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A diffuse interface model and semi-implicit energy stable finite element method for two-phase magnetohydrodynamic flows[☆]

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Abstract

In this paper, we propose a diffuse interface model and finite element approximation for two-phase magnetohydrodynamic (MHD) flows with different viscosities and electric conductivities. An energy stable scheme, which is based on the finite element method for the spatial discretization and first order semi-implicit scheme combined with convex splitting method for the temporal discretization, is proposed to solve this new model. The numerical scheme is proved to be mass-conservative and energy law preserving. By Leray–Schauder fixed point theorem, the existence of solutions to the numerical scheme is shown. The uniqueness of the numerical solutions is obtained. Utilizing the stability of the numerical scheme and the compactness method, the existence of the weak solutions to the two-phase MHD model is established as well. Furthermore, given more regularity on the weak solution, the convergence of the numerical scheme is derived. Finally, numerical experiments are provided to verify the theoretical results and validate the proposed model.

Keywords: Two-phase MHD flows; Diffuse interface models; Finite element method; Energy stability; Semi-implicit scheme

1. Introduction

Magnetohydrodynamic (MHD) system describes the interaction of electromagnetic fields and electrically conducting fluids. The model couples the Navier–Stokes equations of continuum fluid mechanics and the Maxwell equations of electromagnetism via the Lorentz force and Ohm's law. The flow of the conducting fluids in the magnetic field generates electric current which changes the electromagnetic field; meanwhile, the electric current running within the magnetic field induces the Lorentz force which influences the flow of fluids. For some

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comprehensive and detailed modeling description and mathematical theory, we refer to [1–13] and the references therein.

Furthermore, the study on the interaction of electromagnetic fields with two incompressible, immiscible and electrically conducting fluids is of great significance in the engineering, such as the Aluminum electrolysis cells, metallurgical industry, pump accelerators, MHD generators and fusion reactors [14–17]. For example, in metallurgy processes, bubbles are always injected into the molten metal for stirring and homogenizing the liquid metal and the magnetic field is imposed to control the bubble motion in a contactless method. In MHD generators and pump accelerators, some experimental and analytical studies on the flow of two immiscible fluids in a channel under an external magnetic field are carried out [18–21]. One fundamental problem for two-phase MHD problem is the interfacial dynamics between two different incompressible fluids.

In many situations, it may not be convenient or accurate for the classical sharp interface model to describe the topological transitions of interfaces such as self-intersection, pinch-off, reconnection and splitting during the evolution of interface [22–24]. In the last decades, the diffuse interface (phase field) method has been widely applied to model and simulate the topological transitions of interfaces. This method assumes that the fluids are mixed and store the mixing (elastic) energy within the thin layer of finite thickness, therefore the surface tension force on the fluids is derived by using the variational approach, see [23,25–30]. It is shown that the sharp interface model can be recovered in the limit as the interface thickness approaches zero [28,31]. About the extensive study on the phase field approach, we refer to [32–53] and references therein.

In this paper, we propose a diffuse interface model to describe the flow of two incompressible, immiscible and electrically conducting fluids with different viscosities and electric conductivities by combining the physics of MHD fluids and the phase field approach. The model consists of Cahn–Hilliard equation (free interface), Navier–Stokes equations (hydrodynamics) and Maxwell equations (magnetic field) which are nonlinearly coupled through convection, stresses, and Lorentz forces. We propose a fully discrete energy stable finite element method with a semi-implicit scheme in temporal discretization for the model which satisfies the mass conservation and discrete energy law, prove the existence of solutions to the numerical method by Leray–Schauder fixed point theorem, and show the uniqueness of the numerical solutions. Utilizing the stability of the proposed numerical method and the compactness method, there exist subsequences of discrete solutions which converge to weak solution of the model as the mesh size h and time-step τ tend to zero. Therefore, the existence of weak solution follows. Furthermore, we postulate more regularity on the weak solution, and thus obtain the convergence of the numerical scheme.

The paper is organized as follows. In Section 2, a diffuse interface model for two-phase MHD flows is proposed. In Section 3, preliminary knowledge and the definition of weak solution are introduced. In Section 4, we present an energy stable semi-implicit scheme with finite element discretization and show the existence and uniqueness of solutions for the scheme. In Section 5, we prove the existence of the weak solution to the proposed model and the convergence of the scheme. In Section 6, three numerical examples are provided to validate the numerical scheme and the proposed model. In Section 7, a conclusion is drawn.

2. The model for two-phase MHD flows

In this paper, the vector-valued functions and vector-valued function spaces in $\mathbb{R}^d$ ($d = 2, 3$) are denoted in boldface. Let Ω be a bounded and connected domain. Firstly, we introduce single-phase MHD flow and phase field model. Then, we couple them together to propose a phase field model for two-phase incompressible MHD flows.

Single-phase MHD flow. The single-phase MHD model consists of a coupling between the Navier–Stokes equations of continuum fluid mechanics and the Maxwell equations of electromagnetism through the Lorentz force and Ohm's law. The equations for single-phase MHD flow read (see [15])

$$\rho\big(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) - 2\,\mathrm{div}\big(\eta D(\boldsymbol{u})\big) + \nabla p = \frac{1}{\mu}\,\mathrm{curl}\,\boldsymbol{B}\times\boldsymbol{B} + \boldsymbol{f}, \tag{2.1a}$$

$$\mathrm{div}\,\boldsymbol{u} = 0, \tag{2.1b}$$

$$\boldsymbol{B}_t + \frac{1}{\mu}\,\mathrm{curl}\left(\frac{1}{\sigma}\,\mathrm{curl}\,\boldsymbol{B}\right) - \mathrm{curl}(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{0}, \tag{2.1c}$$

$$\mathrm{div}\,\boldsymbol{B} = 0, \tag{2.1d}$$

where $\boldsymbol{u}$, p, $\boldsymbol{B}$ denote the velocity field, the hydrodynamic pressure and the magnetic field, $D(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$ and $\boldsymbol{f}$ is the external force, for example, the gravity force $\boldsymbol{f} = \rho \boldsymbol{g}$. The physical parameters ρ, η, μ and σ, respectively, denote the density of the fluid, hydrodynamic viscosity, magnetic permeability and electric conductivity. The first term on the right-hand side of (2.1a) is the Lorentz force. This term is obtained from Lorentz force $\boldsymbol{j} \times \boldsymbol{B}$ and simplified Maxwell–Ampère equation $\boldsymbol{j} = \frac{1}{\mu} \operatorname{curl} \boldsymbol{B}$ where the displacement current is neglected and $\boldsymbol{j}$ stands for the electric current. Eq. (2.1c) is complemented by coupling Maxwell–Faraday equation $\boldsymbol{B}_t + \operatorname{curl} \boldsymbol{E} = 0$, simplified Maxwell–Ampère equation with Ohm's law $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$, where $\boldsymbol{E}$ is the electric field. For more details of the single-phase MHD model, see [8,14,15,54–64].

Phase field model. For the phase field φ, the free energy of two-phase fluids is

$$E(\varphi) = \int_\Omega \left(\frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon^2} F(\varphi) \right) \mathrm{d}x,$$

where $F(\varphi)$ models the immiscibility of the fluid components and is usually taken to be a double-well polynomial of Ginzburg–Landau type $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$. The two minima of $F(\varphi)$, i.e., $\varphi = \pm 1$, correspond to two stable phases of the fluids. The first term (i.e., the gradient energy) and second term (i.e., the bulk energy) of $E(\varphi)$, respectively, represent the hydrophilic and hydrophobic parts of the free energy. It is well known that Allen–Cahn equation is the L^2-gradient flow of the free energy $E(\varphi)$ and Cahn–Hilliard equation is the H^{-1}-gradient flow of $E(\varphi)$ (see [65]). To preserve the mass conservation, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi(x, t) \mathrm{d}x = 0$, we consider the Cahn–Hilliard equation

$$\varphi_t = \operatorname{div}\left(\gamma \nabla \frac{\partial E}{\partial \varphi} \right) = \gamma \Delta w, \tag{2.2a}$$

$$w = \frac{\partial E}{\partial \varphi} = -\Delta \varphi + \frac{1}{\varepsilon^2} f(\varphi), \tag{2.2b}$$

where w represents the chemical potential which is given by the variational derivative of the energy E with respect to φ, $f(\varphi) = F'(\varphi)$, and γ, ε denote the elastic relaxation time and width of the interfacial layer, respectively.

A new Cahn–Hilliard–MHD model for two-phase MHD flows. Based on the single-phase MHD flow and phase field model, we propose the following Cahn–Hilliard–MHD model for two-phase MHD flows:

$$\varphi_t + \operatorname{div}(\varphi \boldsymbol{u}) = \gamma \Delta w, \tag{2.3a}$$

$$-\Delta \varphi + \frac{1}{\varepsilon^2} f(\varphi) = w, \tag{2.3b}$$

$$\rho\big(\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big) - 2 \operatorname{div}\big(\eta(\varphi) D(\boldsymbol{u})\big) + \nabla p + \lambda \varphi \nabla w = \frac{1}{\mu} \operatorname{curl} \boldsymbol{B} \times \boldsymbol{B} + \boldsymbol{f}, \tag{2.3c}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.3d}$$

$$\boldsymbol{B}_t + \frac{1}{\mu} \operatorname{curl}\left(\frac{1}{\sigma(\varphi)} \operatorname{curl} \boldsymbol{B} \right) - \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0}, \tag{2.3e}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.3f}$$

The identity $\operatorname{div}(\varphi \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\varphi$ follows from the incompressibility of fluids. The left-hand side of (2.3a) expresses the transport property of phase field, i.e., material point does not change type at least in the limit case [23,28]. The term $\lambda \varphi \nabla w$ in (2.3c) is the continuum surface tension force in the potential form [23,24]. This force originates from the phase induced force in the stress form

$$\lambda \operatorname{div}(\nabla \varphi \otimes \nabla \varphi) = \lambda \Delta \varphi \nabla \varphi + \frac{\lambda}{2} \nabla |\nabla \varphi|^2 = \lambda \varphi \nabla w + \nabla \left(\frac{\lambda}{\varepsilon^2} F(\varphi) - \lambda w \varphi + \frac{\lambda}{2} |\nabla \varphi|^2 \right),$$

where $\nabla \varphi \otimes \nabla \varphi$ is the induced elastic stress due to the mixing of the different phases [24,28]. The pressure in (2.3c) is given by $p + \frac{\lambda}{\varepsilon^2} F(\varphi) - \lambda w \varphi + \frac{\lambda}{2} |\nabla \varphi|^2$ (still denote by p for simplicity) [24].

The model (2.3a)–(2.3f) is complemented with the following initial and boundary conditions:

$$\begin{cases} \boldsymbol{u}=\mathbf{0}, & \text{on } \partial\Omega\times[0,T],\\ \boldsymbol{B}\cdot\boldsymbol{n}=0,\ \ \boldsymbol{n}\times\operatorname{curl}\boldsymbol{B}=\mathbf{0}, & \text{on } \partial\Omega\times[0,T],\\ \dfrac{\partial\varphi}{\partial\boldsymbol{n}}=\dfrac{\partial w}{\partial\boldsymbol{n}}=0, & \text{on } \partial\Omega\times[0,T],\\ \varphi(x,0)=\varphi_0, \boldsymbol{u}(x,0)=\boldsymbol{u}_0, \boldsymbol{B}(x,0)=\boldsymbol{B}_0, & \forall\, x\in\Omega, \end{cases}$$

where $\operatorname{div}\boldsymbol{u}_0=\operatorname{div}\boldsymbol{B}_0=0$, $T\in(0,\infty)$ and $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$.

The system (2.3a)–(2.3f) models the interaction of electromagnetic fields and two incompressible, immiscible fluids with different viscosities and electric conductivities. In this paper, we consider two-phase fluids with matching density ρ. For brevity, ρ is taken to be 1. The variable density case will be studied in the future work. $\eta(\varphi)$ and $\sigma(\varphi)$, which depend on φ, are hydrodynamic viscosity and electric conductivity satisfying

$$\begin{aligned} &0<\eta^-:=\min\{\eta_1,\eta_2\}\le\eta(\varphi)\le\max\{\eta_1,\eta_2\}=:\eta^+,\\ &0<\sigma^-:=\min\{\sigma_1,\sigma_2\}\le\sigma(\varphi)\le\max\{\sigma_1,\sigma_2\}=:\sigma^+, \end{aligned}\tag{2.4}$$

where η_i and σ_i $(i=1,2)$ are the viscosity and electric conductivity of the fluid i. Assume $\eta(\varphi)$ and $\frac{1}{\sigma(\varphi)}$ are Lipschitz continuous functions with respect to φ. The phase field φ is almost constants (±1) in bulk regions and smoothly transitions between these values in an interfacial region of thickness ε. In this paper, choose

$$\eta(\varphi)=\eta_1+(\eta_2-\eta_1)\mathcal{H}_\varepsilon(\varphi),\quad \sigma(\varphi)=\sigma_1+(\sigma_2-\sigma_1)\mathcal{H}_\varepsilon(\varphi),\tag{2.5}$$

where $\mathcal{H}_\varepsilon(x)=\frac{1}{1+e^{-\frac{x}{\varepsilon}}}$ is a regularized approximation of the Heaviside step function [46]. It can be shown that $\eta(\varphi)$ and $\frac{1}{\sigma(\varphi)}$ in (2.5) are Lipschitz continuous functions of φ and satisfy (2.4).

3. Preliminary knowledge and definition of weak solution

Consider a bounded domain Ω in $\mathbb{R}^d$ $(d=2,3)$ is a convex polygon/polyhedron. According to Poincaré inequality and Proposition 3.16 of [15], the norms of the spaces $\boldsymbol{H}_0^1(\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\,\boldsymbol{v}\,|_{\partial\Omega}=0\right\}$ and $\boldsymbol{H}_n^1(\Omega)=\left\{\boldsymbol{C}\in\boldsymbol{H}^1(\Omega);\,\boldsymbol{C}\cdot\boldsymbol{n}\,|_{\partial\Omega}=0\right\}$ are defined by $\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}=\|\nabla\boldsymbol{v}\|_{L^2}$ and $\|\boldsymbol{C}\|_{\boldsymbol{H}_n^1(\Omega)}=\left(\|\operatorname{curl}\boldsymbol{C}\|_{L^2}^2+\|\operatorname{div}\boldsymbol{C}\|_{L^2}^2\right)^{\frac12}$, respectively. The spaces $\boldsymbol{H}$, $\boldsymbol{V}$, $\boldsymbol{W}$ and their norms are denoted by

$$\begin{aligned} &\boldsymbol{H}=\left\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\,\operatorname{div}\boldsymbol{v}=0,\,\boldsymbol{v}\cdot\boldsymbol{n}\,|_{\partial\Omega}=0\right\}, && \|\boldsymbol{v}\|_{\boldsymbol{H}}=\|\boldsymbol{v}\|_{L^2};\\ &\boldsymbol{V}=\left\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\,\operatorname{div}\boldsymbol{v}=0,\,\boldsymbol{v}\,|_{\partial\Omega}=\mathbf{0}\right\}, && \|\boldsymbol{v}\|_{\boldsymbol{V}}=\|\nabla\boldsymbol{v}\|_{L^2};\\ &\boldsymbol{W}=\left\{\boldsymbol{C}\in\boldsymbol{H}^1(\Omega);\,\operatorname{div}\boldsymbol{C}=0,\,\boldsymbol{C}\cdot\boldsymbol{n}\,|_{\partial\Omega}=0\right\}, && \|\boldsymbol{C}\|_{\boldsymbol{W}}=\|\operatorname{curl}\boldsymbol{C}\|_{L^2}. \end{aligned}$$

Furthermore, for the function spaces $L^r(0,T;X)$, $1\le r\le\infty$, the norms are denoted as $\|\cdot\|_{L^\infty(X)}:=\operatorname{ess\,sup}_{0\le t\le T}\|\cdot\|_X$ and $\|\cdot\|_{L^r(X)}:=\left(\int_0^T\|\cdot\|_X^r\mathrm{d}t\right)^{\frac1r}$ for $1\le r<\infty$, where X is a real Banach space with the norm $\|\cdot\|_X$. The symbol $(\cdot,\cdot)$ denotes the L^2 inner product over Ω and $\langle\cdot,\cdot\rangle$ stands for the dual product between the space and its dual space, for example $(H^1(\Omega))'$ and $H^1(\Omega)$, $(\boldsymbol{H}_0^1(\Omega))'$ and $\boldsymbol{H}_0^1(\Omega)$, $(\boldsymbol{H}_n^1(\Omega))'$ and $\boldsymbol{H}_n^1(\Omega)$.

Under the assumptions on Ω stated above, there exists the orthogonal decomposition:

$$\boldsymbol{L}^2(\Omega)=\boldsymbol{H}\oplus\nabla H^1(\Omega)/\mathbb{R},\tag{3.1}$$

and $\|P_H\boldsymbol{u}\|_{H^1}\le c_0\|\boldsymbol{u}\|_{H^1}$ holds for any $\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$, where P_H is the Helmholtz projection from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}$ (see Theorems 1.10 and 2.7 of [66]). According to [15,67–69], the following estimates hold:

$$c_0\|\nabla\boldsymbol{u}\|_{L^2}\le\|D(\boldsymbol{u})\|_{L^2}\le\|\nabla\boldsymbol{u}\|_{L^2},\qquad \forall\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega),\tag{3.2}$$

$$\|\boldsymbol{u}\|_{L^p}\le c_0\|\nabla\boldsymbol{u}\|_{L^2},\qquad \forall\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega),\,2\le p\le6,\tag{3.3}$$

$$\|\boldsymbol{u}\|_{L^3}\le c_0\|\boldsymbol{u}\|_{L^2}^{\frac{6-d}{6}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{d}{6}},\qquad \forall\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega),\tag{3.4}$$

$$\|\boldsymbol{u}\|_{L^4}\le c_0\|\boldsymbol{u}\|_{L^2}^{\frac{4-d}{4}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{d}{4}},\qquad \forall\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega),\tag{3.5}$$

$$\|\boldsymbol{B}\|_{L^p}\le c_0\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)},\qquad \forall\boldsymbol{B}\in\boldsymbol{H}_n^1(\Omega),\,2\le p\le6,\tag{3.6}$$

$$\|\boldsymbol{B}\|_{L^3} \le c_0 \|\boldsymbol{B}\|_{L^2}^{\frac{6-d}{6}} \|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^{\frac{d}{6}}, \qquad \forall \boldsymbol{B} \in \boldsymbol{H}_n^1(\Omega), \tag{3.7}$$

$$\|\boldsymbol{B}\|_{L^4} \le c_0 \|\boldsymbol{B}\|_{L^2}^{\frac{4-d}{4}} \|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^{\frac{d}{4}}, \qquad \forall \boldsymbol{B} \in \boldsymbol{H}_n^1(\Omega), \tag{3.8}$$

$$\|\varphi\|_{L^p} \le c_0 \|\varphi\|_{H^1}, \qquad \forall \varphi \in H^1(\Omega), 2 \le p \le 6, \tag{3.9}$$

$$\|\varphi\|_{L^3} \le c_0 \|\varphi\|_{L^2}^{\frac{6-d}{6}} \|\nabla\varphi\|_{L^2}^{\frac{d}{6}} + c_0\|\varphi\|_{L^2}, \qquad \forall \varphi \in H^1(\Omega), \tag{3.10}$$

$$\|\varphi\|_{L^\infty} \le c_0 \|\Delta\varphi\|_{L^2}^{\frac{d}{2(6-d)}} \|\varphi\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} + c_0\|\varphi\|_{L^6}, \qquad \forall \varphi \in H^2(\Omega). \tag{3.11}$$

In this paper, c_0 is a generic positive constant depending only on Ω and c is a generic positive constant depending on $(\Omega, \gamma, \varepsilon, \lambda, \eta, \mu, \sigma)$. c_0 and c may be different at each occurrence.

The definition of a weak solution to the problem (2.3a)–(2.3f) is given as follows.

