DERIVATION OF THE ION EQUATION

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Abstract. We consider the classical Euler-Poisson system for electrons and ions, interacting through an electrostatic field. The mass ratio of an electron and an ion $m_e/M_i \ll 1$ is small and we establish an asymptotic expansion of solutions, where the main term is obtained from a solution to a self-consistent equation involving only the ion variables. Moreover, on $\mathbb{R}^3$, the validity of such an expansion is established even with "ill-prepared" Cauchy data, by including an additional initial layer correction.

This paper is dedicated to Walter Strauss on the occasion of his 80th birthday.

1. presentation

1.1. **From the two fluid model to the ion equation.** Consider the following Euler-Poisson system for ion and electron fluids near a constant equilibrium of $n_e = n_i \equiv n_0 > 0$:

$$\begin{aligned} &\partial_t n_e + \mathrm{div}(n_e v_e) = 0, \\ &n_e m_e \left(\partial_t v_e + v_e \cdot \nabla v_e\right) = -\nabla p(n_e) + e n_e \nabla V, \\ &\partial_t n_i + \mathrm{div}(n_i v_i) = 0, \\ &n_i M_i \left(\partial_t v_i + v_i \cdot \nabla v_i\right) = -\nabla \pi(n_i) - e n_i \nabla V, \\ &-\Delta V = 4\pi e(n_i - n_e), \end{aligned} \tag{1.1}$$

where $e > 0$ is the electron charge, $m_e, T_e \equiv: p'(n_0)$ (resp. $M_i, T_i \equiv: \pi'(n_0)$) are the mass and effective temperature of an electron (res. ion), V is the electric potential ($E = -\nabla V$), n_e and n_i the electronic and ionic densities and v_e and v_i the electronic and ionic velocities. Here pressure laws $p = p(n)$ and $\pi = \pi(\rho)$ are smooth and strictly increasing with $p(0) = \pi(0) = 0$.

This system depends on one small parameter

$$\eta^2 = \frac{m_e}{M_i} \ll 1 \tag{1.2}$$

and the purpose of this paper is to investigate the behavior of solutions as $\eta \to 0$. If one takes this limit while keeping $m_e = O(1)$, the limit is nonsingular and leads to the Euler-Poisson equation for electrons. We consider the case when $M_i = O(1)$, i.e. when we consider timescales where the ions have a nontrivial dynamics.

In this case, formally, the electrons adjust to the Boltzmann equilibrium which reads, in the case of linear pressure law

$$n_e = n_0 e^{eV/T_e} \tag{1.3}$$

and the ion solve a self-consistent equation, the *Euler-Poisson equation for ions*:

$$\begin{aligned} &\partial_t n_i + \operatorname{div}(n_i u_i) = 0, \\ &n_i M_i(\partial_t u_i + u_i \cdot \nabla u_i) + \nabla p_i + n_i e \nabla V = 0, \\ &-\Delta V = 4\pi e(n_i - n_0 e^V). \end{aligned} \tag{1.4}$$

Our purpose is to make this limit rigorous in two different domains: $X = \mathbb{T}^3$ and $X = \mathbb{R}^3$, leaving the interesting question of the influence of boundaries. We first provide uniform energy estimates for solutions of (1.1) in the limit considered, which in particular justify the above system (1.4) as the main order term in the description of the ion density in an expansion in η.

This result covers the case of well and ill-prepared initial data. For ill-prepared initial data, the electron unknowns are not fully described in terms of solutions to (1.4) and their convergence as $\eta \to 0$ is at most in a weak sense. In our second main result, we obtain strong convergence of solutions in the case $X = \mathbb{R}^3$ by adding an initial dispersive layer. The precise statement of the result is done in Subsection 1.3 below.

