (5.1) by $2h\Phi^2$ and integrating the resulting equation over $\mathcal{R} := [-t_0, s] \times \mathbb{T}^3 \times B(0;R_0)$ for some $s \in [-t_1, 0]$, we have

$$\begin{aligned}
&\int_{\mathcal{R}} \partial_t(h^2)\Phi^2 + \int_{\mathcal{R}} v \cdot \nabla_x(h^2)\Phi^2 \\
&\quad \leq 2\int_{\mathcal{R}} \nabla_v \cdot (\sigma_G \nabla_v h)h\Phi^2 + \int_{\mathcal{R}} a_g \cdot \nabla_v(h^2)\Phi^2 - 2\int_{\mathcal{R}} \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \cdot \sigma_G \nabla_v(h^2)\Phi^2,
\end{aligned}$$

where σ_G is defined as in (2.1) with $G = \mu + \mu^{1/2}g$. Using the integration by parts and the positivity of σ_G, we have

$$
\begin{aligned}
&\int_{\mathcal{R}} \partial_t(h^2\Phi^2) + 2\int_{\mathcal{R}} (\nabla_v h \cdot \sigma_G \nabla_v h)\Phi^2 \\
&\quad \le \int_{\mathcal{R}} h^2 \left(\partial_t(\Phi^2) + v\cdot\nabla_x(\Phi^2) - \nabla_v\cdot(\Phi^2 a_g) + 2\nabla_v \cdot \left(\Phi^2 \sigma_G \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right) \right) \\
&\qquad - 4\int_{\mathcal{R}} h\Phi\nabla_v\Phi\cdot\sigma_G\nabla_v h \\
&\quad \le \int_{\mathcal{R}} h^2 \left(\partial_t(\Phi^2) + v\cdot\nabla_x(\Phi^2) - \nabla_v\cdot(\Phi^2 a_g) + 2\nabla_v \cdot \left(\Phi^2 \sigma_G \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right) \right) \\
&\qquad + \int_{\mathcal{R}} (\nabla_v h\cdot\sigma_G\nabla_v h)\Phi^2 + C\int_{\mathcal{R}} (\nabla_v\Phi\cdot\sigma_G\nabla_v\Phi)h^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\int_{\mathcal{R}} \partial_t(h^2\Phi^2) + \min(1,(1+R_0)^{-3})\int_{\mathcal{R}} |\nabla_v h|^2\Phi^2 \\
&\quad \le C' \max(1,(1+R_0)^{-1}) \Bigg(\|\partial_t\Phi\|_\infty\|\Phi\|_\infty + R_0\|\nabla_x\Phi\|_\infty\|\Phi\|_\infty \\
&\qquad + \|\Phi\|_\infty\|a_g\|_\infty\|\nabla_v\Phi\|_\infty + \|\Phi\|_\infty^2\|\nabla_v\cdot a_g\|_\infty \\
&\qquad + \|\nabla_v\Phi\|_\infty^2 + \|\Phi\|_\infty\|\nabla_v\Phi\|_\infty \left\| \sigma\frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right\|_\infty + \|\Phi\|_\infty^2 \left\| \nabla_v\cdot\left(\sigma\frac{\nabla_v(w^\vartheta)}{w^\vartheta}\right) \right\|_\infty \Bigg) \int_{\mathcal{R}} h^2.
\end{aligned}
$$

Choosing Φ such that $\Phi(-t_0,x,v)=0$ and $\Phi=1$ in Q_1, we have

$$
\int_{\mathbb{T}^3\times B(0;R_1)} h^2(s)\mathrm{d}x\mathrm{d}v + \int_{\mathcal{R}} |\nabla_v h|^2 \le \bar{C}(R_0)\left(1+\frac{1}{t_0-t_1}+\frac{1}{R_0-R_1}+\frac{1}{(R_0-R_1)^2}\right)\int_{\mathcal{R}} h^2. \tag{5.5}
$$

Especially,

$$
\sup_{s\in[-t_1,0]} \int_{\mathbb{T}^3\times B(0;R_1)} h^2(s)\mathrm{d}x\mathrm{d}v \le \bar{C}(R_0)\left(1+\frac{1}{t_0-t_1}+\frac{1}{R_0-R_1}+\frac{1}{(R_0-R_1)^2}\right)\int_{Q_0} h^2.
$$

Therefore, we prove (5.4). Choosing $s=0$ in (5.5), we have

$$
\int_{Q_1} |\nabla_v h|^2 \le \bar{C}(R_0)\left(1+\frac{1}{t_0-t_1}+\frac{1}{R_0-R_1}+\frac{1}{(R_0-R_1)^2}\right)\int_{Q_0} h^2,
$$

so we obtain (5.2). Moreover, the Sobolev inequality implies (5.3). ◻

Lemma 5.2 (Lemma 5 in [20]) *Assume* (2.6). *If h is a weak solution of* (3.6), *then*

$$
\begin{aligned}
\|D_x^{1/3}h\|_{L^2(Q_1)}^2 &\le C\|h\|_{L^2(Q_0)}^2, \\
\|D_t^{1/3}h\|_{L^2(Q_1)}^2 &\le C\|h\|_{L^2(Q_0)}^2
\end{aligned} \tag{5.6}
$$

for some $C=\bar{C}(R_0)\left(1+\frac{1}{t_0-t_1}+\frac{1}{R_0-R_1}+\frac{1}{(R_0-R_1)^2}\right)$.

Proof Let $R_{\frac{1}{2}} = \frac{R_1+R_0}{2}$ and $Q_{\frac{1}{2}} = Q_{R_{\frac{1}{2}}}$. Define truncation functions χ_1 and $\chi_{1/2}$ such that

$$\chi_1 = \begin{cases} 1, & \text{if } (t,x,v) \in Q_1, \\ 0, & \text{if } (t,x,v) \in Q^c_{\frac{1}{2}}, \end{cases}$$

$$\chi_{\frac{1}{2}} = \begin{cases} 1, & \text{if } (t,x,v) \in Q_{\frac{1}{2}}, \\ 0, & \text{if } (t,x,v) \in Q^c_0. \end{cases}$$

Let $h_i = h\chi_i$, for $i = 1, \frac{1}{2}$. Then we have

$$(\partial_t + v \cdot \nabla_x)h_1 = \nabla_v \cdot H_1 + H_0 \quad \text{in } (-\infty, 0] \times \mathbb{R}^6,$$

$$\begin{aligned} H_1 &= \chi_1 \sigma_G \nabla_v h_{\frac{1}{2}}, \\ H_0 &= -\nabla_v \chi_1 \cdot \sigma_G \nabla_v h_{\frac{1}{2}} + \alpha_1 h_{\frac{1}{2}} + \chi_1 a_g \cdot \nabla_v h_{1/2} - 2\chi_1 \frac{\nabla_v (w^\vartheta)}{w^\vartheta} \cdot \sigma_G \nabla_v h_{1/2}, \\ \alpha_1 &= (\partial_t + v \cdot \nabla_x)\chi_1, \end{aligned}$$

where σ_G is defined as in (2.1) with $G = \mu + \mu^{1/2} g$. By Lemma 5.1,

$$\|H_0\|_{L^2(\mathbb{R}^7)} + \|H_1\|_{L^2(\mathbb{R}^7)} \leq C\|h\|_{L^2(Q_0)}$$

with C as in the statement. Applying Theorem 1.3 in [4] with $p = 2$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 1$ yields (5.6). □

Lemma 5.3 (Lemma 6 in [20]) *Under the assumptions of* Lemma 5.1, *there exists $p > 2$ such that*

$$\|h\|^2_{L^2_t L^p_x L^2_v(Q_1)} \leq C\|h\|^2_{L^2(Q_0)} \tag{5.7}$$

with the same C as in Lemma 5.1.

Proof The proof is exactly the same as in the proof of Lemma 6 in [20]. We omit the proof. □

The following lemma is a consequence of Lemmas 5.1 and 5.2. We omit the proof.

Lemma 5.4 *Under the assumptions of* Lemma 5.2, *we have*

$$\|h\|_{H^s_{x,v,t}(Q_1)} \leq C\|h\|_{L^2(Q_0)}$$

with the same C as in Lemma 5.1 *and $s = 1/3$.*

Lemma 5.5 (Theorem 2 in [20]) *Under the assumptions of* Lemma 5.1, *there exists $q > 2$ such that*

$$\|h\|^2_{L^q(Q_1)} \le C\|h\|^2_{L^2(Q_0)} \tag{5.8}$$

with the same C as in Lemma 5.1.

Proof The proof is exactly the same as in the proof of Theorem 2 in [20]. We omit the proof. □

Lemma 5.6 (Theorem 7 in [20]) *Assume* (2.6). *Let h be a non-negative subsolution of* (3.6). *Then, there exists $m > 1$ such that*

$$\|h\|_{L^\infty(Q_\infty)} \le \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)}\right)^m \|h\|_{L^2(Q_0)},$$

where $Q_0 = [-t_0, 0] \times \mathbb{T}^3 \times [-R_0, R_0]$ *and* $Q_\infty = [-t_\infty, 0] \times \mathbb{T}^3 \times [-R_\infty, R_\infty]$.

Proof Let $\kappa := q/2 > 1$. Since $|h|^{q_n}$, $q_n > 1$, is also a subsolution of (3.6), by Lemma 5.5

$$\||h|^{q_n}\|^2_{L^q(Q_{n+1})} \le C_n \||h|^{q_n}\|^2_{L^2(Q_n)},$$

where $C_n = \bar{C}(R_0)\left(1 + \frac{1}{t_n - t_{n+1}} + \frac{1}{R_n - R_{n+1}} + \frac{1}{(R_n - R_{n+1})^2}\right)$. Changing $\|\cdot\|_q$ to $\|\cdot\|_2$ yields

$$\||h|^{\kappa q_n}\|^2_{L^2(Q_{n+1})} \le C_n^\kappa \||h|^{q_n}\|^{2\kappa}_{L^2(Q_n)}.$$

Let $q_n := \kappa^n$, then after iteration we have

$$\||h|^{q_n}\|^2_{L^2(Q_n)} \le \prod_{j=1}^n C_{n-j}^{\kappa^j} \|h\|^{2\kappa^n}_{L^2(Q_0)}.$$

Changing $\|\cdot\|_2$ to $\|\cdot\|_{2q_n}$, we have

$$\|h\|^2_{L^{2q_n}(Q_n)} \le \prod_{j=1}^n C_{n-j}^{\kappa^{j-n}} \|h\|^2_{L^2(Q_0)} = \prod_{j=0}^{n-1} C_j^{\kappa^{-j}} \|h\|^2_{L^2(Q_0)}.$$

Choosing $t_n - t_{n+1} = \alpha(t_0 - t_\infty)n^{-4}$ and $R_n - R_{n+1} = \beta(R_0 - R_\infty)n^{-2}$, we have

$$C_j^{\kappa^{-j}} \le \bar{C}^{\kappa^{-j}} \left(C'\left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)}\right)\right)^{j^4\kappa^{-j}}.$$

Thus,

$$\prod_{j=0}^\infty C_j^{\kappa^{-j}} \le C'^m \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)}\right)^m$$

for some $m > 1$. So the proof is complete. □

Lemma 5.7 *Assume* (2.6). *If* $h_+ = \max\{h, 0\}$, *where h is a subsolution of* (3.6), *then* h_+ *is a subsolution.*

Proof Approximate a convex function $Q(h) \to h_+$ and then use the convexity of $Q(h)$ such that $Q'(h) > 0$ and $Q''(h) > 0$. Applying $Q(h)$ to Eq. (3.6), we complete the proof. □

Let h be a weak solution. Then since $|h| = h_+ - h_-$ and $h_+ = \max\{h, 0\}$ are subsolutions (maximum of two subsolutions is a subsolution) and $h_- = \max\{-h, 0\}$ is a supersolution (minimum of two supersolution is a supersolution), we can apply Lemma 5.7 to both h_+ and $-h_-$. Thus we obtain:

Lemma 5.8 (Theorem 7 in [20]) *Assume* (2.6). *Let h be a subsolution of* (3.6). *Then, there exists* $m > 1$ *such that*

$$\|h\|_{L^\infty(Q_\infty)} \le \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)}\right)^m \|h\|_{L^2(Q_0)},$$

where $Q_0 = [-t_0, 0] \times \mathbb{T}^3 \times [-R_0, R_0]$ *and* $Q_\infty = [-t_\infty, 0] \times \mathbb{T}^3 \times [-R_\infty, R_\infty]$.

5.2 $L^2 - L^\infty$ Estimate for (1.22)

We now consider (1.22) and let f be a solution of (1.22). Then we split f into two parts:

$$f = f\mathbf{1}_{\{|v|\le M\}} + f\mathbf{1}_{\{|v|\ge M\}} =: f_1 + f_2.$$

Let $U(t,s)h$ be a solution of (3.6) corresponding to the initial times s with the initial data h. Then

$$\begin{aligned} f_1(t,x,v) &= \mathbf{1}_{\{|v|\le M\}} U(t,0) f_0 + \mathbf{1}_{\{|v|\le M\}} \int_0^t U(t,\tau)\bar{K}_g^\vartheta f(\tau)\mathrm{d}\tau \\ &= \mathbf{1}_{\{|v|\le M\}} U(t,0) f_0 + \int_0^t \mathbf{1}_{\{|v|\le M\}} U(t,\tau)\bar{K}_g^\vartheta f(\tau)\mathrm{d}\tau. \end{aligned}$$

To obtain the L^∞ estimates for f_1, we will give some basic estimates for $\bar{K}_g^\vartheta$.

Lemma 5.9 *Let* $\bar{K}_g^\vartheta$ *be defined as in* (3.4). *Suppose that g satisfies the assumption in* Lemma 2.4. *Then there exists* $C = C_\vartheta > 0$ *such that for every* $N, M > 0$,

$$\|\bar{K}_g^\vartheta f\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le C\| f^\vartheta \|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{5.9}$$

$$\|\bar{K}_g^\vartheta 1_{|v|>M} f\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le C(1+M)^{-1}\| f^\vartheta \|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{5.10}$$

and

$$\|\bar{K}_g^\vartheta f\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} \leq CN^2\| f^\vartheta\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} + \frac{C}{N}\| f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}. \tag{5.11}$$

Proof Note that

$$\begin{aligned}
\bar{K}_g^\vartheta f &= w^\vartheta \bar{K}_g f + \left(2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i\sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right) f^\vartheta \\
&= w^\vartheta Kf + \partial_i\sigma^i f^\vartheta - v\cdot\sigma v f^\vartheta - \partial_i\left\{\phi^{ij} * [\mu^{1/2}\partial_j g]\right\} f^\vartheta + \left\{\phi^{ij} * [v_i\mu^{1/2}\partial_j g]\right\} f^\vartheta \\
&\quad + \left(2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i\sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right) f^\vartheta \\
&= w^\vartheta Kf + (I)\times f^\vartheta,
\end{aligned}$$

where ϕ, K, σ_G, σ^i are defined as in (1.2), (1.10), (2.1), and (2.2) with $G = \mu + \mu^{1/2} g$. Since $w^\vartheta Kf$ satisfies (2.20)–(2.22), it is enough to estimate (I).