Definition 3.1. Let $\varphi_0 \in H^1(\Omega)$, $\boldsymbol{u}_0, \boldsymbol{B}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0, T; (\boldsymbol{H}_0^1(\Omega))')$. $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ is called a weak solution of the problem (2.3a)–(2.3f) if (i) it satisfies

$$\varphi \in L^\infty(0, T; H^1(\Omega)), \qquad \varphi_t \in L^2(0, T; (H^1(\Omega))'), \tag{3.12}$$

$$w \in L^2(0, T; H^1(\Omega)), \tag{3.13}$$

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \cap L^2(0, T; \boldsymbol{H}_0^1(\Omega)), \qquad \boldsymbol{u}_t \in L^{\frac{12}{6+d}}(0, T; (\boldsymbol{H}_0^1(\Omega))'), \tag{3.14}$$

$$p \in L^{\frac{12}{6+d}}(0, T; L_0^2(\Omega)), \tag{3.15}$$

$$\boldsymbol{B} \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \cap L^2(0, T; \boldsymbol{H}_n^1(\Omega)), \qquad \boldsymbol{B}_t \in L^{\frac{4}{d}}(0, T; (\boldsymbol{H}_n^1(\Omega))'), \tag{3.16}$$

where $d = 2, 3$; (ii) there hold

$$\langle \varphi_t, \psi\rangle - (\varphi \boldsymbol{u}, \nabla\psi) + \gamma(\nabla w, \nabla\psi) = 0, \tag{3.17a}$$

$$(\nabla\varphi, \nabla\chi) + \frac{1}{\varepsilon^2}\big(f(\varphi), \chi\big) = (w, \chi), \tag{3.17b}$$

$$\langle \boldsymbol{u}_t, \boldsymbol{v}\rangle + 2\big(\eta(\varphi)D(\boldsymbol{u}), D(\boldsymbol{v})\big) + \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}\big) + \frac{1}{\mu}(\boldsymbol{B}\times \operatorname{curl}\boldsymbol{B}, \boldsymbol{v}) - (\operatorname{div}\boldsymbol{v}, p) + \lambda(\varphi\nabla w, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{3.17c}$$

$$(\operatorname{div}\boldsymbol{u}, q) = 0, \tag{3.17d}$$

$$\langle \boldsymbol{B}_t, \boldsymbol{C}\rangle + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi)}\operatorname{curl}\boldsymbol{B}, \operatorname{curl}\boldsymbol{C}\right) + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi)}\operatorname{div}\boldsymbol{B}, \operatorname{div}\boldsymbol{C}\right) - (\boldsymbol{u}\times\boldsymbol{B}, \operatorname{curl}\boldsymbol{C}) = 0, \tag{3.17e}$$

for almost all $t \in (0, T)$ and any $(\psi, \chi, \boldsymbol{v}, q, \boldsymbol{C}) \in H^1(\Omega) \times H^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_n^1(\Omega)$, and $\varphi(0) = \varphi_0, \boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{B}(0) = \boldsymbol{B}_0$; and (iii) the energy stability

$$\mathcal{J}_{\varepsilon,\lambda,\mu}\big(\boldsymbol{u}(t), \boldsymbol{B}(t), \varphi(t)\big) + \int_0^t \Big\{\lambda\gamma\|\nabla w\|_{L^2}^2 + 2\|\sqrt{\eta(\varphi)}D(\boldsymbol{u})\|_{L^2}^2 + \frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi)}}\operatorname{curl}\boldsymbol{B}\|_{L^2}^2 + \frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi)}}\operatorname{div}\boldsymbol{B}\|_{L^2}^2\Big\}\mathrm{d}s \le \int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle \mathrm{d}s + \mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0), \tag{3.18}$$

is true for almost all $t \in [0, T]$, where $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}, \boldsymbol{B}, \varphi) := \frac{1}{2}\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2\mu}\|\boldsymbol{B}\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}(F(\varphi), 1)$.

Remark 3.1. If $\boldsymbol{B}_t \in L^2(0, T; (\boldsymbol{H}_n^1(\Omega))')$ and $\boldsymbol{B} \in L^2(0, T; \boldsymbol{H}_n^1(\Omega))$, we can prove the divergence-free constraint on $\boldsymbol{B}$ from (3.17e). Consider the backward-in-time equation

$$\begin{cases} \phi_t + \dfrac{1}{\mu\sigma(\varphi)}\Delta\phi = \operatorname{div}\boldsymbol{B}, & \text{in } \Omega \times [0, T], \\ \dfrac{\partial\phi}{\partial\boldsymbol{n}} = 0, & \text{on } \partial\Omega \times [0, T], \\ \phi = 0, & \text{in } \Omega \times \{T\}. \end{cases} \tag{3.19}$$

According to (2.4)–(2.5) and [66,70–72], there exists a solution ϕ to the problem (3.19) satisfying $\phi \in L^2(0,T;H^2(\Omega))$ and $\phi_t \in L^2(0,T;L^2(\Omega))$. Then, taking $\boldsymbol{C} = \nabla\phi \in L^2(0,T;\boldsymbol{H}^1_n(\Omega))$ in (3.17e), integrating with respect to t, and using $\operatorname{div}\boldsymbol{B}(x,0) = \phi(x,T) = 0$ for all $x \in \Omega$, we have

$$\int_0^T \left(\operatorname{div}\boldsymbol{B}, \phi_t + \frac{1}{\mu\sigma(\varphi)}\Delta\phi\right) \mathrm{d}t = \int_0^T \|\operatorname{div}\boldsymbol{B}\|^2_{L^2}\mathrm{d}t = 0.$$

Remark 3.2. For all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{B} \in \boldsymbol{W}$, we can obtain the following equalities

$$\begin{aligned} \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}\big) &= -\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{u}\big), \\ (\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}, \boldsymbol{v}) &= \big((\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \boldsymbol{B}\big). \end{aligned}$$

Therefore, based on (3.17c) and the estimates (3.2)–(3.9), we get

$$\boldsymbol{u}_t \in L^{\frac{4}{d}}(0,T;\boldsymbol{V}').$$

The result will be used in Lemma 5.4.

Remark 3.3. Based on [73] and Chapter III of [74], we have $\boldsymbol{u}, \boldsymbol{B} \in C(0,T;\boldsymbol{L}^2(\Omega))$ for $d = 2$, $\boldsymbol{u}, \boldsymbol{B} \in C_w(0,T;\boldsymbol{L}^2(\Omega))$ for $d = 3$ and $\varphi \in C(0,T;L^2(\Omega))$ for $d = 2,3$. The space $C_w(0,T;\boldsymbol{L}^2(\Omega))$ consists of all weakly continuous functions in $\boldsymbol{L}^2(\Omega)$, i.e., if $\boldsymbol{u}(t) \in C_w(0,T;\boldsymbol{L}^2(\Omega))$, $F(t) = (\boldsymbol{u}(t), \boldsymbol{v})$ is a continuous function for all $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

4. Fully discrete energy stable finite element method

In this section, we propose a fully discrete finite element method, which is energy stable and semi-implicit, to solve the Cahn–Hilliard–MHD model proposed above. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform partition of Ω into triangles in two dimensions or tetrahedra in three dimensions with characteristic mesh size h. Based on the partition $\mathcal{T}_h$, we introduce the finite element spaces $\boldsymbol{X}_h \subset \boldsymbol{H}^1_0(\Omega)$, $M_h \subset L^2_0(\Omega)$, $\boldsymbol{W}_h \subset \boldsymbol{H}^1_n(\Omega)$ for the discrete velocity, pressure and magnetic field, and the finite element space $Y_h \subset H^1(\Omega)$ for the discrete phase field φ and chemical potential w. Assume $\boldsymbol{X}_h$, M_h and Y_h satisfy the following conditions.

Assumption (A). The finite element spaces $(\boldsymbol{X}_h, M_h)$ and (Y_h, Y_h) satisfy the inf–sup conditions:

$$\inf_{q_h\in M_h\setminus\{0\}} \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h\setminus\{\boldsymbol{0}\}} \frac{(\operatorname{div}\boldsymbol{v}_h, q_h)}{\|q_h\|_{L^2}\|\nabla\boldsymbol{v}_h\|_{L^2}} \geq \beta_0, \tag{4.1}$$

$$\inf_{\chi_h\in Y_h\setminus\{0\}} \sup_{\psi_h\in Y_h\setminus\{0\}} \frac{(\nabla\psi_h, \nabla\chi_h)}{\|\psi_h\|_{H^1}\|\chi_h\|_{H^1}} \geq \beta_1, \tag{4.2}$$

where β_0 and β_1 are positive constants depending only on Ω.

Remark 4.1. According to Chapter II of [66] and Chapter IV of [75], there are a variety of spaces $(\boldsymbol{X}_h, M_h)$ satisfying Assumption (A) such as $\boldsymbol{P}_2$-P_0 element, Mini-element ($\boldsymbol{P}_{1b}$-P_1) and Taylor–Hood element. The P_r-P_r ($r \geq 1$) conforming finite element spaces (Y_h, Y_h) are a family of stable mixed finite element spaces for biharmonic problem, that is, these spaces satisfy the inf–sup condition (4.2) (see [76–79]). The finite space $\boldsymbol{W}_h$ is taken to be $\boldsymbol{W}_h = \{\boldsymbol{C}_h \in \boldsymbol{C}^0(\overline{\Omega}) \cap \boldsymbol{H}^1_n(\Omega); \boldsymbol{C}_h|_K \in \boldsymbol{P}_k(K), \forall K \in \mathcal{T}_h\}$, $k \geq 1$.

Remark 4.2. In this paper, we consider the domain is a convex polygon/polyhedron. The classical H^1-conforming finite elements can be used to approximate the magnetic field. For the general domain with re-entrant corners, the magnetic field is in general not in $\boldsymbol{H}^1(\Omega)$. Some numerical methods can be applied to approximate the singular solution, such as Nédélec finite elements [80,81], weighted regularization technique [82,83] and stabilized finite element formulation [84,85]. Based on these methods, the extensions to general domains with re-entrant corners are possible for two-phase MHD flows, which will be studied in the future work. Furthermore, pre-conditioners can be considered for handling the difficult cases with high condition numbers [86–89].

4.1. Description of the scheme and its stability

For arbitrary but fixed $T > 0$ and positive integer $N \in \mathbb{N}$, we denote by $\tau = \frac{T}{N}$ the time step and $d_t v_h^n = \frac{v_h^n - v_h^{n-1}}{\tau}$. For two-phase MHD model (2.3a)–(2.3f), the semi-implicit energy stable finite element scheme is to find $(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ such that

$$(d_t\varphi_h^n, \psi_h) - (\varphi_h^{n-1}\boldsymbol{u}_h^n, \nabla\psi_h) + \gamma(\nabla w_h^n, \nabla\psi_h) = 0, \tag{4.3a}$$

$$(\nabla\varphi_h^n, \nabla\chi_h) + \frac{1}{\varepsilon^2}(f_h^n, \chi_h) = (w_h^n, \chi_h), \tag{4.3b}$$

$$\begin{aligned}(d_t\boldsymbol{u}_h^n, \boldsymbol{v}_h) &+ 2\big(\eta(\varphi_h^{n-1})D(\boldsymbol{u}_h^n), D(\boldsymbol{v}_h)\big) + \big((\boldsymbol{u}_h^{n-1}\cdot\nabla)\boldsymbol{u}_h^n, \boldsymbol{v}_h\big) + \frac{1}{2}\big((\operatorname{div}\boldsymbol{u}_h^{n-1})\boldsymbol{u}_h^n, \boldsymbol{v}_h\big)\\ &+ \frac{1}{\mu}(\boldsymbol{B}_h^{n-1}\times\operatorname{curl}\boldsymbol{B}_h^n, \boldsymbol{v}_h) - (p_h^n, \operatorname{div}\boldsymbol{v}_h) + \lambda(\varphi_h^{n-1}\nabla w_h^n, \boldsymbol{v}_h) = \boldsymbol{f}^n(\boldsymbol{v}_h),\end{aligned} \tag{4.3c}$$

$$(\operatorname{div}\boldsymbol{u}_h^n, q_h) = 0, \tag{4.3d}$$

$$\begin{aligned}(d_t\boldsymbol{B}_h^n, \boldsymbol{C}_h) + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{curl}\boldsymbol{B}_h^n, \operatorname{curl}\boldsymbol{C}_h\right) + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{div}\boldsymbol{B}_h^n, \operatorname{div}\boldsymbol{C}_h\right)&\\ - (\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1}, \operatorname{curl}\boldsymbol{C}_h) = 0,&\end{aligned} \tag{4.3e}$$

$$\varphi_h^0 = Q_h\varphi_0, \quad \boldsymbol{u}_h^0 = P_{0h}\boldsymbol{u}_0, \quad \boldsymbol{B}_h^0 = R_h\boldsymbol{B}_0, \tag{4.3f}$$

for any $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{C}_h) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ and $\boldsymbol{f}^n(\boldsymbol{v}_h) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\langle\boldsymbol{f}, \boldsymbol{v}_h\rangle\mathrm{d}t$. Furthermore, if $\boldsymbol{f} \in C(0, T; (\boldsymbol{H}_0^1(\Omega))')$, $\boldsymbol{f}^n(\boldsymbol{v}_h)$ in (4.3c) can be taken by $\langle\boldsymbol{f}(t_n), \boldsymbol{v}_h\rangle$. $f_h^n := (\varphi_h^n)^3 - \varphi_h^{n-1}$ in (4.3b) is derived from a convex splitting approximation to the non-convex function $F(\varphi)$ (see [90–92]). Denote Q_h as L^2-orthogonal projection operator from $L^2(\Omega)$ into Y_h, P_h (R_h) as L^2-orthogonal projection operator from $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{X}_h$ ($\boldsymbol{W}_h$), and P_{0h} as L^2-orthogonal projection operator from $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{V}_h$, respectively. The space $\boldsymbol{V}_h$ is denoted by $\boldsymbol{V}_h = \{\boldsymbol{u}_h \in \boldsymbol{X}_h; (\operatorname{div}\boldsymbol{u}_h, q_h) = 0, \forall q_h \in M_h\}$. The projection operators Q_h, P_h and R_h have H^1-stability [93–95]. There also holds $W^{1,4}$-stability for these projection operators. In fact, $\boldsymbol{W}^{1,4}(\Omega) \subset \boldsymbol{C}^0(\overline{\Omega})$ with compact injection. For any $\boldsymbol{u} \in \boldsymbol{W}^{1,4}(\Omega)$, we have

$$\begin{aligned}\|P_h\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}} &\le \|P_h\boldsymbol{u} - \pi_h\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}} + \|\pi_h\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}}\\ &\le ch^{-1}(\|P_h\boldsymbol{u} - \boldsymbol{u}\|_{L^4} + \|\boldsymbol{u} - \pi_h\boldsymbol{u}\|_{L^4}) + \|\pi_h\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}}\\ &\le c\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}},\end{aligned}$$

where π_h is the nodal interpolation operator from $\boldsymbol{C}^0(\overline{\Omega})$ to $\boldsymbol{X}_h$. Assume that

$$\lim_{h\to 0}\|\varphi_h^0 - \varphi_0\|_{H^1} = \lim_{h\to 0}\|\boldsymbol{u}_h^0 - \boldsymbol{u}_0\|_{L^2} = \lim_{h\to 0}\|\boldsymbol{B}_h^0 - \boldsymbol{B}_0\|_{L^2} = 0. \tag{4.4}$$

Firstly, the fully discrete scheme (4.3a)–(4.3f) satisfies a discrete energy law.

Theorem 4.1. *Suppose Assumption (A) is valid and let* $\{(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n)\}$ $(n = 1, \ldots, N)$ *be a solution of the scheme (4.3a)–(4.3f). Then for any* $1 \le m \le N$*, there holds the following estimate*

$$\begin{aligned}&\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^m, \boldsymbol{B}_h^m, \varphi_h^m) + \tau\sum_{n=1}^m\left(\frac{\lambda\tau}{2}\|\nabla d_t\varphi_h^n\|_{L^2}^2 + \frac{\lambda\tau}{4\varepsilon^2}\|d_t(\varphi_h^n)^2\|_{L^2}^2\right)\\ &+\tau\sum_{n=1}^m\left(\frac{\lambda\tau}{2\varepsilon^2}\|\varphi_h^n d_t\varphi_h^n\|_{L^2}^2 + \frac{\lambda\tau}{2\varepsilon^2}\|d_t\varphi_h^n\|_{L^2}^2 + \frac{\tau}{2}\|d_t\boldsymbol{u}_h^n\|_{L^2}^2 + \frac{\tau}{2\mu}\|d_t\boldsymbol{B}_h^n\|_{L^2}^2 + \lambda\gamma\|\nabla w_h^n\|_{L^2}^2\right)\\ &+\tau\sum_{n=1}^m\left(2\|\sqrt{\eta(\varphi_h^{n-1})}D(\boldsymbol{u}_h^n)\|_{L^2}^2 + \frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_h^{n-1})}}\operatorname{curl}\boldsymbol{B}_h^n\|_{L^2}^2 + \frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_h^{n-1})}}\operatorname{div}\boldsymbol{B}_h^n\|_{L^2}^2\right)\\ &= \tau\sum_{n=1}^m\boldsymbol{f}^n(\boldsymbol{u}_h^n) + \mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^0, \boldsymbol{B}_h^0, \varphi_h^0).\end{aligned} \tag{4.5}$$

Proof. Taking $\psi_h = \lambda\tau w_h^n$ in (4.3a), $\chi_h = \lambda\tau d_t\varphi_h^n$ in (4.3b), $\boldsymbol{v}_h = \tau\boldsymbol{u}_h^n$ in (4.3c), $q_h = \tau p_h^n$ in (4.3d) and $\boldsymbol{C}_h = \frac{\tau}{\mu}\boldsymbol{B}_h^n$ in (4.3e), and applying the equalities

$$2a(a-b) = a^2 - b^2 + (a-b)^2, \tag{4.6}$$

$$(a^3-b)(a-b) = \frac{1}{4}\left[(a^2-1)^2 - (b^2-1)^2\right] + \frac{1}{4}(a^2-b^2)^2 + \frac{1}{2}a^2(a-b)^2 + \frac{1}{2}(a-b)^2, \tag{4.7}$$

we have (4.5). $\square$

Furthermore, the fully discrete scheme (4.3a)–(4.3f) satisfies the mass conservation and its solution has the following estimates.

Theorem 4.2. *Suppose that Assumption (A) is valid, $\boldsymbol{f} \in L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')$ and $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0,\boldsymbol{B}_0,\varphi_0) < \infty$. Let $\{(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n)\}$ $(n = 1,\ldots,N)$ be a solution of the scheme (4.3a)–(4.3f). Then for any $1 \le m \le N$, the following estimates hold*

$$\int_\Omega \varphi_h^m \mathrm{d}x = \int_\Omega \varphi_h^0 \mathrm{d}x, \quad (\textit{mass conservation}) \tag{4.8}$$

$$\max_{1\le n\le N}\left\{\|\boldsymbol{u}_h^n\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}_h^n\|_{L^2}^2 + \lambda\|\nabla\varphi_h^n\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}(F(\varphi_h^n),1)\right\} \le C, \tag{4.9}$$

$$\tau\sum_{n=1}^m\left(\eta^-\|\nabla\boldsymbol{u}_h^n\|_{L^2}^2 + \frac{1}{\mu^2\sigma^+}\|\boldsymbol{B}_h^n\|_{\boldsymbol{H}_n^1(\Omega)}^2 + \lambda\gamma\|\nabla w_h^n\|_{L^2}^2\right) \le C, \tag{4.10}$$

$$\sum_{n=1}^m\left(\lambda\|\nabla\varphi_h^n - \nabla\varphi_h^{n-1}\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\|\varphi_h^n - \varphi_h^{n-1}\|_{L^2}^2\right) \le C, \tag{4.11}$$

$$\sum_{n=1}^m\left(\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1}\|_{L^2}^2\right) \le C, \tag{4.12}$$

$$\max_{1\le n\le N}\|\varphi_h^n\|_{H^1} \le C, \tag{4.13}$$

$$\tau\sum_{n=1}^m\|w_h^n\|_{H^1}^2 \le C\left(\frac{T}{\varepsilon^4} + T + 1\right), \tag{4.14}$$

$$\tau\sum_{n=1}^m\|d_t\varphi_h^n\|_{(H^1)'}^2 \le C, \tag{4.15}$$

$$\tau\sum_{n=1}^m\left(\|d_t\boldsymbol{u}_h^n\|_{(\boldsymbol{H}_0^1)'}^{\frac{12}{6+d}} + \|p_h^n\|_{L^2}^{\frac{12}{6+d}} + \|d_t\boldsymbol{B}_h^n\|_{(\boldsymbol{H}_n^1)'}^{\frac{4}{d}}\right) \le C(T+1), \tag{4.16}$$

where C is a constant depending on $(\Omega,\lambda,\gamma,\eta,\mu,\sigma,\varphi_0,\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{f})$.

Proof. Letting $\psi_h = 1$ in (4.3a), we have (4.8). Based on (2.4) and (3.2), we get

$$\begin{aligned}\tau\sum_{n=1}^m \boldsymbol{f}^n(\boldsymbol{u}_h^n) &= \sum_{n=1}^m\int_{t_{n-1}}^{t_n}\langle\boldsymbol{f},\boldsymbol{u}_h^n\rangle dt \le \tau^{\frac{1}{2}}\sum_{n=1}^m\|\nabla\boldsymbol{u}_h^n\|_{L^2}\left(\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}^2 dt\right)^{\frac{1}{2}} \\ &\le \tau\sum_{n=1}^m\eta^-\|D(\boldsymbol{u}_h^n)\|_{L^2}^2 + c\int_0^{t_m}\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}^2\mathrm{d}t.\end{aligned}$$

Then, (4.9)–(4.12) follow from (4.5). There holds

$$\big(F(\varphi_h^n),1\big) = \big(\frac{1}{4}((\varphi_h^n)^2-1)^2,1\big) \ge \|\varphi_h^n\|_{L^2}^2 - 2|\Omega|,$$

where $|\Omega|$ stands for the area in two dimensions or volume in three dimensions of Ω. Based on the above inequality and (4.9), we have (4.13).

Taking $\chi_h = \tau w_h^n$ in (4.3b), we have

$$\begin{aligned}\tau\|w_h^n\|_{L^2}^2 &= \tau(\nabla\varphi_h^n, \nabla w_h^n) + \frac{\tau}{\varepsilon^2}(f_h^n, w_h^n) \le \tau\|\nabla\varphi_h^n\|_{L^2}\|\nabla w_h^n\|_{L^2} + \frac{\tau}{\varepsilon^2}(\|\varphi_h^{n-1}\|_{L^2} + \|\varphi_h^n\|_{L^6}^3)\|w_h^n\|_{L^2}\\ &\le \frac{\tau}{2}\|w_h^n\|_{L^2}^2 + \frac{\tau}{2}\|\nabla\varphi_h^n\|_{L^2}^2 + \frac{\tau}{2}\|\nabla w_h^n\|_{L^2}^2 + \frac{c\tau}{\varepsilon^4}(\|\varphi_h^{n-1}\|_{L^2}^2 + \|\varphi_h^n\|_{H^1}^6). \end{aligned} \tag{4.17}$$

Combining (4.9)–(4.10), (4.13) with (4.17), we obtain (4.14).