1.2. **Nondimensionalization and limit system.** We let

$$\begin{aligned} n_e(t,x) =& \frac{M_i}{4\pi e^2} n(t, \sqrt{\frac{M_i}{T_i}} x), \qquad v_e(t,x) = \sqrt{\frac{T_i}{M_i}} v(t, \sqrt{\frac{M_i}{T_i}} x), \\ n_i(t,x) =& \frac{M_i}{4\pi e^2} \rho(t, \sqrt{\frac{M_i}{T_i}} x), \qquad v_i(t,x) = \sqrt{\frac{T_i}{M_i}} \nu(t, \sqrt{\frac{M_i}{T_i}} x), \\ V(t,x) =& \frac{T_i}{e} \phi(t, \sqrt{\frac{M_i}{T_i}} x). \end{aligned}$$

We suppose that the spatial averages and end states of n, ρ, and ϕ are $\overline{n}$, $\overline{\rho} = \overline{n}$, and 0. Then we let

$$\tilde{h}(n) = \int_{\overline{n}}^{n} \frac{p'(s)}{s} ds, \quad \tilde{\gamma}(\rho) = \int_{\overline{n}}^{\rho} \frac{\pi'(s)}{s} ds \tag{1.5}$$

be the electron and ion enthalpies, respectively. The first relation can be inverted to give $n = \tilde{n}(h)$.

The system (1.1) becomes

$$\begin{aligned} \partial_t n + \operatorname{div}[nv] &= 0, \\ \eta^2 \left[\partial_t v + v \cdot \nabla v\right] + \nabla \tilde{h}(n) - \nabla \phi &= 0, \\ \partial_t \rho + \operatorname{div}[\rho \nu] &= 0, \\ \partial_t \nu + \nu \cdot \nabla \nu + \nabla \tilde{\gamma}(\rho) + \nabla \phi &= 0, \\ -\Delta \phi &= \rho - n. \end{aligned} \tag{1.6}$$

From now on, we study the asymptotic behavior of solutions of (1.6) as $\eta \to 0$. Formally, using in the second line in (1.6), we obtain that

$$\nabla \tilde{h}(n) = \nabla \phi$$

hence, using the fact that ϕ is defined up to a constant,

$$\tilde{h}(n) = \phi. \tag{1.7}$$

This is a generalization of the celebrated (normalized) Boltzmann relation, which, for the case $\tilde{h} = \ln n$, takes the form (1.3). Using (1.7), we see that, in the limit

$\eta \to 0$, the ionic equation decouples from the electronic equations. Denoting by ρ_0, ν_0 and ϕ_0 the (formal) limits of the ionic density, ionic velocity and electric potential, we derive the following classical Euler-Poisson system for ion dynamics:

$$\begin{aligned}
&\partial_t \rho_0 + \operatorname{div}(\rho_0 \nu_0) = 0, \\
&\partial_t \nu_0 + \nu_0 \cdot \nabla \nu_0 + \nabla \tilde{\gamma}(\rho_0) + \nabla \phi_0 = 0, \\
&- \Delta \phi_0 = \rho_0 - \tilde{n}(\phi_0).
\end{aligned} \tag{1.8}$$

This equation is classical in plasma physics and various of its properties are studied in [7, 10, 15].

Let us now turn to the electronic quantities. Note that the electronic density is given by the potential through $\tilde{n}(\phi_0)$. The equation on v_0 is more complex. First using the equation on electronic density we get

$$\operatorname{div}(n_0 v_0) = -\partial_t n_0. \tag{1.9}$$

Moreover, taking the curl of electronic velocity equation we get

$$\operatorname{curl}\Big(\partial_t v_0 + v_0 \cdot \nabla v_0\Big) = 0. \tag{1.10}$$

Using (1.9) and (1.10), it is possible to obtain v_0 by a method similar to that for the incompressible Euler equation.