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
|(I)| &\leq |\partial_i\sigma^i(v)| + |v\cdot\sigma v| + \left|\partial_i\left\{\phi^{ij} * [\mu^{1/2}\partial_j g](v)\right\}\right| + \left|\left\{\phi^{ij} * [v_i\mu^{1/2}\partial_j g](v)\right\}\right| \\
&\quad + \left|2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij}\right| + \left|\frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij}\right| + \left|\frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i\sigma_G^{ij}\right| + \left|\frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right| \\
&\leq C(1+|v|)^{-1}.
\end{aligned}$$

So, we complete the proof. $\square$

Now we can obtain the L^∞ estimates for f_1:

Lemma 5.10 *Assume* (2.6). *Let f be a weak solution of* (1.6), (1.7), *and* (1.22) *in a periodic box in the sense of* Definition 4.1, *then there exist* $C, \beta > 0$ *satisfying the following property: for any* $Z, s, k > 1$, *and* $\vartheta, l \in \mathbb{N}\cup\{0\}$, *there exists* $C_{\vartheta,l}$ *such that*

$$\begin{aligned}
&\left\|1_{|v|<Zs^k} f^\vartheta(s)\right\|_\infty \\
&\quad \leq C_{\vartheta,l}\left(Zs^k\right)^\beta (1+s)^{-l}\|f_0\|_{2,\vartheta+l} + \frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\| f^\vartheta(s')\|_\infty.
\end{aligned} \tag{5.12}$$

Proof By the Duhamel principle,

$$\begin{aligned}
\|1_{|v|<Zs^k} f^\vartheta(s)\|_{L^\infty} \leq\ & \|1_{|v|<Zs^k} U(s,s-1) f^\vartheta(s-1)\|_{L^\infty} \\
&+ \int_0^{1-\varepsilon} \|1_{|v|\leq Zs^k} U(s,s-1+\tau)\bar{K}_g^\vartheta f(s-1+\tau)\|_\infty \mathrm{d}\tau \\
&+ \int_{1-\varepsilon}^1 \|1_{|v|\leq Zs^k} U(s,s-1+\tau)\bar{K}_g^\vartheta f(s-1+\tau)\|_\infty \mathrm{d}\tau \\
=\ & (i) + (ii) + (iii),
\end{aligned}$$

where $\bar{K}_g^{\vartheta}$ is defined as in (3.4) and ε is a constant which will be chosen later. By Lemma 5.8, there exists $m > 0$ such that

$$(i) \leq C\big(Zs^k\big)^m \left(\int_0^1 \|U(s', s-1) f^{\vartheta}(s-1)\|_2^2 \mathrm{d}s'\right)^{1/2}.$$

By Theorem 1.2 and Lemma 3.4, for every integer l, there exists C_l such that

$$\begin{aligned}\|U(s', s-1) f^{\vartheta}(s-1)\|_2 &\leq C\| f^{\vartheta}(s-1)\|_2 = C\| f(s-1)\|_{2,\vartheta}\\ &\leq C_{\vartheta,l}\left(1+\frac{s-1}{l}\right)^{-l} \|f_0\|_{2,\vartheta+l}\\ &\leq C_{\vartheta,l}(1+s)^{-l}\|f_0\|_{2,\vartheta+l}.\end{aligned}$$

Thus

$$(i) \leq C_{\vartheta,l}\big(Zs^k\big)^m (1+s)^{-l}\|f_0\|_{2,\vartheta+l}.$$

By the maximum principle and (5.9),

$$(iii) \leq C\varepsilon \sup_{s'\in(s-1,s)} \| f^{\vartheta}(s')\|_\infty.$$

By Lemma 5.8,

$$\begin{aligned}&\|1_{|v|\leq Zs^k} U(s, s-1+\tau)\bar{K}_g^{\vartheta} f(s-1+\tau)\|_\infty\\ &\quad\leq C\big(Zs^k\big)^m\left(1+\frac{1}{1-\tau}\right)^m \left(\int_{s-1+\tau}^s \|1_{|v|<2Zs^k} U(s', s-1+\tau)\bar{K}_g^{\vartheta} f(s-1+\tau)\|_2^2 \mathrm{d}s'\right)^{1/2}.\end{aligned}$$

By (5.11) and Theorem 1.2, for any $N > 0$,

$$\begin{aligned}(ii) &\leq C\int_0^{1-\varepsilon} \big(Zs^k\big)^m\left(1+\frac{1}{1-\tau}\right)^m\left(\int_{s-1+\tau}^s \|\bar{K}_f^{\vartheta} f(s-1+\tau)\|_2^2 \mathrm{d}s'\right)^{1/2}\mathrm{d}\tau\\ &\leq C\big(Zs^k\big)^m\left(1+\frac{1}{\varepsilon}\right)^m \int_0^1 \left(N^2\| f^{\vartheta}(s-1+\tau)\|_2 + \frac{1}{N}\| f^{\vartheta}(s-1+\tau)\|_\infty\right)\mathrm{d}\tau\\ &\leq \big(Zs^k\big)^m\left(1+\frac{1}{\varepsilon}\right)^m\left(C_{\vartheta,l}N^2(1+s)^{-l}\|f_0\|_{2,\vartheta+l} + \frac{C}{N}\sup_{s'\in(s-1,s)}\| f^{\vartheta}(s')\|_\infty\right).\end{aligned}$$

Choose $\varepsilon^{-1} = 1 + Zs^k$ and $N = (1+Zs^k)^{2m+1}$. Then

$$(i) + (ii) + (iii) \leq C_{\vartheta,l}\big(Zs^k\big)^\beta (1+s)^{-l}\|f_0\|_{2,\vartheta+l} + \frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\| f^{\vartheta}(s')\|_\infty,$$

where $\beta = 6m + 2 > 0$. □

Based on the above results, we will prove Theorem 1.3.

Proof of Theorem 1.3 Choose ε as in Lemma 2.4. By Lemma 5.10, there exists l such that

$$\left\|1_{|v|<Zs^k} f^\vartheta(s)\right\|_{L^\infty} \leq C_{\vartheta,l,Z}(1+s)^{-2}\|f_0\|_{2,\vartheta+l} + \frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\|f^\vartheta(s')\|_\infty.$$

Therefore by the Duhamel principle, the maximum principle, and (5.10),

$$\begin{aligned}
\|f^\vartheta(n+1)\|_{L^\infty} \leq{}& \|U(n+1,n)f^\vartheta(n)\|_{L^\infty} \\
&+ \int_0^1 \left\|U(n+1,n+s)\bar{K}_g^\vartheta\left(1_{|v|<Z(n+s)^k} f(n+s) + 1_{|v|>Z(n+s)^k} f(n+s)\right)\right\|_{L^\infty}\mathrm{d}s \\
\leq{}& \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta,l,Z}\int_0^1 \Bigg((1+n+s)^{-2}\|f_0\|_{2,\vartheta+l} \\
&+ \frac{C}{1+Z(n+s)^k}\sup_{s'\in(n+s-1,n+s)}\|f^\vartheta(s')\|_\infty\Bigg)\mathrm{d}s \\
&+ \int_0^1 (1+Z(n+s)^k)^{-1}\|f^\vartheta(n+s)\|_\infty \mathrm{d}s \\
\leq{}& \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta,l,Z}(1+n)^{-2}\|f_0\|_{2,\vartheta+l} + C(Zn^k)^{-1}\sup_{s'\in[n-1,n+1]}\|f^\vartheta(s')\|_\infty.
\end{aligned}$$

After iteration,

$$\begin{aligned}
\|f^\vartheta(n+1)\|_{L^\infty} \leq{}& \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z}\sum_{\bar{n}=1}^n (1+\bar{n})^{-2}\|f_0\|_{2,\vartheta+l} \\
&+ CZ^{-1}\sum_{\bar{n}=1}^n \bar{n}^{-k}\sup_{s\in[0,n+1]}\|f^\vartheta(s)\|_\infty.
\end{aligned}$$

Choose large k and Z such that $-k < -1$ and $CZ^{-1}\sum_{\bar{n}=1}^\infty \bar{n}^{k(-1)} \leq \varepsilon_0$, where ε_0 will be determined later. Then

$$\|f^\vartheta(n+1)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta+l} + \varepsilon_0\sup_{s\in[0,n+1]}\|f^\vartheta(s)\|_\infty.$$

Since n is an arbitrary integer,

$$\sup_{s=1,2,\ldots n+1}\|f^\vartheta(s)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta+l} + \varepsilon_0\sup_{s\in[0,n+1]}\|f^\vartheta(s)\|_\infty.$$

By the Duhamel principle, the maximum principle, and (5.10),

$$\begin{aligned}
\|f^\vartheta(n+t)\|_{L^\infty} \leq{}& \|U(n+t,n)f^\vartheta(n)\|_{L^\infty} + \int_0^t \left\|U(n+t,n+s)\bar{K}_g^\vartheta f(n+s)\right\|_{L^\infty}\mathrm{d}s \\
\leq{}& \|f^\vartheta(n)\|_{L^\infty} + \int_0^t \left\|\bar{K}_g^\vartheta f(n+s)\right\|_{L^\infty}\mathrm{d}s \\
\leq{}& \|f^\vartheta(n)\|_{L^\infty} + C\int_0^t \left\|f^\vartheta(n+s)\right\|_{L^\infty}\mathrm{d}s,
\end{aligned}$$

for $t \in [0,1]$. By the Gronwall inequality,

$$\| f^{\vartheta}(n+t)\|_{L^\infty} \le C\| f^{\vartheta}(n)\|_{\infty}, \text{ for all } t \in [0,1].$$

Therefore,

$$\sup_{s\in[0,n+1]} \| f^{\vartheta}(s)\|_{L^\infty} \le C\| f_0^{\vartheta}\|_{L^\infty} + C_{\vartheta,l}\|f_0\|_{2,\vartheta+l} + C\varepsilon_0 \sup_{s\in[0,n+1]} \| f^{\vartheta}(s)\|_{\infty}.$$

Now, we choose a small ε_0 satisfying $C\varepsilon_0 < 1/2$, and then absorb the last term on the RHS to the LHS. Then, we have (1.25) in case of $\vartheta_0 = 0$ by taking $l_0(0) = l$.

By (5.12), there exist C, $l_1(\vartheta_0)$ such that

$$\|1_{|v|<(1+t)^{\vartheta_0}} f^{\vartheta}(t)\|_\infty \le C(1+t)^{-\vartheta_0}(\|f_0\|_{2,\vartheta+l_1(\vartheta_0)} + \|f\|_{\infty,\vartheta}).$$

Thus, by Proposition 2.1, we have

$$\begin{aligned}
\|f(t)\|_{\infty,\vartheta} &\le \|1_{|v|<(1+t)} f(t)\|_{\infty,\vartheta} + \|1_{|v|\ge(1+t)} f(t)\|_{\infty,\vartheta} \\
&\le C(1+t)^{-\vartheta_0}\left(\|f_0\|_{2,\vartheta+l_1(\vartheta_0)} + \sup_{0\le s\le t} \|f(s)\|_{\infty,\vartheta}\right) + C(1+t)^{-\vartheta_0}\|1_{|v|\ge(1+t)} f(t)\|_{\infty,\vartheta+\vartheta_0} \\
&\le C(1+t)^{-\vartheta_0}\left(\|f_0\|_{2,\vartheta+l_1(\vartheta_0)} + \sup_{0\le s\le t} \|f(s)\|_{\infty,\vartheta+\vartheta_0}\right) \\
&\le C(1+t)^{-\vartheta_0}(\|f_0\|_{2,\vartheta+l_1(\vartheta_0)} + \|f_0\|_{2,\vartheta+\vartheta_0+l_0(0)} + \|f_0\|_{\infty,\vartheta+\vartheta_0}) \\
&\le C(1+t)^{-\vartheta_0}\|f_0\|_{\infty,\vartheta+l_0(\vartheta_0)},
\end{aligned}$$

where $l_0(\vartheta_0) = \max\{l_1(\vartheta_0), \vartheta_0 + l_0(0)\} + 2$. □

Lemma 5.11 *Assume* (2.6). *Let f be a strong solution of* (1.6), (1.7), *and* (1.22) *in a periodic box. Let $\beta > 0$ and $p > 2$ be given constants. Then there exist $l \in \mathbb{N}$ and $C_{\beta,l}$ such that*

$$\begin{aligned}
\left(\int_0^t \|f(s)\|_{p,\beta}^p \mathrm{d}s\right)^{1/p} &\le C_{\beta,l_0}\|f_0\|_{2,\beta+l}^{2/p}\big(\|f_0\|_{\infty,\beta} + \|f_0\|_{2,\beta+l}\big)^{(p-2)/p} \\
&\le C_{\beta,l_0}\big(\|f_0\|_{\infty,\beta} + \|f_0\|_{2,\beta+l}\big).
\end{aligned} \tag{5.13}$$

Proof By Theorems 1.2 and 1.3, there exist $l \in \mathbb{N}$ and $C_{\beta,l}$ such that

$$\begin{aligned}
\|f(s)\|_{2,\beta} &\le C_{\beta,l_0}(1+s)^{-l}\|f_0\|_{2,\beta+l}, \\
\|f(s)\|_{\infty,\beta} &\le C_{\beta,l_0}\big(\|f_0\|_{\infty,\beta} + \|f_0|_{2,\beta+l}\big).
\end{aligned}$$

By the interpolation, we have

$$\|f(s)\|_{p,\beta}^p \le (C_{\beta,l_0})^p(1+s)^{-2l}\|f_0\|_{2,\beta+l}^2\big(\|f_0\|_{\infty,\beta} + \|f_0\|_{2,\beta+l}\big)^{p-2}.$$

Taking the integral over $s \in (0,\infty)$, we have the first inequality of (5.13). The second inequality of (5.13) comes from the Young inequality, then we complete the proof. □

5.3 $L^2 - L^\infty$ Estimate for (4.5)

We will derive another type of $L^2 - L^\infty$ estimate to obtain a uniform Hölder estimate for a weak solution of (1.22) in the sense of Definition 4.1. The proof is similar to the case of Subsect. 5.2.

Let us multiply (4.5) by w^ϑ, then $h := w^\vartheta f = f^\vartheta$ satisfies

$$(\partial_t + v \cdot \nabla_x - \bar{A}_g^\vartheta)h = \tilde{K}_g^\vartheta h, \tag{5.14}$$

where

$$\tilde{K}_g^\vartheta h = \left(2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right)h. \tag{5.15}$$

Similar to Definition 4.1, we can define a weak solution of (5.14).

