LOSS OF REGULARITY OF SOLUTIONS OF THE LIGHTHILL PROBLEM FOR SHOCK DIFFRACTION FOR POTENTIAL FLOW

GUI-QIANG CHEN, MIKHAIL FELDMAN, JINGCHEN HU, AND WEI XIANG

Abstract. We are concerned with the suitability of the main models of compressible fluid dynamics for the Lighthill problem for shock diffraction by a convex corned wedge, by studying the regularity of solutions of the problem, which can be formulated as a free boundary problem. In this paper, we prove that there is no regular solution that is subsonic up to the wedge corner for potential flow. This indicates that, if the solution is subsonic at the wedge corner, at least a characteristic discontinuity (vortex sheet or entropy wave) is expected to be generated, which is consistent with the experimental and computational results. Therefore, the potential flow equation is not suitable for the Lighthill problem so that the compressible Euler system must be considered. In order to achieve the non-existence result, a weak maximum principle for the solution is established, and several other mathematical techniques are developed. The methods and techniques developed here are also useful to the other problems with similar difficulties.

1. Introduction

We are concerned with the regularity of solutions of the Lighthill problem for shock diffraction by a two-dimensional convex cornered wedge, which is not only a longstanding open problem in fluid mechanics but also fundamental in the mathematical theory of multidimensional conservation laws, since the shock diffraction configurations are fundamental for the local structure of general entropy solutions. The Lighthill problem can be formulated as a free boundary problem.

The main objective of this paper is to address the fundamental issue: which of the main models of compressible fluid dynamics is suitable for the shock diffraction problem. There are two main models: the compressible Euler system and the potential flow equation. We prove that there is no regular solution of the shock diffraction problem in the framework of potential flow. In fact, our results show the limitation of the potential flow equation, since solutions of this equation have a high regularity, so that it is too rigid for the Lighthill problem. This shows that the Euler system may be more stable as it allows nonregular solutions.
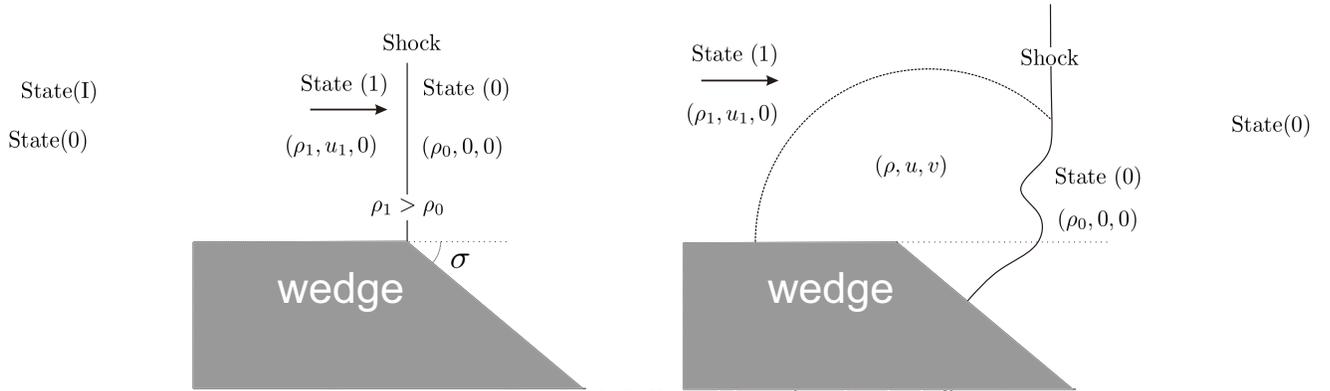
FIGURE 1.1. The Lighthill problem for shock diffraction

The Euler equations for polytropic potential flow consist of

$$\partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \nabla_{\mathbf{x}} \Phi) = 0 \tag{1.1}$$

with Bernoulli's law:

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, ρ is the density, Φ is the velocity potential such that $\nabla_{\mathbf{x}} \Phi =: (u, v) = \mathbf{v}$ is the flow velocity, and B_0 is the Bernoulli constant determined by the incoming flow and/or boundary conditions. For a polytropic gas, by scaling,

$$c^2(\rho) = \rho^{\gamma - 1}, \quad i(\rho) = \frac{\rho^{\gamma - 1} - 1}{\gamma - 1}, \qquad \gamma > 1,$$

where $c(\rho)$ is the sound speed. For the isothermal case, $\gamma = 1$,

$$p(\rho) = \rho, \qquad c^2(\rho) = 1, \qquad i(\rho) = ln\rho$$

as the limiting case when $\gamma \to 1$. In this paper, we focus mainly on the case $\gamma > 1$, since a similar argument works when $\gamma = 1$.

As shown in Fig. 1.1, we consider two piecewise constant Riemann data with the left state, State (1): $(\rho_1, u_1, 0), u_1 > 0$, and the right state, State (0): $(\rho_0, 0, 0)$, separated by a vertical shock S_0 and above the wedge with the corner angle $\theta_w = \pi - \sigma$, where $\sigma \in (0, \pi)$. As this incident shock passes through the wedge, the incident shock interacts with the sonic circle and becomes a transonic shock. On the other hand, physical observation and numerical analysis [26, 27] indicate that, when a shock is diffracted by a convex cornered wedge, at least a characteristic discontinuity (vortex sheet or entropy wave) should be generated. Mathematically, to confirm this phenomenon, one needs to show the nonexistence of regular solutions of the Lighthill problem. Therefore, we are interested in the question of whether any solution of the Lighthill problem, governed by the potential flow equation, is irregular at the wedge corner; that is, a regular solution, which is Lipschitz at the corner, does not exist, where the exact notion of regular solutions will be given in §2.3.

In this paper, we prove the nonexistence of regular solutions to this Riemann data when the initial left state $(\rho_1, u_1, 0)$ is subsonic, *i.e.*, $u_1 < c_1$. Then the pseudo-velocity

$(U, V) = (u_1, 0)$ at the origin is pseudosubsonic, and the degenerate boundary is the sonic circle centered at $(u_1, 0)$ with radius c_1. One of the main ingredients in our analysis is to develop a weak maximum principle at the wedge corner for the velocity in a special direction by the integration method. It is well known that, if a regular solution is assumed to have a C^1-velocity at the wedge corner, then we can directly apply Hopf's maximum principle to the directional derivative of the potential function to reach a contradiction. However, it does not apply directly to the solution that is not C^2-continuous. Therefore, it is the key point in §3 to develop the weak maximum principle without the C^2-continuity of the solution.

The mathematical study of the shock diffraction problem dates back to the 1950s by the work of Lighthill [21, 22] via asymptotic analysis, which is now called the Lighthill problem; also see Bargman [2], Fletcher, Weimer, and Bleakney [17], and Fletcher, Taub, and Bleakney [16] via experimental analysis, as well as Courant, and Friedrichs [12] and Whitham [28]. To date, all efforts for rigorous mathematical analysis of the Lighthill problem have focused on some simplified models. For one of these models, the nonlinear wave system, Kim [19] first studied this problem for the right wedge angle with an additional physical assumption that the transonic shock does not collide with the sonic circle of the right state. More recently, in Chen, Deng, and Xiang [7], this assumption was removed, and the existence and optimal regularity of shock diffraction configurations were established for all angles of the convex wedge via a different approach.

A closely related problem, shock reflection-diffraction by concave cornered wedges, has been systematically analyzed in Chen, and Feldman [8, 9], Bae, Chen, and Feldman [3] and Chen, Feldman, and Xiang [10, 11], where the existence and uniqueness of regular shock reflection-diffraction configurations has been established up to the detachment wedge angle. For the nonsymmetric case, the non-existence of regular solutions has been shown in Feldman, and Hu [15] when the wedge angle is sufficiently close to π. The Prandtl-Meyer reflection for supersonic potential flow impinging onto a solid wedge has also been analyzed first in Elling, and Liu [13] and, most recently, in Bae, Chen, and Feldman [4, 5] for the general case. For other related references, we refer the reader to Serre [25] for Chaplygin gas, Canic, Keyfitz, and Kim [6] for the nonlinear wave system, and Zheng [29] for the pressure-gradient system. See also [14, 24] for steady flow passing through a wedge.