1.3. **The remainder and statement of the results.** We expect that in the limit $\eta \to 0$ the electrons oscillate with high frequency of order $O(\eta^{-1})$, and large amplitude, of order $O(1)$. This leads to the introduction of expansions of the form

$$n = n_0 + \eta n_1, \quad v = v_0 + v_1, \quad \rho = \rho_0 + \eta \rho_1, \quad \nu = \nu_0 + \eta \nu_1, \quad \phi = \phi_0 + \eta \phi_1, \tag{1.11}$$

where

$$n_0 := \tilde{n}(\phi_0). \tag{1.12}$$

The total physical energy associated to (1.6) is

$$\mathcal{E} = \frac{1}{2} \int_X \left\{ n|\eta v|^2 + H(n) + \rho|\nu|^2 + \Gamma(\rho) + |\nabla \phi|^2 \right\} dx$$

where

$$H(x) = \int_{\overline{n}}^x \left\{ \widetilde{h}(s) - \widetilde{h}(\overline{n}) \right\} ds, \qquad \Gamma(x) = \int_{\overline{n}}^x \left\{ \widetilde{\gamma}(s) - \widetilde{\gamma}(\overline{n}) \right\} ds,$$

and we observe that in the case of perturbations of the trivial equilibrium: $\rho_0 \equiv \overline{n}$, $\nu_0 \equiv 0$, initial data of the form (1.11) have $O(\eta^2)$ energy. We note however, that this class of initial data is in general "ill-prepared" in the sense that its time derivative at $t = 0$ is not bounded uniformly in η (cf the singular terms in (1.13b) below).

The system (1.6) becomes

$$\partial_t n_1 + v_0 \cdot \nabla n_1 + \frac{1}{\eta} \operatorname{div}\left[[n_0 + \eta n_1] v_1 \right] = -\sigma_0, \tag{1.13a}$$

$$\partial_t v_1 + [v_0 + v_1] \cdot \nabla v_1 + \frac{1}{\eta} \nabla \left[\frac{\tilde{h}(n_0 + \eta n_1) - \tilde{h}(n_0)}{\eta} - \phi_1 \right] = -\sigma_1, \tag{1.13b}$$

$$\partial_t \rho_1 + [\nu_0 + \eta \nu_1] \cdot \nabla \rho_1 + [\rho_0 + \eta \rho_1] \operatorname{div}[\nu_1] = -\sigma_2, \tag{1.13c}$$

$$\partial_t \nu_1 + [\nu_0 + \eta \nu_1] \cdot \nabla \nu_1 + \nabla \left[\frac{\tilde{\gamma}(\rho_0 + \eta \rho_1) - \tilde{\gamma}(\rho_0)}{\eta} + \phi_1 \right] = -\sigma_3, \tag{1.13d}$$

$$-\Delta \phi_1 = \rho_1 - n_1 \tag{1.13e}$$

with error terms

$$\sigma_0 := n_1 \mathrm{div}[v_0], \qquad \sigma_1 := \partial_t v_0 + [v_0 + v_1] \cdot \nabla v_0,$$
$$\sigma_2 := \nu_1 \cdot \nabla \rho_0 + \rho_1 \mathrm{div}[\nu_0], \qquad \sigma_3 := \nu_1 \cdot \nabla \nu_0.$$

We first prove existence of solutions on a time interval which is independent of η. More precisely we establish

Theorem 1.1. *Let $s > 3/2 + 1$, $X = \mathbb{T}^3$ or $X = \mathbb{R}^3$. Let $(n_0, v_0, \rho_0, \nu_0, \phi_0)$ solve (1.8), (1.9) and (1.10) such that $(n_0 - \bar{n}, v_0, \rho_0 - \bar{\rho}, \nu_0, \phi_0) \in C^s([0,T]; H^{s+1}(X))$. For any choice of initial data $(n_1(0), v_1(0), \rho_1(0), \nu_1(0)) \in H^s(X)$ supposed to satisfy*

$$\rho_1(0) - n_1(0) \in \dot{H}^{-1}(X), \tag{1.14}$$

there exists a small parameter η_0 and a time T_0 such that one has the uniform estimates on $[0, T_0]$ for the solutions of (1.13):

$$\sup_{[0,T_0]} [\|(n_1, v_1, \rho_1, \nu_1, \nabla\phi_1)\|_{H^s} + \|(\eta\partial_t n_1, \eta\partial_t v_1, \partial_t \rho_1, \partial_t \nu_1)\|_{H^{s-1}}] \lesssim 1$$

for all $\eta \in (0, \eta_0)$. Besides T_0 and η_0 can be chosen uniformly on bounded sets of initial data.