Then we split f^ϑ into two parts:

$$f^\vartheta = f^\vartheta \mathbf{1}_{\{|v|\le M\}} + f^\vartheta \mathbf{1}_{\{|v|\ge M\}} =: f_1 + f_2.$$

Let $U^\vartheta(t,s)f_0$ be a weak solution of (5.1) in the sense of Definition 3.2 corresponding to the initial data f_0 with the initial time $t = s$, then we have

$$\begin{aligned} f_1(t) &= \mathbf{1}_{|v|\le M} U^\vartheta(t,0) f_0^\vartheta + \mathbf{1}_{|v|\le M}\int_0^t U^\vartheta(t,\tau)\tilde{K}_g^\vartheta f^\vartheta(\tau)\mathrm{d}\tau \\ &= \mathbf{1}_{|v|\le M} U^\vartheta(t,0) f_0^\vartheta + \int_0^t \mathbf{1}_{|v|\le M} U^\vartheta(t,\tau)\tilde{K}_g^\vartheta f^\vartheta(\tau)\mathrm{d}\tau. \end{aligned}$$

Lemma 5.12 *Assume (2.6). There exists $C = C_\vartheta > 0$ such that*

$$\|\tilde{K}_g^\vartheta f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le C\|f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{5.16}$$

$$\|\tilde{K}_g^\vartheta 1_{|v|>M} f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le C(1+M)^{-1}\|f^\vartheta\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{5.17}$$

and

$$\|\tilde{K}_g^\vartheta f^\vartheta\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} \le C\|f^\vartheta\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}. \tag{5.18}$$

Proof Since

$$\tilde{K}_g^\vartheta f^\vartheta = \left(2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right)f^\vartheta$$

and by Lemma 2.2,

$$\left|2\frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}}\sigma_G^{ij}\right| + \left|\frac{\partial_{ij} w^\vartheta}{w^\vartheta}\sigma_G^{ij}\right| + \left|\frac{\partial_j w^\vartheta}{w^\vartheta}\partial_i\sigma_G^{ij}\right| + \left|\frac{\partial_i w^\vartheta}{w^\vartheta}a_g^i\right| \le C(1+|v|)^{-1}.$$

So the proof is complete. ☐

Lemma 5.13 *Assume* (2.6). *Let f be a weak solution of* (4.5) *in a periodic box in the sense of* Definition 3.2, *then there exist C, $\beta > 0$ satisfying the following property: for any Z, $s > 1$, ϑ, and $k > 0$, and $l \in \mathbb{N}$, there exists $C_{\vartheta,l}$ such that*

$$\begin{aligned}\left\|1_{|v|<Zs^k} f^\vartheta(s)\right\|_\infty \\ \le C_{\vartheta,l}\left(Zs^k\right)^\beta (1+s)^{-l}\|f_0\|_{2,\vartheta+l} + \frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\|f^\vartheta(s')\|_\infty,\end{aligned} \tag{5.19}$$

for any $s \ge 1$.

Proof By the Duhamel principle,

$$\begin{aligned}\|1_{|v|<Zs^k} f^\vartheta(s)\|_{L^\infty} &\le \|1_{|v|<Zs^k} U^\vartheta(s,s-1)f^\vartheta(s-1)\|_{L^\infty} \\ &\quad + \int_0^{1-\varepsilon}\|1_{|v|\le Zs^k} U^\vartheta(s,s-1+\tau)\tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_\infty \mathrm{d}\tau \\ &\quad + \int_{1-\varepsilon}^{1}\|1_{|v|\le Zs^k} U^\vartheta(s,s-1+\tau)\tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_\infty \mathrm{d}\tau \\ &= (i) + (ii) + (iii).\end{aligned}$$

By Lemma 5.8, there exists $m > 0$ such that

$$(i) \le C\left(Zs^k\right)^m \left(\int_{s-1}^{s}\|U^\vartheta(s',s-1)f^\vartheta(s-1)\|_2^2 \mathrm{d}s'\right)^{1/2}.$$

By Theorem 4.6 and Lemma 3.4, for every integer l, there exists C_l such that

$$\begin{aligned}\|U^\vartheta(s',s-1)f^\vartheta(s-1)\|_2 \le C\|f^\vartheta(s-1)\|_2 &= C\|f(s-1)\|_{2,\vartheta} \\ &\le C_{\vartheta,l}\left(1+\frac{s-1}{l}\right)^{-l}\|f_0\|_{2,\vartheta+l} \\ &\le C_{\vartheta,l}(1+s)^{-l}\|f_0\|_{2,\vartheta+l}.\end{aligned}$$

Thus

$$(i) \le C_{\vartheta,l}\left(Zs^k\right)^m (1+s)^{-l}\|f_0\|_{2,\vartheta+l}.$$

By the maximum principle and (5.16),

$$(iii) \le C\varepsilon \sup_{s'\in(s-1,s)}\|f^\vartheta(s')\|_\infty.$$

By Lemma 5.8,

$$\begin{aligned}&\|1_{|v|\le Zs^k}U^\vartheta(s,s-1+\tau)\tilde{K}_g^\vartheta f(s-1+\tau)\|_\infty\\&\quad\le C\big(Zs^k\big)^m\Big(1+\frac{1}{1-\tau}\Big)^m\Big(\int_s^{s-1+\tau}\|1_{|v|<2Zs^k}U^\vartheta(s',s-1+\tau)\tilde{K}_g^\vartheta f(s-1+\tau)\|_2^2\mathrm{d}s'\Big)^{1/2}.\end{aligned}$$

By (5.18) and Theorem 4.6, for any $N>0$,

$$\begin{aligned}(ii)&\le C\int_0^{1-\varepsilon}\big(Zs^k\big)^m\Big(1+\frac{1}{1-\tau}\Big)^m\Big(\int_0^{1-\tau}\|\tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_2^2\mathrm{d}s'\Big)^{1/2}\mathrm{d}\tau\\&\le C\big(Zs^k\big)^m\Big(1+\frac{1}{\varepsilon}\Big)^m\int_0^1 C\|f^\vartheta(s-1+\tau)\|_2\mathrm{d}\tau\\&\le\big(Zs^k\big)^m\Big(1+\frac{1}{\varepsilon}\Big)^m C_{\vartheta,l}(1+s)^{-l}\|f_0\|_{2,\vartheta-l}.\end{aligned}$$

Choose $\varepsilon^{-1}=1+Zs^k$. Then

$$(i)+(ii)+(iii)\le C_{\vartheta,l}\big(Zs^k\big)^\beta(1+s)^{-l}\|f_0\|_{2,\vartheta-l}+\frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\|f^\vartheta(s')\|_\infty,$$

where $\beta=2m+1$. □

Theorem 5.14 *Assume* (2.6). *Let f be a weak solution of* (1.7), (4.5) *in a periodic box in the sense of* Definition 3.2. *Then there exists l such that for every* $\vartheta>0$,

$$\|f^\vartheta(t)\|_{L^\infty}\le C\|f_0^\vartheta\|_{L^\infty}+C_{\vartheta,l}\|f_0\|_{2,\vartheta+l}\le C\|f_0\|_{\infty,\vartheta+l_0},\ \text{ for any } t>0, \tag{5.20}$$

where $l_0=l+2$.

Proof By Lemma 5.13, there exists l_0 such that for $l>l_0$,

$$\Big\|1_{|v|<Zs^k}f^\vartheta(s)\Big\|_{L^\infty}\le C_{\vartheta,l,Z}(1+s)^{-2}\|f_0\|_{2,\vartheta+l}+\frac{C}{1+Zs^k}\sup_{s'\in(s-1,s)}\|f^\vartheta(s')\|_\infty.$$

Therefore by the Duhamel principle, the maximum principle, and (5.17),

$$\begin{aligned}\|f^\vartheta(n+1)\|_{L^\infty}&\le\|U^\vartheta(n+1,n)f^\vartheta(n)\|_{L^\infty}\\&\quad+\int_1^0\Big\|U^\vartheta(n+1,n+s)\tilde{K}_g^\vartheta\big(1_{|v|<Z(n+s)^k}f^\vartheta(n+s)+1_{|v|>Z(n+s)^k}f^\vartheta(n+s)\big)\Big\|_{L^\infty}\mathrm{d}s\\&\le\|f^\vartheta(n)\|_{L^\infty}+C_{\vartheta,l,Z}\int_0^1(1+n+s)^{-2}\|f_0\|_{2,\vartheta+l}\\&\quad+\frac{C}{1+Z(n+s)^k}\sup_{s'\in(n+s-1,n+s)}\|f^\vartheta(s')\|_\infty\mathrm{d}s\\&\quad+\int_0^1(1+Z(n+s)^k)^{-1}\|f^\vartheta(n+s)\|_\infty\mathrm{d}s\\&\le\|f^\vartheta(n)\|_{L^\infty}+C_{\vartheta,l,Z}(1+n)^{-2}\|f_0\|_{2,\vartheta+l}+C(Zn^k)^{-1}\sup_{s'\in[n-1,n+1]}\|f^\vartheta(s')\|_\infty.\end{aligned}$$

After iteration,

$$\|f^{\vartheta}(n+1)\|_{L^\infty} \leq \|f^{\vartheta}(1)\|_{L^\infty} + C_{\vartheta,l,Z}\sum_{\bar{n}=1}^{n}(1+\bar{n})^{-2}\|f_0\|_{2,\vartheta+l}$$
$$+ CZ^{-1}\sum_{\bar{n}=1}^{n}\bar{n}^{k(-1)}\sup_{s\in[0,n+1]}\|f^{\vartheta}(s)\|_\infty.$$

Choose large k and Z such that $k(-1) < -1$ and $CZ^{-1}\sum_{\bar{n}=1}^{\infty}\bar{n}^{k(-1)} \leq \varepsilon$, where ε will be determined later. Then

$$\|f^{\vartheta}(n+1)\|_{L^\infty} \leq \|f^{\vartheta}(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta+l} + \varepsilon\sup_{s\in[0,n+1]}\|f^{\vartheta}(s)\|_\infty.$$

Since n is an arbitrary integer,

$$\sup_{s=1,2,\ldots n+1}\|f^{\vartheta}(s)\|_{L^\infty} \leq \|f^{\vartheta}(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta,+l} + \varepsilon\sup_{s\in[0,n+1]}\|f^{\vartheta}(s)\|_\infty.$$

By the Duhamel principle, the maximum principle, and (5.17),

$$\begin{aligned}
\|f^{\vartheta}(n+t)\|_{L^\infty} &\leq \|U^{\vartheta}(n+t,n)f^{\vartheta}(n)\|_{L^\infty} + \int_0^t \left\|U^{\vartheta}(n+t,n+s)\tilde{K}_g^{\vartheta}f^{\vartheta}(n+s)\right\|_{L^\infty}\mathrm{d}s \\
&\leq \|f^{\vartheta}(n)\|_{L^\infty} + \int_0^t \left\|\tilde{K}_g^{\vartheta}f^{\vartheta}(n+s)\right\|_{L^\infty}\mathrm{d}s \\
&\leq \|f^{\vartheta}(n)\|_{L^\infty} + C\int_0^t \left\|f^{\vartheta}(n+s)\right\|_{L^\infty}\mathrm{d}s.
\end{aligned}$$

By the Gronwall inequality,

$$\|f^{\vartheta}(n+t)\|_{L^\infty} \leq C\|f^{\vartheta}(n)\|_\infty, \text{ for all } t\in[0,1].$$

Therefore,

$$\sup_{s\in[0,n+1]}\|f^{\vartheta}(s)\|_{L^\infty} \leq C\|f_0^{\vartheta}\|_{L^\infty} + C_{\vartheta,l}\|f_0\|_{2,\vartheta+l} + C\varepsilon\sup_{s\in[0,n+1]}\|f^{\vartheta}(s)\|_\infty.$$

Now, we choose small ε satisfying $C\varepsilon < 1/2$, and then absorb the last term on the RHS to the LHS. Thus, we obtain the first inequality of (5.20). The second inequality of (5.20) is a consequence of Proposition 2.1. ☐

6 L^∞ to Hölder Estimate

6.1 Local Hölder Estimate

In this section, we will derive a local Hölder estimate for (4.5). We redefine $Q_R(z_0) := (t_0-R^2,t_0]\times B(x_0;R^3)\times B(v_0;R)$, $z_0=(t_0,x_0,v_0)$, and $Q_R := Q_R((0,0,0))$.

Since we consider the local properties of the solution on the interior part, we can use the technique in [10] for our modified operator $\tilde{A_g}$. In this subsection, we assume that g satisfies the conditions in Lemma 2.4.

First, we introduce a De Giorgi-type lemma.

Lemma 6.1 (Lemma 13 in [10]) *Assume* (2.6). *Let* $\hat{Q} := Q_{1/4}(0,0,-1)$. *For any (universal) constants* $\delta_1 \in (0,1)$ *and* $\delta_2 \in (0,1)$ *there exist* $\nu > 0$ *and* $\vartheta \in (0,1)$ *(both universal) such that for any solution* f *of* (4.5) *in* Q_2 *with* $|f| \le 1$ *and*

$$|\{f \ge 1-\vartheta\} \cap Q_{1/4}| \ge \delta_1 |Q_{1/4}|,$$
$$|\{f \le 0\} \cap \hat{Q}| \ge \delta_2 |\hat{Q}|,$$

we have

$$|\{0 < f < 1-\vartheta\} \cap B_1 \times B_1 \times (-2,0]| \ge \nu.$$

Proof The proof is exactly the same as [10]. We omit the proof. □

Lemma 6.2 (Lemma 17 in [10]) *Assume* (2.6). *Let* $\hat{Q} := Q_{1/4}(0,0,-1)$ *and* f *be a weak solution of* (4.5) *in* Q_2 *in the sense of* Definition 3.2 *with* $|f| \le 1$. *If*

$$|\{f \le 0\} \cap \hat{Q}| \ge \delta_2 |\hat{Q}|,$$

then

$$\sup_{Q_{1/8}} f \le 1-\lambda$$

for some $\lambda \in (0,1)$, *depending only on dimension and the eigenvalue of* σ.

Proof The proof is exactly the same as [10]. We omit the proof. □

The following lemma can be derived by the previous lemma.

Lemma 6.3 *Assume* (2.6). *Let* f *be a weak solution of* (4.5) *in* Q_2 *in the sense of* Definition 3.2 *with* $|f| \le 1$. *Then*

$$\sup_{Q_{1/8}} f - \inf_{Q_{1/8}} f \le 2-\lambda$$

for some $\lambda \in (0,2)$, *depending only on dimension and the eigenvalue of* σ.

By the scaling argument, Q_2 and $Q_{1/8}$ can be replaced by Q_{2r} and $Q_{r/8}$.

Lemma 6.4 *Assume* (2.6). *Let* f *be a weak solution of* (4.5) *in* Q_{2r} *in the sense of* Definition 3.2 *with* $|f| \le 1$. *For any subset* $Q \subset \mathbb{R}^7$, *define*

$$\operatorname{Osc}_Q f := \sup_{(t',x',v') \in Q} f(t',x',v') - \inf_{(t',x',v') \in Q} f(t',x',v').$$

Then for every $r \le 1$,

$$\operatorname*{Osc}_{Q_{r/8}} f \le \left(1 - \frac{\lambda}{2}\right) \operatorname*{Osc}_{Q_{2r}} f$$

for some $\lambda \in (0, 2)$, depending only on dimension and the eigenvalue of σ.

Proof Define

$$\bar{F}(t, x, v) := \frac{2}{\operatorname*{Osc}_{Q_{2r}} f} \left(f(r^2 t, r^3 x, rv) - \frac{\sup_{Q_{2r}} f + \inf_{Q_{2r}} f}{2} \right).$$

Then $\bar{F}$ satisfies

$$\bar{F}_t + v \cdot \partial_x \bar{F} = \tilde{A_g}^r \bar{F},$$
$$\tilde{A_g}^r \bar{F}(t, x, v) := \nabla_v(\sigma_G(r^2 t, r^3 x, rv) \nabla_v \bar{F}(t, x, v)) + r a_g(r^2 t, r^3 x, rv) \cdot \nabla_v \bar{F}(t, x, v),$$

and then apply Lemma 6.3. □

Now we establish the Hölder continuity at $v = 0$.