The organization of this paper is as follows: In §2, we formulate the Lighthill problem as a free boundary problem for a nonlinear equation of mixed elliptic-hyperbolic type, prove the important fact that the speed of incident shock is subsonic and the incident shock hits the sonic circle in the first quadrant, and then introduce the notion of regular solutions of the free boundary problem and state the main theorem. In §3, we prove the monotonicity property of regular solutions with respect to a special direction. This means that, if the regular solution exists, the speed along some direction cannot achieve the negative minimum anywhere in $\overline{\Omega}$. Then we reach a contradiction in §3 from the monotonicity property, actually via a weak version of Hopf's maximum principle. In Appendix A, we prove that a solution of some type of linear elliptic equations, which is only assumed to be L^∞ at the wedge corner, is actually continuous.

2. Mathematical formulation of the Lighthill problem

In this section, we first formulate the Lighthill problem as an initial-boundary value problem (Problem 2.1) for the potential flow equation, then reduce it to a boundary value problem (Problem 2.2) and further to a free boundary problem (Problem 2.3) for a nonlinear equation of mixed elliptic-hyperbolic type, prove that the speed of incident shock is subsonic and the incident shock hits the sonic circle in the first quadrant, and finally introduce the notion of regular solutions of the free boundary problem and state the main theorem.

2.1. **The Lighthill problem.** When a plane shock in the $(t, \mathbf{x})$-coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left state $(\rho, u, v) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0)$ satisfying $u_1 > 0$ and $\rho_0 < \rho_1$, passes a wedge

$$W := \{(x_1, x_2) \ : \ x_2 < 0, \ x_1 < x_2 \cot \theta_{\mathrm{w}}\}$$

stepping down, the shock diffraction phenomenon occurs. Mathematically, this problem can be formulated as the following initial-boundary value problem.
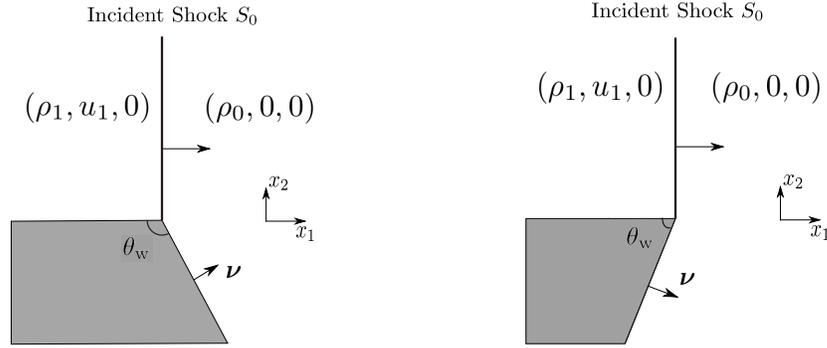

Figure 2.1. Initial-boundary value problem

Problem 2.1 (initial-boundary value problem) (see Fig. 2.1). Seek a solution of system (1.1)–(1.2) with the initial condition at $t = 0$,

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_1, u_1 x_1) & \text{in } \{x_1 < 0, x_2 > 0\}, \\ (\rho_0, 0) & \text{in } \mathbb{R}^2 \setminus (\overline{W} \cup \{x_1 \leq 0, x_2 \geq 0\}), \end{cases} \tag{2.1}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla \Phi \cdot \boldsymbol{\nu} = 0, \tag{2.2}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to ∂W.

The initial-boundary value problem (1.1)–(1.2) and (2.1)–(2.2) is invariant under the self-similar scaling

$$(t, \mathbf{x}) \to (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \to (\rho, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0.$$

Thus, we seek self-similar solutions with the form

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\phi(\xi, \eta) \qquad \text{for } (\xi, \eta) = \frac{\mathbf{x}}{t}, \tag{2.3}$$

where $\varphi := \phi - \frac{\xi^2 + \eta^2}{2}$ is called a pseudovelocity potential with $(\varphi_\xi, \varphi_\eta) = (u - \xi, v - \eta) = (U, V)$ as the corresponding pseudovelocity. In the self-similar plane, the domain outside the wedge is

$$\Lambda := \mathbb{R}^2 \setminus \overline{W}. \tag{2.4}$$

Then the pseudovelocity potential function φ satisfies the following equation for self-similar solutions:

$$\text{div}(\rho D\varphi) + 2\rho = 0 \tag{2.5}$$

with

$$\frac{1}{2}|D\varphi|^2 + \varphi + \frac{c^2}{\gamma - 1} = \frac{c_0^2}{\gamma - 1}, \tag{2.6}$$

where the divergence div and gradient D are with respect to the self-similar variables (ξ, η) and $c_0 = c(\rho_0)$. Therefore, the potential function φ is governed by the following second-order potential flow equation:

$$\text{div}(\rho(D\varphi, \varphi)D\varphi) + 2\rho(D\varphi, \varphi) = 0 \tag{2.7}$$

with

$$\rho(D\varphi, \varphi) = \left(c_0^2 - (\gamma - 1)\left(\varphi + \frac{1}{2}|D\varphi|^2\right)\right)^{\frac{1}{\gamma - 1}}. \tag{2.8}$$

Equation (2.7) is of mixed hyperbolic-elliptic type: It is elliptic if and only if $|D\varphi| < c(D\varphi, \varphi) := \rho(D\varphi, \varphi)^{\frac{\gamma - 1}{2}}$, which is equivalent to

$$|D\varphi| < c_\star(\varphi, \gamma) := \sqrt{\frac{2}{\gamma + 1}\left(c_0^2 - (\gamma - 1)\varphi\right)}. \tag{2.9}$$

Since the problem involves shock waves, the solutions of (2.7)–(2.8) have to be understood as weak solutions in distributional sense. Based on the argument in §2.1 in [8], for a piecewise C^1 solution φ separated by a shock S, if $\varphi \in W^{1,\infty}_{\text{loc}}(\Lambda)$, it is easy to verify that φ satisfies (2.7)–(2.8) in the distributional sense if and only if it is a classic solution of (2.7)–(2.8) in each smooth subregion and satisfies the following Rankine–Hugoniot conditions across S:

$$[\rho(D\varphi, \varphi)D\varphi \cdot \boldsymbol{\nu}]_S = 0, \tag{2.10}$$

$$[\varphi]_S = 0, \tag{2.11}$$

where $[w]_S$ denote the difference between the right and left traces of quantity w along S.

In fact, a discontinuity of $D\varphi$ satisfying the Rankine–Hugoniot conditions (2.10)–(2.11) is called a shock if it satisfies the following physical entropy condition: *The density function ρ increases across a shock in the pseudoflow direction.*

2.2. **Location of the incident shock and the boundary value problem in the self-similar coordinates.** Consider the left state (1): $(\rho, u, v) = (\rho_1, u_1, 0)$ with $\rho_1 > 0$ and $u_1 > 0$, and the right state (0): $(\rho, u, v) = (\rho_0, 0, 0)$ with $\rho_1 > \rho_0 > 0$ such that the entropy condition and the Rankine–Hugoniot conditions are satisfied on the incident shock separating these two states.

We note that, from (2.6), the potentials of states (1) and (0) are

$$\varphi_1 = -\frac{\xi^2 + \eta^2}{2} + u_1\xi + \frac{c_0^2 - c_1^2}{\gamma - 1} - \frac{1}{2}u_1^2, \qquad \varphi_0 = -\frac{\xi^2 + \eta^2}{2}. \tag{2.12}$$

From (2.11), $[D\varphi]_S$ is the normal direction to the shock. Then we have the following two equations as the Rankine-Hugoniot conditions along the flat shock L separating state (0) from a constant state (u, v, ρ) with potential φ:

$$u\big(\rho(u - \xi) + \rho_0\xi\big) + v\big(\rho(v - \eta) + \rho_0\eta\big) = 0 \tag{2.13}$$

and

$$\varphi = \varphi_0. \tag{2.14}$$

Thus, on the incident shock separating states (1) and (2), equations (2.13)–(2.14) hold with $(\rho, u, v) = (\rho_1, u_1, 0)$ and $\varphi = \varphi_1$. From (2.14), we find that the incident shock is line $\{\xi = \xi_1\}$ for $\xi_1 = \frac{1}{2}u_1 - \frac{c_0^2 - c_1^2}{u_1(\gamma - 1)}$. From (2.13) with $(\rho, u, v, \xi) = (\rho_1, u_1, 0, \xi_1)$, we have

$$0 < u_1 = \frac{\rho_1 - \rho_0}{\rho_1}\xi_1 < \xi_1. \tag{2.15}$$

Then (2.13)–(2.14) imply

$$\xi_1 = \sqrt{\frac{2\rho_1^2(c_1^2 - c_0^2)}{(\gamma - 1)(\rho_1^2 - \rho_0^2)}} > 0. \tag{2.16}$$

We now show that the incident shock interacts with the sonic circle of the left state (1) through the following relation:

$$0 < \xi_1 - u_1 < c_1. \tag{2.17}$$

In fact, the positivity follows directly from (2.15). From (2.15)–(2.16), we have

$$c_1^2 - (\xi_1 - u_1)^2 = c_1^2 - \frac{2\rho_0^2(c_1^2 - c_0^2)}{(\gamma - 1)(\rho_1^2 - \rho_0^2)} = \frac{(\gamma - 1)c_1^2\rho_1^2 - (\gamma + 1)\rho_0^2 c_1^2 + 2\rho_0^2 c_0^2}{(\gamma - 1)(\rho_1^2 - \rho_0^2)}.$$

Since $c^2 = \rho^{\gamma - 1}$, it suffices for the second inequality in (2.17) to prove that

$$f(s) := 2s^{\gamma + 1} - (\gamma + 1)s^2 + (\gamma - 1) > 0 \qquad \text{for } s = \tfrac{\rho_0}{\rho_1} \in (0, 1).$$

This can be seen by the fact that $f'(s) < 0$ and $f(1) = 0$. Then we have the following lemma.