Setting $\psi_h \in Q_h\psi$, for any $\psi \in H^1(\Omega)$ in (4.3a) and using the H^1 stability of L^2 projection Q_h, we have

$$\begin{aligned}(d_t\varphi_h^n, \psi) &= (d_t\varphi_h^n, Q_h\psi) = (\varphi_h^{n-1}\boldsymbol{u}_h^n, \nabla Q_h\psi) - \gamma(\nabla w_h^n, \nabla Q_h\psi)\\ &\le (\|\varphi_h^{n-1}\|_{L^3}\|\boldsymbol{u}_h^n\|_{L^6} + \gamma\|\nabla w_h^n\|_{L^2})\|\nabla Q_h\psi\|_{L^2}\\ &\le c_0(\|\varphi_h^{n-1}\|_{H^1}\|\nabla\boldsymbol{u}_h^n\|_{L^2} + \gamma\|\nabla w_h^n\|_{L^2})\|\nabla\psi\|_{L^2}. \end{aligned} \tag{4.18}$$

From (4.10), (4.13) and the above inequality, we have (4.15).

Next, define the discrete inverse Stokes operator S_h from $(\boldsymbol{H}_0^1(\Omega))'$ to X_h as follows: for all $\boldsymbol{v} \in (\boldsymbol{H}_0^1(\Omega))'$, $(S_h(\boldsymbol{v}), r_h) \in \boldsymbol{X}_h \times M_h$ satisfies

$$\begin{aligned}&(\nabla S_h(\boldsymbol{v}), \nabla\boldsymbol{v}_h) + (\operatorname{div}\boldsymbol{v}_h, r_h) = \langle\boldsymbol{v}, \boldsymbol{v}_h\rangle, \qquad \forall\boldsymbol{v}_h \in \boldsymbol{X}_h,\\ &(\operatorname{div} S_h(\boldsymbol{v}), q_h) = 0, \qquad \forall q_h \in M_h.\end{aligned}$$

If $\boldsymbol{u}_h \in \boldsymbol{V}_h$, there exists a constant $c > 0$ independent of h such that (see Lemma 4.12 of [96])

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h\setminus\{\boldsymbol{0}\}} \frac{\langle\boldsymbol{u}_h, \boldsymbol{v}_h\rangle}{\|\nabla\boldsymbol{v}_h\|_{L^2}} \le c\|\nabla S_h(\boldsymbol{u}_h)\|_{L^2}.$$

Hence, for $\boldsymbol{u}_h \in \boldsymbol{V}_h$, there holds

$$\|\boldsymbol{u}_h\|_{(\boldsymbol{H}_0^1)'} = \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\langle\boldsymbol{u}_h, \boldsymbol{v}\rangle}{\|\nabla\boldsymbol{v}\|_{L^2}} = \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\langle\boldsymbol{u}_h, P_h\boldsymbol{v}\rangle}{\|\nabla P_h\boldsymbol{v}\|_{L^2}}\cdot\frac{\|\nabla P_h\boldsymbol{v}\|_{L^2}}{\|\nabla\boldsymbol{v}\|_{L^2}} \le c\|\nabla S_h(\boldsymbol{u}_h)\|_{L^2}. \tag{4.19}$$

From (4.3d), we know $d_t\boldsymbol{u}_h^n \in \boldsymbol{V}_h$. Using the definition of discrete inverse Stokes operator S_h and setting $\boldsymbol{v}_h = S_h(d_t\boldsymbol{u}_h^n)$ in (4.3c), we have

$$\begin{aligned}\|\nabla S_h(d_t\boldsymbol{u}_h^n)\|_{L^2}^2 &= \big(d_t\boldsymbol{u}_h^n, S_h(d_t\boldsymbol{u}_h^n)\big) = -2\big(\eta(\varphi_h^{n-1})D(\boldsymbol{u}_h^n), D(S_h(d_t\boldsymbol{u}_h^n))\big)\\ &-\big((\boldsymbol{u}_h^{n-1}\cdot\nabla)\boldsymbol{u}_h^n, S_h(d_t\boldsymbol{u}_h^n)\big) - \frac{1}{2}\big((\operatorname{div}\boldsymbol{u}_h^{n-1})\boldsymbol{u}_h^n, S_h(d_t\boldsymbol{u}_h^n)\big)\\ &-\frac{1}{\mu}\big(\boldsymbol{B}_h^{n-1}\times\operatorname{curl}\boldsymbol{B}_h^n, S_h(d_t\boldsymbol{u}_h^n)\big) - \lambda\big(\varphi_h^{n-1}\nabla w_h^n, S_h(d_t\boldsymbol{u}_h^n)\big) + \boldsymbol{f}^n(S_h(d_t\boldsymbol{u}_h^n))\\ &\le c\left(\|\boldsymbol{u}_h^{n-1}\|_{L^2}^{\frac{6-d}{6}}\|\nabla\boldsymbol{u}_h^{n-1}\|_{L^2}^{\frac{d}{6}}\|\nabla\boldsymbol{u}_h^n\|_{L^2} + \|\boldsymbol{B}_h^{n-1}\|_{L^2}^{\frac{6-d}{6}}\|\boldsymbol{B}_h^{n-1}\|_{\boldsymbol{H}_n^1(\Omega)}^{\frac{d}{6}}\|\boldsymbol{B}_h^n\|_{\boldsymbol{H}_n^1(\Omega)}\right)\|\nabla S_h(d_t\boldsymbol{u}_h^n)\|_{L^2}\\ &+c\left(\|\nabla\boldsymbol{u}_h^n\|_{L^2} + \|\varphi_h^{n-1}\|_{H^1}\|\nabla w_h^n\|_{L^2} + \tau^{-\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}^2\mathrm{d}t\right)^{\frac{1}{2}}\right)\|\nabla S_h(d_t\boldsymbol{u}_h^n)\|_{L^2}. \end{aligned} \tag{4.20}$$

Setting $\boldsymbol{C}_h = R_h\boldsymbol{C}$ for any $\boldsymbol{C} \in H_n^1(\Omega)$ in (4.3e) and using the H^1 stability of L^2 projection R_h, we have

$$\begin{aligned}(d_t\boldsymbol{B}_h^n, \boldsymbol{C}) &= (d_t\boldsymbol{B}_h^n, R_h\boldsymbol{C}) = -\frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{curl}\boldsymbol{B}_h^n, \operatorname{curl} R_h\boldsymbol{C}\right)\\ &-\frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{div}\boldsymbol{B}_h^n, \operatorname{div} R_h\boldsymbol{C}\right) + (\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1}, \operatorname{curl} R_h\boldsymbol{C})\\ &\le c\left(\|\boldsymbol{B}_h^n\|_{\boldsymbol{H}_n^1(\Omega)} + \|\boldsymbol{u}_h^n\|_{L^2}^{\frac{4-d}{4}}\|\nabla\boldsymbol{u}_h^n\|_{L^2}^{\frac{d}{4}}\|\boldsymbol{B}_h^{n-1}\|_{L^2}^{\frac{4-d}{4}}\|\boldsymbol{B}_h^{n-1}\|_{\boldsymbol{H}_n^1(\Omega)}^{\frac{d}{4}}\right)\|\boldsymbol{C}\|_{\boldsymbol{H}_n^1(\Omega)}. \end{aligned} \tag{4.21}$$

The estimate (4.16) follows from (4.9)–(4.10), (4.13) and (4.19)–(4.21). □

Remark 4.3. Based on the stability results of [53,97,98], the discrete phase variable can be bounded in $L^\infty(0,T;L^\infty(\Omega))$ norm and the discrete chemical potential can be bounded in $L^\infty(0,T;L^2(\Omega))$ norm for any time and space step sizes in two and three dimensions. This is an interesting and important future work for the target model of this article.

4.2. Existence and uniqueness of solutions of the scheme

In this subsection, we prove the existence of solutions of the scheme (4.3a)–(4.3f) by Leray–Schauder fixed point theorem and obtain the uniqueness of the numerical solutions.

Lemma 4.3 ([99]). *Let $\mathcal{G}$ be a compact mapping of a Banach space B into itself, and suppose there exists a constant M such that*

$$\|x\|_B < M \tag{4.22}$$

for all $x \in B$ and $\alpha \in [0,1]$ satisfying $x = \alpha\mathcal{G}x$. Then $\mathcal{G}$ has a fixed point.

Theorem 4.4. *Suppose Assumption (A) is valid and initial data $\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0$ satisfy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0) < \infty$. For any given $\tau > 0$ and $h > 0$, there exists a solution $\{(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n)\}$ $(n = 1, \ldots, N)$ to the scheme (4.3a)–(4.3f).*

Proof. Firstly, we define a map $\mathcal{G}$: $Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h \to Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ by

$$\mathcal{G}(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n) = (\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n),$$

where $(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ satisfies

$$\left(\frac{\widehat{\varphi}_h^n - \varphi_h^{n-1}}{\tau}, \psi_h\right) - (\varphi_h^{n-1}\boldsymbol{u}_h^n, \nabla\psi_h) + \gamma(\nabla\widehat{w}_h^n, \nabla\psi_h) = 0, \tag{4.23a}$$

$$(\widehat{w}_h^n, \chi_h) - (\nabla\widehat{\varphi}_h^n, \nabla\chi_h) - \frac{1}{\varepsilon^2}\big((\varphi_h^n)^3 - \varphi_h^{n-1}, \chi_h\big) = 0, \tag{4.23b}$$

$$\begin{aligned}\left(\frac{\widehat{\boldsymbol{u}}_h^n - \boldsymbol{u}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) &+ 2\big(\eta(\varphi_h^{n-1})D(\widehat{\boldsymbol{u}}_h^n), D(\boldsymbol{v}_h)\big) + \big((\boldsymbol{u}_h^{n-1}\cdot\nabla)\boldsymbol{u}_h^n, \boldsymbol{v}_h\big) + \frac{1}{2}\big((\operatorname{div}\boldsymbol{u}_h^{n-1})\boldsymbol{u}_h^n, \boldsymbol{v}_h\big)\\ &+\frac{1}{\mu}(\boldsymbol{B}_h^{n-1}\times\operatorname{curl}\boldsymbol{B}_h^n, \boldsymbol{v}_h) - (\widehat{p}_h^n, \operatorname{div}\boldsymbol{v}_h) + \lambda(\varphi_h^{n-1}\nabla w_h^n, \boldsymbol{v}_h) = \boldsymbol{f}^n(\boldsymbol{v}_h),\end{aligned} \tag{4.23c}$$

$$(\operatorname{div}\widehat{\boldsymbol{u}}_h^n, q_h) = 0, \tag{4.23d}$$

$$\begin{aligned}\left(\frac{\widehat{\boldsymbol{B}}_h^n - \boldsymbol{B}_h^{n-1}}{\tau}, \boldsymbol{C}_h\right) + \left(\frac{1}{\mu\sigma(\varphi_h^{n-1})}\operatorname{curl}\widehat{\boldsymbol{B}}_h^n, \operatorname{curl}\boldsymbol{C}_h\right) + \left(\frac{1}{\mu\sigma(\varphi_h^{n-1})}\operatorname{div}\widehat{\boldsymbol{B}}_h^n, \operatorname{div}\boldsymbol{C}_h\right)&\\ -(\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1}, \operatorname{curl}\boldsymbol{C}_h) = 0,&\end{aligned} \tag{4.23e}$$

for given $(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ and any $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{C}_h) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$. Next, we will prove the map $\mathcal{G}$ satisfies the conditions of Lemma 4.3 and then has a fixed point which is a solution of the scheme (4.3a)–(4.3f).

Given $\varphi_h^{n-1}, \varphi_h^n \in Y_h$ and $\boldsymbol{u}_h^n \in \boldsymbol{X}_h$, the Cahn–Hilliard equation (4.23a)–(4.23b) can be viewed as the following problem: find $(\widehat{w}_h^n, \widehat{\varphi}_h^n) \in Y_h \times Y_h$ satisfying

$$\begin{cases} a(\widehat{w}_h^n, \chi_h) + b(\chi_h, \widehat{\varphi}_h^n) = \langle f, \chi_h\rangle,\\ b(\widehat{w}_h^n, \psi_h) - c(\widehat{\varphi}_h^n, \psi_h) = \langle g, \psi_h\rangle, \end{cases} \tag{4.24}$$

for any $(\chi_h, \psi_h) \in Y_h \times Y_h$, where

$$a(\widehat{w}_h^n, \chi_h) = (\widehat{w}_h^n, \chi_h), \quad b(\chi_h, \psi_h) = -(\nabla\chi_h, \nabla\psi_h), \quad c(\widehat{\varphi}_h^n, \psi_h) = \frac{1}{\gamma\tau}(\widehat{\varphi}_h^n, \psi_h),$$

$$\langle f, \chi_h\rangle = \frac{1}{\varepsilon^2}\big((\varphi_h^n)^3 - \varphi_h^{n-1}, \chi_h\big), \quad \langle g, \psi_h\rangle = -\frac{1}{\gamma\tau}(\varphi_h^{n-1}, \psi_h) - \frac{1}{\gamma}(\varphi_h^n\boldsymbol{u}_h^n, \nabla\psi_h).$$

Denoting $Y_{0h} = \{\chi_h \in Y_h; b(\chi_h, \psi_h) = 0, \forall \psi_h \in Y_h\}$, we deduce $\|\nabla \chi_h\|_{L^2} = 0$ for any $\chi_h \in Y_{0h}$ and $a(\cdot,\cdot)$ is coercive on Y_{0h}, i.e.,

$$a(\chi_h, \chi_h) = \|\chi_h\|_{L^2}^2 = \|\chi_h\|_{H^1}^2,$$

for any $\chi_h \in Y_{0h}$. Moreover, one can easily show $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, and $c(\cdot,\cdot)$ is continuous, positive semi-definite and symmetric. Under the inf–sup condition (4.2) and given $\varphi_h^{n-1}, \varphi_h^n \in Y_h$ and $\boldsymbol{u}_h^n \in \boldsymbol{X}_h$, the problem (4.23a)–(4.23b) is well-posed (see Section II.1.2 of [75]). Meanwhile, one can easily prove Stokes problem (4.23c)–(4.23d) and Maxwell problem (4.23e) are well-posed. Furthermore, since the spaces $Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ are finite dimensional spaces, it follows that $\mathcal{G}$ is a compact map.

Next, we prove the boundedness of $(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n)$ in $Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$,

$$\|\widehat{\varphi}_h^n\|_{H^1} + \|\widehat{w}_h^n\|_{H^1} + \|\widehat{\boldsymbol{u}}_h^n\|_{\boldsymbol{H}_0^1(\Omega)} + \|\widehat{p}_h^n\|_{L^2} + \|\widehat{\boldsymbol{B}}_h^n\|_{\boldsymbol{H}_n^1(\Omega)} \le M, \tag{4.25}$$

where M is a positive constant independent of α and $(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n)$, if there holds

$$\mathcal{G}(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n) = \frac{1}{\alpha}(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n),$$

that is

$$\left(\frac{\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}}{\tau}, \psi_h\right) - \alpha(\varphi_h^{n-1}\widehat{\boldsymbol{u}}_h^n, \nabla\psi_h) + \gamma(\nabla\widehat{w}_h^n, \nabla\psi_h) = 0, \tag{4.26a}$$

$$(\widehat{w}_h^n, \chi_h) - (\nabla\widehat{\varphi}_h^n, \nabla\chi_h) - \frac{\alpha}{\varepsilon^2}\big((\widehat{\varphi}_h^n)^3 - \varphi_h^{n-1}, \chi_h\big) = 0, \tag{4.26b}$$

$$\begin{aligned}\left(\frac{\widehat{\boldsymbol{u}}_h^n - \alpha\boldsymbol{u}_h^{n-1}}{\tau}, \boldsymbol{v}_h\right) &+ 2\big(\eta(\varphi_h^{n-1})D(\widehat{\boldsymbol{u}}_h^n), D(\boldsymbol{v}_h)\big) + \alpha\big((\boldsymbol{u}_h^{n-1}\cdot\nabla)\widehat{\boldsymbol{u}}_h^n, \boldsymbol{v}_h\big) + \frac{\alpha}{2}\big((\operatorname{div}\boldsymbol{u}_h^{n-1})\widehat{\boldsymbol{u}}_h^n, \boldsymbol{v}_h\big)\\ &+\frac{\alpha}{\mu}(\boldsymbol{B}_h^{n-1}\times\operatorname{curl}\widehat{\boldsymbol{B}}_h^n, \boldsymbol{v}_h) - (\widehat{p}_h^n, \operatorname{div}\boldsymbol{v}_h) + \lambda\alpha(\varphi_h^{n-1}\nabla\widehat{w}_h^n, \boldsymbol{v}_h) = \alpha\boldsymbol{f}^n(\boldsymbol{v}_h),\end{aligned} \tag{4.26c}$$

$$(\operatorname{div}\widehat{\boldsymbol{u}}_h^n, q_h) = 0, \tag{4.26d}$$

$$\begin{aligned}\left(\frac{\widehat{\boldsymbol{B}}_h^n - \alpha\boldsymbol{B}_h^{n-1}}{\tau}, \boldsymbol{C}_h\right) &+ \left(\frac{1}{\mu\sigma(\varphi_h^{n-1})}\operatorname{curl}\widehat{\boldsymbol{B}}_h^n, \operatorname{curl}\boldsymbol{C}_h\right) + \left(\frac{1}{\mu\sigma(\varphi_h^{n-1})}\operatorname{div}\widehat{\boldsymbol{B}}_h^n, \operatorname{div}\boldsymbol{C}_h\right)\\ &-\alpha(\widehat{\boldsymbol{u}}_h^n\times\boldsymbol{B}_h^{n-1}, \operatorname{curl}\boldsymbol{C}_h) = 0,\end{aligned} \tag{4.26e}$$

for any $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{C}_h) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$.

Setting $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{C}_h) = 2(\lambda\tau\widehat{w}_h^n, -\lambda\widehat{\varphi}_h^n + \alpha\lambda\varphi_h^{n-1}, \tau\widehat{\boldsymbol{u}}_h^n, \tau\widehat{p}_h^n, \frac{\tau}{\mu}\widehat{\boldsymbol{B}}_h^n)$ in (4.26a)–(4.26e), taking sum of the obtained equalities and using (3.2) and (4.6)–(4.7), we have

$$\begin{aligned}&2\gamma\lambda\tau\|\nabla\widehat{w}_h^n\|_{L^2}^2 + \lambda\left(\|\nabla\widehat{\varphi}_h^n\|_{L^2}^2 - \alpha^2\|\nabla\varphi_h^{n-1}\|_{L^2}^2 + \|\nabla(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\|_{L^2}^2\right)\\ &\quad+\frac{\alpha\lambda}{\varepsilon^2}\left(\frac{1}{2}\|(\widehat{\varphi}_h^n)^2 - 1\|_{L^2}^2 - \frac{1}{2}\|(\alpha\varphi_h^{n-1})^2 - 1\|_{L^2}^2 + \frac{1}{2}\|(\widehat{\varphi}_h^n)^2 - (\alpha\varphi_h^{n-1})^2\|_{L^2}^2\right)\\ &\quad+\frac{\alpha\lambda}{\varepsilon^2}\left(\|\widehat{\varphi}_h^n(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\|_{L^2}^2 + \|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2\right) + \|\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2 - \alpha^2\|\boldsymbol{u}_h^{n-1}\|_{L^2}^2 + \|\widehat{\boldsymbol{u}}_h^n - \alpha\boldsymbol{u}_h^{n-1}\|_{L^2}^2\\ &\quad+4c_0^2\eta^-\tau\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2 + \frac{1}{\mu}\left(\|\widehat{\boldsymbol{B}}_h^n\|_{L^2}^2 - \alpha^2\|\boldsymbol{B}_h^{n-1}\|_{L^2}^2 + \|\widehat{\boldsymbol{B}}_h^n - \alpha\boldsymbol{B}_h^{n-1}\|_{L^2}^2\right) + \frac{2\tau}{\mu^2\sigma^+}\|\widehat{\boldsymbol{B}}_h^n\|_{\boldsymbol{H}_n^1(\Omega)}^2\\ &\le 2\tau\alpha\boldsymbol{f}^n(\widehat{\boldsymbol{u}}_h^n) + \frac{2\alpha(1-\alpha)\lambda}{\varepsilon^2}(\varphi_h^{n-1}, \widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\\ &\le c_0^2\eta^-\tau\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2 + \frac{\alpha\lambda}{2\varepsilon^2}\|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2 + \frac{\alpha^2}{c_0^2\eta^-}\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}^2\mathrm{d}t + \frac{2\alpha(1-\alpha)^2\lambda}{\varepsilon^2}\|\varphi_h^{n-1}\|_{L^2}^2.\end{aligned} \tag{4.27}$$

Then, according to (4.27) and $\alpha \in [0, 1]$, we get

$$\begin{aligned}&2\gamma\lambda\tau\|\nabla\widehat{w}_h^n\|_{L^2}^2 + \lambda\|\nabla\widehat{\varphi}_h^n\|_{L^2}^2 + \lambda\|\nabla(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\|_{L^2}^2 + \frac{\alpha\lambda}{2\varepsilon^2}\|(\widehat{\varphi}_h^n)^2 - 1\|_{L^2}^2\\ &\quad+\frac{\alpha\lambda}{\varepsilon^2}\left\{\frac{1}{2}\|(\widehat{\varphi}_h^n)^2 - (\alpha\varphi_h^{n-1})^2\|_{L^2}^2 + \|\widehat{\varphi}_h^n(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\|_{L^2}^2 + \frac{1}{2}\|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2\right\}\end{aligned}$$

$$
\begin{aligned}
&+\|\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2 + c_0^2\eta^-\tau\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2 + \frac{1}{\mu}\|\widehat{\boldsymbol{B}}_h^n\|_{L^2}^2 + \frac{\tau}{\mu^2\sigma^+}\|\widehat{\boldsymbol{B}}_h^n\|_{\boldsymbol{H}_n^1(\Omega)}^2\\
&\le \frac{1}{c_0^2\eta^-}\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}^2\mathrm{d}t + \frac{2\lambda^2}{\varepsilon^2}\|\varphi_h^{n-1}\|_{L^2}^2 + \lambda\|\nabla\varphi_h^{n-1}\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\left(\|\varphi_h^{n-1}\|_{L^4}^4 + 2\|\varphi_h^{n-1}\|_{L^2}^2 + |\Omega|\right)\\
&+\|\boldsymbol{u}_h^{n-1}\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}_h^{n-1}\|_{L^2}^2 =: M_1,
\end{aligned}
\tag{4.28}
$$

where M_1 is a positive constant independent of $(\widehat{\varphi}_h^n, \widehat{w}_h^n, \widehat{\boldsymbol{u}}_h^n, \widehat{p}_h^n, \widehat{\boldsymbol{B}}_h^n)$ and α.