Lemma 6.5 (Hölder continuity near $v = 0$) *Assume* (2.6). *Let f be a weak solution of* (4.5) *in $Q_R(t_0, x_0, 0)$ in the sense of* Definition 3.2. *Then there exist a uniform constant $C > 0$ and a constant $\alpha \in (0, 1)$ depending only on dimension and the eigenvalue of σ_G such that*

$$\| f \|_{C^\alpha(Q_{R/128}(t_0, x_0, 0))} \le \frac{C}{R^{3\alpha}} \| f \|_{L^\infty(Q_R(t_0, x_0, 0))},$$

for every $R < 1$.

Proof We first prove

$$\sup_{(s,y,w) \in Q_{R/16}} \frac{|f(s, y, w) - f(0, 0, 0)|}{(|s| + |y| + |w|)^\alpha} \le \frac{C}{R^{3\alpha}} \| f \|_{L^\infty(Q_R)}.$$

Define $\mathrm{Osc}_Q f$ as in Lemma 6.4 and

$$\varphi(r) := r^{-\alpha_0} \operatorname*{Osc}_{Q_r} f,$$

where $\alpha_0 > 0$ can be chosen later. By Lemma 6.4,

$$\operatorname*{Osc}_{Q_{r/16}} f \le \left(1 - \frac{\lambda}{2}\right) \operatorname*{Osc}_{Q_r} f. \tag{6.1}$$

Choose α_0 such that $16^{\alpha_0} \left(1 - \frac{\lambda}{2}\right) < 1$. Then by (6.1),

$$\varphi\left(\frac{r}{16}\right) = r^{-\alpha_0} 16^{\alpha_0} \operatorname*{Osc}_{Q_{r/4}(t,x,v)} f \le 16^{\alpha_0}\left(1-\frac{\lambda}{2}\right) r^{-\alpha_0} \operatorname*{Osc}_{Q_r} f < \varphi(r).$$

Therefore, we have

$$\sup_{0<r\le R/16} \varphi(r) \le \sup_{\frac{R}{16}<r\le R} \varphi(r) \le 2\frac{16^{\alpha_0}}{R^{\alpha_0}} \sup_{(t,x,v)\in Q_R} |f(t,x,v)|. \tag{6.2}$$

If $(t,x,v)\in \partial Q_r$ then $|t|+|x|+|v|\ge r^3$. Therefore, for $3\alpha=\alpha_0$ and $r\le R/16$, by (6.2),

$$\begin{aligned}\sup_{(s,y,w)\in Q_{R/16}} \frac{|f(s,y,w)-f(0,0,0)|}{(|s|+|y|+|w|)^\alpha} &= \sup_{(s,y,w)\in \partial Q_r, r\in(0,R/16)} \frac{|f(s,y,w)-f(0,0,0)|}{(|s|+|y|+|w|)^\alpha} \\ &\le \sup_{(s,y,w)\in Q_r, r\in(0,R/16)} \frac{|f(s,y,w)-f(0,0,0)|}{r^{\alpha_0}} \\ &\le \sup_{r\in(0,R/16)} \varphi(r) \\ &\le \frac{C}{R^{\alpha_0}} \sup_{(t,x,v)\in Q_R} |f(t,x,v)|.\end{aligned} \tag{6.3}$$

Now we consider the general case. For any $(t_*,x_*,v_*)\in Q_{R/32}(t_0,x_0,0)$, define the translated function

$$F(T,X,V)=f(t,x,v),$$
$$T=t-t_*,\quad X=x-x_*-Tv_*,\quad V=v-v_*.$$

Then F satisfies

$$\partial_T F + V\cdot\nabla_X F = \nabla_V\cdot(\Sigma_G(t,x,v)\nabla_V F) + a_g(t,x,v)\cdot\nabla_V F.$$

Therefore, by (6.3),

$$\sup_{(s,y,w)\in Q_{R_1/16}} \frac{|F(s,y,w)-F(0,0,0)|}{(|s|+|y|+|w|)^\alpha} \le \frac{C}{R_1{}^{\alpha_0}} \sup_{(t,x,v)\in Q_{R_1}} |F(t,x,v)|$$

for every $R_1<1$. Since $|v_*|\le R/128$,

$$(t,x,v)\in Q_{R/64}(t_*,x_*,v_*) \text{ implies } (T,X,V)\in Q_{R/32}$$

and

$$(T,X,V)\in Q_{R/2} \text{ implies } (t,x,v)\in Q_R(t_*,x_*,v_*).$$

Therefore, by (6.3)

$$
\begin{aligned}
&\sup_{(t,x,v)\in Q_{R/64}(t_*,x_*,v_*)} \frac{|f(t,x,v)-f(t_*,x_*,v_*)|}{(|t-t_*|+|x-x_*|+|v-v_*|)^\alpha}\\
&\quad\le (1+|v_*|)^\alpha \sup_{(t,x,v)\in Q_{R/64}(t_*,x_*,v_*)} \frac{|f(t,x,v)-f(t_*,x_*,v_*)|}{((1+|v_*|)|t-t_*|+|x-x_*|+|v-v_*|)^\alpha}\\
&\quad\le C \sup_{(T,X,V)\in Q_{R/32}} \frac{|F(T,X,V)-F(0,0,0)|}{((1+|v_*|)|T|+|X+v_*T|+|V|)^\alpha}\\
&\quad\le C \sup_{(T,X,V)\in Q_{R/32}} \frac{|F(T,X,V)-F(0,0,0)|}{(|T|+|X|+|V|)^\alpha}\\
&\quad\le \frac{C}{R^{\alpha_0}} \sup_{(T,X,V)\in Q_{R/2}} |F(T,X,V)|\\
&\quad\le \frac{C}{R^{\alpha_0}} \sup_{(t,x,v)\in Q_R} |f(t,x,v)|.
\end{aligned}
$$

So the proof is complete. □

6.2 Global Hölder Estimate

In this subsection, we will derive a Hölder continuity for the solution of (1.22). Let $f(t,x,v)$ be a weak solution of (1.22) in the sense of Definition 4.1. Then

$$
\tilde f(t,x,v) := \begin{cases} f(t,x,v), & \text{if } t\ge 0,\\ f_0(x,v), & \text{if } -1\le t<0 \end{cases}
$$

satisfies

$$
\tilde f_t + v\cdot\nabla_x \tilde f - \bar A_g \tilde f = \tilde S(t,x,v),
$$

where $\bar A_g$ and $\bar K_g$ is defined as in (3.2), and (3.3),

$$
\tilde S(t,x,v) = \begin{cases} (v\cdot\nabla_x - \bar A_{f_0})f_0(x,v), & \text{if } t\le 0,\\ \bar K_g f(t,x,v), & \text{if } t>0. \end{cases}
$$

Since $U(t,s)$ is the solution operator of (4.5). Then f satisfies

$$
f(t) = U(t,-1)f_0 + \int_{-1}^t U(t,s)\tilde S(s)\mathrm{d}s.
$$

Fisrt, we will obtain a uniform Hölder continuity of $U(t,s)f$. Finally, we will derive a uniform Hölder continuity of $f(t)$.

As a starting point, we introduce a technical lemma to obtain a uniform Hölder continuity of $U(t,s)f$.

Lemma 6.6 *Let* $(t_*, x_*, v_*) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, $N - 1/2 \le |v_*| \le N + 1/2$, $m > 9$ *and* O *be an orthonormal matrix. Define*

$$D := \begin{bmatrix} (1+|v_*|)^{-3/2} & 0 & 0 \\ 0 & (1+|v_*|)^{-1/2} & 0 \\ 0 & 0 & (1+|v_*|)^{-1/2} \end{bmatrix}, \tag{6.4}$$

$$X := D^{-1}O^T(x - v_*(t - t_*)), \quad X_* := D^{-1}O^T x_*, \quad V := D^{-1}O^T(v - v_*),$$
$$r_0 := (2+N)^{-m}, \quad r_1 := (2+N)^{-\frac{2m}{3}+\frac{5}{6}}, \quad r_2 := (2+N)^{-\frac{4m}{9}+\frac{13}{18}}.$$

Then if $(t, x, v) \in Q_{r_0}(t_*, x_*, v_*)$, *then* $(t, X, V) \in Q_{r_1}(t_*, X_*, 0)$. *Moreover, if* $(t, X, V) \in Q_{128r_1}(t, X_*, 0)$, *then* $(t, x, v) \in Q_{128r_2}(t_*, x_*, v_*)$.

Proof If $(t, x, v) \in Q_{r_0}(t_*, x_*, v_*)$, then

$$|t - t_*| \le r_0^2 \le r_1^2,$$

$$\begin{aligned} |X - X_*| &= |D^{-1}O^T(x - x_* - v_*(t - t_*))| \\ &\le (2+N)^{3/2}(r_0^3 + (N+1/2)r_0^2) \\ &\le (2+N)^{3/2}\left((2+N)^{-3m} + (N+1/2)(2+N)^{-2m}\right) \\ &\le (2+N)^{3/2}(2+N)^{1-2m} \\ &\le r_1^3, \end{aligned}$$

and

$$\begin{aligned} |V| &= |D^{-1}O^T(v - v_*)| \\ &\le (2+N)^{3/2} r_0 \\ &\le r_1. \end{aligned}$$

Conversely, if $(t, X, V) \in Q_{128r_1}(t, X_*, 0)$, then

$$|t - t_*| \le (128r_1)^2 \le (128r_2)^2$$

and

$$|v - v*| = |ODV| \le (1/2+N)^{-1/2} r_1 \le 128 r_2.$$

Since $128r_1 \le 1$ and $(1/2+N)^{-1/2}(1+N) \le 128(2+N)^{1/2}$, we have

$$\begin{aligned} |x - x_*| &= |OD(X - X_*) + v_*(t - t_*)| \\ &\le (1/2+N)^{-1/2}(r_1^3 + N r_1^2) \\ &\le (1/2+N)^{-1/2}(1+N) r_1^2 \\ &\le (128)^3 (2+N)^{\frac{1}{2}-\frac{4m}{3}+\frac{5}{3}} \\ &\le (128 r_2)^3. \end{aligned}$$

So the proof is complete. □

Lemma 6.7 (Uniform Hölder for (4.5)) *Assume* (2.6). *Let f be a solution of* (4.5) *in* $Q_1(t_0,x_0,v_0)$. *Then there exist* $\vartheta>0$, $\vartheta_0>0$, C_ϑ, *and* $\alpha\in(0,1)$ *depending only on dimension such that*

$$\sup_{(t,x,v),(t',x',v')\in Q_1(t_0,x_0,v_0)}\frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\le C\|f\|_{\infty,\vartheta}\le C_\vartheta\|f_0\|_{\infty,\vartheta+\vartheta_0}. \tag{6.5}$$

Proof By the integration by parts,

$$\begin{aligned}
a_g\cdot\nabla_v f&=-\{\phi^{ij}*[v_i\mu^{1/2}g]\}\partial_j f-\{\phi^{ij}*[\mu^{1/2}\partial_j g]\}\partial_i f\\
&=-\big(\{\phi^{ij}*[v_i\mu^{1/2}g]\}+\{\phi^{ij}*[\mu^{1/2}\partial_i g]\}\big)\partial_j f\\
&=-\big(2\{\phi^{ij}*[v_i\mu^{1/2}g]\}+\{\phi^{ij}*[\partial_i(\mu^{1/2}g)]\}\big)\partial_j f\\
&=-\big(2\{\phi^{ij}*[v_i\mu^{1/2}g]\}+\{\partial_i\phi^{ij}*[\mu^{1/2}g]\}\big)\partial_j f\\
&=-2v\cdot(\sigma_{\sqrt{\mu}g}\nabla_v f)-\partial_i\sigma^{ij}_{\sqrt{\mu}g}\partial_j f.
\end{aligned}$$

Let $N:=|v_0|$.

To obtain (6.5), we split the proof in two cases; $|(t,x,v)-(t',x',v')|\le(2+N)^{-3m}$ or $|(t,x,v)-(t',x',v')|>(2+N)^{-3m}$ for some $m>0$ to be determined later. For the first case, we will consider a new center $(t_*,x_*,v_*)\in Q_1$, such that $(t,x,v),(t',x',v')\in Q_{(2+N)^{-m}}(t_*,x_*,v_*)$. Note that $N-1/2\le|v_*|\le N+1/2$.

Therefore, it is enough to prove that for every $(t_*,x_*,v_*)\in Q_1(t_0,x_0,v_0)$,

$$\sup_{(t,x,v),(t',x',v')\in Q_{(2+N)^{-m}}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)}\frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\le C\|f\|_{\infty,\vartheta} \tag{6.6}$$

and

$$\sup_{\substack{(t,x,v),(t',x',v')\in Q_1(t_0,x_0,v_0),\\ |t-t'|+|x-x'|+|v-v'|>(2+N)^{-3m}}}\frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\le C\|f\|_{\infty,\vartheta}. \tag{6.7}$$

We first focus on (6.6). Consider the the following translation

$$\bar f(t,y,w):=f(t,x,v),$$

where $x=y+v_*(t-t_*)$, $v=v_*+w$. Then it is easy to check that $\bar f$ satisfies

$$\partial_t\bar f+w\cdot\nabla_y\bar f=\nabla_w\cdot(\bar\sigma_G\nabla_w\bar f)+(v_*+w)\cdot(\bar\sigma_{\sqrt{\mu}g}\nabla_w\bar f)-\sum_{ij}\partial_i\bar\sigma^{ij}_{\sqrt{\mu}g}\partial_j\bar f,$$

where $\bar\sigma_G(t,y,w):=\sigma_G(t,x,v)$, $\bar\sigma_\mu(t,y,w):=\sigma_\mu(t,x,v)$, and $\bar\sigma_{\sqrt{\mu}g}(t,y,w):=\sigma_{\sqrt{\mu}g}(t,x,v)$. Let O be an orthonormal constant matrix which will be determined later. Next consider

$$\tilde{f}(t,\xi,v) := \bar{f}(t,y,w),$$

where $y = O\xi$, $w = Ov$. Then we have

$$\begin{aligned}
\partial_t \bar{f}(t,y,w) &= \partial_t \tilde{f}(t,\xi,v),\\
w \cdot \nabla_y \bar{f}(t,y,w) &= \sum_i w_i \partial_{y_i}(\tilde{f}(t,\xi,v))\\
&= \sum_{i,k} w_i \partial_{\xi_k} \tilde{f}(t,\xi,v) \frac{\partial \xi_k}{\partial y_i}\\
&= \sum_{i,k} O_{ik} w_i \partial_{\xi_k} \tilde{f}(t,\xi,v)\\
&= \sum_k (O^T w)_k \cdot \partial_{\xi_k} \tilde{f}(t,\xi,v)\\
&= v \cdot \nabla_\xi \tilde{f}(t,\xi,v),
\end{aligned} \tag{6.8}$$

where O_{ik} is the (i,k) component of O. We used the following formula to derive the third equality in (6.8),

$$\frac{\partial \xi_k}{\partial y_i} = \frac{\partial \sum_l O_{lk} y_l}{\partial y_i} = O_{ik}.$$

Similarly,

$$\begin{aligned}
\bar{\sigma}_G(t,y,w)\nabla_w \bar{f}(t,y,w) &= \sum_j \bar{\sigma}_G^{ij}(t,y,w)\partial_{w_j}\tilde{f}(t,\xi,v)\\
&= \sum_{j,k} \bar{\sigma}_G^{ij}(t,y,w) O_{jk} \partial_{v_k} \tilde{f}(t,\xi,v).
\end{aligned} \tag{6.9}$$