Lemma 2.1. *Consider the left state $(\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0)$ with $\rho_1 > \rho_0 > 0$ and $u_1 > 0$. Then the location of the incident shock ξ_1 satisfies (2.17), and angle θ_1 determined by $\tan\theta_1 = \frac{\eta_1}{\xi_1 - u_1}$ for $P_1 = (\xi_1, \eta_1)$ must be in the interval $(0, \frac{\pi}{2})$.*

Therefore, the incident shock interacts with the pseudosonic circle of state (1) to become a transonic shock. Moreover, initial-boundary value problem (1.1)–(1.2) and (2.1)–(2.2) in the $(t, \mathbf{x})$-coordinates can be formulated as the following boundary value problem in the self-similar coordinates (ξ, η).

Problem 2.2 (boundary value problem). Seek a regular solution φ (see Definition 2.1 below) of (2.7) in the self-similar domain Λ with the slip boundary condition on the wedge boundary $\partial\Lambda$,

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0, \tag{2.18}$$

and the asymptotic boundary condition at infinity:

$$(\rho, \varphi) \to (\bar{\rho}, \bar{\varphi}) = \begin{cases} (\rho_1, \varphi_1) & \text{in } \{\xi < \xi_1, \ \eta \geq 0\}, \\ (\rho_0, \varphi_0) & \text{in } \{\xi > \xi_1, \ \eta \geq 0\} \cup (\{\eta < 0\} \cap \Lambda), \end{cases}$$

when $\xi^2 + \eta^2 \to \infty$ in the sense that

$$\lim_{R \to \infty} \|\varphi - \bar{\varphi}\|_{C^1(\Lambda \setminus B_R(0))} = 0.$$

The solution of Problem 2.2 can be shown to be the solution of Problem 2.1. Therefore, if the nonexistence of the regular solution of Problem 2.2 can be shown, then the nonexistence of the regular self-similar solution of Problem 2.1 follows.

2.3. **Regular solutions.** Since $\bar{\varphi}$ does not satisfy the slip boundary condition for $\xi \geq 0$, the solution must differ from state (1) in $\{\xi < \xi_1\} \cap \Lambda$ near the wedge corner, which forces the shock to be diffracted by the wedge. Inspired by the "no diffraction" solution when the wedge is flat ($\sigma = 0$) and based on Lemma 2.1, in an open domain Ω bounded by shock Γ_{shock}, the pseudosonic circle Γ_{sonic} of the left state (1) with center $(u_1, 0)$ and radius $c_1 > 0$, and the cornered wedge Γ_{wedge}, the regular solution is expected to be pseudosubsonic and smooth, to satisfy the slip boundary condition along the wedge, and to be $C^{1,1}$-continuous across the pseudosonic circle to become pseudosupersonic.

Let $\mathcal{C}$ be the corner of the wedge. Let $\phi := \varphi + \frac{1}{2}(\xi^2 + \eta^2)$. Note that $\phi_0 \equiv 0$. To show the nonexistence of regular solutions of Problem 2.2, we will show the nonexistence of regular solutions of the following free boundary problem.

Problem 2.3 (free boundary value problem) (see Fig. 2.2). Seek a function $\phi \in \text{Lip}(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \mathcal{C}) \cap C^3(\overline{\Omega} \setminus (\overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sonic}}} \cup \mathcal{C}))$ that satisfies

$$\left(c^2 - (\phi_\xi - \xi)^2\right)\phi_{\xi\xi} - 2(\phi_\xi - \xi)(\phi_\eta - \eta)\phi_{\xi\eta} + \left(c^2 - (\phi_\eta - \eta)^2\right)\phi_{\eta\eta} = 0 \quad \text{in } \Omega, \tag{2.19}$$

with the following boundary conditions.

(i) *Boundary conditions on the shock*:

$$\phi = 0 \qquad \text{on } \Gamma_{\text{shock}}; \tag{2.20}$$

$$\rho(D\varphi, \varphi)D\varphi \cdot \boldsymbol{\nu} = \rho_0 D\varphi_0 \cdot \boldsymbol{\nu} \qquad \text{on } \Gamma_{\text{shock}}, \tag{2.21}$$

where $\varphi := \phi - \frac{1}{2}(\xi^2 + \eta^2)$.
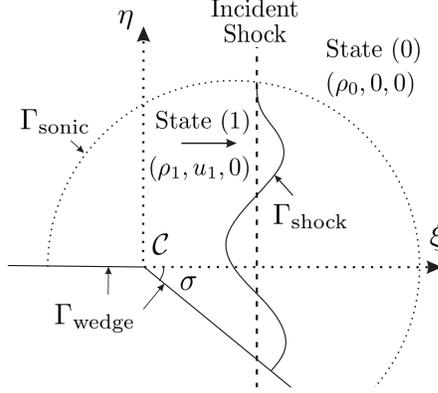
FIGURE 2.2. Free boundary value problem

(ii) *The sonic circle is a weak discontinuity*:

$$\phi = \phi_1, \quad D\phi = D\phi_1 = (u_1, 0) \qquad \text{on } \overline{\Gamma_{\text{sonic}}}.$$

(iii) *Slip boundary condition along the wedge*:

$$D\phi \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \Gamma_{\text{wedge}} \backslash \mathcal{C}. \tag{2.22}$$

Moreover, the function ϕ is subsonic in $\Omega \cup \Gamma_{\text{wedge}}^0$, *i.e.*,

$$c^2 - |D\varphi|^2 > 0 \qquad \text{in } \Omega \cup \Gamma_{\text{wedge}}^0, \tag{2.23}$$

where Γ_{wedge}^0 is the relative interior of Γ_{wedge}.

Since the sonic circle is a weak discontinuity, any solution of Problem 2.3 is a solution of Problem 2.2, provided that

$$\varphi = \begin{cases} \varphi_1 & \text{in } \{\xi < \xi_1, \ (\xi - u_1)^2 + \eta^2 \geq c_1^2\}, \\ \varphi_0 & \text{in } \Lambda \backslash (\{\xi < \xi_1, \ (\xi - u_1)^2 + \eta^2 \geq c_1^2\} \cup \Omega). \end{cases}$$

Now we define the notion of regular solutions of the Lighthill problem. We use the notation $\sigma = \pi - \theta_{\text{w}}$.

Definition 2.1. *A function ϕ is called a regular solution of the Lighthill problem if ϕ is a solution of Problem 2.3 with the following additional condition: There exists $\widetilde{\sigma} \in (\sigma, \pi)$ such that vector $\boldsymbol{\gamma} := (\sin\widetilde{\sigma}, \cos\widetilde{\sigma})$ is not tangential to Γ_{shock} at any point on Γ_{shock}.*

Remark 2.1. *Note that, by the C^1-regularity of ϕ in $\overline{\Omega} \backslash \mathcal{C}$, (2.20), and the condition of Definition 2.1, it follows that Γ_{shock} is a C^1-curve up to its ends. Also, in the proof of the main result of this paper, Theorem 2.1, the boundary condition (2.21) is not used, i.e. we use only (2.20) on Γ_{shock}.*

For the condition of Definition 2.1, we have the following remarks.