Next, taking $(\psi_h, \chi_h) = (2\widehat{\varphi}_h^n\tau, 2\gamma\widehat{w}_h^n\tau)$ in (4.26a)–(4.26b) and adding the obtained equalities, we have

$$
\begin{aligned}
&\|\widehat{\varphi}_h^n\|_{L^2}^2 - \alpha^2\|\varphi_h^{n-1}\|_{L^2}^2 + \|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2 + 2\gamma\tau\|\widehat{w}_h^n\|_{L^2}^2\\
&= 2\alpha\tau(\varphi_h^{n-1}\widehat{\boldsymbol{u}}_h^n, \nabla\widehat{\varphi}_h^n) + \frac{2\gamma\alpha\tau}{\varepsilon^2}\left((\widehat{\varphi}_h^n)^3 - \varphi_h^{n-1}, \widehat{w}_h^n\right).
\end{aligned}
\tag{4.29}
$$

From Hölder inequality, (3.3), (3.9) and the following equality

$$
(\widehat{\varphi}_h^n)^4 = \left((\widehat{\varphi}_h^n)^2 - 1\right)^2 + 2(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})^2 + 4\alpha(\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1})\varphi_h^{n-1} - 1 + 2\alpha^2(\varphi_h^{n-1})^2,
$$

we obtain

$$
\begin{aligned}
&2\alpha\tau(\varphi_h^{n-1}\widehat{\boldsymbol{u}}_h^n, \nabla\widehat{\varphi}_h^n) \le 2\alpha\tau\|\varphi_h^{n-1}\|_{L^3}\|\widehat{\boldsymbol{u}}_h^n\|_{L^6}\|\nabla\widehat{\varphi}_h^n\|_{L^2},\\
&\frac{2\gamma\alpha\tau}{\varepsilon^2}\left((\widehat{\varphi}_h^n)^3 - \varphi_h^{n-1}, \widehat{w}_h^n\right) \le \frac{2\gamma\alpha\tau}{\varepsilon^2}\left(\|\widehat{\varphi}_h^n\|_{L^4}^3\|\widehat{w}_h^n\|_{L^4} + \|\varphi_h^{n-1}\|_{L^2}\|\widehat{w}_h^n\|_{L^2}\right)\\
&\quad\le \gamma\tau\|\widehat{w}_h^n\|_{L^2}^2 + \gamma\tau\|\nabla\widehat{w}_h^n\|_{L^2}^2 + \frac{c_0\gamma\alpha^2\tau}{\varepsilon^4}\left(\|(\widehat{\varphi}_h^n)^2 - 1\|_{L^2}^2 + \|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2\right)^{\frac{3}{2}}\\
&\qquad+\frac{c_0\gamma\alpha^2\tau}{\varepsilon^4}\left(|\Omega|^{\frac{3}{2}} + \alpha^3\|\varphi_h^{n-1}\|_{L^2}^3 + \|\varphi_h^{n-1}\|_{L^2}^2\right).
\end{aligned}
$$

Combining (4.29) with the above inequalities, we get

$$
\begin{aligned}
\|\widehat{\varphi}_h^n\|_{L^2}^2 + \gamma\tau\|\widehat{w}_h^n\|_{L^2}^2 \le{}& \|\varphi_h^{n-1}\|_{L^2}^2 + 2c_0\tau\|\varphi_h^{n-1}\|_{H^1}\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2}\|\nabla\widehat{\varphi}_h^n\|_{L^2}\\
&+\gamma\tau\|\nabla\widehat{w}_h^n\|_{L^2}^2 + \frac{c_0\gamma\alpha^2\tau}{\varepsilon^4}\left(\|(\widehat{\varphi}_h^n)^2 - 1\|_{L^2}^2 + \|\widehat{\varphi}_h^n - \alpha\varphi_h^{n-1}\|_{L^2}^2\right)^{\frac{3}{2}}\\
&+\frac{c_0\gamma\tau}{\varepsilon^4}\left(|\Omega|^{\frac{3}{2}} + \|\varphi_h^{n-1}\|_{L^2}^3 + \|\varphi_h^{n-1}\|_{L^2}^2\right).
\end{aligned}
\tag{4.30}
$$

Based on the inf–sup condition (4.1) and (4.26c), we have

$$
\begin{aligned}
\beta_0\|\widehat{p}_h^n\|_{L^2} \le{}& \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h\setminus\{\boldsymbol{0}\}}\frac{(\widehat{p}_h^n, \operatorname{div}\boldsymbol{v}_h)}{\|\nabla\boldsymbol{v}_h\|_{L^2}} \le \frac{c_0}{\tau}(\|\widehat{\boldsymbol{u}}_h^n\|_{L^2} + \|\boldsymbol{u}_h^{n-1}\|_{L^2}) + \eta^+\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2} + c_0\|\nabla\widehat{\boldsymbol{u}}_h^n\|_{L^2}^2\\
&+\frac{c_0}{\mu}\|\widehat{\boldsymbol{B}}_h^n\|_{\boldsymbol{H}_n^1(\Omega)}^2 + \lambda c_0\|\varphi_h^{n-1}\|_{H^1}\|\nabla\widehat{w}_h^n\|_{L^2} + c\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}^n\|_{(\boldsymbol{H}_0^1(\Omega))'}^2\mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.31}
$$

Combining (4.30)–(4.31) with (4.28), we deduce (4.25).

According to the above analysis and Lemma 4.3, we obtain $\mathcal{G}$ has a fixed point which is a solution to the scheme (4.3a)–(4.3f). $\square$

Theorem 4.5. *Under the conditions of Theorem 4.4, there exists a unique solution* $\{(\varphi_h^n, w_h^n, \boldsymbol{u}_h^n, p_h^n, \boldsymbol{B}_h^n)\}$ $(n = 1, \ldots, N)$ *to the scheme (4.3a)–(4.3f).*

Proof. Suppose that $\{(\varphi_{h1}^n, w_{h1}^n, \boldsymbol{u}_{h1}^n, p_{h1}^n, \boldsymbol{B}_{h1}^n)\}$ and $\{(\varphi_{h2}^n, w_{h2}^n, \boldsymbol{u}_{h2}^n, p_{h2}^n, \boldsymbol{B}_{h2}^n)\}$ are two solutions of the scheme (4.3a)–(4.3f). Denote

$$
\overline{\varphi}_h^n = \varphi_{h1}^n - \varphi_{h2}^n, \quad \overline{w}_h^n = w_{h1}^n - w_{h2}^n, \quad \overline{\boldsymbol{u}}_h^n = \boldsymbol{u}_{h1}^n - \boldsymbol{u}_{h2}^n, \quad \overline{p}_h^n = p_{h1}^n - p_{h2}^n, \quad \overline{\boldsymbol{B}}_h^n = \boldsymbol{B}_{h1}^n - \boldsymbol{B}_{h2}^n.
$$

From (4.3a)–(4.3f), we have

$$
\left(\frac{\overline{\varphi}_h^n}{\tau}, \psi_h\right) - (\varphi_h^{n-1}\overline{\boldsymbol{u}}_h^n, \nabla\psi_h) + \gamma(\nabla\overline{w}_h^n, \nabla\psi_h) = 0,
\tag{4.32a}
$$

$$(\nabla\overline{\varphi}_h^n, \nabla\chi_h) + \frac{1}{\varepsilon^2}\left(((\varphi_{h1}^n)^2 + \varphi_{h1}^n\varphi_{h2}^n + (\varphi_{h2}^n)^2)\overline{\varphi}_h^n, \chi_h\right) = (\overline{w}_h^n, \chi_h), \tag{4.32b}$$

$$\begin{aligned}\left(\frac{\overline{\boldsymbol{u}}_h^n}{\tau}, \boldsymbol{v}_h\right) + 2\left(\eta(\varphi_h^{n-1})D(\overline{\boldsymbol{u}}_h^n), D(\boldsymbol{v}_h)\right) + \left((\boldsymbol{u}_h^{n-1}\cdot\nabla)\overline{\boldsymbol{u}}_h^n, \boldsymbol{v}_h\right) + \frac{1}{2}\left((\operatorname{div}\boldsymbol{u}_h^{n-1})\overline{\boldsymbol{u}}_h^n, \boldsymbol{v}_h\right)&\\ + \frac{1}{\mu}(\boldsymbol{B}_h^{n-1}\times\operatorname{curl}\overline{\boldsymbol{B}}_h^n, \boldsymbol{v}_h) - (\overline{p}_h^n, \operatorname{div}\boldsymbol{v}_h) + \lambda(\varphi_h^{n-1}\nabla\overline{w}_h^n, \boldsymbol{v}_h) = 0,&\end{aligned} \tag{4.32c}$$

$$(\operatorname{div}\overline{\boldsymbol{u}}_h^n, q_h) = 0, \tag{4.32d}$$

$$\begin{aligned}\left(\frac{\overline{\boldsymbol{B}}_h^n}{\tau}, \boldsymbol{C}_h\right) + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{curl}\overline{\boldsymbol{B}}_h^n, \operatorname{curl}\boldsymbol{C}_h\right) + \frac{1}{\mu}\left(\frac{1}{\sigma(\varphi_h^{n-1})}\operatorname{div}\overline{\boldsymbol{B}}_h^n, \operatorname{div}\boldsymbol{C}_h\right)&\\ - (\overline{\boldsymbol{u}}_h^n\times\boldsymbol{B}_h^{n-1}, \operatorname{curl}\boldsymbol{C}_h) = 0.&\end{aligned} \tag{4.32e}$$

Taking $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{B}_h) = (\lambda\tau\overline{w}_h^n, \lambda\overline{\varphi}_h^n, \tau\overline{\boldsymbol{u}}_h^n, \tau\overline{p}_h^n, \frac{\tau}{\mu}\overline{\boldsymbol{B}}_h^n)$ in (4.32a)–(4.32e), we get

$$\begin{aligned}&\lambda\gamma\tau\|\nabla\overline{w}_h^n\|_{L^2}^2 + \lambda\|\nabla\overline{\varphi}_h^n\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\left((\varphi_{h1}^n)^2 + \varphi_{h1}^n\varphi_{h2}^n + (\varphi_{h2}^n)^2, (\overline{\varphi}_h^n)^2\right) + \|\overline{\boldsymbol{u}}_h^n\|_{L^2}^2 + \frac{1}{\mu}\|\overline{\boldsymbol{B}}_h^n\|_{L^2}^2\\ &+2\tau\|\sqrt{\eta(\varphi_h^{n-1})}D(\overline{\boldsymbol{u}}_h^n)\|_{L^2}^2 + \frac{\tau}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_h^{n-1})}}\operatorname{curl}\overline{\boldsymbol{B}}_h^n\|_{L^2}^2 + \frac{\tau}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_h^{n-1})}}\operatorname{div}\overline{\boldsymbol{B}}_h^n\|_{L^2}^2 = 0.\end{aligned} \tag{4.33}$$

According to

$$\left((\varphi_{h1}^n)^2 + \varphi_{h1}^n\varphi_{h2}^n + (\varphi_{h2}^n)^2, (\overline{\varphi}_h^n)^2\right) = \left(\frac{(\varphi_{h1}^n)^2 + (\varphi_{h2}^n)^2}{2} + \frac{(\varphi_{h1}^n + \varphi_{h2}^n)^2}{2}, (\overline{\varphi}_h^n)^2\right) \geq 0,$$

it follows from (4.33) that

$$\|\nabla\overline{w}_h^n\|_{L^2}^2 + \|\overline{\varphi}_h^n\|_{H^1}^2 + \|\overline{\boldsymbol{u}}_h^n\|_{L^2}^2 + \|\overline{\boldsymbol{B}}_h^n\|_{L^2}^2 = 0. \tag{4.34}$$

Combining (4.34) with (4.32b), we have

$$\|\overline{w}_h^n\|_{H^1} = 0.$$

The uniqueness of pressure p_h^n can be obtained from (4.1). Therefore, the theorem is proved. □

5. Existence of weak solution and convergence of the numerical scheme

The purpose of this section is to prove the existence of weak solutions to the two-phase MHD problem (2.3a)–(2.3f) by a compactness argument and obtain the convergence of the numerical scheme (4.3a)–(4.3f).

Let $\{\varphi_{h\tau}(x,t), \boldsymbol{u}_{h\tau}(x,t), \boldsymbol{B}_{h\tau}(x,t)\}$ be the piecewise linear interpolation of the fully discrete finite element solution $\{\varphi_h^m, \boldsymbol{u}_h^m, \boldsymbol{B}_h^m\}$, $m = 1, \dots, N$, i.e., for any $t \in [t_{m-1}, t_m]$

$$\begin{aligned}&\varphi_{h\tau}(\cdot,t) := \frac{t - t_{m-1}}{\tau}\varphi_h^m(\cdot) + \frac{t_m - t}{\tau}\varphi_h^{m-1}(\cdot), \qquad \boldsymbol{u}_{h\tau}(\cdot,t) := \frac{t - t_{m-1}}{\tau}\boldsymbol{u}_h^m(\cdot) + \frac{t_m - t}{\tau}\boldsymbol{u}_h^{m-1}(\cdot),\\ &\boldsymbol{B}_{h\tau}(\cdot,t) := \frac{t - t_{m-1}}{\tau}\boldsymbol{B}_h^m(\cdot) + \frac{t_m - t}{\tau}\boldsymbol{B}_h^{m-1}(\cdot).\end{aligned}$$

Let $\left\{\overline{\varphi}_{h\tau}(x,t), \overline{w}_{h\tau}(x,t), \overline{\boldsymbol{u}}_{h\tau}(x,t), \overline{p}_{h\tau}(x,t), \overline{\boldsymbol{B}}_{h\tau}(x,t)\right\}$ and $\left\{\overline{\overline{\varphi}}_{h\tau}(x,t), \overline{\overline{\boldsymbol{u}}}_{h\tau}(x,t), \overline{\overline{\boldsymbol{B}}}_{h\tau}(x,t)\right\}$ be the piecewise constant extensions of $\left\{\varphi_h^m, w_h^m, \boldsymbol{u}_h^m, p_h^m, \boldsymbol{B}_h^m\right\}$ and $\left\{\varphi_h^{m-1}, \boldsymbol{u}_h^{m-1}, \boldsymbol{B}_h^{m-1}\right\}$, $m = 1, \dots, N$, respectively. That is, for any $t \in (t_{m-1}, t_m]$,

$$\begin{aligned}&\overline{\varphi}_{h\tau}(\cdot,t) := \varphi_h^m(\cdot), \qquad \overline{w}_{h\tau}(\cdot,t) := w_h^m(\cdot), \qquad \overline{p}_{h\tau}(\cdot,t) := p_h^m(\cdot),\\ &\overline{\boldsymbol{u}}_{h\tau}(\cdot,t) := \boldsymbol{u}_h^m(\cdot), \qquad \overline{\boldsymbol{B}}_{h\tau}(\cdot,t) := \boldsymbol{B}_h^m(\cdot), \qquad \overline{\overline{\varphi}}_{h\tau}(\cdot,t) := \varphi_h^{m-1}(\cdot),\\ &\overline{\overline{\boldsymbol{u}}}_{h\tau}(\cdot,t) := \boldsymbol{u}_h^{m-1}(\cdot), \qquad \overline{\overline{\boldsymbol{B}}}_{h\tau}(\cdot,t) := \boldsymbol{B}_h^{m-1}(\cdot).\end{aligned}$$

Moreover, denote

$$\overline{\boldsymbol{f}}_\tau(t) := \boldsymbol{f}^m(\boldsymbol{v}_h) = \frac{1}{\tau}\int_{t_{m-1}}^{t_m} \langle \boldsymbol{f}(t), \boldsymbol{v}_h\rangle \mathrm{d}t,$$

for any $t \in (t_{m-1}, t_m]$.

It is well known that Hilbert spaces are reflexive Banach spaces, and if X is reflexive and $1 < p < \infty$, the space $L^p(0,T;X)$ is reflexive and $[L^p(0,T;X)]' = L^{p'}(0,T;X')$, where p' is the conjugate of p (see Propositions 3.55 and 3.59 of [100]). Therefore, according to Theorems 1.18 and 1.26 of [100], Corollary 4 of [73] and Theorem 4.2, we have the following convergence. For the convenience, the convergent subsequences are denoted by the same symbols.

Lemma 5.1 (*Weak Convergence*)**.** *For the sequences* $\left\{\overline{\overline{\varphi}}_{h\tau}, \overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\overline{\boldsymbol{B}}}_{h\tau}\right\}$, $\left\{\overline{\varphi}_{h\tau}, \overline{w}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \overline{p}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}\right\}$ *and* $\{\varphi_{h\tau}, w_{h\tau}, \boldsymbol{u}_{h\tau}, p_{h\tau}, \boldsymbol{B}_{h\tau}\}$*, there exist convergent subsequences satisfying*

$$\overline{\overline{\varphi}}_{h\tau}, \overline{\varphi}_{h\tau}, \varphi_{h\tau} \rightharpoonup *\varphi \quad \text{in } L^\infty(0,T;H^1(\Omega)), \tag{5.1}$$

$$(\varphi_{h\tau})_t \rightharpoonup \varphi_t \quad \text{in } L^2(0,T;(H^1(\Omega))'), \tag{5.2}$$

$$\overline{w}_{h\tau} \rightharpoonup w \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{5.3}$$

$$\overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \boldsymbol{u}_{h\tau} \rightharpoonup *\boldsymbol{u} \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)), \tag{5.4}$$

$$\overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \boldsymbol{u}_{h\tau} \rightharpoonup \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{H}_0^1(\Omega)), \tag{5.5}$$

$$(\boldsymbol{u}_{h\tau})_t \rightharpoonup \boldsymbol{u}_t \quad \text{in } L^{\frac{12}{6+d}}(0,T;(\boldsymbol{H}_0^1(\Omega))'), \tag{5.6}$$

$$\overline{p}_{h\tau} \rightharpoonup p \quad \text{in } L^{\frac{12}{6+d}}(0,T;L_0^2(\Omega)), \tag{5.7}$$

$$\overline{\overline{\boldsymbol{B}}}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}, \boldsymbol{B}_{h\tau} \rightharpoonup *\boldsymbol{B} \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)), \tag{5.8}$$

$$\overline{\overline{\boldsymbol{B}}}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}, \boldsymbol{B}_{h\tau} \rightharpoonup \boldsymbol{B} \quad \text{in } L^2(0,T;\boldsymbol{H}_n^1(\Omega)), \tag{5.9}$$

$$(\boldsymbol{B}_{h\tau})_t \rightharpoonup \boldsymbol{B}_t \quad \text{in } L^{\frac{4}{d}}(0,T;(\boldsymbol{H}_n^1(\Omega))'), \tag{5.10}$$

as $h,\tau \to 0$*. Here,* $\rightharpoonup *$ *means weak* $*$ *convergence.*

Proof. We only give the proof of (5.1). Based on the fact that $\{\overline{\overline{\varphi}}_{h\tau}\}$, $\{\overline{\varphi}_{h\tau}\}$ and $\{\varphi_{h\tau}\}$ are bounded sequences in $L^\infty(0,T;H^1(\Omega))$, the sequences $\{\varphi_{h\tau}\}$, $\{\overline{\varphi}_{h\tau}\}$ and $\{\overline{\overline{\varphi}}_{h\tau}\}$ weakly $*$ converge to φ, φ_1 and φ_2 in $L^\infty(0,T;H^1(\Omega))$, respectively. Therefore, there holds

$$\lim_{h,\tau\to 0}\int_0^T \langle \varphi_{h\tau} - \overline{\varphi}_{h\tau}, \psi\rangle \mathrm{d}t = \int_0^T \langle \varphi - \varphi_1, \psi\rangle \mathrm{d}t, \quad \forall \psi \in L^1(0,T;(H^1(\Omega))'). \tag{5.11}$$

According to $L^2(\Omega) \subset (H^1(\Omega))'$ with continuous injection, $L^\infty(0,T;H^1(\Omega)) \subset L^1(0,T;(H^1(\Omega))')$ and (4.11), we have

$$\begin{aligned}
\int_0^T \langle \varphi_{h\tau} - \overline{\varphi}_{h\tau}, \varphi - \varphi_1\rangle \mathrm{d}t &\le \int_0^T \|\varphi_{h\tau} - \overline{\varphi}_{h\tau}\|_{H^1(\Omega)} \|\varphi - \varphi_1\|_{(H^1(\Omega))'} \mathrm{d}t \\
&\le c\int_0^T \|\varphi_{h\tau} - \overline{\varphi}_{h\tau}\|_{H^1(\Omega)} \|\varphi - \varphi_1\|_{L^2} \mathrm{d}t \\
&\le c\|\varphi - \varphi_1\|_{L^\infty(L^2)} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{t_n - t}{\tau} \|\varphi_h^n - \varphi_h^{n-1}\|_{H^1(\Omega)} \mathrm{d}t \\
&\le c\tau^{\frac{1}{2}} T^{\frac{1}{2}} \|\varphi - \varphi_1\|_{L^\infty(L^2)} \left(\sum_{n=1}^N \|\varphi_h^n - \varphi_h^{n-1}\|_{H^1(\Omega)}^2\right)^{\frac{1}{2}} \xrightarrow{\tau\to 0} 0.
\end{aligned} \tag{5.12}$$

Taking $\psi = \varphi - \varphi_1$ in (5.11) and using (5.12), we get $\varphi = \varphi_1$. Similar to the above analysis, we also have $\varphi = \varphi_2$. $\square$

Lemma 5.2 (*Strong Convergence*)**.** *For the sequences* $\left\{\overline{\overline{\varphi}}_{h\tau}, \overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\overline{\boldsymbol{B}}}_{h\tau}\right\}$, $\left\{\overline{\varphi}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}\right\}$ *and* $\{\varphi_{h\tau}, \boldsymbol{u}_{h\tau}, \boldsymbol{B}_{h\tau}\}$, *there exist convergent subsequences satisfying*

$$\varphi_{h\tau} \to \varphi \qquad in\ C(0,T;L^p(\Omega)), \tag{5.13}$$

$$\overline{\overline{\varphi}}_{h\tau}, \overline{\varphi}_{h\tau} \to \varphi \qquad in\ L^\infty(0,T;L^p(\Omega)), \tag{5.14}$$

$$\overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \boldsymbol{u}_{h\tau} \to \boldsymbol{u} \qquad in\ L^2(0,T;L^p(\Omega)), \tag{5.15}$$

$$\overline{\overline{\boldsymbol{B}}}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}, \boldsymbol{B}_{h\tau} \to \boldsymbol{B} \qquad in\ L^2(0,T;L^p(\Omega)), \tag{5.16}$$

as $h, \tau \to 0$, *where* $p \in [1, \frac{2d}{d-2})$.