Define $\tilde{\sigma}_G(t,\xi,v) := O^T \bar{\sigma}_G(t,y,w) O$. Note that

$$O\tilde{\sigma}_G(t,\xi,v) = \bar{\sigma}_G(t,y,w)O. \tag{6.10}$$

Then by (6.9) and (6.10), we have

$$\begin{aligned}
\nabla_w \cdot (\bar{\sigma}_G(t,y,w)\nabla_w \bar{f}(t,y,w)) &= \sum_{i,j} \partial_{w_i}(\bar{\sigma}_G^{ij}(t,y,w)\partial_{w_j}\tilde{f}(t,\xi,v))\\
&= \sum_{i,j,k} \partial_{w_i}(O_{jk}\bar{\sigma}_G^{ij}(t,y,w)\partial_{v_k}\tilde{f}(t,\xi,v))\\
&= \sum_{i,j,k} \partial_{w_i}(O_{ij}\tilde{\sigma}_G^{jk}(t,\xi,v)\partial_{v_k}\tilde{f}(t,\xi,v))\\
&= \sum_{i,j,k,l} O_{il}O_{ij}\partial_{v_*}(\tilde{\sigma}_G^{jk}(t,\xi,v)\partial_{v_k}\tilde{f}(t,\xi,v))\\
&= \sum_{k,l} \partial_{v_*}(\tilde{\sigma}_G^{lk}(t,\xi,v)\partial_{v_k}\tilde{f}(t,\xi,v))\\
&= \nabla_v \cdot (\tilde{\sigma}_G(t,\xi,v)\nabla_v \bar{f}(t,\xi,v)).
\end{aligned}$$

In the last equality, we used $O^T O = I$. Similarly, define $\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) := O^T \bar{\sigma}_{\sqrt{\mu}g}(t, y, w) O$, then

$$\begin{aligned} v_* \cdot (\bar{\sigma}_{\sqrt{\mu}g}(t, y, w) \nabla_w \bar{f}(t, y, w)) &= v_* \cdot (O \tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)) \\ &= (O^T v_*) \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)) \\ &= v_* \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)), \end{aligned}$$

where $v_* = O^T v_*$,

$$\begin{aligned} w \cdot (\bar{\sigma}_{\sqrt{\mu}g}(t, y, w) \nabla_w \bar{f}(t, y, w)) &= w \cdot (O \tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)) \\ &= (O^T w) \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)) \\ &= v \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, v) \nabla_v \tilde{f}(t, \xi, v)), \end{aligned}$$

and

$$\begin{aligned} \sum_{i,j} \partial_{w_i} \bar{\sigma}^{ij}_{\sqrt{\mu}g}(t, y, w) \partial_{w_j} \bar{f}(t, y, w) &= \sum_{i,j,k} \partial_{w_i} \bar{\sigma}^{ij}_{\sqrt{\mu}g}(t, y, w) \partial_{w_j} \tilde{f}(t, \xi, v) \\ &= \sum_{i,j,k} \partial_{w_i} \bar{\sigma}^{ij}_{\sqrt{\mu}g}(t, y, w) O_{jk} \partial_{v_k} \tilde{f}(t, \xi, v) \\ &= \sum_{i,j,k,l} O_{il} \partial_{v_*} \bar{\sigma}^{ij}_{\sqrt{\mu}g}(t, y, w) O_{jk} \partial_{v_k} \tilde{f}(t, \xi, v) \\ &= \sum_{k,l} \partial_{v_*} \tilde{\sigma}^{lk}_{\sqrt{\mu}g}(t, \xi, v) \partial_{v_k} \tilde{f}(t, \xi, v). \end{aligned}$$

Therefore $\tilde{f}$ satisfies

$$\partial_t \tilde{f} + v \cdot \nabla_\xi \tilde{f} = \nabla_v \cdot (\tilde{\sigma}_G \nabla_v \tilde{f}) + (v_* + v) \cdot (\tilde{\sigma}_{\sqrt{\mu}g} \nabla_v \tilde{f}) - \sum_{k,l} \partial_l \tilde{\sigma}^{lk}_{\sqrt{\mu}g} \partial_k \tilde{f}.$$

We split $\tilde{\sigma}_G(t, \xi, v)$ in three parts.

$$\begin{aligned} \tilde{\sigma}_G(t, \xi, v) = &\, O^T \bar{\sigma}_\mu(0) O \\ &+ O^T \left(\bar{\sigma}_\mu(w) - \bar{\sigma}_\mu(0) \right) O \\ &+ O^T \bar{\sigma}_{\sqrt{\mu}g}(t, y, w) O \\ = &\, \tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3. \end{aligned}$$

Choose orthonormal vectors $o_1 = v_*/|v_*|, o_2, o_3$ and

$$O := \begin{bmatrix} o_1 & o_2 & o_3 \end{bmatrix}.$$

Note that

$$v_* = O^T v_* = \begin{bmatrix} |v_*| \\ 0 \\ 0 \end{bmatrix}.$$

Moreover $\bar{\sigma}_\mu(0)$ has a simple eigenvalue $\lambda_1(v_*)$ associated with the vector v_*, and a double eigenvalue $\lambda_2(v_*)$ associated with $v^\perp$. Therefore,

$$\tilde{\sigma}_1 = \begin{bmatrix} \lambda_1(v_*) & 0 & 0 \\ 0 & \lambda_2(v_*) & 0 \\ 0 & 0 & \lambda_2(v_*) \end{bmatrix}.$$

Note that $\lambda_1(v_*)$ and $\lambda_2(v_*)$ satisfy

$$\begin{aligned} &\frac{1}{C}(1+N)^{-3} \leq \lambda_1(v_*) \leq C(1+N)^{-3}, \\ &\frac{1}{C}(1+N)^{-1} \leq \lambda_2(v_*) \leq C(1+N)^{-1}. \end{aligned} \tag{6.11}$$

Since $\partial_{v_k}(\sigma_\mu)^{ij}(v) \leq C(1+|v|)^{-2}$, by the mean value theorem,

$$|(\bar{\sigma}_\mu)^{ij}(w) - (\bar{\sigma}_\mu(0))| \leq C(1+N)^{-3+1}(2+N)^{-m}.$$

Therefore,

$$|(\tilde{\sigma}_2)^{ij}| \leq C(1+N)^{-2}(2+N)^{-m}.$$

Define

$$D_u(v, v'; v) := v^T \sigma_u(v) v'.$$

Then we can easily check that

$$|D_u(v, v'; v)| \leq |D_u(v, v; v)|^{1/2} |D_u(v', v'; v)|^{1/2}.$$

Since $|v - v_*| < (2+N)^{-m}$ and $o_1 = v_*/|v_*|$, we have

$$\begin{aligned} |(I - P_v)o_1| &= \frac{|(I - P_v)v_*|}{|v_*|} \\ &= \frac{|-v + v_* + v - P_v v_*|}{|v_*|} \\ &\leq \frac{|v - v_*| + |v - P_v v_*|}{|v_*|} \\ &= \frac{|v - v_*| + |P_v(v - v_*)|}{|v_*|} \\ &= 2\frac{|v - v_*|}{|v_*|} \leq C(2+N)^{-m}. \end{aligned} \tag{6.12}$$

Note that

$$(\tilde{\sigma}_3)^{ij} = o_i^T \sigma_{\sqrt{\mu} g}(v) o_j. \tag{6.13}$$

Therefore, by (2.7),

$$\begin{aligned}|(\tilde{\sigma}_3)^{11}| &= |D_{\sqrt{\mu}g}(o_1;v)|\\ &\le C\|g\|_\infty\big((1+|v|)^{-3}|P_v o_1|^2+(1+|v|)^{-1}|(I-P_v)o_1|^2\big)\\ &\le C\|g\|_\infty\big((1+N)^{-3}+(1+N)^{-1}(2+N)^{-2m}\big)\\ &\le C\|g\|_\infty(1+N)^{-3},\end{aligned} \tag{6.14}$$

and for $(i,j)\in\{(1,2),(1,3),(2,1),(3,1)\}$,

$$\begin{aligned}|(\tilde{\sigma}_3)^{ij}| &\le |D_{\sqrt{\mu}g}(o_1;v)|^{1/2}|D_{\sqrt{\mu}g}(o_k;v)|^{1/2}\\ &\le C\|g\|_\infty(1+N)^{-3/2}\big((1+|v|)^{-3}|P_v o_k|^2+(1+|v|)^{-1}|(I-P_v)o_k|^2\big)^{1/2}\\ &\le C\|g\|_\infty(1+N)^{-3/2}(1+N)^{-1/2}\\ &= C\|g\|_\infty(1+N)^{-2},\end{aligned}$$

where $k=2$ or 3. Finally, for $i,j=2$ or 3,

$$\begin{aligned}|(\tilde{\sigma}_3)^{ij}| &\le |D_{\sqrt{\mu}g}(o_i;v)|^{1/2}|D_{\sqrt{\mu}g}(o_j;v)|^{1/2}\\ &= C\|g\|_\infty\big((1+|v|)^{-3}|P_v o_i|^2+(1+|v|)^{-1}|(I-P_v)o_i|^2\big)^{1/2}\\ &\quad\times\big((1+|v|)^{-3}|P_v o_j|^2+(1+|v|)^{-1}|(I-P_v)o_j|^2\big)^{1/2}\\ &\le C\|g\|_\infty(1+N)^{-1}.\end{aligned}$$

Finally, consider the dilation matrix D as in (6.4) and the dilated function

$$F(t,X,V) := \tilde{f}(t,\xi,v),$$

where $\xi = DX$, $v = DV$. Then we can easily check that F satisfies

$$\partial_t F + V\cdot\nabla_X F = \nabla_V\cdot(\Sigma\nabla_V F) + (v_l+v)^T D\Sigma_3\nabla_V F + \sum_{k,l}\partial_{V_l}\Sigma_3^{lk}\partial_{V_k}F,$$

where $\Sigma=\Sigma_1+\Sigma_2+\Sigma_3$, $\Sigma_i(t,X,V)=D^{-1}\tilde{\sigma}_i(t,\xi,v)D^{-1}$ for $i=1,2,3$. Then by (6.11), we have

$$\frac{1}{C}\le(\Sigma_1)^{ii}\le C,\quad (\Sigma_1)^{ij}=0 \text{ for } i\ne j$$

and

$$\begin{aligned}|(\Sigma_2)^{ij}| &\le C(1+N)(2+N)^{-m},\\ |(\Sigma_3)^{ij}| &\le C\|g\|_\infty.\end{aligned}$$

Moreover, since $|v|\le(2+N)^{-m}$,

$$\begin{aligned}|D(v_*+v)| &\le |Dv_*|+|Dv|\\ &\le (1+N)^{-1/2}+C(1+N)^{-1/2}(2+N)^{-m},\end{aligned}$$

and since $\partial_{v_k}\tilde{\sigma}_3(v) \le C\|g\|_\infty(1+N)^{-2}$, we have

$$\begin{aligned}|\partial_{V_l}\Sigma_3^{lk}| &= |d_l^{-1}d_k^{-1}\partial_{V_l}\tilde{\sigma}_3(v)|\\ &\le d_k^{-1}|\partial_{v_l}\tilde{\sigma}_3(v)|\\ &\le \|g\|_\infty(1+N)^{-1/2},\end{aligned}$$

where d_ks are the kth diagonal element of D. Choose $m>4$ such that $|(\Sigma_2)^{ij}|, |(v_*+v)^T D\Sigma_3| \le \varepsilon \ll 1$. If $\|g\|_\infty \le \varepsilon$, then any eigenvalue of Σ is bounded above and below uniformly in N. Therefore, by Lemma 6.5, there exist a constant $C>0$ uniformly in N and a constant $\alpha \in (0,1)$, depending only on dimension such that

$$\begin{aligned}&\sup_{(t,X,V),(t',X',V')\in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V)-F(t',X',V')|}{(|t-t'|+|X-X'|+|V-V'|)^\alpha}\\ &\le \frac{C}{(r_1)^{3\alpha}}\|f\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))},\end{aligned} \tag{6.15}$$

where $X_* = D^{-1}\xi_*$, $\xi_* = O^T y_l$, $y_l = x_* - v_* t_*$ and r_1 is defined as in Lemma 6.5. Note that

$$\begin{aligned}&\frac{1}{(1+N)^\alpha}\sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\\ &\le \sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{((2+N)|t-t'|+|x-x'|+|v-v'|)^\alpha},\end{aligned} \tag{6.16}$$

where r_0 and r_1 are defined as in Lemma 6.6. Moreover, we have

$$\begin{aligned}&(2+N)|t-t'|+|x-x'|+|v-v'|\\ &= (2+N)|t-t'| + |ODX + v_*(t-t_*) - (ODX' + v_*(t'-t_*))| + |ODV + v_* - (ODV' + v_*)|\\ &\ge (2+N)|t-t'| + |OD(X-X')| - |v_*||t-t'| + |OD(V-V')|\\ &\ge |t-t'| + |OD(X-X')| + |OD(V-V')|\\ &\ge (1+N)^{-3/2}(|t-t'| + |(X-X')| + |(V-V')|).\end{aligned} \tag{6.17}$$

By (6.15), (6.16), (6.17), and Lemma 6.6, we have

$$\begin{aligned}&\sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{((2+N)|t-t'|+|x-x'|+|v-v'|)^\alpha}\\ &\le \sup_{(t,X,V),(t',X',V')\in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V)-f(t',X',V')|}{(1+N)^{-3\alpha/2}(|t-t'|+|(X-X')|+|(V-V')|)^\alpha}.\end{aligned} \tag{6.18}$$

Combine (6.15), (6.16), and (6.18). Then we have

$$
\begin{aligned}
&\sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\\
&\le (2+N)^\alpha \sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{((2+N)|t-t'|+|x-x'|+|v-v'|)^\alpha}\\
&\le \frac{(2+N)^\alpha}{(1+N)^{-3\alpha/2}} \sup_{(t,X,V),(t',X',V')\in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V)-f(t',X',V')|}{(|t-t'|+|X-X'|+|(V-V'|)^\alpha}\\
&\le \frac{C(2+N)^\alpha}{(1+N)^{-3\alpha/2}(r_1)^{3\alpha}}\|f\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))}.
\end{aligned}
\tag{6.19}
$$

By Lemma 6.6,

$$
\|f\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))} \le \|f\|_{L^\infty(Q_{128r_2}(t_*,x_*,v_*))}. \tag{6.20}
$$

Choose $m > 9$ such that $128r_2 < 1$. Then we have

$$
\begin{aligned}
\|f\|_{L^\infty(Q_{128r_2}(t_*,x_*,v_*))} &\le C(1+N)^{-\vartheta}\|(1+|v|)^\vartheta f\|_{L^\infty(Q_{128r_2}(t_*,x_*,v_*))}\\
&\le C(1+N)^{-\vartheta}\|(1+|v|)^\vartheta f\|_{L^\infty},
\end{aligned}
\tag{6.21}
$$

for every $\vartheta > 0$. Finally combining (6.19)–(6.21), we have

$$
\begin{aligned}
&\sup_{(t,x,v),(t',x',v')\in Q_{r_0}(t_*,x_*,v_*)\cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\\
&\le \frac{C(1+N)^{\alpha-\vartheta}}{(1+N)^{-3\alpha/2}(r_1)^{3\alpha}}\|(1+|v|)^\vartheta f\|_{L^\infty}\\
&\le C(1+N)^{\alpha-\vartheta+\frac{3\alpha}{2}+2m\alpha-\frac{5\alpha}{2}}\|(1+|v|)^\vartheta f\|_{L^\infty}\\
&= C(1+N)^{-\vartheta+2m\alpha}\|(1+|v|)^\vartheta f\|_{L^\infty},
\end{aligned}
\tag{6.22}
$$

where $\vartheta > 0$ will be determined later.