Note that condition (1.14) is trivially true in the case $X = \mathbb{T}^3$. The structure of equation (1.13) is similar to the structure isolated in [11] and appears to be crucial in obtaining good energy estimates.

Theorem 1.1 immediately justifies the expansion in the derivation of the ion equations.

Corollary 1.1. *The variables (ρ, ν, ϕ) converge strongly in $C([0,T], H^s)$ to (ρ_0, ν_0, ϕ_0) as $\eta \to 0$, where (ρ_0, ν_0, ϕ_0) solves (1.8) with initial data $(\rho_0(0), \nu_0(0))$.*

In the case $X = \mathbb{R}^3$, we can also justify the expansion in the derivation of the electron equations for general "ill-prepared" Cauchy data by adding an initial dispersive layer responsible for the singular terms in the initial time derivatives. Our second main theorem is as follows

Theorem 1.2. *There exists $\delta > 0$ such that the following holds: assume that $(\rho_0, \nu_0, \phi_0) \in C^s([0, T_0] : H^{s+1})$ are a given solution to (1.8) satisfying*

$$\|(\langle x\rangle^3[\rho_0 - \bar{\rho}], \langle x\rangle^3 \nu_0)(0)\|_{H^3} \le \delta, \tag{1.15}$$

and consider initial data $(n_1(0), v_1(0)) \in H^s$. Then there exists $\eta_0 > 0$, and $0 < T_\ell \le T_0$ such that, uniformly in $0 < \eta < \eta_0$, the solution of (1.6) with initial data

$$n(0) = \widetilde{n}(\phi_0(0)) + \eta n_1(0), \quad v(0) = v_0(0) + v_1(0), \quad \rho(0) = \rho_0(0), \quad \nu(0) = \nu_0(0)$$

exists on a uniform time interval $(0, T_\ell)$ and moreover, one can obtain the following expansion of the solutions

$$n = n_0 + \eta n_\ell + \eta n_r, \qquad v = v_0 + v_\ell + v_r, \qquad \phi = \phi_0 + \eta\phi_\ell + \eta\phi_r,$$
$$\rho = \rho_0 + \eta\rho_r, \qquad \nu = \nu_0 + \eta\nu_r,$$

where (n_0, v_0) are defined in (1.9), (1.10), $(n_\ell, v_\ell, \phi_\ell)$ is an initial dispersive layer, which converges locally to 0 in $L^2_{x,t}(K)$ for any compact region of space-time and the remainder converges strongly to 0:

$$\|(n_r, v_r, \rho_r, \nu_r, \phi_r)\|_{L^\infty([0,T_\ell]:H^{s-1})} \to 0, \quad \text{as } \eta \to 0. \tag{1.16}$$

The main information is that $v_r \to 0$ since the convergence of n toward n_0 can be obtained by Theorem 1.1.

The initial layer is described below in more details. We note that Theorem 1.2 also applies for well-prepared initial data in the case of $X = \mathbb{T}^3$ by defining $(n_\ell, v_\ell, \phi_\ell) = (0,0,0)$. Finally, by assuming further localization, one can make the rate of decay in (1.16) quantitative in η.