Proof. According to Sobolev embedding theorem, Corollary 4 of [73] and Theorem 4.2, we obtain $\{\varphi_{h\tau}\}$ strongly converges to φ in $C(0,T;L^p(\Omega))$, and $\{\boldsymbol{u}_{h\tau}\}$ and $\{\boldsymbol{B}_{h\tau}\}$ strongly converge to $\boldsymbol{u}$ and $\boldsymbol{B}$ in $L^2(0,T;\ L^p(\Omega))$, $1 \le p < \frac{2d}{d-2}$. Next, we prove the convergence of $\left\{\overline{\varphi}_{h\tau}, \overline{\boldsymbol{u}}_{h\tau}, \overline{\boldsymbol{B}}_{h\tau}\right\}$ and $\left\{\overline{\overline{\varphi}}_{h\tau}, \overline{\overline{\boldsymbol{u}}}_{h\tau}, \overline{\overline{\boldsymbol{B}}}_{h\tau}\right\}$. Since $\{\varphi_{h\tau}\}$ is relatively compact in $C(0,T;L^p(\Omega))$, $\{\varphi_{h\tau}\}$ is uniformly equicontinuous, i.e., for all $\epsilon > 0$, there is $\delta > 0$ such that for all $h, \tau > 0$, $\|\varphi_{h\tau}(t_1') - \varphi_{h\tau}(t_2')\|_{L^p} \le \epsilon$, where t_1' and t_2' are in $[0,T]$ with $|t_2' - t_1'| \le \delta$. Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\overline{\overline{\varphi}}_{h\tau} - \varphi_{h\tau}\|_{L^\infty(L^p)} = \|\overline{\varphi}_{h\tau} - \varphi_{h\tau}\|_{L^\infty(L^p)} = \operatorname*{ess\,sup}_{1\le m\le N} \|\varphi_h^m - \varphi_h^{m-1}\|_{L^p} \le \epsilon,$$

for $\tau \le \delta$. The estimate (5.14) holds.

For any $p \in (1, \frac{2d}{d-2})$, taking $p_1 \in (p, \frac{2d}{d-2})$ and using an interpolation inequality (see Theorem 2.11 of [67]), Hölder inequality, (4.10) and (4.12), we have

$$\begin{aligned}
\|\overline{\boldsymbol{u}}_{h\tau} - \boldsymbol{u}_{h\tau}\|^2_{L^2(L^p)} &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\frac{t_n - t}{\tau}\right)^2 \mathrm{d}t \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{L^p} \le c\tau \sum_{n=1}^N \left(\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^\theta_{L^1} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^{1-\theta}_{L^{p_1}}\right)^2 \\
&\le c\left(\sum_{n=1}^N \tau \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{L^1}\right)^\theta \left(\sum_{n=1}^N \tau \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{L^{p_1}}\right)^{1-\theta} \\
&\le c\tau^\theta \left(\sum_{n=1}^N \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{L^2}\right)^\theta \left(\sum_{n=1}^N \tau \|\nabla(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})\|^2_{L^2}\right)^{1-\theta} \xrightarrow{\tau\to 0} 0,
\end{aligned} \tag{5.17}$$

where $\theta = \frac{p_1 - p}{p(p_1 - 1)}$. Consequently, $\{\overline{\boldsymbol{u}}_{h\tau}\}$ strongly converges to $\boldsymbol{u}$ in $L^2\,(0,T;L^p(\Omega))$, $p \in [1, \frac{2d}{d-2})$. Similarly, we also obtain the convergence of $\left\{\overline{\overline{\boldsymbol{u}}}_{h\tau}\right\}$, $\left\{\overline{\boldsymbol{B}}_{h\tau}\right\}$ and $\left\{\overline{\overline{\boldsymbol{B}}}_{h\tau}\right\}$. □

In addition, to prove the existence of weak solution, we give the following remark.

Remark 5.1. If $\boldsymbol{u}, \boldsymbol{v} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$ and $\boldsymbol{B}, \boldsymbol{C} \in L^2(0,T;\boldsymbol{H}_n^1(\Omega))$, we can define the bounded linear functionals $\boldsymbol{f}_{\boldsymbol{v},\varphi} \in L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')$ and $\boldsymbol{f}_{\boldsymbol{C},\varphi} \in L^2(0,T;(\boldsymbol{H}_n^1(\Omega))')$ satisfying

$$\langle \boldsymbol{f}_{\boldsymbol{v},\varphi}, \boldsymbol{u}\rangle = \big(\eta(\varphi)D(\boldsymbol{u}), D(\boldsymbol{v})\big), \quad \langle \boldsymbol{f}_{\boldsymbol{C},\varphi}, \boldsymbol{B}\rangle = \left(\frac{1}{\sigma(\varphi)} \operatorname{curl} \boldsymbol{B}, \operatorname{curl} \boldsymbol{C}\right) + \left(\frac{1}{\sigma(\varphi)} \operatorname{div} \boldsymbol{B}, \operatorname{div} \boldsymbol{C}\right),$$

based on the fact that

$$\begin{aligned}
&\big(\eta(\varphi)D(\boldsymbol{u}), D(\boldsymbol{v})\big) \le \eta^+ \|\boldsymbol{u}\|_{\boldsymbol{H}_0^1(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}, \\
&\left(\frac{1}{\sigma(\varphi)} \operatorname{curl} \boldsymbol{B}, \operatorname{curl} \boldsymbol{C}\right) + \left(\frac{1}{\sigma(\varphi)} \operatorname{div} \boldsymbol{B}, \operatorname{div} \boldsymbol{C}\right) \le \frac{1}{\sigma^-} \|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)} \|\boldsymbol{C}\|_{\boldsymbol{H}_n^1(\Omega)}.
\end{aligned}$$

Also, if $\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$, $\boldsymbol{v} \in L^{\frac{12}{6-d}}(0,T;\boldsymbol{H}_0^1(\Omega))$ and $q \in L^2(0,T;L^2(\Omega))$, $p \in L^{\frac{12}{6+d}}(0,T;L^2(\Omega))$, we can define the bounded linear functionals $\boldsymbol{f}_{\boldsymbol{v}} \in L^{\frac{12}{6-d}}(0,T;L^2(\Omega))$ and $\boldsymbol{f}_q \in L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')$ satisfying

$$\langle \boldsymbol{f}_{\boldsymbol{v}}, p\rangle = (\operatorname{div} \boldsymbol{v}, p), \quad \langle \boldsymbol{f}_q, \boldsymbol{u}\rangle = (\operatorname{div} \boldsymbol{u}, q),$$

based on $(\operatorname{div} \boldsymbol{v}, p) \le \sqrt{d}\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)} \|p\|_{L^2}$.

Theorem 5.3. *Suppose Assumption (A) and (4.4) are valid and the initial data $\boldsymbol{u}_0$, $\boldsymbol{B}_0$, φ_0 satisfy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0) < \infty$. There exists a subsequence of $\{(\varphi_{h\tau}, \overline{w}_{h\tau}, \boldsymbol{u}_{h\tau}, \overline{p}_{h\tau}, \boldsymbol{B}_{h\tau})\}$ which has an accumulation point $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$. And $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ is a weak solution to the problem (2.3a)–(2.3f).*

Proof. For any $(\psi, \chi, \boldsymbol{v}, q, \boldsymbol{C}) \in \mathcal{C}^\infty(\overline{\Omega}) \times \mathcal{C}^\infty(\overline{\Omega}) \times \boldsymbol{\mathcal{C}}_c^\infty(\Omega) \times \mathcal{C}_c^\infty(\Omega) \cap L_0^2(\Omega) \times \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}) \cap \boldsymbol{H}_n^1(\Omega)$, where $\boldsymbol{\mathcal{C}}_c^\infty(\Omega)$ represents the space of real infinitely differentiable functions with compact support in Ω, we can choose $(\psi_h, \chi_h, \boldsymbol{v}_h, q_h, \boldsymbol{C}_h) = (Q_h\psi, Q_h\chi, P_h\boldsymbol{v}, I_h q, R_h\boldsymbol{C}) \in Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ such that

$$
\begin{aligned}
&\psi_h \xrightarrow{h\to 0} \psi \quad \text{in } H^1(\Omega), \qquad \chi_h \xrightarrow{h\to 0} \chi \quad \text{in } H^1(\Omega),\\
&\boldsymbol{v}_h \xrightarrow{h\to 0} \boldsymbol{v} \quad \text{in } \boldsymbol{H}_0^1(\Omega), \qquad q_h \xrightarrow{h\to 0} q \quad \text{in } L^2(\Omega), \qquad \boldsymbol{C}_h \xrightarrow{h\to 0} \boldsymbol{C} \quad \text{in } \boldsymbol{H}_n^1(\Omega),
\end{aligned}
$$

where I_h is the L^2 orthogonal projection operator from $L^2(\Omega)$ to M_h.

Then, taking these test functions in (4.3a)–(4.3e), multiplying by $\xi(t) \in \mathcal{C}^\infty([0, T])$, and integrating the obtained equations with respect to t from 0 to T, we have

$$
\int_0^T \left\{\big((\varphi_{h\tau})_t, \psi_h\big) - (\overline{\varphi}_{h\tau}\overline{\boldsymbol{u}}_{h\tau}, \nabla\psi_h) + \gamma(\nabla\overline{w}_{h\tau}, \nabla\psi_h)\right\}\xi(t)\mathrm{d}t = 0, \tag{5.18a}
$$

$$
\int_0^T \left\{(\nabla\overline{\varphi}_{h\tau}, \nabla\chi_h) + \frac{1}{\varepsilon^2}(\overline{f}_{h\tau}, \chi_h)\right\}\xi(t)\mathrm{d}t = \int_0^T (\overline{w}_{h\tau}, \chi_h)\xi(t)\mathrm{d}t, \tag{5.18b}
$$

$$
\begin{aligned}
\int_0^T \Big\{ &\big((\boldsymbol{u}_{h\tau})_t, \boldsymbol{v}_h\big) + 2\big(\eta(\overline{\overline{\varphi}}_{h\tau})D(\overline{\boldsymbol{u}}_{h\tau}), D(\boldsymbol{v}_h)\big) + \big((\overline{\overline{\boldsymbol{u}}}_{h\tau}\cdot\nabla)\overline{\boldsymbol{u}}_{h\tau}, \boldsymbol{v}_h\big) + \frac{1}{2}\big((\operatorname{div}\overline{\overline{\boldsymbol{u}}}_{h\tau})\overline{\boldsymbol{u}}_{h\tau}, \boldsymbol{v}_h\big)\\
&+\frac{1}{\mu}(\overline{\overline{\boldsymbol{B}}}_{h\tau} \times \operatorname{curl}\overline{\boldsymbol{B}}_{h\tau}, \boldsymbol{v}_h) - (\overline{p}_{h\tau}, \operatorname{div}\boldsymbol{v}_h) + \lambda(\overline{\overline{\varphi}}_{h\tau}\nabla\overline{w}_{h\tau}, \boldsymbol{v}_h)\Big\}\xi(t)\mathrm{d}t = \int_0^T \overline{\boldsymbol{f}}_\tau(t)\xi(t)\mathrm{d}t,
\end{aligned} \tag{5.18c}
$$

$$
\int_0^T (\operatorname{div}\overline{\boldsymbol{u}}_{h\tau}, q_h)\xi(t)\mathrm{d}t = 0, \tag{5.18d}
$$

$$
\begin{aligned}
\int_0^T \Big\{ &\big((\boldsymbol{B}_{h\tau})_t, \boldsymbol{C}_h\big) + \left(\frac{1}{\mu\sigma(\overline{\overline{\varphi}}_{h\tau})}\operatorname{curl}\overline{\boldsymbol{B}}_{h\tau}, \operatorname{curl}\boldsymbol{C}_h\right) + \left(\frac{1}{\mu\sigma(\overline{\overline{\varphi}}_{h\tau})}\operatorname{div}\overline{\boldsymbol{B}}_{h\tau}, \operatorname{div}\boldsymbol{C}_h\right)\\
&-(\overline{\boldsymbol{u}}_{h\tau} \times \overline{\overline{\boldsymbol{B}}}_{h\tau}, \operatorname{curl}\boldsymbol{C}_h)\Big\}\xi(t)\mathrm{d}t = 0,
\end{aligned} \tag{5.18e}
$$

where $\overline{f}_{h\tau} := \overline{\varphi}_{h\tau}^3 - \overline{\overline{\varphi}}_{h\tau}$. Next, we pass to the limit term by term in (5.18a)–(5.18e) as $h, \tau \to 0$. For time derivative terms, it follows from (5.2), (5.6) and (5.10) that

$$
\begin{aligned}
&\int_0^T \big((\varphi_{h\tau})_t, \psi_h\big)\xi(t)\mathrm{d}t \xrightarrow{h,\tau\to 0} \int_0^T \langle\varphi_t, \psi\rangle\xi(t)\mathrm{d}t,\\
&\int_0^T \big((\boldsymbol{u}_{h\tau})_t, \boldsymbol{v}_h\big)\xi(t)\mathrm{d}t \xrightarrow{h,\tau\to 0} \int_0^T \langle\boldsymbol{u}_t, \boldsymbol{v}\rangle\xi(t)\mathrm{d}t, \qquad \int_0^T \big((\boldsymbol{B}_{h\tau})_t, \boldsymbol{C}_h\big)\xi(t)\mathrm{d}t \xrightarrow{h,\tau\to 0} \int_0^T \langle\boldsymbol{B}_t, \boldsymbol{C}\rangle\xi(t)\mathrm{d}t.
\end{aligned}
$$

For elliptic term, there holds

$$
\begin{aligned}
&\int_0^T \big(\eta(\overline{\overline{\varphi}}_{h\tau})D(\overline{\boldsymbol{u}}_{h\tau}), D(\boldsymbol{v}_h)\big)\xi(t)\mathrm{d}t - \int_0^T \big(\eta(\varphi)D(\boldsymbol{u}), D(\boldsymbol{v})\big)\xi(t)\mathrm{d}t\\
&\quad\le \|\eta(\overline{\overline{\varphi}}_{h\tau}) - \eta(\varphi)\|_{L^4(L^4)}\|D(\overline{\boldsymbol{u}}_{h\tau})\|_{L^2(L^2)}\|D(\boldsymbol{v}_h)\xi(t)\|_{L^4(L^4)}\\
&\qquad+\eta^+\|D(\overline{\boldsymbol{u}}_{h\tau})\|_{L^2(L^2)}\|\big(D(\boldsymbol{v}_h) - D(\boldsymbol{v})\big)\xi(t)\|_{L^2(L^2)}\\
&\qquad+\left|\int_0^T \big(\eta(\varphi)(D(\overline{\boldsymbol{u}}_{h\tau}) - D(\boldsymbol{u})), D(\boldsymbol{v})\big)\xi(t)\mathrm{d}t\right| \xrightarrow{h,\tau\to 0} 0.
\end{aligned} \tag{5.19}
$$

In fact, because of $\mathcal{H}_\varepsilon'(x) \le \frac{1}{4\varepsilon}$, we have

$$
|\eta(\overline{\overline{\varphi}}_{h\tau}) - \eta(\varphi)| = |\eta_2 - \eta_1|\,\big|\mathcal{H}_\varepsilon(\overline{\overline{\varphi}}_{h\tau}) - \mathcal{H}_\varepsilon(\varphi)\big| \le \frac{|\eta_2 - \eta_1|}{4\varepsilon}|\overline{\overline{\varphi}}_{h\tau} - \varphi|.
$$

Then (5.19) follows from (4.10), (5.5), (5.14), Remark 5.1 and the definition of weak convergence.

Similarly, using (4.10), (5.9), (5.14), Remark 5.1 and the definition of weak convergence, we also obtain

$$\int_0^T \left\{ \left(\frac{1}{\mu\sigma(\overline{\overline{\varphi}}_{h\tau})} \operatorname{curl} \overline{\boldsymbol{B}}_{h\tau}, \operatorname{curl} \boldsymbol{C}_h \right) + \left(\frac{1}{\mu\sigma(\overline{\overline{\varphi}}_{h\tau})} \operatorname{div} \overline{\boldsymbol{B}}_{h\tau}, \operatorname{div} \boldsymbol{C}_h \right) \right\} \xi(t) \mathrm{d}t$$
$$\xrightarrow{h,\tau\to 0} \int_0^T \left\{ \left(\frac{1}{\mu\sigma(\varphi)} \operatorname{curl} \boldsymbol{B}, \operatorname{curl} \boldsymbol{C} \right) + \left(\frac{1}{\mu\sigma(\varphi)} \operatorname{div} \boldsymbol{B}, \operatorname{div} \boldsymbol{C} \right) \right\} \xi(t) \mathrm{d}t.$$

Next, we show the convergence of the trilinear terms. From (4.10), (5.9) and (5.16), we deduce

$$\int_0^T \frac{1}{\mu} (\overline{\overline{\boldsymbol{B}}}_{h\tau} \times \operatorname{curl} \overline{\boldsymbol{B}}_{h\tau}, \boldsymbol{v}_h) \xi(t) \mathrm{d}t - \int_0^T \frac{1}{\mu} (\boldsymbol{B} \times \operatorname{curl} \boldsymbol{B}, \boldsymbol{v}) \xi(t) \mathrm{d}t$$
$$\le \frac{1}{\mu} \|\overline{\overline{\boldsymbol{B}}}_{h\tau} - \boldsymbol{B}\|_{L^2(L^4)} \|\operatorname{curl} \overline{\boldsymbol{B}}_{h\tau}\|_{L^2(L^2)} \|\boldsymbol{v}_h \xi(t)\|_{L^\infty(L^4)}$$
$$+ \frac{1}{\mu} \|\boldsymbol{B}\|_{L^2(L^4)} \|\operatorname{curl} \overline{\boldsymbol{B}}_{h\tau}\|_{L^2(L^2)} \|(\boldsymbol{v}_h - \boldsymbol{v}) \xi(t)\|_{L^\infty(L^4)}$$
$$+ \left| \int_0^T \frac{1}{\mu} \big(\boldsymbol{B} \times \operatorname{curl}(\overline{\boldsymbol{B}}_{h\tau} - \boldsymbol{B}), \boldsymbol{v}\big) \xi(t) \mathrm{d}t \right| \xrightarrow{h,\tau\to 0} 0.$$

By (4.10) and (5.14)–(5.15), there holds

$$\int_0^T (\overline{\overline{\varphi}}_{h\tau} \overline{\boldsymbol{u}}_{h\tau}, \nabla \psi_h) \xi(t) \mathrm{d}t - \int_0^T (\varphi \boldsymbol{u}, \nabla \psi) \xi(t) \mathrm{d}t$$
$$\le \big(\|\overline{\overline{\varphi}}_{h\tau} - \varphi\|_{L^\infty(L^4)} \|\overline{\boldsymbol{u}}_{h\tau}\|_{L^2(L^4)} + \|\varphi\|_{L^\infty(L^4)} \|\overline{\boldsymbol{u}}_{h\tau} - \boldsymbol{u}\|_{L^2(L^4)} \big) \|\nabla \psi_h \xi(t)\|_{L^2(L^2)}$$
$$+ \|\varphi\|_{L^\infty(L^4)} \|\boldsymbol{u}\|_{L^2(L^4)} \|(\nabla \psi_h - \nabla \psi) \xi(t)\|_{L^2(L^2)} \xrightarrow{h,\tau\to 0} 0.$$

Similar to the above analysis, we can get the convergence of other trilinear terms. Moreover, we have

$$\int_0^T \big(\overline{f}_{h\tau}, \chi_h\big) \xi(t) \mathrm{d}t - \int_0^T \big(f(\varphi), \chi\big) \xi(t) \mathrm{d}t$$
$$\le c \left\{ \|\overline{\varphi}_{h\tau} - \varphi\|_{L^\infty(L^4)} \left(\|\overline{\varphi}_{h\tau}\|^2_{L^4(L^4)} + \|\varphi\|^2_{L^4(L^4)} \right) + \|\overline{\overline{\varphi}}_{h\tau} - \varphi\|_{L^2(L^2)} \right\} \|\chi_h \xi(t)\|_{L^2(L^4)}$$
$$+ c(\|\varphi\|^3_{L^\infty(L^4)} + \|\varphi\|_{L^2(L^2)}) \|(\chi_h - \chi) \xi(t)\|_{L^2(L^4)} \xrightarrow{h,\tau\to 0} 0.$$

Based on Remark 5.1 and the definition of weak convergence, there hold

$$\int_0^T (\operatorname{div} \overline{\boldsymbol{u}}_{h\tau}, q_h) \xi(t) \mathrm{d}t \xrightarrow{h,\tau\to 0} \int_0^T (\operatorname{div} \boldsymbol{u}, q) \xi(t) \mathrm{d}t, \qquad \int_0^T (\operatorname{div} \boldsymbol{v}_h, \overline{p}_{h\tau}) \xi(t) \mathrm{d}t \xrightarrow{h,\tau\to 0} \int_0^T (\operatorname{div} \boldsymbol{v}, p) \xi(t) \mathrm{d}t.$$

It is well known that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, $\boldsymbol{\mathcal{C}}_c^\infty(\Omega)$ is dense in $\boldsymbol{H}_0^1(\Omega)$, $\mathcal{C}_c^\infty(\Omega) \cap L_0^2(\Omega)$ is dense in $L_0^2(\Omega)$ and $\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}) \cap \boldsymbol{H}_n^1(\Omega)$ is dense in $\boldsymbol{H}_n^1(\Omega)$ and $\mathcal{C}^\infty([0,T])$ is dense in $L^p([0,T])$, $1 \le p < \infty$. Consequently, letting h and τ converge to 0 in (5.18a)–(5.18e), (3.17a)–(3.17e) hold in the sense of distributions.