To prove (6.7),

$$
\begin{aligned}
&\sup_{\substack{(t,x,v),(t',x',v')\in Q_1(t_0,x_0,v_0)\\ |t-t'|+|x-x'|+|v-v'|>(2+N)^{-3m}}} \frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha}\\
&\le 2(2+N)^{3\alpha m}\|f\|_{L^\infty(Q_1(t_0,x_0,v_0))}\\
&\le 2C(1+N)^{3\alpha m-\vartheta}\|(1+|v|)^\vartheta f\|_{L^\infty(Q(t_0,x_0,v_0;1))}\\
&\le 2C(1+N)^{3\alpha m-\vartheta}\|(1+|v|)^\vartheta f\|_{L^\infty}.
\end{aligned}
\tag{6.23}
$$

Now choose

$$
\vartheta > 3\alpha m. \tag{6.24}
$$

Then from (6.22) to (6.24), we prove (6.6) and (6.7). Therefore, we have

$$\sup_{(t,x,v),(t',x',v')\in Q_1(t_0,x_0,v_1)} \frac{|f(t,x,v)-f(t',x',v')|}{(|t-t'|+|x-x'|+|v-v'|)^\alpha} \le C\|f\|_{\infty,\vartheta}.$$

By Theorem 5.14, we have (6.5). □

Now we will prove Theorem 1.4.

Proof of Theorem 1.4 Since f satisfies (1.22), we have

$$f_t + v\cdot\nabla_x f - \bar{A}_g f = \bar{K}_g f. \tag{6.25}$$

Define

$$\tilde{f}(t,x,v) = \begin{cases} f(t,x,v), & \text{if } t \ge 0, \\ f_0(x,v), & \text{if } -1 \le t < 0. \end{cases}$$

Consider $\tilde{S}(t,x,v) = (\partial_t + v\cdot\nabla_x - \bar{A}_g)\tilde{f}$. Then for $t \le 0$,

$$\begin{aligned}\tilde{S}(t,x,v) &= (\partial_t + v\cdot\nabla_x - \bar{A}_g)\tilde{f} \\ &= (v\cdot\nabla_x - \bar{A}_{f_0})f_0(x,v) \\ &= -f_{0t},\end{aligned}$$

where f_{0t} was defined in Theorem 1.1. Since f is a weak solution of (1.22) in the sense of Definition 4.1, for $t>0$, $\tilde{S}(t,x,v) = \bar{K}_g f(t,x,v)$. Thus, $\tilde{f}$ satisfies

$$\tilde{f}_t + v\cdot\nabla_x\tilde{f} - \bar{A}_g\tilde{f} = \tilde{S}(t,x,v).$$

Since $U(t,s)$ is the solution operator for $\partial_t + v\cdot\nabla_x - \bar{A}_g = 0$. Then $\tilde{f}$ satisfies

$$\tilde{f}(t) = U(t,-1)f_0 + \int_{-1}^t U(t,s)\tilde{S}(s)\mathrm{d}s.$$

Let $0<\bar{\varepsilon}\ll 1$ be given. Note that by Lemma 6.5, there exists $\alpha>0$ such that $U(t,-1)\tilde{f}(-1)$ is uniformly Hölder continuous on $(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$.

For every $0\le t_2\le t_1$, $|t_1-t_2|+|x_2-x_1|+|v_2-v_1|=\bar{\varepsilon}$,

$$\begin{aligned}&|f(t_1,x_1,v_1)-f(t_2,x_2,v_2)| \\ &\quad\le \big|(U(t_1,-1)f_0)(x_1,v_1)-(U(t_2,-1)f_0)(x_2,v_2)\big| \\ &\qquad + \left|\int_{t_2}^{t_1}(U(t_1,s)\tilde{S}(s))(x_1,v_1)\mathrm{d}s\right| \\ &\qquad + \left|\int_{t_2-\bar{\varepsilon}^\alpha}^{t_2}\big((U(t_1,s)\tilde{S}(s))(x_1,v_1)-(U(t_2,s)\tilde{S}(s))(x_2,v_2)\big)\mathrm{d}s\right| \\ &\qquad + \left|\int_{-1}^{t_2-\bar{\varepsilon}^\alpha}\big((U(t_1,s)\tilde{S}(s))(x_1,v_1)-(U(t_2,s)\tilde{S}(s))(x_2,v_2)\big)\mathrm{d}s\right| \\ &\quad\le (I)+(II)+(III)+(IV).\end{aligned}$$

By Lemma 6.7, there exist ϑ, l, C and $C_{\vartheta,l}$ such that

$$(I) \leq (C\|f_0\|_{\infty,\vartheta} + C_{\vartheta,l}\|f_0\|_{2,\vartheta+l})\bar{\varepsilon}^{\alpha}.$$

By Lemma 3.8, (1.25), and (5.16), there exists l_0 such that

$$(II) \leq \bar{\varepsilon} \sup_{s>0} \|\tilde{K}_g f(s)\|_\infty \leq C\bar{\varepsilon} \sup_{s>0} \|f(s)\| \leq C\bar{\varepsilon}(\|f_0\|_\infty + \|f_0\|_{2,l_0}).$$

Note that for $s \leq 0$,

$$\|\tilde{S}(s)\|_{\infty,\vartheta} \leq C\|f_{0t}\|_{\infty,\vartheta}, \tag{6.26}$$

$$\|\tilde{S}(s)\|_{2,\vartheta} \leq C\|f_{0t}\|_{2,\vartheta}, \tag{6.27}$$

and for $s \geq 0$, by Lemma 5.9 and Theorems 1.2 and 1.3 there exists l_0 such that

$$\|\tilde{S}(s)\|_{\infty,\vartheta} \leq C(1+s)^{-2}(\|f_0\|_{2,\vartheta+l_0} + \|f_0\|_{\infty,\vartheta+l_0}) \tag{6.28}$$

and

$$\|\tilde{S}(s)\|_{2,\vartheta+l} \leq C(1+s)^{-2}(\|f_0\|_{2,\vartheta+l_0} + \|f_0\|_{\infty,\vartheta+l_0}). \tag{6.29}$$

Therefore, by (6.26) and (6.29), we have

$$\begin{aligned}(III) &\leq \bar{\varepsilon}^{\alpha} \sup_{s\in(t_2-\bar{\varepsilon}^{\alpha},t_2)} \|\tilde{S}(s)\|_\infty \\ &\leq C\bar{\varepsilon}^{\alpha}(\|f_{0t}\|_\infty + \|f_0\|_{2,l_0} + \|f_0\|_{\infty,l_0}).\end{aligned}$$

For $-1 \leq s \leq t_2 - \bar{\varepsilon}^{\alpha}$, we have

$$\begin{aligned}&|t_1 - (t_2 + 1 - \bar{\varepsilon}^{\alpha})| + |x_1 - x_2| + |v_1 - v_2| \\ &\quad = |t_1 - t_2 - 1 + \bar{\varepsilon}^{\alpha}| + |x_1 - x_2| + |v_1 - v_2| \\ &\quad \leq 1 - \bar{\varepsilon}^{\alpha} + |t_1 - t_2| + |x_1 - x_2| + |v_1 - v_2| \\ &\quad \leq 1, \\ &|t_2 - (t_2 + 1 - \bar{\varepsilon}^{\alpha})| + |x_2 - x_2| + |v_2 - v_2| = 1 - \bar{\varepsilon}^{\alpha} \leq 1.\end{aligned}$$

Therefore,

$$(t_i, x_i, v_i) \in Q_1(t_2 + 1 - \bar{\varepsilon}^{\alpha}, x_2, v_2) \subset (s, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3,$$

for each $i = 1, 2$. Therefore by Lemma 6.7, there exist ϑ, l, C and $C_{\vartheta,l}$ such that

$$(U(t_1,s)\tilde{S}(s))(x_1,v_1) - (U(t_2,s)\tilde{S}(s))(x_2,v_2) \leq (C\|\tilde{S}(s)\|_{\infty,\vartheta} + C_{\vartheta,l}\|\tilde{S}(s)\|_{2,\vartheta+l})\bar{\varepsilon}^{\alpha},$$

for $-1 \leq s \leq t_2 - \bar{\varepsilon}^{\alpha}$. Therefore, by Lemma 6.7, (6.26), (6.28), and (6.27), (6.29), we have

$$
\begin{aligned}
(IV) \le & \left| \int_{-1}^{0} \big((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2) \big) \mathrm{d}s \right| \\
& + \left| \int_{0}^{t_2 - \bar{\varepsilon}^\alpha} \big((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2) \big) \mathrm{d}s \right| \\
& \le C\bar{\varepsilon}^\alpha \left(\sup_{s\in[-1,0]} (\|\tilde{S}(s)\|_{\infty,\vartheta} + \|\tilde{S}(s)\|_{2,\vartheta+l}) + \int_0^\infty (\|\tilde{S}(s)\|_{\infty,\vartheta} + \|\tilde{S}(s)\|_{2,\vartheta+l}) \mathrm{d}s \right) \\
& \le C\bar{\varepsilon}^\alpha \left(\|f_{0t}\|_{\infty,\vartheta} + \|f_{0t}\|_{2,\vartheta+l} + (\|f_0\|_{2,\vartheta+l_0} + \|f_0\|_{\infty,\vartheta+l_0}) \int_0^\infty (1+s)^{-2} \mathrm{d}s \right) \\
& \le C\bar{\varepsilon}^\alpha \big(\|f_{0t}\|_{\infty,\vartheta} + \|f_{0t}\|_{2,\vartheta+l} + \|f_0\|_{2,\vartheta+l_0} + \|f_0\|_{\infty,\vartheta+l_0} \big).
\end{aligned}
$$

Now we update ϑ to $\vartheta + 2 + \max\{l, l_0\}$. Then, by Proposition 2.1, we have

$$
\begin{aligned}
|f(t_1, x_1, v_1) - f(t_2, x_2, v_2)| & \le (I) + (II) + (III) + (IV) \\
& \le C\bar{\varepsilon}^\alpha \big(\|f_{0t}\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta} \big).
\end{aligned}
$$

Thus we complete the proof. □

7 Hölder Estimate and S^p Bound

Let f be a weak solution of (1.22) in the sense of Definition 4.1. Define

$$
\bar{f}(t, x, v) = \begin{cases} f(t, x, v), & \text{if } t \ge 0, \\ f_0(x, v), & \text{if } -1 \le t < 0. \end{cases}
$$

Then $\bar{f}$ satisfies

$$
\begin{aligned}
\partial_t \bar{f} + v \cdot \nabla_x \bar{f} - \sigma_G^{ij} \partial_{v_i, v_j} \bar{f} & = \begin{cases} -\partial_{v_i} \sigma_G^{ij} \partial_{v_j} f + a_g \cdot \nabla_v f + K_1 f + J_g f, & \text{if } t \ge 0, \\ (v \cdot \nabla_x - \sigma^{ij}_{\mu + \mu^{1/2} f_0} \partial_{v_i, v_j}) f_0, & \text{if } -1 \le t < 0 \end{cases} \\
& = \begin{cases} -\partial_{v_i} \sigma_G^{ij} \partial_{v_j} f + a_g \cdot \nabla_v f + K_1 f + J_g f, & \text{if } t \ge 0, \\ -f_{0t} + \partial_{v_i} \sigma^{ij}_{\mu + \mu^{1/2} f_0} \partial_{v_j} f_0, & \text{if } -1 \le t < 0, \end{cases}
\end{aligned}
\tag{7.1}
$$

where σ_G is defined as in (2.1) with $G = \mu + \mu^{1/2} g$ and

$$
\begin{aligned}
K_1 f & = -\mu^{-1/2} \partial_i \big\{ \mu \big[\phi^{ij} * \big\{ \mu^{1/2} [\partial_j f + v_j f] \big\} \big] \big\} \\
& = 2 v_i \mu \big[\phi^{ij} * \big\{ \mu^{1/2} [\partial_j f + v_j f] \big\} \big] - \mu^{1/2} \big[\partial_i \phi^{ij} * \big\{ \mu^{1/2} [\partial_j f + v_j f] \big\} \big],
\end{aligned}
\tag{7.2}
$$

and

$$
J_g f = -v \cdot \sigma v f - \partial_i \big\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \big\} f + \big\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \big\} f + \partial_i \sigma^i f. \tag{7.3}
$$

Lemma 7.1 *For every $\beta > 0$ and $p > 3$,*

$$\|K_1 f\|_{L^p(n\le|v|\le n+1)} \le \frac{C_{p,\beta}}{n^\beta}\big(\|f\|_{L^p} + \|D_v f\|_{L^p}\big), \tag{7.4}$$

where K_1 is defined as in (7.2).

Proof Since K_1 is defined as in (7.2), it is enough to show that

$$\left\|\mu(v)\int_{\mathbb{R}^3}|v-v'|^\vartheta \mu(v')h(v')\mathrm{d}v'\right\|_{L^p} \le \|h\|_{L^p}.$$

By the Hölder inequality,

$$\begin{aligned}
&\int_{(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3}\mu^p(v)\left(\int_{\mathbb{R}^3}|v-v'|^\vartheta\mu(v')h(v')\mathrm{d}v'\right)^p \mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\le \int_{(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3}\mu^p(v)\left(\int_{\mathbb{R}^3}|v-v'|^{\vartheta p'}\mu^{p'}(v')\mathrm{d}v'\right)^{p/p'}\left(\int_{\mathbb{R}^3}h^p(v')\mathrm{d}v'\right)\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\le \int_{\mathbb{R}^3}\mu^p(v)(1+|v|)^{\vartheta p}\mathrm{d}v\int_{(0,\infty)\times\mathbb{T}^3}\left(\int_{\mathbb{R}^3}h^p(v')\mathrm{d}v'\right)\mathrm{d}x\mathrm{d}t\\
&= C_p\|h\|^p_{L^p}.
\end{aligned}$$

□

Clearly, we have

$$\|J_g f\|_{L^p(n\le|v|\le n+1)} \le (C+\|g\|_\infty)\|f\|_{L^p(n\le|v|\le n+1)} \le Cn^{-\beta}\|(1+|v|)^\beta f\|_{L^p(n\le|v|\le n+1)}. \tag{7.5}$$

Theorem 7.2 (Theorem 3.3 in [5]) *Let Ω be a bounded open set in $\mathbb{R}^7$ and let f be a strong solution in Ω to the equation*

$$\sum_{i,j=1}^{3}\sigma^{ij}(t,x,v)\partial_{v_i,v_j}f + Yf = h,$$

where $Y = -\partial_t - v\cdot\nabla_x$. Suppose that σ is uniformly elliptic,

$$\|\sigma^{ij}\|_{C^\alpha(\Omega)} \le C, \tag{7.6}$$

and $f, h \in L^p$. Then $\partial_{v_i,v_j}f \in L^p_{\mathrm{loc}}$, $Yf \in L^p_{\mathrm{loc}}$ and for every open set $\Omega' \subset\subset \Omega$ there exists a positive constant c_1 depending only on $p, \Omega', \Omega, \alpha, C$ and elliptic constant of σ such that

$$\begin{aligned}
\|\partial_{v_i,v_j}f\|_{L^p(\Omega')} &\le c_1(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}),\\
\|Yf\|_{L^p(\Omega')} &\le c_1(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}).
\end{aligned}$$

Remark 7.3 Especially, from [5]

$$c_1 = c_2(\lambda_1, \lambda_2)c_3(\mathrm{dist}(\Omega', \Omega), \alpha, C, p),$$

where λ_1, λ_2 are the smallest and the largest eigenvalues, respectively. More precisely, there exist $C > 0$ and $\alpha' > 0$ such that

$$c_2(\lambda_1, \lambda_2) := \max_{|x|^2+|v|^2+|t|^2=1} |\Gamma^0_{v_i,v_j}(x, v, t)| \le C(\lambda_1^{-1} + \lambda_2)^{\alpha'},$$

where $\Gamma^0(\zeta^{-1} \circ z)$ is a fundamental solution of

$$\sum_{i,j=1}^{3} \sigma^{ij}(\tau, \xi, v)\partial_{v_i,v_j} f + Yf = 0 \tag{7.7}$$

and $\circ$ is a Lie group operation corresponding to (7.7) for some $C > 0$.