Remark 2.2. *When the shock is convex as observed in the experimental results, the additional condition in Definition 2.1 follows directly.*

Remark 2.3. *When the wedge is flat ($\sigma = 0$), there is "no diffraction" solution, i.e., φ consists of only two uniform states φ_0 and φ_1 separated by the incident shock. Then, when the wedge is almost flat (i.e., σ is very small), one expects that φ should be close to φ_1 in the sense of $C^1(\Omega)$. In this case, from the Dirichlet condition (2.20), the curved shock Γ_{shock} is a C^1-perturbation of the incident shock. This implies the condition of Definition 2.1.*

Remark 2.4. *If Γ_{shock} is a graph of a C^1-function of η and the wedge angle is obtuse (i.e., $\sigma < \frac{\pi}{2}$), then the additional condition in Definition 2.1 holds, since we can choose $\widetilde{\sigma} = \frac{\pi}{2}$.*

Now we can present our main theorem.

Theorem 2.1 (main theorem). *The Lighthill problem does not permit a regular solution in the sense of Definition 2.1.*

With the estimates and properties developed in §3 and Appendix A, the main theorem (Theorem 2.1) will be proved in §3.

Based on Remark 2.3, one can obtain the following corollary from Theorem 2.1.

Corollary 2.2. *The "no diffraction" solution is not stable in the sense of $C^1(\Omega)$ with respect to the wedge angle as $\sigma \to 0$.*

3. Proof of the main theorem — Theorem 2.1

In order to prove the main theorem, we need to derive several estimates on the derivatives of regular solutions in the sense of Definition 2.1, based on Lemma A.1. Let $r_0 := \frac{1}{2}\text{dist}(\mathcal{C}, \Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}})$.

Lemma 3.1. *Let ϕ be a regular solution in the sense of Definition 2.1. Assume that, for any $r \in (0, r_0)$,*

$$\|D\phi\|_{C(B_r(\mathcal{C}) \cap \Omega)} \leq \omega(r),$$

where $\omega(r)$ is a nondecreasing continuous function of r on $[0, r_0]$. Then, for any $r \in (0, r_0)$,

$$|D^2\phi| \leq C\frac{\omega(2r)}{r} \qquad \text{on } \partial B_r \cap \Omega,$$

where C depends only on the elliptic ratio λ and bound $\|D\phi\|_{0;B_{r_0}(\mathcal{C}) \cap \Omega}$.

Proof. For any fixed $r \in (0, r_0)$, we scale $B_r(\mathcal{C})$ to $B_2(\mathcal{C})$ by the change of coordinates as follows:

$$(x, y) := \frac{2}{r}(\xi, \eta), \qquad \psi(x, y) = \phi(\xi, \eta).$$

In the (x, y)-coordinates, ψ satisfies

$$\sum_{i,j=1}^{2} \tilde{a}_{ij}\psi_{ij} = 0 \qquad \text{in } B_2(\mathcal{C}) \backslash (B_1(\mathcal{C}) \cup W), \tag{3.1}$$

where $\tilde{a}_{ij}(x, y) = a_{ij}(\frac{r}{2}x, \frac{r}{2}y)$. Equation (3.1) is uniformly elliptic in $B_2(\mathcal{C}) \backslash (B_1(\mathcal{C}) \cup W)$. Then we can apply Nirenberg's estimate (cf. Chapter 12 of [18] or Nirenberg [23]) to obtain

$$\|D\psi\|_{C^\alpha(B_{7/4}(\mathcal{C}) \backslash (B_{5/4}(\mathcal{C}) \cup W))} \leq C\|D\psi\|_{L^\infty(B_2(\mathcal{C}))} \leq C\frac{r}{2}\|D\phi\|_{C(B_r(\mathcal{C}) \cap \Omega)} \leq \frac{C}{2}r\omega(r), \quad (3.2)$$

where C depends only on the elliptic ratio of (3.1). In the (x, y)-coordinates, (3.1) can be rewritten as

$$\left(c^2 - \left(\frac{\psi_x}{r} - rx\right)^2\right)\psi_{xx} - 2\left(\frac{\psi_x}{r} - rx\right)\left(\frac{\psi_y}{r} - ry\right)\psi_{xy} + \left(c^2 - \left(\frac{\psi_y}{r} - ry\right)^2\right)\psi_{yy} = 0,$$
$$(3.3)$$

where the sonic speed c can be represented as

$$c^2 = c_0^2 - (\gamma - 1)\left(\psi - \psi_x x - \psi_y y + \frac{\psi_x^2 + \psi_y^2}{2r^2}\right). \quad (3.4)$$

With (3.2)–(3.4), we obtain the estimate of the Hölder norm of $\tilde{a}_{ij}$ as

$$\|\tilde{a}_{ij}\|_{C^\alpha(B_{7/4}(\mathcal{C}) \backslash (B_{5/4}(\mathcal{C}) \cup W))} \leq C(\lambda, \frac{1}{r}\|D\psi\|_{C^\alpha(B_{7/4}(\mathcal{C}) \backslash (B_{5/4}(\mathcal{C}) \cup W))})$$
$$\leq C(\lambda, \|D\phi\|_{C(B_{r_0}(\mathcal{C}) \cap \Omega)}),$$

where $C(\lambda, \|D\phi\|_{C(B_{r_0}(\mathcal{C}) \cap \Omega)})$ depends only on the elliptic ratio λ and on the bound $\|D\phi\|_{C(B_{r_0}(\mathcal{C}) \cap \Omega)}$. Then we can apply the Schauder estimate to (3.1) to obtain

$$\|D^2\psi\|_{C(B_{13/8}(\mathcal{C}) \backslash (B_{11/8}(\mathcal{C}) \cup W))} \leq C\|D\psi\|_{C(B_2(\mathcal{C}) \backslash (B_1(\mathcal{C}) \cup W))} \leq C\omega(r)r,$$

where C is a universal constant that depends only on the elliptic ratio λ and bound $\|D\phi\|_{0;B_{r_0}(\mathcal{C}) \cap \Omega}$ and may be different at each occurrence. Scaling back to the (ξ, η)-coordinates, we have

$$\|D^2\phi\|_{C(\partial B_{3r/4})} \leq C\frac{\omega(r)}{r}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Based on Lemma 3.1 and Lemma A.1, we have the following lemma.

Lemma 3.2. *If ϕ is a regular solution in the sense of Definition 2.1, then*

$$\phi \in C^1(\overline{B_{r_0}(\mathcal{C}) \cap \Lambda}).$$

Proof. Let $v_1 = \phi_\eta$ and $v_2 = \sin\sigma\phi_\xi + \cos\sigma\phi_\eta$. Similar to (3.9)–(3.12), we can derive the equation and the boundary conditions of v_1 and v_2, respectively, near the wedge corner. The only difference is that $v_1 = 0$ on $\Gamma_{\text{wedge}} \cap \{\eta = 0\}$ and $v_2 = 0$ on $\Gamma_{\text{wedge}} \cap \{\eta < 0\}$.

Without loss of the generality, let us only consider the case for $w := v_1 = \phi_\eta$, since the argument for v_2 is similar. Then, in order to apply Lemma A.1, we must show

$$|b_i| \leq \frac{C}{r}, \quad (3.5)$$

where r is the distance to the wedge corner. It suffices to show

$$|D^2\phi| \le \frac{C}{r},$$

which can be achieved by Lemma 3.1 and (2.23). More precisely, for any $r \in (0, r_0)$,

$$|D^2\phi| \le \frac{M(\|D\phi\|_{C^0(\Omega)})}{r} \qquad \text{on } \partial B_r(\mathcal{C}) \cap \Omega.$$

By a straightforward calculation, we have

$$0 = \left(\phi_{\xi\xi} + \frac{2a_{12}}{a_{11}}\phi_{\xi\eta} + \frac{a_{22}}{a_{11}}\phi_{\eta\eta}\right)_\eta$$

$$= w_{\xi\xi} + \frac{2a_{12}}{a_{11}}w_{\xi\eta} + \frac{a_{22}}{a_{11}}w_{\eta\eta} + \left(\frac{2a_{12}}{a_{11}}\right)_\eta w_\xi + \left(\frac{a_{22}}{a_{11}}\right)_\eta w_\eta,$$

where $a_{11} = c^2 - (\phi_\xi - \xi)^2$, $a_{12} = -(\phi_\xi - \xi)(\phi_\eta - \eta)$, $a_{22} = c^2 - (\phi_\eta - \eta)^2$, and $|(\frac{2a_{12}}{a_{11}}, \frac{a_{22}}{a_{11}})_\eta| \le C|D^2\phi|$. Therefore, w satisfies the equation of form

$$\sum_{i,j=1}^2 a_{ij}w_{ij} + \sum_{i=1}^2 b_i w_i = 0$$

with the estimate that $|b_i(\xi, \eta)| \le \frac{C}{r}$. This completes the proof. $\square$

Lemma 3.3. *If ϕ is a regular solution in the sense of Definition 2.1, then $D\phi = (0,0)$ at the wedge corner.*

This lemma follows from Lemma 3.2, the boundary condition $\phi_{\boldsymbol{\nu}}$ on $\Gamma_{\text{wedge}} \backslash \mathcal{C}$, and the fact that the unit normal on $\Gamma_{\text{wedge}} \cap \{\eta = 0\}$ is different from the one on $\Gamma_{\text{wedge}} \cap \{\eta < 0\}$.