Next, we prove $\varphi(0) = \varphi_0, \boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{B}(0) = \boldsymbol{B}_0$. Based on the fact that $\mathcal{C}^\infty([0,T])$ is dense in $H^1([0,T])$, we choose

$$\xi(t) = \begin{cases} 1 - \dfrac{t}{s}, & 0 \le t \le s, \\ 0, & s < t \le T, \end{cases}$$

in (5.18a), (5.18c) and (5.18e). Next, as h and τ converge 0, we obtain

$$-(\varphi_0, \psi) + \frac{1}{s} \int_0^s (\varphi(t), \psi) \mathrm{d}t = \int_0^s \big\{ (\varphi \boldsymbol{u}, \nabla \psi) - \gamma(\nabla w, \nabla \psi) \big\} \xi(t) \mathrm{d}t, \tag{5.20a}$$

$$-(\boldsymbol{u}_0, \boldsymbol{v}) + \frac{1}{s} \int_0^s (\boldsymbol{u}(t), \boldsymbol{v}) \mathrm{d}t = -\int_0^s \Big\{ 2\big(\eta(\varphi) D(\boldsymbol{u}), D(\boldsymbol{v})\big) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) + \frac{1}{2}\big((\operatorname{div} \boldsymbol{u})\boldsymbol{u}, \boldsymbol{v}\big)$$
$$+ \frac{1}{\mu}(\boldsymbol{B} \times \operatorname{curl} \boldsymbol{B}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) + \lambda(\varphi \nabla w n, \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle \Big\} \xi(t) \mathrm{d}t, \tag{5.20b}$$

$$-(\boldsymbol{B}_0, \boldsymbol{C}) + \frac{1}{s}\int_0^s (\boldsymbol{B}(t), \boldsymbol{C})\mathrm{d}t = -\int_0^s \left\{ \left(\frac{1}{\mu\sigma(\varphi)} \operatorname{curl} \boldsymbol{B}, \operatorname{curl} \boldsymbol{C} \right) + \left(\frac{1}{\mu\sigma(\varphi)} \operatorname{div} \boldsymbol{B}, \operatorname{div} \boldsymbol{C} \right) - (\boldsymbol{u} \times \boldsymbol{B}, \operatorname{curl} \boldsymbol{C}) \right\} \xi(t)\mathrm{d}t. \tag{5.20c}$$

Then, as s converges 0 in (5.20a)–(5.20c), we can get from Remark 3.3

$$\begin{aligned} (\varphi_0, \psi) &= (\varphi(0), \psi), && \forall\, \psi \in H^1(\Omega), \\ (\boldsymbol{u}_0, \boldsymbol{v}) &= (\boldsymbol{u}(0), \boldsymbol{v}), && \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ (\boldsymbol{B}_0, \boldsymbol{C}) &= (\boldsymbol{B}(0), \boldsymbol{C}), && \forall\, \boldsymbol{C} \in \boldsymbol{H}_n^1(\Omega). \end{aligned}$$

Since $H^1(\Omega)$ is dense in $L^2(\Omega)$, and $\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{H}_n^1(\Omega)$ are dense in $\boldsymbol{L}^2(\Omega)$, there hold $\varphi(0) = \varphi_0$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{B}(0) = \boldsymbol{B}_0$. From (4.5), we have

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{u}_{h\tau}(t_m)\|_{L^2}^2 + \frac{1}{2\mu}\|\boldsymbol{B}_{h\tau}(t_m)\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla\varphi_{h\tau}(t_m)\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\big(F(\varphi_{h\tau}(t_m)), 1\big) + \int_0^{t_m} \lambda\gamma\|\nabla\overline{w}_{h\tau}\|_{L^2}^2\mathrm{d}t \\ &\quad + \int_0^{t_m} \left\{ 2\|\sqrt{\eta(\overline{\overline{\varphi}}_{h\tau})}D(\overline{\boldsymbol{u}}_{h\tau})\|_{L^2}^2 + \frac{1}{\mu^2}\Big\|\frac{1}{\sqrt{\sigma(\overline{\overline{\varphi}}_{h\tau})}}\operatorname{curl}\overline{\boldsymbol{B}}_{h\tau}\Big\|_{L^2}^2 + \frac{1}{\mu^2}\Big\|\frac{1}{\sqrt{\sigma(\overline{\overline{\varphi}}_{h\tau})}}\operatorname{div}\overline{\boldsymbol{B}}_{h\tau}\Big\|_{L^2}^2 \right\}\mathrm{d}t \\ &\le \int_0^{t_m} \langle \boldsymbol{f}, \overline{\boldsymbol{u}}_{h\tau}\rangle\mathrm{d}t + \mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^0, \boldsymbol{B}_h^0, \varphi_h^0). \end{aligned}$$

According to lower semi-continuity of norms and (4.4), an accumulation point $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ satisfies the energy inequality (3.18).

From the above analysis and Definition 3.1, an accumulation point $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ is a weak solution to the problem (2.3a)–(2.3f). □

Next, we give the following estimates of weak solution.

Lemma 5.4. *Suppose that $\boldsymbol{f} \in L^2(0, T; (\boldsymbol{H}_0^1(\Omega))')$ and $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0) < \infty$. Then, for almost all $t \in (0, T)$, there hold*

$$\int_\Omega \varphi(x, t)\mathrm{d}x = \int_\Omega \varphi_0(x)\mathrm{d}x, \quad (\textit{mass conservation}) \tag{5.21}$$

$$\lambda\|\nabla\varphi(t)\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\big(F(\varphi(t)), 1\big) + \|\boldsymbol{u}(t)\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}(t)\|_{L^2}^2 \le C, \tag{5.22}$$

$$\int_0^t \left(\|\nabla w\|_{L^2}^2 + \eta^-\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\mu^2\sigma^+}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^2 \right)\mathrm{d}s \le C, \tag{5.23}$$

$$\|\varphi(t)\|_{H^1} + \int_0^t \|\varphi_t\|_{(H^1)'}^2\mathrm{d}s \le C, \tag{5.24}$$

$$\int_0^t \|w\|_{H^1}^2\mathrm{d}s \le C\left(\frac{T}{\varepsilon^4} + T + 1\right), \tag{5.25}$$

$$\int_0^t \left(\|\boldsymbol{u}_t\|_{\boldsymbol{V}'}^{\frac{4}{d}} + \|\boldsymbol{B}_t\|_{(\boldsymbol{H}_n^1)'}^{\frac{4}{d}} + \|p\|_{L^2}^{\frac{12}{6+d}} \right)\mathrm{d}s \le C(T + 1), \tag{5.26}$$

$$\int_0^t \left(\|\Delta\varphi\|_{L^2}^2 + \|\varphi\|_{L^\infty}^{\frac{4(6-d)}{d}} \right)\mathrm{d}s \le C\left(\frac{T}{\varepsilon^2} + T + 1\right). \tag{5.27}$$

Proof. Taking $\psi = 1$ in (3.17a), we have (5.21). By Hölder inequality, (2.4), (3.2) and (3.18), we have (5.22)–(5.23). Based on (5.22)–(5.23) and the proof of Theorem 4.2, we can prove (5.24)–(5.25).

According to (3.1), there holds $\boldsymbol{v} = P_H\boldsymbol{v} + \nabla v_0$ for any $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, where $v_0 \in H^1(\Omega)/\mathbb{R} \subset L_0^2(\Omega)$. Then, from (3.17d), we have $\boldsymbol{u} \in \boldsymbol{V} \subset \boldsymbol{H}$ and

$$\langle \boldsymbol{u}_t, \boldsymbol{v}\rangle = \langle \boldsymbol{u}_t, P_H\boldsymbol{v}\rangle + \langle \boldsymbol{u}_t, \nabla v_0\rangle = \langle \boldsymbol{u}_t, P_H\boldsymbol{v}\rangle + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \nabla v_0) = \langle \boldsymbol{u}_t, P_H\boldsymbol{v}\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

It follows from (3.17c) and the H^1 stability of P_H that

$$\|\boldsymbol{u}_t\|_{(\boldsymbol{H}_0^1)'} = \sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}_t, \boldsymbol{v}\rangle}{\|\nabla \boldsymbol{v}\|_{L^2}} = \sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}_t, P_H\boldsymbol{v}\rangle}{\|\nabla P_H \boldsymbol{v}\|_{L^2}} \cdot \frac{\|\nabla P_H \boldsymbol{v}\|_{L^2}}{\|\nabla \boldsymbol{v}\|_{L^2}}$$
$$\leq c_0\eta^+\|\nabla \boldsymbol{u}\|_{L^2} + c_0\|\boldsymbol{u}\|_{L^2}^{\frac{6-d}{6}}\|\nabla \boldsymbol{u}\|_{L^2}^{\frac{6+d}{6}} + \frac{c_0}{\mu}\|\boldsymbol{B}\|_{L^2}^{\frac{6-d}{6}}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^{\frac{6+d}{6}} + \lambda c_0\|\varphi\|_{H^1}\|\nabla w\|_{L^2} + \|\boldsymbol{f}\|_{(\boldsymbol{H}_0^1)'}.$$

Therefore, from the above inequality, (5.22)–(5.24), (3.17c), (3.17e) and Remark 3.2, we obtain (5.26).

Letting $\chi = -\Delta\varphi$ in (3.17b) and using $f'(\varphi) = 3\varphi^2 - 1 \geq -1$, (3.9) and (3.11), we have

$$\|\Delta\varphi\|_{L^2}^2 \leq (\frac{1}{2} + \frac{1}{\varepsilon^2})\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2,$$
$$\|\varphi\|_{L^\infty}^{\frac{4(6-d)}{d}} \leq c_0\|\Delta\varphi\|_{L^2}^2\|\varphi\|_{L^6}^{\frac{6(4-d)}{d}} + c_0\|\varphi\|_{L^6}^{\frac{4(6-d)}{d}} \leq c_0\|\Delta\varphi\|_{L^2}^2\|\varphi\|_{H^1}^{\frac{6(4-d)}{d}} + c_0\|\varphi\|_{H^1}^{\frac{4(6-d)}{d}}.$$

Hence, we get (5.27) from the above inequality and (5.22)–(5.24). □

To guarantee the uniqueness of weak solutions, more regularity on weak solutions is needed. Then the space $\mathcal{S}$ is introduced as follows

$$\mathcal{S} = \{(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B}) : (\varphi, w, \boldsymbol{u}, p, \boldsymbol{B}) \text{ satisfies (3.12)–(3.16) and } \nabla\boldsymbol{u}, \nabla\boldsymbol{B} \in L^2(0, T; \boldsymbol{L}^q(\Omega))\},$$

where $q = 3$ for $d = 3$ and $q > 2$ for $d = 2$. In fact, the weak solutions $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ of the problem (2.3a)–(2.3f) belong to the function space $\mathcal{S}$, if weak solutions $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ satisfy the additional regularity conditions $\nabla\boldsymbol{u} \in L^2(0, T; \boldsymbol{L}^q(\Omega))$, $\nabla\boldsymbol{B} \in L^2(0, T; \boldsymbol{L}^q(\Omega))$, where $q = 3$ for $d = 3$ and $q > 2$ for $d = 2$. From Remark 4.1 and [66], Y_h, $\boldsymbol{X}_h$ and $\boldsymbol{W}_h$ are finite-dimensional subspaces of $\boldsymbol{W}^{1,\infty}(\Omega)$. Therefore, $Y_h \times Y_h \times \boldsymbol{X}_h \times M_h \times \boldsymbol{W}_h$ is a subset of $\mathcal{S}$.

Theorem 5.5. *Suppose that $\boldsymbol{f} \in L^2(0, T; (\boldsymbol{H}_0^1(\Omega))')$ and initial data $\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0$ satisfy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0, \boldsymbol{B}_0, \varphi_0) < \infty$. If the weak solutions of the problem (2.3a)–(2.3f) belong to the function space $\mathcal{S}$, the weak solutions of the problem (2.3a)–(2.3f) in the function space $\mathcal{S}$ are unique for $d = 2, 3$.*

Proof. Assume $(\varphi_i, w_i, \boldsymbol{u}_i, p_i, \boldsymbol{B}_i)$, $i = 1, 2$, are two weak solutions to the problem (2.3a)–(2.3f) which belong to the function space $\mathcal{S}$. Denote $\varphi := \varphi_1 - \varphi_2$, $w := w_1 - w_2$, $\boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{B} := \boldsymbol{B}_1 - \boldsymbol{B}_2$. Then, there hold

$$\langle\varphi_t, \psi\rangle - (\varphi_1\boldsymbol{u}, \nabla\psi) - (\varphi\boldsymbol{u}_2, \nabla\psi) + \gamma(\nabla w, \nabla\psi) = 0, \tag{5.28a}$$

$$(\nabla\varphi, \nabla\chi) + \frac{1}{\varepsilon^2}\big(g(\varphi_1, \varphi_2)\varphi, \chi\big) = (w, \chi), \tag{5.28b}$$

$$\begin{aligned}&\langle\boldsymbol{u}_t, \boldsymbol{v}\rangle + 2\big(\eta(\varphi_1)D(\boldsymbol{u}), D(\boldsymbol{v})\big) + 2\big((\eta(\varphi_1) - \eta(\varphi_2))D(\boldsymbol{u}_2), D(\boldsymbol{v})\big) + \big((\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}\big)\\ &+\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2, \boldsymbol{v}\big) + \frac{1}{\mu}(\boldsymbol{B}_1 \times \operatorname{curl}\boldsymbol{B}, \boldsymbol{v}) + \frac{1}{\mu}(\boldsymbol{B} \times \operatorname{curl}\boldsymbol{B}_2, \boldsymbol{v}) + \lambda(\varphi_1\nabla w, \boldsymbol{v}) + \lambda(\varphi\nabla w_2, \boldsymbol{v}) = 0,\end{aligned} \tag{5.28c}$$

$$\begin{aligned}\langle\boldsymbol{B}_t, \boldsymbol{C}\rangle &+ \left(\left(\frac{1}{\mu\sigma(\varphi_1)} - \frac{1}{\mu\sigma(\varphi_2)}\right)\operatorname{curl}\boldsymbol{B}_2, \operatorname{curl}\boldsymbol{C}\right) + \left(\left(\frac{1}{\mu\sigma(\varphi_1)} - \frac{1}{\mu\sigma(\varphi_2)}\right)\operatorname{div}\boldsymbol{B}_2, \operatorname{div}\boldsymbol{C}\right)\\ &+ \left(\frac{1}{\mu\sigma(\varphi_1)}\operatorname{curl}\boldsymbol{B}, \operatorname{curl}\boldsymbol{C}\right) + \left(\frac{1}{\mu\sigma(\varphi_1)}\operatorname{div}\boldsymbol{B}, \operatorname{div}\boldsymbol{C}\right)\\ &- (\boldsymbol{u} \times \boldsymbol{B}_1, \operatorname{curl}\boldsymbol{C}) - (\boldsymbol{u}_2 \times \boldsymbol{B}, \operatorname{curl}\boldsymbol{C}) = 0,\end{aligned} \tag{5.28d}$$

for any $(\psi, \chi, \boldsymbol{v}, \boldsymbol{C}) \in H^1(\Omega) \times H^1(\Omega) \times \boldsymbol{V} \times \boldsymbol{H}_n^1(\Omega)$, where $g(\varphi_1, \varphi_2) := \varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2 - 1$.

Setting $(\psi, \chi, \boldsymbol{v}, \boldsymbol{C}) = (\lambda w, \lambda\varphi_t, \boldsymbol{u}, \frac{1}{\mu}\boldsymbol{B})$ in (5.28a)–(5.28d) and adding the resulted equalities, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2\mu}\|\boldsymbol{B}\|_{L^2}^2\right) + \gamma\lambda\|\nabla w\|_{L^2}^2 + 2\|\sqrt{\eta(\varphi_1)}D(\boldsymbol{u})\|_{L^2}^2\\ &+\frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_1)}}\operatorname{curl}\boldsymbol{B}\|_{L^2}^2 + \frac{1}{\mu^2}\|\frac{1}{\sqrt{\sigma(\varphi_1)}}\operatorname{div}\boldsymbol{B}\|_{L^2}^2\\ &= -\frac{\lambda}{\varepsilon^2}\langle\varphi_t, g(\varphi_1, \varphi_2)\varphi\rangle + \lambda(\varphi\boldsymbol{u}_2, \nabla w) - \lambda(\varphi\nabla w_2, \boldsymbol{u}) - \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2, \boldsymbol{u}\big)\end{aligned}$$

$$-\frac{1}{\mu}(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}_2,\boldsymbol{u})+\frac{1}{\mu}(\boldsymbol{u}_2\times\boldsymbol{B},\operatorname{curl}\boldsymbol{B})-2\big((\eta(\varphi_1)-\eta(\varphi_2))D(\boldsymbol{u}_2),D(\boldsymbol{u})\big)$$
$$-\frac{1}{\mu^2}\left(\left(\frac{1}{\sigma(\varphi_1)}-\frac{1}{\sigma(\varphi_2)}\right)\operatorname{curl}\boldsymbol{B}_2,\operatorname{curl}\boldsymbol{B}\right)-\frac{1}{\mu^2}\left(\left(\frac{1}{\sigma(\varphi_1)}-\frac{1}{\sigma(\varphi_2)}\right)\operatorname{div}\boldsymbol{B}_2,\operatorname{div}\boldsymbol{B}\right). \tag{5.29}$$

To estimate the first term on the right-hand side of (5.29), we test $\psi=\frac{\lambda}{\varepsilon^2}g(\varphi_1,\varphi_2)\varphi\in L^2(0,T;H^1(\Omega))$ in (5.28a) and have

$$\begin{aligned}-\frac{\lambda}{\varepsilon^2}\langle\varphi_t,g(\varphi_1,\varphi_2)\varphi\rangle&=-\frac{\lambda}{\varepsilon^2}\big(\varphi_1\boldsymbol{u},\nabla(g(\varphi_1,\varphi_2)\varphi)\big)-\frac{\lambda}{\varepsilon^2}\big(\varphi\boldsymbol{u}_2,\nabla(g(\varphi_1,\varphi_2)\varphi)\big)+\frac{\lambda\gamma}{\varepsilon^2}\big(\nabla w,\nabla(g(\varphi_1,\varphi_2)\varphi)\big)\\&\le\frac{c_0^2\eta^-}{8}\|\nabla\boldsymbol{u}\|_{L^2}^2+\frac{\gamma\lambda}{4}\|\nabla w\|_{L^2}^2+c\|\nabla\boldsymbol{u}_2\|_{L^2}^2\|\varphi\|_{H^1}^2\\&\quad+c\left(\|\varphi_1\|_{H^1}^2+1\right)\left(\|g(\varphi_1,\varphi_2)\|_{L^\infty}^2+\|\nabla g(\varphi_1,\varphi_2)\|_{L^3}^2\right)\|\varphi\|_{H^1}^2,\end{aligned} \tag{5.30}$$

where $g(\varphi_1,\varphi_2)\in L^2(0,T;L^\infty(\Omega))$ and $\nabla g(\varphi_1,\varphi_2)\in L^2(0,T;L^3(\Omega))$ can be obtained from (5.24) and (5.27).