Remark 7.4 Since $f(t, \cdot, v)$ is a periodic function on $\mathbb{T}^3$, we extend it to a periodic function on $(3\mathbb{T})^3$. Note that

$$\| f\|^p_{L^p(3\mathbb{T}^3)} = 27\| f\|^p_{L^p(\mathbb{T}^3)}.$$

Define

$$\| f\|_{S^p(\Omega)} := \| f\|_{L^p(\Omega)} + \|D_v f\|_{L^p(\Omega)} + \|D_{vv} f\|_{L^p(\Omega)} + \|Yf\|_{L^p(\Omega)},$$

where $Y = -\partial_t - v \cdot \nabla_x$.

Lemma 7.5 *Assume* (2.6). *Let f be a weak solution of* (1.6), (1.7), *and* (1.22) *in the sense of* Definition 4.1. *Suppose that g satisfies* $\|g\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C'$ *for some* $0 < \alpha < 1$ *and* $C > 0$. *Then there exist* $\vartheta > 0$, $p > 3$, $C_{\vartheta,\alpha,C',p}$ *such that*

$$\| f\|_{S^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C\big(\|f_{0t}\|_{\infty,\vartheta} + \|D_v f_0\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}\big). \tag{7.8}$$

Proof Since $\|g\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C$, σ_G satisfies (7.6). Apply Theorem 7.2 to (7.1) with $\Omega' = \Omega'_n$ and $\Omega = \Omega_n$, where

$$\begin{aligned} \Omega' = \Omega'_n &:= \{t \ge 0, x \in \mathbb{T}^3, n \le |v| \le n+1\}, \\ \Omega = \Omega_n &:= \{t \ge -1, x \in 3\mathbb{T}^3, n - 1/2 \le |v| \le n + 3/2\}. \end{aligned}$$

Let σ_G, K_1, and J_g be defined as in (2.1), (7.2), (7.3). Then, we have

$$\begin{aligned} \| f\|^p_{S^p(\Omega'_n)} \le (c_1)^p\Big(&\|\partial_{v_i}\sigma^{ij}_G\partial_{v_j} f\|^p_{L^p(\Omega'_n)} + \|a_g \cdot \nabla_v f\|^p_{L^p(\Omega'_n)} + \|K_1 f + J_g f\|^p_{L^p(\Omega'_n)} \\ &+\|(v \cdot \nabla_x - \sigma^{ij}_G\partial_{v_i,v_j})f_0\|^p_{L^p((-1,0)\times 3\mathbb{T}^3\times B(0;n-1/2,n+3/2))}\Big), \end{aligned}$$

where $B(z;r_1, r_2) := B(z;r_2)\backslash B(z;r_1)$. By Lemma 2.3 and Remark 7.3,

$$c_2(\lambda_1, \lambda_2) = Cn^a$$

for some $a > 0$ and

$$c_3(\mathrm{dist}(\Omega', \Omega), \alpha, C, p) = C_{\alpha,C,p}.$$

Therefore,

$$c_1 = C_{\alpha,C,p} n^a$$

for some $a > 0$. Let

$$\tilde{\Omega}'_n = \{t \geq 0, x \in \mathbb{T}^3, n - 1/2 \leq |v| \leq n + 3/2\}.$$

Then by Remark 7.4,

$$\begin{aligned}\| f\|^p_{S^p(\Omega'_n)} \leq (C_{\alpha,C,p} n^a)^p \Big(&\|\partial_{v_i} \sigma^{ij}_G \partial_{v_j} f\|^p_{L^p(\tilde{\Omega}'_n)} + \|a_g \cdot \nabla_v f\|^p_{L^p(\tilde{\Omega}'_n)} + \|K_1 f + J_g f\|^p_{L^p(\tilde{\Omega}'_n)} \\ &+ \|(v \cdot \nabla_x - \sigma^{ij}_G \partial_{v_i,v_j}) f_0\|^p_{L^p\left((-1,0)\times\mathbb{T}^3\times B(0;n-1/2,n+3/2)\right)} \Big),\end{aligned} \tag{7.9}$$

where $\tilde{\Omega}'_n = \{t \geq 0, x \in \mathbb{T}^3, n - 1/2 \leq |v| \leq n + 3/2\}$. Note that by the standard interpolation, we have

$$\|D_v f\|^p_{L^p(\tilde{\Omega}'_n)} \leq \varepsilon' \|D_{vv} f\|^p_{L^p(\tilde{\Omega}'_n)} + \frac{C}{\varepsilon'} \| f\|^p_{L^p(\tilde{\Omega}'_n)}. \tag{7.10}$$

Let $\beta > 2a + 4$ and $\varepsilon' = \varepsilon_0$ which can be determined later. Then by (7.4), (7.5), and (7.10), we have

$$\begin{aligned}&\sum (C_{\alpha,C,p} n^a)^p \|K_1 f + J_g f\|^p_{L^p(\tilde{\Omega}'_n)} \\ &\leq \sum (C_{\beta,\alpha,C,p})^p n^{p(a-\beta)} \Big(\| f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)} + \|D_v f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)} + \|(1+|v|)^\beta f\|^p_{L^p(\tilde{\Omega}'_n)} \Big) \\ &\leq (C_{\beta,\alpha,C,p})^p \Big(\|(1+|v|)^\beta f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)} + \|D_v f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)} \Big) \\ &\leq (C_{\beta,\alpha,C,p,\varepsilon_0})^p \|(1+|v|)^\beta f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)} + \varepsilon_0 \|D_{vv} f\|^p_{L^p\left((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3\right)}\end{aligned} \tag{7.11}$$

for some ε_0. Similarly,

$$\begin{aligned}&\sum (C_{\alpha,C,p} n^a)^p \|(v \cdot \nabla_x - \sigma^{ij}_G \partial_{v_i,v_j}) f_0(x,v)\|^p_{L^p\left((-1,0)\times\mathbb{T}^3\times B(0;n-1/2,n+3/2)\right)} \\ &\leq \sum (C_{\alpha,C,p})^p n^{p(a-\beta)} \|(1+|v|)^\beta (v \cdot \nabla_x - \sigma^{ij}_G \partial_{v_i,v_j}) f_0(x,v)\|^p_{L^p\left(\mathbb{T}^3\times B(0;n-1/2,n+3/2)\right)} \\ &\leq (C_{\alpha,C,p})^p \|(v \cdot \nabla_x - \sigma^{ij}_G \partial_{v_i,v_j}) f_0\|^p_{p,\beta}.\end{aligned} \tag{7.12}$$

Choose $\varepsilon' = \varepsilon n^{-p(a+2)}$ small enough. Since $\|\partial_{v_i}\sigma_G^{ij}\|_\infty < C$, $\|a_g\|_\infty \le C$, by (7.10) we have

$$\begin{aligned}
&\sum (C_{\alpha,C,p} n^a)^p \Big(\|\partial_{v_i}\sigma_G^{ij}\partial_{v_j} f)\|^p_{L^p(\tilde\Omega_n')} + \|a_g \cdot \nabla_v f\|^p_{L^p(\tilde\Omega_n')} \Big) \\
&\quad \le \sum (C_{\alpha,C,p})^p \Big(\varepsilon n^{-2p} \|D_{vv} f\|^p_{L^p(\tilde\Omega_n')} + \varepsilon^{-1} n^{2p(a+1)} \|f\|^p_{L^p(\tilde\Omega_n')} \Big) \\
&\quad \le \sum (C_{\alpha,C,p})^p \Big(\varepsilon n^{-2p} \|D_{vv} f\|^p_{L^p(\tilde\Omega_n')} + n^{p(2a+2-\beta)} \|(1+|v|)^\beta f\|^p_{L^p(\tilde\Omega_n')} \Big) \\
&\quad \le (C_{\beta,\alpha,C,p})^p \Big(\varepsilon \|D_{vv} f\|^p_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} + C_\varepsilon \|(1+|v|)^\beta f\|^p_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \Big).
\end{aligned} \tag{7.13}$$

Combining (7.9)–(7.13) and absorbing $\|D_{vv} f\|$ term on RHS to the LHS, we have

$$\begin{aligned}
&\|f\|^p_{S^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \\
&\quad \le (C_{\beta,\alpha,C,p})^p \Big(\|(1+|v|)^\beta f\|^p_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} + \|(v\cdot\nabla_x - \sigma_F^{ij}\partial_{v_i,v_j}) f_0\|^p_{p,\beta} \Big) \\
&\quad \le (C_{\beta,\alpha,C,p})^p \Big(\|(1+|v|)^\beta f\|^p_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} + \|f_{0t}\|^p_{p,\beta} + \|D_v f_0\|^p_{p,\beta-2} \Big).
\end{aligned}$$

Finally, by Lemmas 2.4 and 5.11, the standard interpolation, and Proposition 2.1, we have (7.8). □

Here we introduce some regularity results in S^p norm.

Theorem 7.6 (Theorem 2.1 in [24]) *Let $f \in S^p(\mathbb{R}^7)$, with $1 < p < \infty$.*

(1) *If $2p > 14$ and $p < 14$, then $f \in C^\gamma(\mathbb{R}^7)$, with $\gamma = \frac{2p-14}{p}$;*
(2) *if $p > 14$, then $\partial_{v_i} f \in C^\delta(\mathbb{R}^7)$, with $\delta = \frac{p-14}{p}$.*

Define $|||(t,x,v)||| = \rho$, where ρ is a unique positive solution to the equation

$$\frac{t^2}{\rho^4} + \frac{|x|^2}{\rho^6} + \frac{|v|^2}{\rho^2} = 1$$

and

$$(\tau,\xi,\nu)^{-1} \circ (t,x,v) = (t-\tau, x-\xi+(t-\tau)\nu, v-\nu).$$

Now we can deduce the following lemma.

Lemma 7.7 *Assume* (2.6). *Let f be a weak solution of* (1.6), (1.7), *and* (1.22) *in the sense of* Definition 4.1. *Suppose that g satisfies $\|g\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C$ for some $0 < \alpha < 1$ and $C > 0$.*

If $2p > 14$ then, letting $\alpha_1 = \min\left\{1, \frac{2p-14}{p}\right\}$, there exist $\vartheta > 0$ and $C = C_{\vartheta,\alpha,C,p}$ such that

$$\frac{|f(t,x,v)-f(\tau,\xi,v)|}{|||(\tau,\xi,v)^{-1}\circ(t,x,v)|||^{\alpha_1}} \le C\big(\|f_{0t}\|_{\infty,\vartheta}+\|D_v f_0\|_{\infty,\vartheta}+\|f_0\|_{\infty,\vartheta}\big)$$

for every $(t,x,v),(\tau,\xi,v)\in(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$, $(t,x,v)\neq(\tau,\xi,v)$.

If $p>14$ *then, letting* $\alpha_2=\frac{p-14}{p}$, *there exist* $\vartheta>0$ *and* $C=C_{\vartheta,\alpha,C,p}$ *such that*

$$\frac{|\partial_{v_i} f(t,x,v)-\partial_{v_i} f(\tau,\xi,v)|}{|||(\tau,\xi,v)^{-1}\circ(t,x,v)|||^{\alpha_2}} \le C\big(\|f_{0t}\|_{\infty,\vartheta}+\|D_v f_0\|_{\infty,\vartheta}+\|f_0\|_{\infty,\vartheta}\big)$$

for every $(t,x,v),(\tau,\xi,v)\in(0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$, $(t,x,v)\neq(\tau,\xi,v)$.

Proof It immediately follows from Lemma 7.5 and Theorem 7.6. □

Remark 7.8 If $\tau=t$ and $\xi=x$, then $|||(t,x,v)^{-1}\circ(t,x,v)|||=|v-v|$.

By Remark 7.8 and Lemma 7.7, we have

Lemma 7.9 *Assume* (2.6). *Let f be a weak solution of* (1.6), (1.7), *and* (1.22) *in the sense of* Definition 4.1. *Suppose that* g *satisfies* $\|g\|_{C^\alpha((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}\le C$ *for some* $0<\alpha<1$ *and* $C>0$. *Let* $p=14$, *then there exist* $\vartheta>0$ *and* $C=C_{\vartheta,\alpha,C,p}$ *such that*

$$\|D_v f\|_{L^\infty((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)} \le C\big(\|f_{0t}\|_{\infty,\vartheta}+\|D_v f_0\|_{\infty,\vartheta}+\|f_0\|_{\infty,\vartheta}\big).$$

8 Proof of Theorem 1.1

In this section we will use an iteration argument to prove the existence and the uniqueness of the weak solution of (1.5)–(1.7) in the sense of Definition 4.1. We first construct the function sequence as follows. Define $f^{(0)}(t,x,v):=f_0(x,v)$. Since $f_0(x,v)$ satisfies (1.16), by Lemma 4.2, we can define $f^{(1)}$ as a solution of (1.6), (1.7), and (1.22) with $g=f^{(0)}$. Moreover, by Theorem 1.3, $f^{(1)}$ satisfies the assumption in Lemma 2.4. Thus we can also define $f^{(2)}$ as a solution of (1.6), (1.7), and (1.22) with $g=f^{(1)}$. Inductively we can define a function sequence $f^{(n)}$ for $n\ge 0$.