By Lemmas 3.2–3.3, there exists $f(r)$ that is a nondecreasing continuous function on $[0, 1]$ with $f(r) \to 0$ as $r \to 0$ such that, for any $r \in (0, r_0)$,

$$\|D\phi\|_{C(B_r(\mathcal{C}) \cap \Omega)} \le f(r). \tag{3.6}$$

Then, by Lemma 3.1, we have the following lemma.

Lemma 3.4. *If ϕ is a regular solution in the sense of Definition 2.1, then, for any $r \in (0, r_0)$,*

$$|D^2\phi| \le M(\|D\phi\|_{C(B_{r_0}(\mathcal{C}) \cap \Omega)})\frac{f(2r)}{r} \qquad \text{on } \partial B_r(\mathcal{C}) \cap \Omega, \tag{3.7}$$

where $f(r)$ is a nondecreasing continuous function on $[0, r_0)$ with $f(r) \to 0$ as $r \to 0$.

We now prove the following key estimate for the sign of a special directional velocity, which is the weak maximum principle for the directional velocity.

Lemma 3.5. *Let $\phi \in C^3(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C})) \cap C^1(\overline{\Omega})$ be a nonconstant solution of the elliptic equation*

$$a_{11}\phi_{\xi\xi} + 2a_{12}\phi_{\xi\eta} + a_{22}\phi_{\eta\eta} = 0 \qquad \text{in } \Omega, \tag{3.8}$$

where $a_{ij} \in C^1(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C})), i, j = 1, 2$. Assume that (3.8) *is uniformly elliptic near the wedge corner and strictly elliptic in* $\Omega \cup \Gamma^0_{\text{wedge}}$*, and* $\phi_{\boldsymbol{\nu}} := D\phi \cdot \boldsymbol{\nu} = 0$ *on* Γ_{wedge}*. Let* $\boldsymbol{\gamma} := (\sin \widetilde{\sigma}, \cos \widetilde{\sigma})$ *for* $\widetilde{\sigma} \in (\sigma, \pi)$*. If*

$$\phi_{\boldsymbol{\gamma}} := D\phi \cdot \boldsymbol{\gamma} \geq 0 \qquad on \ \Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}},$$

then

$$\phi_{\boldsymbol{\gamma}} > 0 \qquad in \ (\Omega \cup \Gamma_{\text{wedge}}) \setminus \mathcal{C}.$$

Proof. We divide the proof into three steps.

1. Let $w := \phi_{\boldsymbol{\gamma}}$. For any $\varepsilon > 0$, by assumption, we can see that $w \geq -\varepsilon$ in a neighborhood of $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}$, since $\phi \in C^1(\overline{\Omega} \setminus \mathcal{C})$. This ε can be arbitrarily small if the neighborhood is chosen to be very small.

Note that $a_{ij} \in C^1(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C}))$. We can differentiate (3.8) to obtain

$$0 = \left(\frac{a_{11}}{a_{22}} \phi_{\xi\xi} + \frac{2a_{12}}{a_{22}} \phi_{\xi\eta} + \phi_{\eta\eta} \right)_{\boldsymbol{\gamma}}$$

$$= \frac{a_{11}}{a_{22}} w_{\xi\xi} + \frac{2a_{12}}{a_{22}} w_{\xi\eta} + w_{\eta\eta} + \left(\frac{a_{11}}{a_{22}} \right)_{\boldsymbol{\gamma}} \phi_{\xi\xi} + \left(\frac{2a_{12}}{a_{22}} \right)_{\boldsymbol{\gamma}} \phi_{\xi\eta}. \tag{3.9}$$

Then, using again (3.8), we can write $\phi_{\xi\eta}$ and $\phi_{\xi\xi}$ as a linear combination of w_ξ and w_η. More precisely, we have

$$\begin{pmatrix} \phi_{\xi\xi} \\ \phi_{\xi\eta} \end{pmatrix} = J^{-1} \begin{pmatrix} \sin \widetilde{\sigma} - \frac{2a_{12}}{a_{22}} \cos \widetilde{\sigma} & -\cos \widetilde{\sigma} \\ \frac{a_{11}}{a_{22}} \cos \widetilde{\sigma} & \sin \widetilde{\sigma} \end{pmatrix} \begin{pmatrix} w_\xi \\ w_\eta \end{pmatrix}, \tag{3.10}$$

where $J = \sin^2 \widetilde{\sigma} - \frac{a_{12}}{a_{22}} \sin(2\widetilde{\sigma}) + \frac{a_{11}}{a_{22}} \cos^2 \widetilde{\sigma} > 0$ in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C})$, thanks to the ellipticity of (3.8).

Plugging (3.10) into (3.9), by a straightforward calculation, we find that w satisfies a linear strictly elliptic equation with coefficients locally bounded in Ω and without zeroth order term. Thus, w cannot achieve the local minimum anywhere in Ω by the maximum principle.

2. Now we show that w cannot achieve the local minimum on Γ_{wedge}.

First, $\phi_{\boldsymbol{\nu}} = \phi_\eta = 0$ on $\Gamma^+_{\text{wedge}} := \Gamma_{\text{wedge}} \cap \{\eta = 0\}$. Taking the tangential derivative along the wedge boundary, we have

$$\phi_{\xi\eta} = 0 \qquad on \ \Gamma^+_{\text{wedge}}.$$

Plugging it into (3.10) yields

$$w_{\boldsymbol{\beta}_+} = 0 \qquad on \ \Gamma^+_{\text{wedge}}, \tag{3.11}$$

where $\boldsymbol{\beta}_+ := (a_{11} \cos \widetilde{\sigma}, a_{22} \sin \widetilde{\sigma})$. Notice that $\widetilde{\sigma} \in (\sigma, \pi)$ so that

$$\boldsymbol{\beta}_+ \cdot \boldsymbol{\nu} = a_{22} \sin \widetilde{\sigma} \neq 0.$$

Next, on $\Gamma^-_{\text{wedge}} := \Gamma_{\text{wedge}} \cap \{\eta < 0\}$, the argument is similar. We rotate coordinates by angle σ clockwise so that, in the new coordinates, the boundary part Γ^-_{wedge} lies on the positive ξ-axis, i.e. $\boldsymbol{\nu} = (0, 1)$ on Γ^-_{wedge}. Equation (3.8) in the rotated coordinates has the

similar form with the new coefficients $\hat{a}_{ij}$, satisfying the ellipticity condition with the same constants. Vector $\boldsymbol{\gamma}$ in the rotated coordinates is $\boldsymbol{\gamma} := (\sin(\widetilde{\sigma} - \sigma), \cos(\widetilde{\sigma} - \sigma))$. Now we can repeat the previous calculation working the rotated coordinates and obtain

$$w_{\boldsymbol{\beta}_-} = 0 \qquad \text{on } \Gamma^-_{\text{wedge}}, \tag{3.12}$$

where $\boldsymbol{\beta}_- := (\hat{a}_{11} \cos(\widetilde{\sigma} - \sigma), \hat{a}_{22} \sin(\widetilde{\sigma} - \sigma))$ in these coordinates. Noting that $\widetilde{\sigma} - \sigma \in (0, \pi)$, we have

$$\boldsymbol{\beta}_- \cdot \boldsymbol{\nu} = \hat{a}_{22} \sin(\widetilde{\sigma} - \sigma) \neq 0 \qquad \text{on } \Gamma^-_{\text{wedge}}.$$

Therefore, the boundary conditions (3.11)–(3.12) are oblique. Then, by Hopf's lemma, we conclude that w cannot achieve its minimum on Γ_{wedge}, except $\mathcal{C}$.