From (3.3) and (3.9), we get

$$\begin{aligned}\lambda(\varphi\boldsymbol{u}_2,\nabla w)-\lambda(\varphi\nabla w_2,\boldsymbol{u})&\le\lambda\|\varphi\|_{L^3}\|\boldsymbol{u}_2\|_{L^6}\|\nabla w\|_{L^2}+\lambda\|\varphi\|_{L^3}\|\nabla w_2\|_{L^2}\|\boldsymbol{u}\|_{L^6}\\&\le\frac{\gamma\lambda}{4}\|\nabla w\|_{L^2}^2+\frac{c_0^2\eta^-}{8}\|\nabla\boldsymbol{u}\|_{L^2}^2+c\left(\|\nabla\boldsymbol{u}_2\|_{L^2}^2+\|\nabla w_2\|_{L^2}^2\right)\|\varphi\|_{H^1}^2.\end{aligned} \tag{5.31}$$

Using the fact that $\eta(\varphi)$ and $\frac{1}{\sigma(\varphi)}$ are Lipschitz-continuous functions of φ, we have

$$\|\eta(\varphi_1)-\eta(\varphi_2)\|_{L^p}+\|\frac{1}{\sigma(\varphi_1)}-\frac{1}{\sigma(\varphi_2)}\|_{L^p}\le c\|\varphi_1-\varphi_2\|_{L^p}\le c\|\varphi\|_{H^1},$$

where $p\in[1,6]$ if $d=3$ and $p\in[1,+\infty)$ if $d=2$. So, by Hölder inequality, there hold

$$2\big((\eta(\varphi_1)-\eta(\varphi_2))D(\boldsymbol{u}_2),D(\boldsymbol{u})\big)\le\frac{c_0^2\eta^-}{8}\|\nabla\boldsymbol{u}\|_{L^2}^2+c\|\nabla\boldsymbol{u}_2\|_{L^q}^2\|\varphi\|_{H^1}^2, \tag{5.32}$$

$$\begin{aligned}&\frac{1}{\mu^2}\left(\left(\frac{1}{\sigma(\varphi_1)}-\frac{1}{\sigma(\varphi_2)}\right)\operatorname{curl}\boldsymbol{B}_2,\operatorname{curl}\boldsymbol{B}\right)+\frac{1}{\mu^2}\left(\left(\frac{1}{\sigma(\varphi_1)}-\frac{1}{\sigma(\varphi_2)}\right)\operatorname{div}\boldsymbol{B}_2,\operatorname{div}\boldsymbol{B}\right)\\&\quad\le\frac{1}{4\mu^2\sigma^+}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^2+c\|\nabla\boldsymbol{B}_2\|_{L^q}^2\|\varphi\|_{H^1}^2,\end{aligned} \tag{5.33}$$

in which $q=3$ if $d=3$ and $q>2$ if $d=2$. Based on (3.3), (3.5)–(3.6), (3.8) and Remark 3.2, we obtain

$$\begin{aligned}&-\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{u}\big)-\frac{1}{\mu}(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}_2,\boldsymbol{u})+\frac{1}{\mu}(\boldsymbol{u}_2\times\boldsymbol{B},\operatorname{curl}\boldsymbol{B})\\&\quad\le\begin{cases}\|\boldsymbol{u}\|_{L^4}\|\nabla\boldsymbol{u}_2\|_{L^2}\|\boldsymbol{u}\|_{L^4}+\frac{1}{\mu}\|\boldsymbol{B}\|_{L^4}\|\operatorname{curl}\boldsymbol{B}_2\|_{L^2}\|\boldsymbol{u}\|_{L^4}+\frac{1}{\mu}\|\nabla\boldsymbol{u}_2\|_{L^2}\|\boldsymbol{B}\|_{L^4}\|\boldsymbol{B}\|_{L^4}, & d=2,\\ \|\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}_2\|_{L^3}\|\boldsymbol{u}\|_{L^6}+\frac{1}{\mu}\|\boldsymbol{B}\|_{L^6}\|\operatorname{curl}\boldsymbol{B}_2\|_{L^3}\|\boldsymbol{u}\|_{L^2}+\frac{c_0}{\mu}\|\nabla\boldsymbol{u}_2\|_{L^3}\|\boldsymbol{B}\|_{L^2}\|\boldsymbol{B}\|_{L^6}, & d=3,\end{cases}\\&\quad\le\frac{c_0^2\eta^-}{8}\|\nabla\boldsymbol{u}\|_{L^2}^2+\frac{1}{4\mu^2\sigma^+}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^2+c(\|\nabla\boldsymbol{u}_2\|_{L^d}^2+\|\operatorname{curl}\boldsymbol{B}_2\|_{L^d}^2)(\|\boldsymbol{u}\|_{L^2}^2+\|\boldsymbol{B}\|_{L^2}^2).\end{aligned} \tag{5.34}$$

Combining (5.30)–(3.2) and (5.34) with (5.29), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2}\|\nabla\varphi\|_{L^2}^2+\frac{1}{2}\|\boldsymbol{u}\|_{L^2}^2+\frac{1}{2\mu}\|\boldsymbol{B}\|_{L^2}^2\right)+\frac{\gamma\lambda}{2}\|\nabla w\|_{L^2}^2+\frac{c_0^2\eta^-}{2}\|\nabla\boldsymbol{u}\|_{L^2}^2+\frac{1}{2\mu^2\sigma^+}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^2\\&\quad\le c\big(\|\nabla w_2\|_{L^2}^2+\|g(\varphi_1,\varphi_2)\|_{L^\infty}^2+\|\nabla g(\varphi_1,\varphi_2)\|_{L^3}^2\big)\|\varphi\|_{H^1}^2\\&\quad\quad+c\left(\|\nabla\boldsymbol{u}_2\|_{L^q}^2+\|\nabla\boldsymbol{B}_2\|_{L^q}^2\right)\left(\|\boldsymbol{u}\|_{L^2}^2+\|\boldsymbol{B}\|_{L^2}^2+\|\varphi\|_{H^1}^2\right),\end{aligned} \tag{5.35}$$

where $q=3$ if $d=3$ and $q>2$ if $d=2$.

Next, we estimate $\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{L^2}^2$ and $\|\Delta\varphi\|_{L^2}^2$. Setting $\psi=\varphi$ in (5.28a) and $\chi=-\gamma\Delta\varphi$ in (5.28b), and adding the obtained equalities, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{L^2}^2+\gamma\|\Delta\varphi\|_{L^2}^2=\frac{\gamma}{\varepsilon^2}\big(g(\varphi_1,\varphi_2)\varphi,\Delta\varphi\big)+(\varphi_1\boldsymbol{u},\nabla\varphi)+(\varphi\boldsymbol{u}_2,\nabla\varphi)$$

$$
\begin{aligned}
&\le \frac{\gamma}{\varepsilon^2}\|g(\varphi_1,\varphi_2)\|_{L^\infty}\|\varphi\|_{L^2}\|\Delta\varphi\|_{L^2} + \|\varphi_1\|_{L^3}\|\boldsymbol{u}\|_{L^2}\|\nabla\varphi\|_{L^6} + \|\varphi\|_{L^2}\|\boldsymbol{u}_2\|_{L^3}\|\nabla\varphi\|_{L^6}\\
&\le \frac{\gamma}{4}\|\Delta\varphi\|_{L^2}^2 + c\big(\|g(\varphi_1,\varphi_2)\|_{L^\infty}^2 + \|\nabla\boldsymbol{u}_2\|_{L^2}^2\big)\|\varphi\|_{L^2}^2 + c\|\varphi_1\|_{H^1}^2\|\boldsymbol{u}\|_{L^2}^2. \qquad (5.36)
\end{aligned}
$$

Taking sum of (5.35)–(5.36), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_{L^2}^2 + \lambda\|\nabla\varphi\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}\|_{L^2}^2\right)\\
&\qquad +\gamma\lambda\|\nabla w\|_{L^2}^2 + c_0^2\eta^-\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\mu^2\sigma^+}\|\boldsymbol{B}\|_{\boldsymbol{H}_n^1(\Omega)}^2 + \gamma\|\Delta\varphi\|_{L^2}^2\\
&\le c\left(\|\nabla w_2\|_{L^2}^2 + \|g(\varphi_1,\varphi_2)\|_{L^\infty}^2 + \|\nabla g(\varphi_1,\varphi_2)\|_{L^3}^2\right)\left(\|\varphi\|_{L^2}^2 + \lambda\|\nabla\varphi\|_{L^2}^2\right)\\
&\qquad +c\left(\|\nabla\boldsymbol{u}_2\|_{L^q}^2 + \|\nabla\boldsymbol{B}_2\|_{L^q}^2 + \|\varphi_1\|_{H^1}^2\right)\left(\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\mu}\|\boldsymbol{B}\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + \lambda\|\nabla\varphi\|_{L^2}^2\right). \qquad (5.37)
\end{aligned}
$$

Then, making use of Gronwall lemma, $\varphi(0)=0$, $\boldsymbol{u}(0)=\boldsymbol{B}(0)=\boldsymbol{0}$ and $\nabla\boldsymbol{u},\nabla\boldsymbol{B}\in L^2(0,T;\boldsymbol{L}^q(\Omega))$ where $q=3$ in case of $d=3$ and $q>2$ for $d=2$, we deduce

$$
\|\varphi\|_{H^1}^2 + \|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2 = 0. \qquad (5.38)
$$

Moreover, the uniqueness of pressure p follows from the inf–sup condition (see Corollary 2.4 and Lemma 4.1 of [66]). Hence, the theorem is proved. □

Remark 5.2. Based on the assumption $\nabla\boldsymbol{u},\nabla\boldsymbol{B}\in L^2(0,T;\boldsymbol{L}^3(\Omega))$ for $d=3$ and Remark 3.2, we can show $\boldsymbol{u}_t\in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{B}_t\in L^2(0,T;(\boldsymbol{H}_n^1)')$ for $d=2,3$. Therefore, under the assumption $\nabla\boldsymbol{u},\nabla\boldsymbol{B}\in L^2(0,T;\boldsymbol{L}^3(\Omega))$ for $d=3$, the problem (2.3a)–(2.3f) satisfies the energy law for almost all $t\in[0,T]$

$$
\begin{aligned}
&\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}(t),\boldsymbol{B}(t),\varphi(t)) + \int_0^t\left\{\lambda\gamma\|\nabla w\|_{L^2}^2 + 2\|\sqrt{\eta(\varphi)}D(\boldsymbol{u})\|_{L^2}^2 + \frac{1}{\mu^2}\Big\|\frac{1}{\sqrt{\sigma(\varphi)}}\operatorname{curl}\boldsymbol{B}\Big\|_{L^2}^2\right.\\
&\qquad\left.+\frac{1}{\mu^2}\Big\|\frac{1}{\sqrt{\sigma(\varphi)}}\operatorname{div}\boldsymbol{B}\Big\|_{L^2}^2\right\}\mathrm{d}s = \int_0^t(\boldsymbol{f},\boldsymbol{v})\mathrm{d}s + \mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_0,\boldsymbol{B}_0,\varphi_0).
\end{aligned}
$$

Theorem 5.6. *Under the conditions of Theorem 5.5, the whole sequence $\{(\varphi_{h\tau},\overline{w}_{h\tau},\boldsymbol{u}_{h\tau},\overline{p}_{h\tau},\boldsymbol{B}_{h\tau})\}$ converges to the unique weak solution.*

Proof. Based on Theorems 5.3 and 5.5, each convergent subsequence of $\{(\varphi_{h\tau},\overline{w}_{h\tau},\boldsymbol{u}_{h\tau},\overline{p}_{h\tau},\boldsymbol{B}_{h\tau})\}$ has the same limit $(\varphi,w,\boldsymbol{u},p,\boldsymbol{B})$ which is the weak solution to the problem (2.3a)–(2.3f). Therefore, the whole sequence $\{(\varphi_{h\tau},\overline{w}_{h\tau},\boldsymbol{u}_{h\tau},\overline{p}_{h\tau},\boldsymbol{B}_{h\tau})\}$ converges to $(\varphi,w,\boldsymbol{u},p,\boldsymbol{B})$. □

6. Numerical examples

In this section, we provide three numerical experiments to validate the proposed numerical scheme and Cahn–Hilliard–MHD model. For spatial discretization, the finite element space

$$
Y_h = \left\{\psi_h\in C^0(\overline{\Omega});\ \psi_h|_K\in P_2(K),\forall K\in\mathcal{T}_h\right\},
$$

is chosen to approximate φ and w, and the finite element spaces

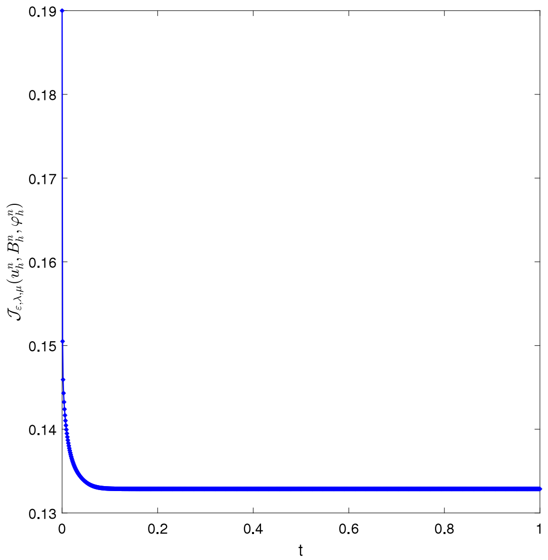
$$
\begin{aligned}
\boldsymbol{X}_h &= \left\{\boldsymbol{v}_h\in\boldsymbol{C}^0(\overline{\Omega})\cap\boldsymbol{H}_0^1(\Omega);\ \boldsymbol{v}_h|_K\in\boldsymbol{P}_2(K),\forall K\in\mathcal{T}_h\right\},\\
M_h &= \left\{q_h\in C^0(\overline{\Omega})\cap L_0^2(\Omega);\ q_h|_K\in P_1(K),\forall K\in\mathcal{T}_h\right\},\\
\boldsymbol{W}_h &= \left\{\boldsymbol{C}_h\in\boldsymbol{C}^0(\overline{\Omega})\cap\boldsymbol{H}_n^1(\Omega);\ \boldsymbol{C}_h|_K\in\boldsymbol{P}_2(K),\forall K\in\mathcal{T}_h\right\}
\end{aligned}
$$

are used to approximate $\boldsymbol{u}$, p and $\boldsymbol{B}$, respectively.

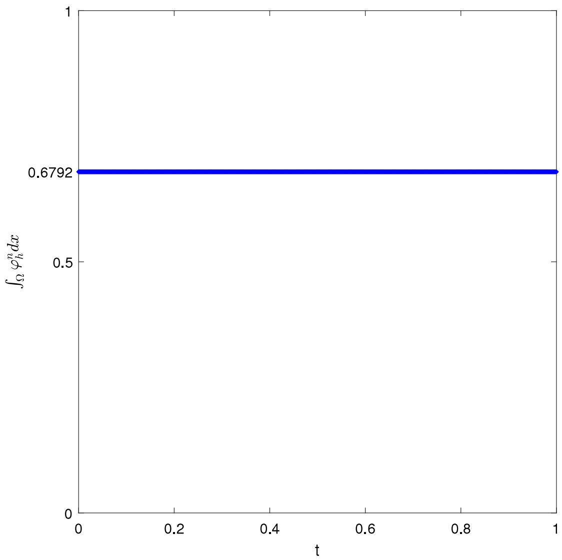
6.1. Energy dissipation and mass conservation

In this test, the initial profile of the phase φ is taken as

$$
\varphi_0 = \tanh\left(\frac{|x+y-1|+|x-y|-0.4}{\sqrt{2}\varepsilon}\right).
$$



(a) The energy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^n, \boldsymbol{B}_h^n, \varphi_h^n)$



(b) The total mass $\int_\Omega \varphi_h^n dx$

Fig. 1. The energy dissipation and mass conservation.

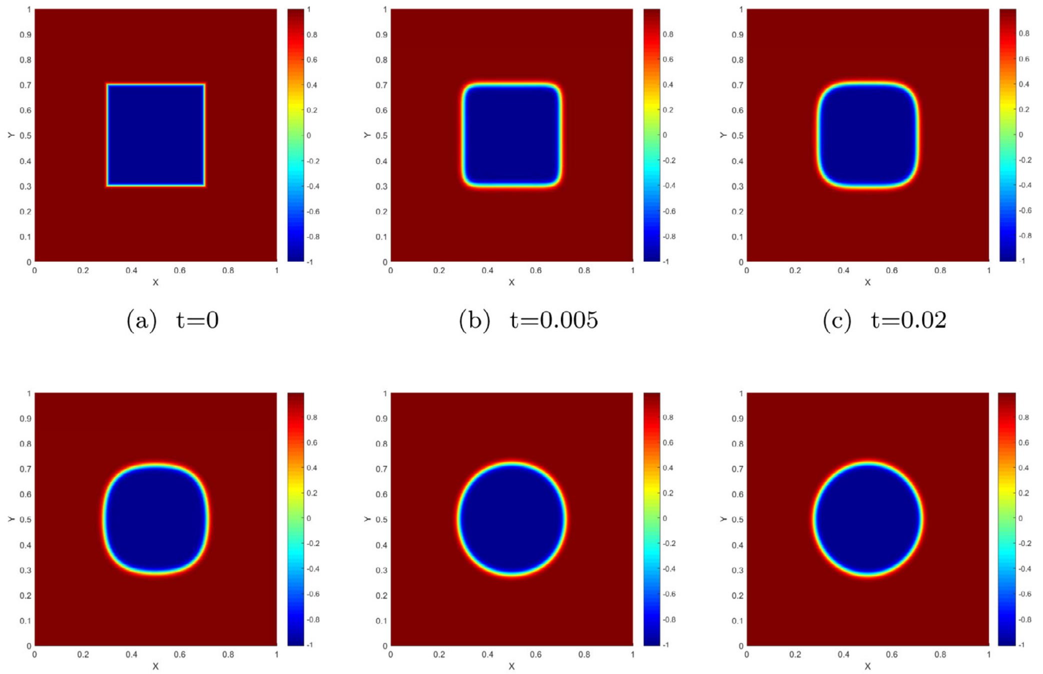
The zero level set of φ_0 is the square $|x + y - 1| + |x - y| = 0.4$ located in the middle of the domain $\Omega = (0, 1)\times(0, 1)$ (see Fig. 2a). We take both the initial conditions and boundary conditions for the velocity and magnetic field to be zero. The source term $\boldsymbol{f}$ is taken to be zero and homogeneous Neumann boundary conditions are imposed for φ and w. Based on Remark 5.2, the energy of two-phase MHD system without source terms and the exchange of external energy is dissipative. Setting the parameters $\varepsilon = 0.01, \lambda = 0.001, \gamma = 0.001, \eta = \mu = \sigma = 1$ and $h = 1/64, \tau = 0.001$ and using the scheme (4.3a)–(4.3f), the energy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^n, \boldsymbol{B}_h^n, \varphi_h^n)$ and the mass $\int_\Omega \varphi_h^n \mathrm{d}x$ are calculated. Fig. 1a shows that the energy $\mathcal{J}_{\varepsilon,\lambda,\mu}(\boldsymbol{u}_h^n, \boldsymbol{B}_h^n, \varphi_h^n)$ decays with time. During the evolution, the mass $\int_\Omega \varphi_h^n \mathrm{d}x$ remains constant (see Fig. 1b). Considering zero initial data and homogeneous boundary conditions for velocity and magnetic field, the isolated square relaxes to a circular shape under the effect of surface tension and the isotropy of the mobility (see Fig. 2).

6.2. Convergence of the scheme

In the domain $\Omega = (0, 1) \times (0, 1)$ and time interval $(0, 1)$, consider the model with the following analytical solution

$$\begin{aligned}
&\varphi = 256x^2(x-1)^2y^2(y-1)^2\cos(t), \qquad w = 256x^2(x-1)^2y^2(y-1)^2\cos(t),\\
&\boldsymbol{u} = \big(x^2(x-1)^2y(y-1)(2y-1)\cos(t),\ -y^2(y-1)^2x(x-1)(2x-1)\cos(t)\big),\\
&p = (2x-1)(2y-1)\cos(t),\\
&\boldsymbol{B} = \big(\sin(\pi x)\cos(\pi y)\cos(t),\ -\sin(\pi y)\cos(\pi x)\cos(t)\big).
\end{aligned}$$

The initial conditions, boundary conditions and source terms are determined by the analytical solution. Set the physical parameters $\gamma = \lambda = \varepsilon = \eta = \sigma = \mu = 1$. Since the first order Euler semi-implicit treatment in time and the finite elements $(P_2 - P_2 - \boldsymbol{P}_2 - P_1 - \boldsymbol{P}_2)$ for $(\varphi, w, \boldsymbol{u}, p, \boldsymbol{B})$ in space are applied to solve the model, the L^2 errors of $(\varphi, w, \boldsymbol{u}, \boldsymbol{B})$ are expected to be $\mathcal{O}(h^3) + \mathcal{O}(\tau)$, and the H^1 errors of $(\varphi, w, \boldsymbol{u}, \boldsymbol{B})$ and the L^2 errors of p are expected to be $\mathcal{O}(h^2) + \mathcal{O}(\tau)$. We test the convergence of the proposed scheme with $\tau = 8h^3$ and $\tau = 4h^2$, respectively. The corresponding convergent results are displayed in Tables 1–2, which show the optimal convergence of the proposed numerical scheme.