Lemma 8.1 *There exist* $C,\vartheta_0>0$, *and* $0<\varepsilon_0\ll 1$ *such that if* f_0 *satisfies*

$$\|f_0\|_{\infty,\vartheta}\le\varepsilon_0,$$

then

$$\sup_{n\in\mathbb{N},t\ge 0}\|f^{(n)}(t)\|_{\infty,\vartheta}\le C\|f_0\|_{\infty,\vartheta+\vartheta_0}.$$

Sketch of proof It is clear by applying Theorem 1.3 to $f^{(n)}$ inductively on n. □

Lemma 8.2 *Let f_0 be a given function satisfying* (1.16). *Let f_1 and f_2 be weak solutions of* (1.6), (1.7), *and* (1.22) *in the sense of* Definition 4.1 *with $g = g_1$, $g = g_2$ respectively. Suppose that g_1 and g_2 are uniformly Hölder continuous functions satisfying* (2.6). *Then we have*

$$\begin{aligned}
&\frac{1}{2}\|(f_1-f_2)(t)\|^2_{2,\bar{\vartheta}} + \left(\frac{1}{2} - C\varepsilon\right)\int_0^t \|(f_1-f_2)(s)\|^2_{\sigma,\bar{\vartheta}}\mathrm{d}s \\
&\leq \int_0^t C(\|f_2\|_\infty + \|\nabla_v f_2\|_\infty)\min\left\{\|(g_1-g_2)(s)\|^2_{2,\bar{\vartheta}}, \|(g_1-g_2)(s)\|^2_{\sigma,\bar{\vartheta}}\right\}\mathrm{d}s + C\int_0^t \|(f_1-f_2)\|^2_{2,\bar{\vartheta}}\mathrm{d}s,
\end{aligned} \tag{8.1}$$

where $\bar{\vartheta} < 0$ is a constant defined as in Theorem 2.8. *Therefore by the Gronwall inequality for every $t_0 > 0$,*

$$\sup_{t\in(0,t_0)} \frac{1}{2}\|(f^{(n+1)} - f^{(n+2)})(t)\|_{2,\bar{\vartheta}} \leq e^{Ct_0}C_\varepsilon t_0 \sup_{s\in(0,t_0)} \|(f^{(n)} - f^{(n+1)})(s)\|_{2,\bar{\vartheta}}. \tag{8.2}$$

Proof Note that f_1, f_2, g_1, and g_2 satisfy

$$\partial_t(f_1 - f_2) + v\cdot\nabla_x(f_1-f_2) + L(f_1-f_2) = \Gamma(g_1, f_1 - f_2) + \Gamma(g_1 - g_2, f_2).$$

Multiplying the above equation by $w^{2\bar{\vartheta}}(f_1 - f_2)$ and integrating both sides of the resulting equation yields

$$\begin{aligned}
&\frac{1}{2}\|(f_1-f_2)(t)\|_{2,\bar{\vartheta}} + \int_0^t \left(w^{2\bar{\vartheta}}L(f_1-f_2)(s), (f_1-f_2)(s)\right)\mathrm{d}s \\
&= \int_0^t \left(w^{2\bar{\vartheta}}\Gamma(g_1(s), (f_1-f_2)(s)), (f_1-f_2)(s)\right)\mathrm{d}s + \int_0^t \left(w^{2\bar{\vartheta}}\Gamma((g_1-g_2)(s), f_2(s)), (f_1-f_2)(s)\right)\mathrm{d}s \\
&= \int_0^t ((I) + (II))\mathrm{d}s.
\end{aligned} \tag{8.3}$$

By Lemma 2.7, we have

$$\left(w^{2\vartheta_0}L(f_1-f_2)(s), (f_1-f_2)(s)\right) \geq \frac{1}{2}\|(f_1-f_2)(s)\|^2_{\sigma,\bar{\vartheta}} - C\|(f_1-f_2)(s)\|^2_{2,\bar{\vartheta}}.$$

Since g_1 satisfies (2.6), by Theorem 2.8, we have

$$\begin{aligned}
(I) &\leq C\|g_1(s)\|_\infty \|(f_1-f_2)(s)\|^2_{\sigma,\bar{\vartheta}} \\
&\leq C\varepsilon\|(f_1-f_2)(s)\|^2_{\sigma,\bar{\vartheta}}.
\end{aligned}$$

Since we want to control (II) in terms of $\|g_1 - g_2\|_{2,\bar{\vartheta}}$, $\|g_1 - g_2\|_{\sigma,\bar{\vartheta}}$, and $\|(f_1-f_2)(s)\|_{\sigma,\bar{\vartheta}}$, we have to show that

$$\|\nabla_v f_2\|_\infty < \infty.$$

Since g_2 is uniformly Hölder continuous by Lemma 7.9, $D_v f_2$ is uniformly bounded. Therefore by Theorems 2.8 and 1.3, Lemma 7.9, and Young's inequality, we have

$$\begin{aligned}(II) &\le C\big(\|f_2\|_\infty + \|\nabla_v f_2\|_\infty\big)\min\{\|(g_1-g_2)(s)\|_{2,\bar\vartheta}, \|(g_1-g_2)(s)\|_{\sigma,\bar\vartheta}\}\|(f_1-f_2)(s)\|_{\sigma,\bar\vartheta}\\ &\le C_\varepsilon\big(\|f_2\|_\infty + \|\nabla_v f_2\|_\infty\big)\Big(\min\Big\{\|(g_1-g_2)(s)\|^2_{2,\bar\vartheta}, \|(g_1-g_2)(s)\|^2_{\sigma,\bar\vartheta}\Big\} + C\varepsilon\|(f_1-f_2)(s)\|^2_{\sigma,\bar\vartheta}\Big).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{2}\|(f_1-f_2)(t)\|^2_{2,\bar\vartheta} + \frac{1}{2}\int_0^t \|(f_1-f_2)(s)\|^2_{\sigma,\bar\vartheta}\mathrm{d}s - C\int_0^t \|(f_1-f_2)(s)\|^2_{2,\bar\vartheta}\mathrm{d}s\\ &\quad\le \int_0^t \Big(C\varepsilon\|(f_1-f_2)(s)\|^2_{\sigma,\bar\vartheta} + C_\varepsilon\big(\|f_2\|_\infty + \|\nabla_v f_2\|_\infty\big)\\ &\qquad\times\min\Big\{\|(g_1-g_2)(s)\|^2_{2,\vartheta_0}, \|(g_1-g_2)(s)\|^2_{\sigma,\vartheta_0}\Big\}\Big)\mathrm{d}s.\end{aligned}$$

Therefore we have (8.1). □

Proof of Theorem 1.1 **Proof of (1)**

Existence Let t_0 be a given positive constant and $\bar\vartheta < 0$ be a constant defined as in Theorem 2.8. Since f_0 satisfies (1.16), by Theorem 1.4, $f^{(n)}$ is Hölder continuous uniformly in n. Therefore, by (8.1) we have

$$\begin{aligned}&\|(f^{(n+1)}-f^{(n)})(t)\|^2_{2,\vartheta_0}\\ &\quad\le C'\int_0^t \|(f^{(n)}-f^{(n-1)})(s)\|^2_{2,\vartheta_0}\mathrm{d}s + C\int_0^t \|(f^{(n+1)}-f^{(n)})(s)\|^2_{2,\vartheta_0}\mathrm{d}s.\end{aligned}$$

for some C and C'. Then we will show that

$$\|(f^{(n)}-f^{(n-1)})(t)\|^2_{2,\vartheta_0} \le \frac{e^{Cnt_0}(C't)^n}{n!} \tag{8.4}$$

for every $t\in(0,t_0)$ and $n\ge 1$ by the induction on n. Suppose that (8.4) holds for $n=k$, then

$$C'\int_0^t \|(f^{(k)}-f^{(k-1)})(s)\|^2_{2,\vartheta_0}\mathrm{d}s \le \frac{e^{Ckt_0}(C't)^{k+1}}{(k+1)!}$$

for $t\in(0,t_0)$. Therefore, we have

$$\|(f^{(k+1)}-f^{(k)})(t)\|^2_{2,\vartheta_0} \le \frac{e^{Ckt_0}(C't)^{k+1}}{(k+1)!} + C\int_0^t \|(f^{(k+1)}-f^{(k)})(s)\|^2_{2,\vartheta_0}\mathrm{d}s$$

for every $t\in(0,t_0)$. Then by the Gronwall inequality, we have

$$\|(f^{(k+1)}-f^{(k)})(t)\|^2_{2,\vartheta_0} \le \frac{e^{Ckt_0}(C't)^{k+1}}{(k+1)!}e^{Ct} \le \frac{e^{C(k+1)t_0}(C't)^{k+1}}{(k+1)!}$$

for every $t \in (0, t_0)$. Thus we have (8.4) for every $n \in \mathbb{N}$. Moreover, we have

$$\lim_{N\to\infty} \sum_{n>N} \sup_{0\le t\le t_0} \|(f^{(n)} - f^{(n-1)})(t)\|_{2,\vartheta_0} = 0.$$

Thus $f^{(n)}$ is a Cauchy sequence in $L^2([0,t_0]\times\mathbb{T}^3\times\mathbb{R}^3, w^{\vartheta_0}\mathrm{d}t\mathrm{d}x\mathrm{d}v)$. Let $f = \lim_{n\to\infty} f^{(n)}$. Then by (8.2), f is a weak solution of (1.5)–(1.7) in the sense of Definition 4.1.

Uniqueness Suppose that f and g are weak solutions of (1.5)–(1.7) in the sense of Definition 4.1. Then by (8.1), we have

$$\frac{1}{2}\|(f-g)(t)\|^2_{2,\vartheta_0} + \left(\frac{1}{2} - C\varepsilon\right)\int_0^t \|(f-g)(s)\|^2_{\sigma,\vartheta_0}\mathrm{d}s \le C\int_0^t \|(f-g)(s)\|^2_{2,\vartheta_0}\mathrm{d}s.$$

Since $C\varepsilon' < 1/4$, we have

$$\frac{1}{2}\|(f-g)(t)\|^2_{2,\vartheta_0} \le C\int_0^t \|(f-g)(s)\|^2_{2,\vartheta_0}\mathrm{d}s.$$

Therefore, by the Gronwall inequality, we have

$$\|(f-g)(t)\|^2_{2,\vartheta_0} = 0$$

for every $t \in (0, t_0)$.

Since t_0 is arbitrary, we conclude that the weak solution of (1.5)–(1.7) in the sense of Definition 4.1 uniquely exists globally in time.

Proof of (3) We can apply f to Theorems 1.2–1.4, and Lemma 7.9. Then we have (1.17)–(1.21).

Proof of (2) Let $F = \mu + \sqrt{\mu}f$, where f is the weak solution of (1.5)–(1.7) in the sense of Definition 4.1. Consider

$$\partial_t F + v\cdot\nabla_x F = Q(F,F) = \sigma_F^{ij}\partial_{v_iv_j}F + 8\pi F^2. \tag{8.5}$$

Similar to Definition 4.1, we can define a weak solution to (8.5) and we can easily check that F is a weak solution to (8.5). Since f satisfies (1.19), by Lemma 2.4, σ_F is a non-negative definite matrix. Therefore, in a similar manner to Sect. 3, we can obtain a weak minimum principle for (8.5). Thus, if $F(0) \ge 0$, then $F(t) \ge 0$. □

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

References

1. Alexandre, R., Villani, C.: On the Landau approximation in plasma physics. Ann. Inst. H. Poincaré Anal. Non Linéaire **21**(1), 61–95 (2004)
2. Boblylev, A.V., Pulvirenti, M., Saffirio, C.: From particle systems to the Landau equation: a consistency result. Commun. Math. Phys. **319**(3), 683–702 (2013)
3. Bobylev, A., Gamba, I., Potapenko, I.: On some properties of the Landau kinetic equation. J. Stat. Phys. **161**(6), 1327–1338 (2015)
4. Bouchut, F.: Hypoelliptic regularity in kinetic equations. J. Math. Pures Appl. (9) **81**(11), 1135–1159 (2002)
5. Bramanti, M., Cerutti, M.C., Manfredini, M.: L^p estimates for some ultraparabolic operators with discontinuous coefficients. J. Math. Anal. Appl. **200**(2), 332–354 (1996)
6. Carrapatoso, K., Tristani, I., Wu, K.-C.: Cauchy problem and exponential stability for the inhomogeneous Landau equation. Arch. Ration. Mech. Anal. **221**(1), 363–418 (2016)
7. Chen, Y., Desvillettes, L., He, L.: Smoothing effects for classical solutions of the full Landau equation. Arch. Ration. Mech. Anal. **193**(1), 21–55 (2009)
8. Desvillettes, L.: Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. J. Funct. Anal. **269**(5), 1359–1403 (2015)
9. Esposito, R., Guo, Y., Kim, C., Marra, R.: Non-isothermal boundary in the Boltzmann theory and Fourier law. Commun. Math. Phys. **323**(1), 177–239 (2013)
10. Golse, F., Imbert, C., Mouhot, C., Vasseur, A.: Harnack inequality for kinetic Fokker–Planck equations with rough coefficients and application to the Landau equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19**(1), 253–295 (2019)
11. Golse, F., Vasseur, A.: Hölder regularity for hypoelliptic kinetic equations with rough diffusion coefficients. arXiv:1506.01908 (2015)
12. Gressman, P.T., Strain, R.M.: Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production. Adv. Math. **227**(6), 2349–2384 (2011)
13. Guo, Y.: The Landau equation in a periodic box. Commun. Math. Phys. **231**(3), 391–434 (2002)
14. Guo, Y.: Decay and continuity of the Boltzmann equation in bounded domains. Arch. Ration. Mech. Anal. **197**(3), 713–809 (2010)
15. Ha, S.-Y., Xiao, Q.: L^2-stability of the Landau equation near global Maxwellians. J. Math. Phys. **56**(8), 081505 (2015)
16. Hérau, F., Pravda-Starov, K.: Anisotropic hypoelliptic estimates for Landau-type operators. J. Math. Pures Appl. (9) **95**(5), 513–552 (2011)
17. Hérau, F., Li, W.-X.: Global hypoelliptic estimates for Landau-type operators with external potential. Kyoto J. Math. **53**(3), 533–565 (2013)
18. Hwang, H.J., Jang, J., Jung, J.: On the kinetic Fokker–Planck equation in a half-space with absorbing barriers. Indiana Univ. Math. J. **64**(6), 1767–1804 (2015)
19. Hwang, H.J., Jang, J., Velázquez, J.J.L.: The Fokker–Planck equation with absorbing boundary conditions. Arch. Ration. Mech. Anal. **214**(1), 183–233 (2014)
20. Imbert, C., Mouhot, C.: Hölder continuity of solutions to hypoelliptic equations with bounded measurable coefficients. arXiv:1505.04608 (2015)
21. Krieger, J., Strain, R.M.: Global solutions to a non-local diffusion equation with quadratic non-linearity. Commun. Partial Differ. Equ. **37**(4), 647–689 (2012)
22. Liu, S., Ma, X.: Regularizing effects for the classical solutions to the Landau equation in the whole space. J. Math. Anal. Appl. **417**(1), 123–143 (2014)
23. Luo, L., Yu, H.: Spectrum analysis of the linearized relativistic Landau equation. J. Stat. Phys. **163**(4), 914–935 (2016)
24. Polidoro, S., Ragusa, M.A.: Sobolev–Morrey spaces related to an ultraparabolic equation. Manuscr. Math. **96**(3), 371–392 (1998)
25. Strain, R.M., Guo, Y.: Almost exponential decay near Maxwellian. Commun. Partial Differ. Equ. **31**(1–3), 417–429 (2006)
26. Strain, R.M., Zhu, K.: The Vlasov–Poisson–Landau system in $\mathbb{R}^3_x$. Arch. Ration. Mech. Anal. **210**(2), 615–671 (2013)
27. Wu, K.-C.: Global in time estimates for the spatially homogeneous Landau equation with soft potentials. J. Funct. Anal. **266**(5), 3134–3155 (2014)
28. Wu, K.-C.: Pointwise behavior of the linearized Boltzmann equation on a torus. SIAM J. Math. Anal. **46**(1), 639–656 (2014)