3. By the $C^1(\overline{\Omega})$-regularity of ϕ, we see that $D\phi = 0$ at $\mathcal{C}$. This means that $w = 0$ at $\mathcal{C}$. Thus, we finally obtain

$$w \geq -\epsilon \qquad \text{in } \overline{\Omega}. \tag{3.13}$$

Let $\varepsilon \to 0$. Then we have

$$w \geq 0 \qquad \text{in } \overline{\Omega}.$$

Therefore, by the strong maximum principle, if ϕ is not a constant, then $w > 0$ in $(\Omega \cup \Gamma_{\text{wedge}}) \backslash \mathcal{C}$. This completes the proof. □

Based on Lemma 3.5, we have the following corollary for the regular solutions defined in Definition 2.1.

Corollary 3.6. *Let $\boldsymbol{\gamma}$ be as in Definition* 2.1 *for regular solutions. Then*

$$\phi_{\boldsymbol{\gamma}} > 0 \qquad \text{in } (\Omega \cup \Gamma_{\text{wedge}}) \backslash \mathcal{C}. \tag{3.14}$$

Proof. Equation (3.8) is strictly elliptic in $\Omega \cup \Gamma^0_{\text{wedge}}$ due to (2.23). We also note that Hopf's lemma can be applied at every point of Γ_{wedge} including the corner point $\mathcal{C}$ because the interior angle at $\mathcal{C}$ is $\pi + \sigma > \pi$, i.e., the interior ball condition holds at $\mathcal{C}$, and ϕ is C^1 up to $\mathcal{C}$ by Lemma 3.3. Thus, by the strong maximum principle and Hopf's lemma,

$$\sup_{\Omega} \phi = \sup_{\partial\Omega \backslash \Gamma_{\text{wedge}}} \phi = \sup_{\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}}} \phi.$$

On $\overline{\Gamma_{\text{sonic}}}$, we have $\phi = \phi_1$ by condition (ii) of Problem 2.3. Note that $\Omega \subset \{\xi < \xi_1\}$ as discussed at the beginning of §2.3, and thus $\Gamma_{\text{sonic}} \subset \{\xi < \xi_1\}$. Also, $\phi_1 < \phi_0 = 0$ in $\{\xi < \xi_1\}$. Thus $\phi < 0$ on Γ_{sonic}.

By the boundary condition (2.20), $\phi = 0$ on Γ_{shock}.

This shows that $\max_{\overline{\Omega}} \phi = 0$, and the maximum value $\phi = 0$ is attained at every point of Γ_{shock}. From this,

$$D\phi = -\phi_{\boldsymbol{\nu}} \boldsymbol{\nu} \qquad \text{and} \quad \phi_{\boldsymbol{\nu}} \leq 0 \qquad \text{on } \Gamma_{\text{shock}}. \tag{3.15}$$

Also, recall that Γ_{shock} is C^1 up to its endpoints by Remark 2.1, and $\phi \in C^1(\overline{\Omega})$ by conditions of Problem 2.3 and Lemma 3.2. Then, at the point A of intersection of $\overline{\Gamma_{\text{shock}}}$ with $\overline{\Gamma_{\text{sonic}}}$, we have $D\phi = D\phi_1$ by condition (ii) of Problem 2.3, which shows that the interior (for Ω) unit normal to Γ_{shock} at A is $\boldsymbol{\nu}(A) = \frac{D(\phi_0 - \phi_1)}{|D(\phi_0 - \phi_1)|} = (-1, 0)$, where we used
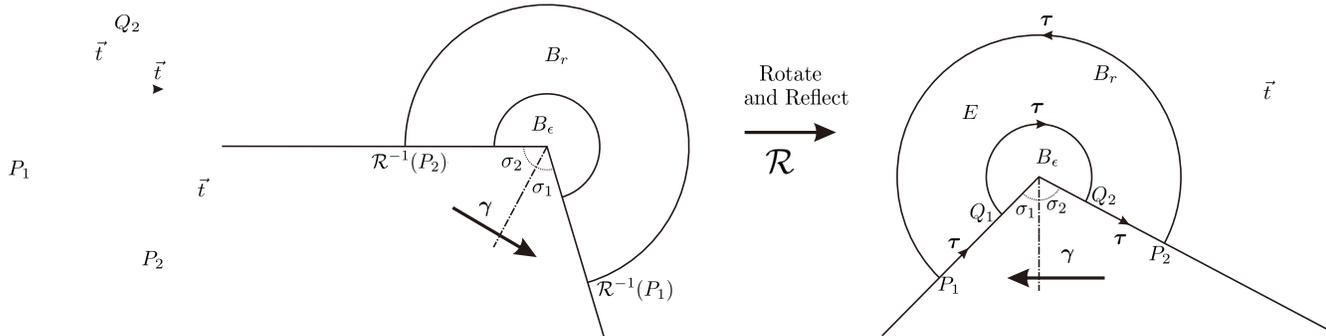
FIGURE 3.1. The potential function ϕ cannot be monotonic in the $\boldsymbol{\gamma}$ direction

(2.12) in the last equality. Thus $\boldsymbol{\nu}(A) \cdot \boldsymbol{\gamma} = -\sin\widetilde{\sigma} < 0$, where we have used that $\widetilde{\sigma} \in (0, \pi)$. Since Γ_{shock} is not tangential to $\boldsymbol{\gamma}$ at any point on Γ_{shock} and $\phi \in C^1(\overline{\Omega}\backslash\mathcal{C})$, we have

$$\boldsymbol{\nu} \cdot \boldsymbol{\gamma} < 0 \qquad \text{on } \Gamma_{\text{shock}}.$$

Combining this with (3.15), we obtain

$$w := \phi_{\boldsymbol{\gamma}} \geq 0 \qquad \text{on } \Gamma_{\text{shock}}.$$

On Γ_{sonic}, $w = (u_1, 0) \cdot \boldsymbol{\gamma} = u_1 \sin\widetilde{\sigma} > 0$.
Finally, since $\phi \in C^3(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C}))$, $a_{ij} \in C^1(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}}} \cup \overline{\Gamma_{\text{shock}}} \cup \mathcal{C}))$. Then Lemma 3.5 applies. $\qquad\square$

Based on these lemmas, we are now going to establish the main theorem, Theorem 2.1.

Proof of the main theorem — Theorem 2.1. We prove the main theorem by deriving a direct contradiction to (3.14).

Since the Euler equations are invariant with respect to the Galilean transformation, we rotate and reflect the coordinates such that $\boldsymbol{\gamma}$ in the new coordinates is the horizontal direction pointing to the left, *i.e.*, $\mathcal{R}_*\boldsymbol{\gamma} = -\partial_\xi$, where $\mathcal{R}$ is the corresponding rotation and reflection operator. After this transformation, ϕ still satisfies a uniform elliptic equation of second order in $(B_r(\mathcal{C})\cap\Omega)\backslash\mathcal{C}$ for any $r \in (0, r_0)$. For notational simplicity, we still use (ξ, η) as the coordinates after this transformation. Furthermore, we use notation $(u, v) = (\phi_\xi, \phi_\eta)$.

As before, let $w := \phi_{\boldsymbol{\gamma}}$. Then, as shown in Fig. 3.1, w defined on the left-hand side of Fig. 3.1 is transformed into $-u = -\phi_\xi$ corresponding to that on the right-hand side of Fig. 3.1. Moreover, the result from (3.14) in the old coordinates that $w \geq \delta$ on $\partial B_r(\mathcal{C}) \cap \Omega$ is transformed into

$$u \leq -\delta \qquad \text{on } \partial B_r(\mathcal{C}) \cap \Omega$$

in the new coordinates.

As shown in Fig. 3.1, the dashed line is defined to pass corner $\mathcal{C}$ and to be perpendicular to $\boldsymbol{\gamma}$. The angles between the dashed line and the two sides of Γ_{wedge} are σ_1 and σ_2, where both σ_1 and σ_2 belong to $(0, \pi - \sigma)$. In fact, $\tilde\sigma$ in Definition 2.1 equals to $\sigma + \sigma_1 = \pi - \sigma_2$.

Now we start to work in the new coordinates and derive a contradiction from an integral estimate of u.