The localization required on the initial data for the ion in (1.15), used to guarantee (1.20), seems stronger than the condition in [11], which is essentially at the level $|\nabla n_0| = o(|x|^{-2})$. This is due to two reasons: (i) in our context, we are considering a Klein-Gordon equation and not a wave equation and spatial decay in v is related to both decay in n and in $\nabla\phi \simeq |\nabla|^{-1} n$, which leads us to require a stronger decay $|x|^3|\nabla n_0| \ll 1$, (ii) the electrostatic equation in the ion equation (1.8) leads to similar spatial decay for ρ and ϕ and their derivatives, thus the homogeneous norm in (1.15).

The initial layer $(n_\ell, v_\ell, \phi_\ell)$ satisfies

$$\partial_t n_\ell + \frac{1}{\eta}\mathrm{div}\left[[n_0 + \eta n_\ell] v_\ell\right] = 0, \tag{1.17a}$$

$$\partial_t v_\ell + v_\ell \cdot \nabla v_\ell + \frac{1}{\eta}\nabla\left[\frac{\tilde h(n_0+\eta n_\ell) - \tilde h(n_0)}{\eta} - \phi_\ell\right] = -\left[\partial_t v_0 + v_0\cdot\nabla v_0\right], \tag{1.17b}$$

$$\mathrm{curl}[v_\ell] = 0, \tag{1.17c}$$

$$\Delta\phi_\ell = n_\ell. \tag{1.17d}$$

This introduces a fast dynamics (see the unbounded time derivatives) which corresponds to a dispersive, Klein-Gordon-type equation in time t/η on a variable, slower, background defined by $n_0(x,t)$. The properties of initial layers are of independent interest and are summarized in the following theorem.

Theorem 1.3. *Let $s > 3/2 + 1$ and $X = \mathbb{R}^3$. Given $(n_0, v_0, \rho_0, \nu_0, \phi_0)$ as above and $(n_0 - \bar n, v_0, \rho_0 - \bar\rho, \nu_0, \phi_0) \in C^s([0,T]; H^{s+1}(\mathbb{R}^3))$. For any choice of initial data $(n_\ell(0), v_\ell(0)) = (n_1(0), v_1(0)) \in H^s(\mathbb{R}^3)$ supposed to satisfy*

$$n_1(0) \in \dot H^{-1}(\mathbb{R}^3), \quad \mathrm{curl}[v_1(0)] = 0, \tag{1.18}$$

there exists a small parameter η_ℓ and a time T_ℓ such that one has the uniform estimates on $[0,T_\ell]$ for the solutions of (1.17):

$$\sup_{[0,T_\ell]}\left[\|(n_\ell, v_\ell, \nabla\phi_\ell)\|_{H^s} + \|(\eta\partial_t n_\ell, \eta\partial_t v_\ell)\|_{H^{s-1}}\right] \lesssim 1 \tag{1.19}$$

for all $\eta \in (0, \eta_\ell)$. Besides T_ℓ and η_ℓ can be chosen uniformly on bounded set of initial data. Furthermore, there exists a positive constant ϵ such that if

$$\|\langle x\rangle^3 \nabla n_0\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le \epsilon, \tag{1.20}$$

then the following local energy decay holds

$$\int_0^T \int_{B(0,|\log\eta|^{\frac14})} \left\{n_\ell^2 + |v_\ell|^2 + |\nabla\phi_\ell|^2\right\} dxdt \lesssim |\log\eta|^{-\frac12}(1+T). \tag{1.21}$$

The problem of the limit of (1.1) as $\eta \to 0$ is related to the classical problem of the low-Mach number limit, see e.g. [8, 9, 17] and many of the results above are motivated by related works [1, 3, 11]. We also refer to [5, 6] for other works on the two-fluid Euler-Poisson system.

Finally, we would like to mention some interesting problems that are left open by this work:

- It would be interesting to understand how the derivation changes in the presence of boundary.
- It seems a very difficult question to consider the is situation with a full electromagnetic field, when Poisson equation is replaced by Maxwell's equation.