(a) t=0 (b) t=0.005 (c) t=0.02

(d) t=0.05 (e) t=0.1 (f) t=1

Fig. 2. Phase field φ at different time t.

Table 1
The convergent results for two-phase MHD model at $t_n = 1$, $\tau = 8h^3$.

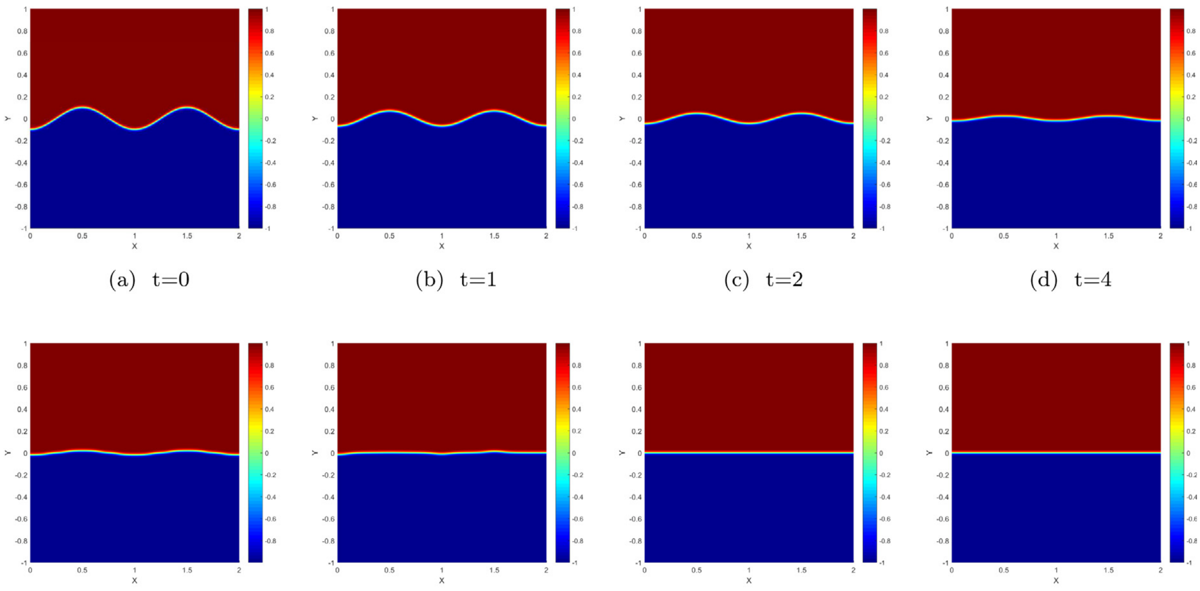
h	$\|\varphi - \varphi_h^n\|_{L^2}$	$\|w - w_h^n\|_{L^2}$	$\|\boldsymbol{u} - \boldsymbol{u}_h^n\|_{L^2}$	$\|\boldsymbol{B} - \boldsymbol{B}_h^n\|_{L^2}$
1/4	1.50457e−002	1.88881e−002	9.4626e−005	3.82049e−003
1/8	1.95976e−003	2.30104e−003	1.05811e−005	4.38114e−004
1/16	2.48352e−004	2.83989e−004	1.2817e−006	5.22506e−005
1/32	3.12249e−005	3.52665e−005	1.58824e−007	6.37922e−006
1/48	9.26867e−006	1.04260e−005	4.69746e−008	1.87560e−006
Order	2.9767	3.0182	3.0561	3.0640

Table 2
The convergent results for two-phase MHD model at $t_n = 1$, $\tau = 4h^2$.

h	$\|\varphi - \varphi_h^n\|_{H^1}$	$\|w - w_h^n\|_{H^1}$	$\|\boldsymbol{u} - \boldsymbol{u}_h^n\|_{H^1}$	$\|\boldsymbol{B} - \boldsymbol{B}_h^n\|_{H^1}$	$\|p - p_h^n\|_{L^2}$
1/4	1.61589e−001	1.62720e−001	3.03279e−003	9.85952e−002	2.03051e−002
1/8	4.51858e−002	4.53299e−002	7.01334e−004	2.55171e−002	5.69076e−003
1/16	1.16641e−002	1.16925e−002	1.76829e−004	6.45235e−003	1.45523e−003
1/32	2.94311e−003	2.94975e−003	4.44182e−005	1.61944e−003	3.65721e−004
1/48	1.31072e−003	1.31363e−003	1.97608e−005	7.20491e−004	1.62698e−004
Order	1.9431	1.9448	2.0188	1.9811	1.9491

6.3. Two-phase hartmann flows

Hartmann flows are the MHD version of the classical Poiseuille flows [15]. In this subsection, we consider two-phase Hartmann flows which describe the internal flow of two immiscible, incompressible and electrically conducting fluids between the parallel insulated and steady plates in the presence of a transverse magnetic field $\boldsymbol{B}^d = (0, B)$. The initial phase field is given in Fig. 3a. The red part of the figure stands for one fluid with the



(a) t=0 (b) t=1 (c) t=2 (d) t=4

(e) t=5 (f) t=10 (g) t=15 (h) t=20

Fig. 3. Phase field φ at different time t for $\alpha = 0.05$.

viscosity η_1 and electric conductivity σ_1; the blue part represents another fluid with the viscosity η_2 and electric conductivity σ_2. By introducing a characteristic velocity U, a characteristic magnetic field B, a characteristic length L, and non-dimensional variables $\tilde{x} = \frac{x}{L}, \tilde{t} = \frac{tU}{L}, \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \tilde{p} = \frac{p}{\rho U^2}, \tilde{\boldsymbol{B}} = \frac{\boldsymbol{B}}{B}, \tilde{\boldsymbol{B}}^d = \frac{\boldsymbol{B}^d}{B}, R_{e_i} = \frac{\rho U L}{\eta_i}, s = \frac{B^2}{\mu\rho U^2}, R_{m_i} = LU\mu\sigma_i, \tilde{\gamma} = \frac{\gamma}{UL^3}, \tilde{\lambda} = \frac{\lambda}{\rho U^2 L^2}, \tilde{\varepsilon} = \frac{\varepsilon}{L}, \tilde{w} = L^2 w$, two-phase MHD model (2.3a)–(2.3f) and the numerical scheme (4.3a)–(4.3f) can be nondimensionalized correspondingly. The boundary conditions are

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, & \text{on } y = \pm 1, \\ \dfrac{2}{R_{e_i}} D(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} = -p_d \boldsymbol{n}, & \text{on } x = 0, L_0, \\ \boldsymbol{B} \times \boldsymbol{n} = \boldsymbol{B}^d \times \boldsymbol{n}, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where R_{e_i} and R_{m_i} are the fluid Reynolds numbers and magnetic Reynolds numbers of the fluid i ($i = 1, 2$), and $\partial\Omega$ is the boundary of $\Omega = (0, L_0) \times (-1, 1)$.

In the numerical test, we apply the nondimensional form of the scheme (4.3a)–(4.3f) with the boundary conditions (6.1) to simulate two-phase Hartmann flows by taking the width ε of interface, mesh size h and time-step τ are small enough. Choose $\varepsilon = 0.01$, $h = 0.01$, $\tau = 0.01$ and $\lambda = 0.01$, $\gamma = 100$, $s = 1$, $T = 20$, $L_0 = 2$. Taking $R_{e_1} = 20$ and $R_{e_2} = R_{m_1} = R_{m_2} = 1$, the evolutions of the phase field are displayed in Fig. 3. From the figure, we observe that the zero level set of the phase field changes from the initial curve interface to a straight line interface and reaches steady state finally. The phenomenon can be explained by the fact that two-phase Hartmann flows are laminar.

In the following, we compare the numerical solutions with the analytical ones for the velocity and magnetic field of two-phase Hartmann flows. The domain of two-phase Hartmann flows in the steady state is illustrated in Fig. 4. The flow of the fluids is driven by the gradient of a pressure p_d and is laminar. The velocity, magnetic field and shear stress are continuous across the interface $y = 0$, that is,

$$\begin{cases} u_1 = u_2, & \dfrac{\mathrm{d}}{\mathrm{d}y} u_1 = \dfrac{1}{\alpha} \dfrac{\mathrm{d}}{\mathrm{d}y} u_2, & \text{on } y = 0, \\ b_1 = b_2, & \dfrac{\mathrm{d}}{\mathrm{d}y} b_1 = \beta \dfrac{\mathrm{d}}{\mathrm{d}y} b_2, & \text{on } y = 0, \end{cases} \tag{6.2}$$

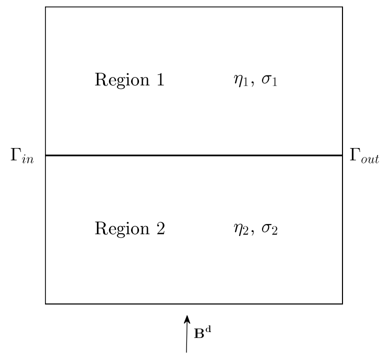


Fig. 4. The region of two-phase Hartmann flows.

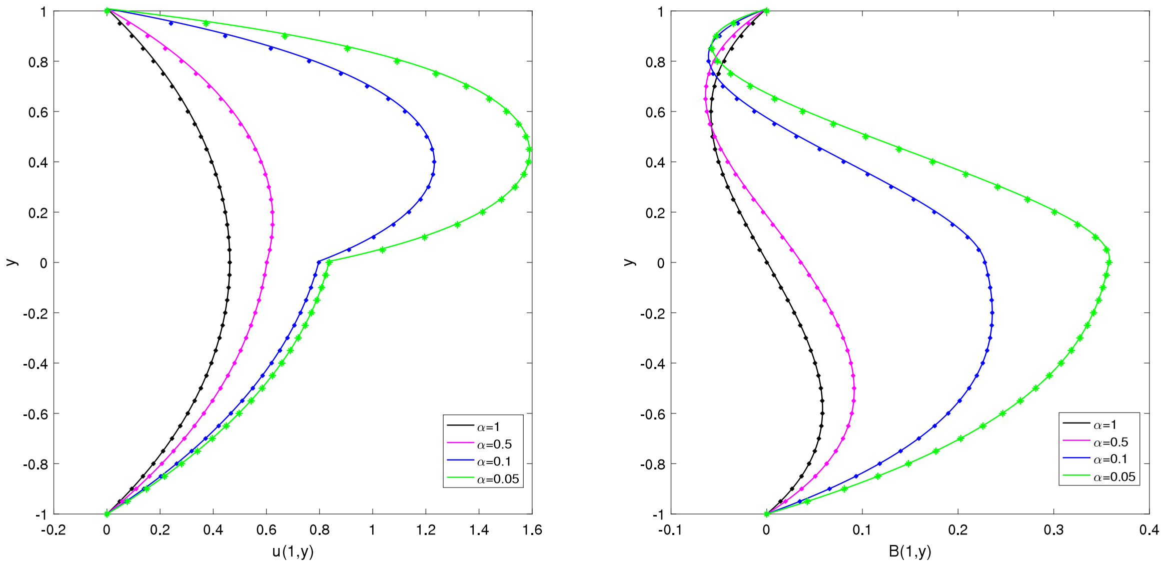
where $\alpha = \frac{\eta_1}{\eta_2} = \frac{R_{e_2}}{R_{e_1}}$, $\beta = \frac{\sigma_1}{\sigma_2} = \frac{R_{m_1}}{R_{m_2}}$. Based on the above information, the exact solution to two-phase Hartmann flows has the following form: $\boldsymbol{u} = (u_i(y), 0)$, $\boldsymbol{B} = (b_i(y), 1)$ in region i, denoted by Ω_i, (see [101])

$$
\begin{aligned}
&u_i(y) = D_{1i}\cosh(H_{a_i}y) + D_{2i}\sinh(H_{a_i}y) + F_i, \qquad p_i(x, y) = -Gx - \frac{s\big(b_i(y)\big)^2}{2} + p_0 = p_d,\\
&b_i(y) = -\frac{R_{m_i}}{H_{a_i}}\big[D_{1i}\sinh(H_{a_i}y) + D_{2i}\cosh(H_{a_i}y)\big] + Q_{1i}y + Q_{2i},
\end{aligned}
$$

where Hartmann numbers are denoted by $H_{a_i} := R_{e_i}R_{m_i}s$ $(i = 1, 2)$. Therefore, plugging the above identities into the nondimensional form of (2.1a)–(2.1c) and the boundary and interface conditions (6.1)–(6.2), undetermined coefficients can be obtained

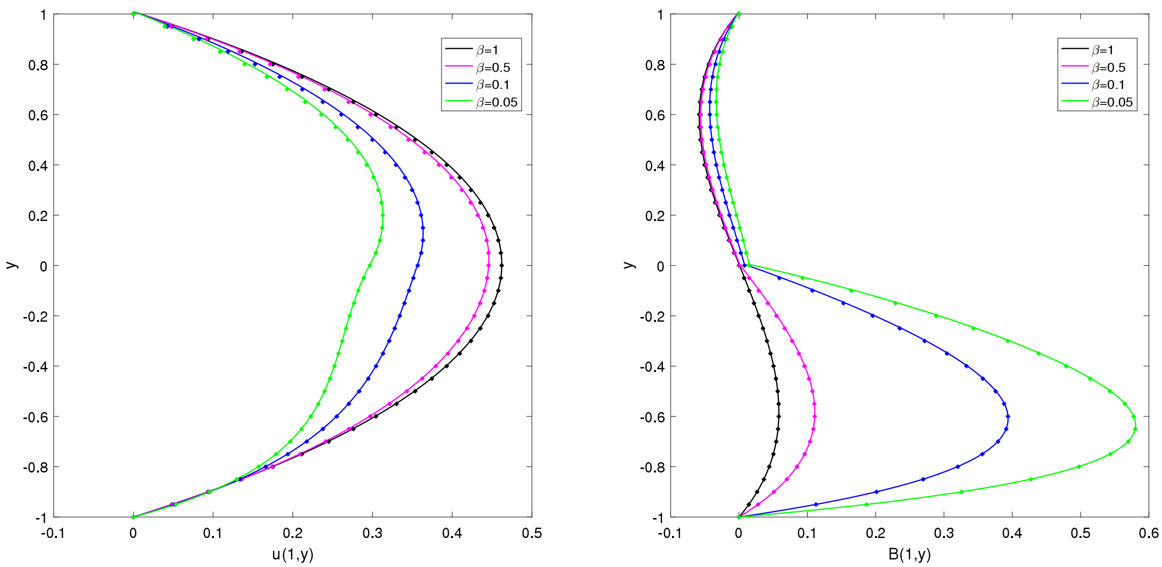
$$
\begin{aligned}
&Q_{11} = Q_{12} = -\frac{G}{s},\\
&a_{12} = \sinh(H_{a_1}) + \frac{\alpha H_{a_1}}{H_{a_2}}\sinh(H_{a_2}), && a_{11} = \cosh(H_{a_1}) - 1 - \frac{R_{m_1}}{\beta R_{m_2}}\big(\cosh(H_{a_2}) - 1\big),\\
&k_1 = \frac{1}{R_{m_2}}\left(-\frac{Q_{11}}{\beta} + Q_{12}\right)\big(\cosh(H_{a_2}) - 1\big), && k_2 = -Q_{11} - Q_{12} + \frac{1}{H_{a_2}}\left(-\frac{Q_{11}}{\beta} + Q_{12}\right)\sinh(H_{a_2}),\\
&a_{21} = -\frac{R_{m_1}}{H_{a_1}}\sinh(H_{a_1}) - \frac{R_{m_1}}{\beta H_{a_2}}\sinh(H_{a_2}), && a_{22} = -\frac{R_{m_1}}{H_{a_1}}\big(\cosh(H_{a_1}) - 1\big) + \frac{\alpha R_{m_2}H_{a_1}}{(H_{a_2})^2}\big(\cosh(H_{a_2}) - 1\big),\\
&D_{11} = \frac{k_1a_{22} - k_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, && D_{21} = \frac{k_1a_{21} - k_2a_{11}}{a_{12}a_{21} - a_{22}a_{11}},\\
&D_{12} = \frac{1}{R_{m_2}}\left(\frac{1}{\beta}(R_{m_1}D_{11} - Q_{11}) + Q_{12}\right), && D_{22} = \frac{\alpha H_{a_1}}{H_{a_2}}D_{21},\\
&F_1 = -D_{11}\cosh(H_{a_1}) - D_{21}\sinh(H_{a_1}), && F_2 = D_{11} - D_{12} + F_1,\\
&Q_{22} = -\frac{R_{m_1}}{H_{a_1}}D_{21} + \frac{R_{m_2}}{H_{a_2}}D_{22} + Q_{21}, && Q_{21} = \frac{R_{m_1}}{H_{a_1}}\big(D_{11}\sinh(H_{a_1}) + D_{21}\cosh(H_{a_1})\big) - Q_{11}.
\end{aligned}
$$

The effect of the ratio of viscosity α of fluids on the velocity and magnetic field is explored. Taking $\alpha = 1, 0.5, 0.1, 0.05$ and fixing $\beta = 1$, $R_{e_2} = R_{m_2} = 1$, the numerical solutions at $t_n = 20$ are in accordance with the exact solutions which are shown in Fig. 5. As the ratio of viscosity decreases, the velocity in region 1 changes greatly.



(a) velocity (b) magnetic field

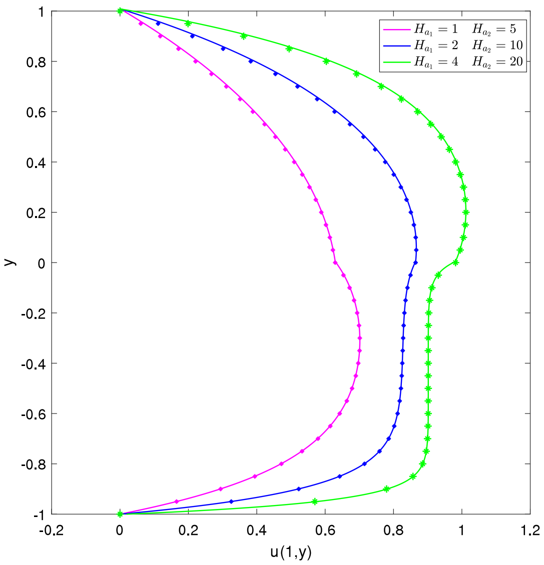
Fig. 5. Horizontal component of velocity and magnetic field along $x = 1$ for different ratios of viscosity, computed (points) and theoretical (lines).



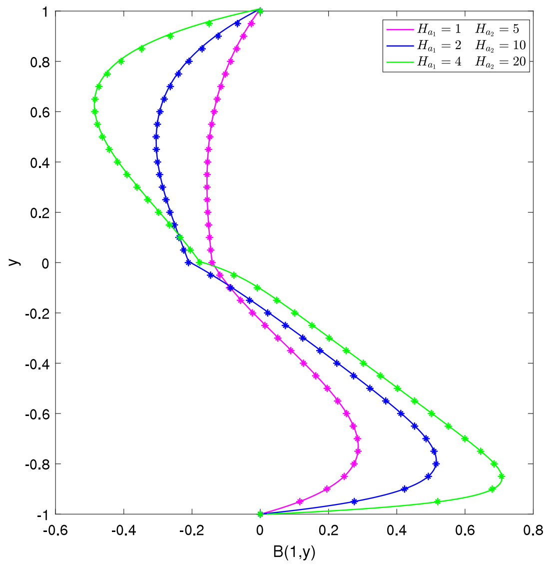
(a) velocity (b) magnetic field

Fig. 6. Horizontal component of velocity and magnetic field along $x = 1$ for different ratios of electric conductivity, computed (points) and theoretical (lines).

Next, we study the effect of the ratio of electric conductivity β of fluids on the velocity and magnetic field. Fixing $\alpha = 1$ and $R_{e_1} = R_{m_1} = 1$, Fig. 6 shows the numerical solutions at $t_n = 20$ coincide with the analytical ones for $\beta = 1, 0.5, 0.1, 0.05$. With the decrease of the ratio of electric conductivity, the induced magnetic field in region 2 becomes greater.



(a) velocity



(b) magnetic field

Fig. 7. Horizontal component of velocity and magnetic field along $x = 1$ for different Hartmann numbers, computed (points) and theoretical (lines).

Finally, we study the effect of Hartmann numbers on the velocity and magnetic field. Fixing $\alpha = 5, \beta = 0.2$, we take the following Hartmann numbers in regions 1 and 2

$$H_{a_1} = 1\ (R_{e_1} = 1, R_{m_1} = 1), H_{a_2} = 5 \quad (R_{e_2} = 5, \quad R_{m_2} = 5);$$

$$H_{a_1} = 2\ (R_{e_1} = 2, R_{m_1} = 2), H_{a_2} = 10\ (R_{e_2} = 10, R_{m_2} = 10);$$

$$H_{a_1} = 4\ (R_{e_1} = 4, R_{m_1} = 4), H_{a_2} = 20\ (R_{e_2} = 20, R_{m_2} = 20).$$

With the increase of the Hartmann number, the velocity profile becomes flatter and velocity gradient near the plates becomes steeper, as shown in Fig. 7.

7. Conclusion

In this paper, we proposed a diffuse-interface Cahn–Hilliard–MHD model to govern the two-phase MHD flows. The model is based on incompressible MHD equations and Cahn–Hilliard phase field model. A semi-implicit energy stable finite element method is proposed for solving this new model. The existence of weak solutions for this new model and the convergence of the numerical scheme are rigorously analyzed. Numerical examples are provided to validate the proposed model, numerical method, and theory.

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