Define
$$h = (u + \delta)^+.$$
Since $u < -\delta$ on $\partial B_r(\mathcal{C}) \cap \Omega$, we have
$$h = 0 \qquad \text{on } \partial B_r(\mathcal{C}) \cap \Omega.$$
Moreover, it is direct to see that
$$h \le \delta + \|D\phi\|_{C(\Omega)} \qquad \text{in } B_r(\mathcal{C}) \cap \Omega.$$
For any $\epsilon \in (0, \frac{r}{2})$, define domain E as
$$E := (B_r(\mathcal{C}) \backslash B_\epsilon(\mathcal{C})) \cap \Omega$$
with vertices
$$Q_1 = \partial B_\epsilon(\mathcal{C}) \cap \Gamma_{\text{wedge}} \cap \{\xi < 0\}, \qquad Q_2 = \partial B_\epsilon(\mathcal{C}) \cap \Gamma_{\text{wedge}} \cap \{\xi > 0\},$$
$$P_1 = \partial B_r(\mathcal{C}) \cap \Gamma_{\text{wedge}} \cap \{\xi < 0\}, \qquad P_2 = \partial B_r(\mathcal{C}) \cap \Gamma_{\text{wedge}} \cap \{\xi > 0\}.$$
Then define I as follows:
$$I = \int_{\{u > -\delta\} \cap E} \left(\frac{a_{11}}{a_{22}} u_\xi^2 + \frac{2a_{12}}{a_{22}} u_\xi u_\eta + u_\eta^2 \right) d\xi d\eta. \tag{3.16}$$
By a straightforward calculation, we have
$$I = \int_E \left(\frac{a_{11}}{a_{22}} u_\xi h_\xi + \frac{2a_{12}}{a_{22}} u_\eta h_\xi + u_\eta h_\eta \right) d\xi d\eta$$
$$= \int_E \left(\nabla \cdot \left(\frac{a_{11}}{a_{22}} u_\xi h + \frac{2a_{12}}{a_{22}} u_\eta h, u_\eta h \right) - \nabla \cdot \left(\frac{a_{11}}{a_{22}} u_\xi + \frac{2a_{12}}{a_{22}} u_\eta, u_\eta \right) h \right) d\xi d\eta$$
$$= -\int_E \nabla \cdot (v_\eta h, -v_\xi h) d\xi d\eta$$
$$= -\int_{\partial E} v_\tau h \, ds$$
$$= -\int_{\partial B_\epsilon(\mathcal{C}) \cap \Omega} v_\tau h \, ds - \int_{\overline{P_1 Q_1}} v_\tau h \, ds - \int_{\overline{Q_2 P_2}} v_\tau h \, ds,$$
where τ is the unit tangential vector of ∂E, with the direction illustrated in Fig. 3.1, and the line integral is respect to the arc length s. Here, for the third identity, we have used the equation that $\frac{a_{11}}{a_{22}} u_\xi + \frac{2a_{12}}{a_{22}} u_\eta = -v_\eta$, while, for the fifth identity, we have used the fact that $h = 0$ on $\partial B_r(\mathcal{C}) \cap \Omega$.

Using the slip boundary condition (2.22) on Γ_{wedge}, we have
$$v = \frac{u}{\tan \sigma_1} \qquad \text{on } \overline{P_1 Q_1}$$
and
$$v = -\frac{u}{\tan \sigma_2} \qquad \text{on } \overline{P_2 Q_2}.$$

Applying the relations above, we have

$$
\begin{aligned}
&- \int_{\overline{P_1 Q_1}} v_{\boldsymbol{\tau}} h \, ds - \int_{\overline{Q_2 P_2}} v_{\boldsymbol{\tau}} h \, ds \\
&= - \int_{\overline{P_1 Q_1}} \frac{h_{\boldsymbol{\tau}} h}{\tan \sigma_1} \, ds + \int_{\overline{Q_2 P_2}} \frac{h_{\boldsymbol{\tau}} h}{\tan \sigma_2} \, ds \\
&= - \int_{\overline{P_1 Q_1}} \frac{(h^2)_{\boldsymbol{\tau}}}{2 \tan \sigma_1} \, ds + \int_{\overline{Q_2 P_2}} \frac{(h^2)_{\boldsymbol{\tau}}}{2 \tan \sigma_2} \, ds \\
&= - \frac{h^2(Q_1)}{2 \tan \sigma_1} + \frac{h^2(P_1)}{2 \tan \sigma_1} - \frac{h^2(Q_2)}{2 \tan \sigma_2} + \frac{h^2(P_2)}{2 \tan \sigma_2}.
\end{aligned}
$$

By Lemma 3.2, ϕ is C^1 at the wedge corner and then, using Lemma 3.3, $D\phi = 0$ at the wedge corner. Thus, $h(Q_1)$ and $h(Q_2)$ converge to δ as ϵ converges to zero. Therefore, as $\epsilon \to 0$,

$$
- \int_{\overline{P_1 Q_1}} v_{\boldsymbol{\tau}} h \, ds - \int_{\overline{Q_2 P_2}} v_{\boldsymbol{\tau}} h \, ds \to -\frac{\delta^2}{2} \left(\frac{1}{\tan \sigma_1} + \frac{1}{\tan \sigma_2} \right) = -\frac{\delta^2 \sin(\pi - \sigma)}{2 \sin \sigma_1 \sin \sigma_2}. \tag{3.17}
$$

By Lemma 3.4, we conclude that there exists a function $f(\epsilon)$ with the properties that $f(\epsilon)$ is a nondecreasing continuous function of $\epsilon \in [0, r_0)$, and $f(\epsilon) \to 0$ as $\epsilon \to 0$, such that

$$
|D^2 \phi| \leq C \frac{f(2\epsilon)}{\epsilon} \qquad \text{on } \partial B_{\epsilon}(\mathcal{C}) \cap \Omega.
$$

Then, as $\epsilon \to 0$,

$$
\int_{\partial B_{\epsilon}(\mathcal{C}) \cap \Omega} v_{\boldsymbol{\tau}} h \, ds \leq \left(\delta + \|D\phi\|_{C(\Omega)} \right) \int_{\partial B_{\epsilon}(\mathcal{C}) \cap \Omega} |D^2 \phi| \, ds \leq 2\pi\epsilon \, C \frac{f(2\epsilon)}{\epsilon} \to 0. \tag{3.18}
$$

Estimates (3.17)–(3.18) together imply that, as $\epsilon \to 0$,

$$
I \to -\frac{\delta^2 \sin(\pi - \sigma)}{2 \sin \sigma_1 \sin \sigma_2} < 0.
$$

On the other hand, I is nonnegative from (3.16) and the ellipticity. This is a contradiction, so that the regular solution defined in Definition 2.1 does not exist indeed. $\qquad \square$

Appendix A. Continuity estimate at the corner

In this appendix, we show that a solution of some type of linear elliptic equations, which is only assumed to be L^{∞} at the wedge corner $\mathcal{C}$, is actually continuous.

Lemma A.1. *Let $v \in L^{\infty}(\overline{B_1 \cap \Lambda}) \cap C^2(\overline{B_1 \cap \Lambda} \setminus \mathcal{C})$ for the wedge corner $\mathcal{C}$ satisfy*

$$
\sum_{i,j=1}^{2} a_{ij} v_{ij} + \sum_{i=1}^{2} b_i v_i = 0 \qquad \text{in } B_1 \cap \Lambda,
$$

$$
v = 0 \qquad \text{on } \partial\Lambda \cap B_1 \cap \{\xi > 0\},
$$

$$
v_{\boldsymbol{\beta}} = 0 \qquad \text{on } \partial\Lambda \cap B_1 \cap \{\xi < 0\},
$$

where $\sum_{i,j=1}^{2} a_{ij}y_iy_j \geq \lambda|\mathbf{y}|^2$ for each $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, $|a_{ij}| \leq \lambda^{-1}$, $|b_i| \leq \frac{C_E}{r}$, $|\boldsymbol{\beta}\cdot\boldsymbol{\nu}| > \frac{1}{C_E}$, and $|\boldsymbol{\beta}| = 1$ for some positive constants λ and C_E, and $r = \sqrt{\xi^2 + \eta^2}$ denotes the distance to the wedge corner. Then v is continuous in $\overline{\mathcal{B}} := \overline{B_1 \cap \Lambda}$. Moreover,

$$|v(\xi, \eta)| \leq Cr^\alpha \|v\|_{L^\infty(\mathcal{B})} \qquad in \ \mathcal{B},$$

where $C > 0$ and $\alpha \in (0, 1)$ depend only on λ and C_E.

Remark A.1. *The proof of this lemma follows the methods in Lieberman [20] and the references therein, more specifically, their versions in [9, 13].*

Proof. We use the comparison function considered in step 3 of the proof of Theorem 4.3.16 in [9].