1.4. **Outline of the proof.**

1.4.1. *Energy estimates.* In order to obtain existence on a uniform interval of time, one needs to control the regularity of the unknowns. However, a look at (1.13) shows that one cannot expect uniform control on some time derivatives. In the case $X = \mathbb{R}^3$, this can be understood since the initial layer is essentially a function of t/η. Fortunately, the equation for electron and ions are only weakly coupled and we can control them in different ways.

For electrons, we use the fact that $\eta\partial_t$ preserves the symmetric structure of the equations to control large *fast-time derivative*, $\eta\partial_t$ of the unknowns and then use the equations to transfer time regularity to spatial regularity ($\eta\partial_t \to \partial_x$) for the electron variables. This does not work for the electron vorticity and we control it independently, which is straightforward since it is essentially transported by the flow. This approach is adapted from the work of Métivier and Schochet [11], see also [4].

For electrons, we make usual energy estimates to get directly spatial regularity for the ion variables. Indeed, there is a priori little to gain from singular time-regularity of the ion variables, as we expect the ions to have time derivatives of size $O(1)$.

1.4.2. *Control of the initial layer.* The proof of Theorem 1.3 relies on the local energy decay. To illustrate this, we consider a model problem for (1.17) in the system

$$\begin{aligned} \eta\partial_t n_\ell + \operatorname{div}[n_0 v_\ell] &= 0, \\ \eta\partial_t v_\ell + \nabla\left[\tilde{h}'(n_0)n_\ell\right] - \nabla\phi_\ell &= 0, \quad \operatorname{curl}[v_\ell] = 0, \\ \Delta\phi_\ell &= n_\ell. \end{aligned} \tag{1.22}$$

This system has a similar structure to the Klein–Gordon equation. For semilinear Klein–Gordon equations, the local energy decay was obtained in [13] by C. S. Morawetz. Since then, there has been a long litterature devoted to its precise study and we refer to [2, 12, 16] and the references therein for more history and context. In our case, the system is quasilinear with variable coefficients and we adapt her method based on a combination of the classical momentum conservation law:

$$\eta\partial_t[n_\ell v_{\ell j}] + \operatorname{div} L_j = -\frac{|v_\ell|^2}{2}\partial_j n_0 + \frac{n_\ell^2}{2}\partial_j \tilde{h}'(n_0) \tag{1.23}$$

for $j = 1, 2, 3$, where $L_j := (L_{j1}, L_{j2}, L_{j3})$ and

$$L_{jk} := n_0 v_{\ell j} v_{\ell k} - \partial_j\phi_\ell\partial_k\phi_\ell + \delta_{jk}\left[-n_0\frac{|v_\ell|^2}{2} + \tilde{h}'(n_0)\frac{n_\ell^2}{2} + \frac{|\nabla\phi_\ell|^2}{2}\right] \quad \text{for } k = 1, 2, 3,$$

and a new *equirepartition of energy* estimate with a key multiplier $\nabla\phi_\ell$, associated to the work of the electrostatic forces:

$$\begin{aligned}&\eta\partial_t[v_\ell\cdot\nabla\phi_\ell]+\mathrm{div}[\tilde{h}'(n_0)n_\ell\nabla\phi_\ell]\\&=\tilde{h}'(n_0)n_\ell^2+|\nabla\phi_\ell|^2-n_0|v_\ell|^2+v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]v_\ell.\end{aligned}\tag{1.24}$$

1.4.3. *Decay of the remainder.* Once the strong limit and the initial layer are extracted, the proof of the decay of the remainder follows from a relatively simple Gronwall estimates. There are only two issues that arise:

(1) One needs to show that the strong limit and the initial layer do not have strong interactions, i.e. to control terms of the form

$$\left|\int_0^T\int_{\mathbb{R}^3}n_\ell v_0\,dxdt\right|\to 0.$$

This is done using orthogonality in space arguments: the strong solution remains localized close to the origin for the times $T\lesssim 1$ that we consider, while the initial layer rapidly exits any compact space.

(2) One also needs to verify that the ion density is not amplified by the forcing of the new electrostatic field $\nabla\phi_\ell$ due to the initial layer. Fortunately, in this case, we can use the fast time oscillations to approximately integrate this interaction through a normal form transformation.

The details are presented in Section 4.

1.5. **Notations.** Let us define some function spaces. We first define $\dot{H}^1(X)$ to be $\dot{H}^1(\mathbb{R}^3)$, the usual homogeneous Sobolev space if $X=\mathbb{R}^3$ and if $X=\mathbb{T}^3$, we define it as the space of mean-zero functions in the usual Sobolev space,

$$\dot{H}^1(\mathbb{T}^3)=\{f\in H^1(\mathbb{T}^3);\int_{\mathbb{T}^3}f\,dx=0\}.$$

1.6. **Outline of this paper.** The structure of this paper is as follows: In Section 2, we obtain energy estimates. Section 3 provides the localization of the limit solution. In Section 4 we prove the decays of the remainder by assuming the local decay of the initial layer. Section 5 deals with the local decay of the initial layer. Finally, in an appendix in Section 6, we recall some results used along the way.

2. Energy estimates; methodology "à la Métivier and Schochet"

In this section, we obtain energy estimates. We start with the case $X=\mathbb{T}^3$ of the torus, where the estimates are slightly easier. In the last subsection, we adapt the proof to the case $X=\mathbb{R}^3$.

2.1. **Principle.** To obtain energy estimates, we closely follow the strategy of G. Métivier and S. Schochet. First, as (1.13) is an hyperbolic system, we note that it admits a solution over a time interval $[0,T(\eta)]$ which may depend on η. We then use a bootstrap argument uniform in η to obtain uniform a priori bounds on a fixed time interval independent of η.

Let us fix an integer $s>3/2+1$. For η fixed, but small, we introduce

$$K(T):=\sup_{t\in[0,T]}\|(n_1,v_1,\rho_1,\nu_1)\|_{H^s}$$

for $T\leq T(\eta)$. Note that K takes into account only part of the unknowns.

The first step is to prove that K also controls time derivatives of the various unknowns. More precisely we will show that

$$L := \sup_{t\in[0,T]} \sum_{k=0}^{s} \left[\| (\eta\partial_t)^k (n_1, u_1, \nabla\phi_1)\|_{H^{s-k}} + \eta^{-\min\{k,1\}} \| (\eta\partial_t)^k (\rho_1, \nu_1)\|_{H^{s-k}} \right] \tag{2.1}$$

is bounded by K. The idea of the proof of this inequality is to use the equations to express time derivatives of the unknowns. At each step we loose one space derivative and a η factor, except for "slow" variables where there is no loss at the first step. This is detailed in Section 2.2.

The second step is to fulfill energy estimates. This involves two phases. First we make classical L^2 energy estimates on the linearized system. Then we use refined energy estimates for the ion variables. This gives a control of fast variables by a function of K.

Combining all these estimates we obtain a Gronwall-type estimate

$$K(T) \le C_0 + (T+\eta)C(K)$$

where, here and later, C_0 denotes a constant depending only on the initial data and $C(K)$ denotes an explicit nondecreasing finite function of K. The upshot is that we have control of K for small time uniformly in η and that solutions exist so long as K remains finite.

2.2. **First step: control of the other unknowns.** We first obtain that

Lemma 2.1. *With L defined in (2.1), there holds that $L \le C(K)$.*

Proof. The proof is a routine but lengthly work. We iteratively use the equation to express time derivatives as functions of spatial derivatives. At each step we gain one time derivative, loose one spatial derivative and an extra factor η^{-1} appears. Let us now go into the details.