We use the polar coordinates (r, θ) centered at the origin and choose the orientation of θ and the direction of ray $\theta = 0$ so that

$$\mathcal{B} = \{|\theta| \leq \hat{\theta}\} \cap B_1,$$
$$\partial\Lambda \cap B_1 \cap \{\xi > 0\} = \partial\mathcal{B} \cap \{\theta = -\hat{\theta}\},$$
$$\partial\Lambda \cap B_1 \cap \{\xi < 0\} = \partial\mathcal{B} \cap \{\theta = \hat{\theta}\},$$

where $\hat{\theta} := \frac{1}{2}(2\pi - \theta_{\mathrm{w}}) \in (0, \pi)$. We show the existence of $w \in C^\infty(\overline{\Omega} \setminus \{\mathbf{0}\})$ such that

$$\sum_{i,j=1}^{2} a_{ij}D_{ij}w + \sum_{i=1}^{2} b_i D_i w < 0 \qquad in \ \mathcal{B}, \tag{A.1}$$

$$w_{\boldsymbol{\beta}} < 0 \qquad on \ \partial\mathcal{B} \cap \{\theta = \hat{\theta}\}. \tag{A.2}$$

Then we consider the function

$$w(r, \theta) = r^\alpha \bar{h}(\theta) \qquad in \ \mathcal{B} \ with \ \bar{h}(\theta) = 1 - \mu e^{-L\theta}, \tag{A.3}$$

where the constants $\alpha \in (-1, 1)$, $\mu \geq 0$, and $L \geq 1$ will be fixed below.

For each $L \geq 1$, we choose $\mu = \frac{1}{2}e^{-\pi L}$ so that $\bar{h}(\theta) \geq \frac{1}{2}$ for all $\theta \in [-\pi, \pi]$. We use such a constant $\mu = \mu(L)$ from now on.

Now we show that w is a supersolution of the equation after a certain choice of L, α. Fix a point $(\xi, \eta) \in \mathcal{B}$, and rotate the Cartesian coordinates in $\mathbb{R}^2$ to become the radial and tangential coordinates at (ξ, η). We denote by a_{rr}, $a_{r\theta}$, etc., the coefficients of the equation in these rotated coordinates. Below, C is a universal constant that may be different at each occurrence, depending only on λ and C_E. We use that $\alpha \in (-1, 1)$ and $\frac{1}{2} \leq h(\theta) \leq 1$ to compute at (ξ, η)

$$\sum_{i,j=1}^{2} a_{ij}D_{ij}w = a_{rr}w_{rr} + 2a_{r\theta}(r^{-1}w_{r\theta} - r^{-2}w_\theta) + a_{\theta\theta}(r^{-2}w_{\theta\theta} + r^{-1}w_r)$$

$$= \left(\mu L e^{-L\theta}\left(-a_{\theta\theta}L + 2a_{r\theta}(\alpha - 1)\right) + \alpha\left((\alpha - 1)a_{rr} + a_{\theta\theta}\right)\bar{h}(\theta)\right)r^{\alpha-2}$$

$$\leq \left(\mu L e^{-L\theta}\left(-\frac{L}{C} + C\right) + C|\alpha|\right)r^{\alpha-2}.$$

Then, for any $L \geq 10C^2$, using that $\mu = \frac{1}{2}e^{-\pi L} > 0$, we can choose $\hat{\alpha} > 0$ small so that, for any $|\alpha| \leq \hat{\alpha}$,

$$\sum_{i,j=1}^{2} a_{ij} D_{ij} w \leq -\mu \frac{L^2}{2C} e^{-L\theta} r^{\alpha - 2}.$$

Then we have

$$\sum_{i,j=1}^{2} a_{ij} D_{ij} w + \sum_{i=1}^{2} b_i D_i w \leq -\mu \frac{L^2}{2C} e^{-L\theta} r^{\alpha - 2} + \frac{C}{r} |Dw|$$

$$\leq \big(-\mu \frac{L^2}{2C} e^{-L\theta} + C(\alpha + \mu L e^{-L\theta}) \big) r^{\alpha - 2}$$

$$= \big(\mu L e^{-L\theta} (-\frac{L}{2C} + C) + C\alpha \big) r^{\alpha - 2}.$$

We first choose L large so that $-\frac{L}{2C} + C \leq -\frac{L}{4C}$, i.e., $L \geq 4C^2$. Then, for such L, using that $\mu = \frac{1}{2}e^{-\pi L} > 0$ and $\theta \in [-\pi, \pi]$, we can choose $\alpha(L)$ small so that the last expression is negative, i.e., that (A.1) holds.

To show (A.2), let $\boldsymbol{\nu}$ be the interior unit normal on $\partial \mathcal{B} \cap \{\theta = \hat{\theta}\}$ with respect to $\mathcal{B}$, and let $\boldsymbol{\tau}$ be the unit tangent vector pointing away from the corner. Then

$$w_{\boldsymbol{\beta}} = -\frac{\boldsymbol{\beta} \cdot \boldsymbol{\nu}}{r} w_\theta + \boldsymbol{\beta} \cdot \boldsymbol{\tau} \, w_r$$

$$\leq \big(-(\boldsymbol{\beta} \cdot \boldsymbol{\nu}) \mu L e^{-L\theta} + C|\alpha| \big) r^{\alpha - 1}$$

$$\leq \big(-\frac{\mu L}{C_E} e^{-L\theta} + C|\alpha| \big) r^{\alpha - 1}.$$

Therefore, for any $L \geq 1$, using that $\mu = \frac{1}{2}e^{-\pi L} > 0$, we can choose $\hat{\alpha} > 0$ small such that, for any $|\alpha| \leq \hat{\alpha}$, the last expression is negative for any $\theta = \hat{\theta} \in [0, \pi]$. Then (A.2) holds.

Now we fix L sufficiently large to satisfy all the conditions stated above. This also fixes $(\mu, \hat{\alpha})$.

Let $w^{\pm} = r^{\pm \hat{\alpha}} \bar{h}(\theta)$, where $\bar{h}(\theta)$ is from (A.3). Then, using that $\bar{h}(\theta) \geq \frac{1}{2}$,

$$w^+ \geq \frac{1}{2} \qquad \text{on } \partial \mathcal{B} \cap \partial B_1,$$
$$w^- \geq \frac{1}{2} r^{-\hat{\alpha}} \qquad \text{on } \mathcal{B} \cap \partial B_r. \tag{A.4}$$

For $\varepsilon \in (0, 1)$, let

$$V_\varepsilon = 2 \|v\|_{L^\infty(\mathcal{B})} w^+ + \varepsilon w^-.$$

Then V_ε satisfies (A.1)–(A.2). Also, by (A.4) and since $w^{\pm} > 0$ in $\mathcal{B}$, for each $\varepsilon \in (0, 1)$, there exists $R_\varepsilon \in (0, 1)$ so that, for each $R \in (0, R_\varepsilon]$,

$$V_\varepsilon \geq \|v\|_{L^\infty(\mathcal{B})} \qquad \text{on } (\partial \mathcal{B} \cap \partial B_1) \cup (\mathcal{B} \cap \partial B_R).$$

Then, by the comparison principle, for each $R \in (0, R_\varepsilon]$,

$$v \leq V_\varepsilon \qquad \text{in } \mathcal{B} \setminus \overline{B}_R.$$

By a similar argument,

$$v \geq -V_\varepsilon \qquad \text{in } \mathcal{B} \setminus \overline{B}_R.$$

Combining these two estimates together, sending $R \to 0+$ for each ε, and then sending $\varepsilon \to 0+$, we obtain

$$|v| \leq w^+ \qquad \text{in } \mathcal{B}.$$

Then, using that $\frac{1}{2} \leq \bar{h}(\theta) \leq 1$, we have

$$|v(r, \theta)| \leq 2\|v\|_{L^\infty(\mathcal{B})} r^{\hat{\alpha}} \qquad \text{in } \mathcal{B}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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GUI-QIANG G. CHEN, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD, OX2 6GG, UK; AMSS AND UCAS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA
E-mail address: `chengq@maths.ox.ac.uk`

MIKHAIL FELDMAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706-1388, USA
E-mail address: `feldman@math.wisc.edu`

JINGCHEN HU, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, CHINA
E-mail address: `jchu@mail.ustc.edu.cn`

WEI XIANG, DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, KOWLOON, HONG KONG, P. R. CHINA
E-mail address: `weixiang@cityu.edu.hk`