We prove by induction on j that

$$\sup_{t\in[0,T]} \sum_{k=0}^{j} \left[\| (\eta\partial_t)^k (n_1, u_1, \nabla\phi_1)\|_{H^{s-k}} + \eta^{-\min\{k,1\}} \| (\eta\partial_t)^k (\rho_1, \nu_1)\|_{H^{s-k}} \right] \lesssim_{j,K} 1.$$

The more precise estimate for ρ_1 and ν_1 comes from the fact that for the first time derivative, no singular term appears (subsequent singular terms will appear as soon as there is a term like $\partial_t\phi_1$).

Case $j = 0$. We only need to bound ϕ_1 in terms of K, which follows from (1.13e).

Induction. From (1.13a), we have

$$\eta\partial_t n_1 = -\eta v_0 \cdot \nabla n_1 - \mathrm{div}\,[[n_0 + \eta n_1]v_1] - \eta\sigma_0.$$

Looking term by term, taking $(\eta\partial_t)^{j-1}$ derivatives, estimating in H^{s-j}, and using Lemma 6.1 to handle the nonlinear term $\mathrm{div}\,[[\eta n_1]v_1]$, we can bound $(\eta\partial_t)^j n_1$. Similarly, for v_1, from (1.13b), we have

$$\eta\partial_t v_1 = -\eta\,[v_0 + v_1] \cdot \nabla v_1 - \nabla \left[\frac{\tilde h(n_0 + \eta n_1) - \tilde h(n_0)}{\eta} - \phi_1 \right] - \eta\sigma_1.$$

The bound is then propagated by estimating each term separately. Here, to handle the nonlinearities of $\tilde{h}$, we have changed $\eta\partial_t$ to ∂_s by $\eta t = s$, then used Lemma 6.2, and changed ∂_s to $\eta\partial_t$ again.

Similarly, one can also have the bound of $(\eta\partial_t)^j\rho_1$ and $(\eta\partial_t)^j\nu_1$ from

$$\begin{aligned}
\partial_t\rho_1 &= -[\nu_0+\eta\nu_1]\cdot\nabla\rho_1 - [\rho_0+\eta\rho_1]\mathrm{div}(\nu_1) - \sigma_2,\\
\partial_t\nu_1 &= -[\nu_0+\eta\nu_1]\cdot\nabla\nu_1 - \nabla\left[\frac{\tilde\gamma(\rho_0+\eta\rho_1)-\tilde\gamma(\rho_0)}{\eta}+\phi_1\right]-\sigma_3.
\end{aligned}$$

What is left is to show the bound of $(\eta\partial_t)^j\,\phi_1$. Applying a standard estimate of elliptic equations to (1.13e) with the fact $(\eta\partial_t)^j\phi_1\in\dot{H}^{s-j+1}$, we have

$$\|(\eta\partial_t)^j\nabla\phi_1\|_{H^{s-j}}\lesssim\left\|(\eta\partial_t)^j\,(\rho_1-n_1)\right\|_{H^{s-j}}\lesssim_K 1.$$

The proof is complete. □

2.3. **Second step: energy estimates by symmetrization.** Here we aim at controlling the most singular terms appearing in (1.13). In order to do this, we first control their time derivatives through energy estimates.

Let us define

$$(N_k,V_k,R_k,U_k,\Phi_k)=(\eta\partial_t)^k\,(n_1,v_1,\rho_1,\nu_1,\phi_1)$$

for $k=0,\dots s$. We first write the equations satisfied by N_k, V_k, and Φ_k.

Lemma 2.2. *For $k=1,\dots s$, there holds that*

$$\begin{aligned}
\partial_tN_k+[v_0+v_1]\cdot\nabla N_k+\frac{1}{\eta}\mathrm{div}\,[[n_0+\eta n_1]V_k]&=r_k^1,\\
\partial_tV_k+[v_0+v_1]\cdot\nabla V_k+\frac{1}{\eta}\nabla\left[\tilde{H}N_k-\Phi_k\right]&=r_k^2,\\
-\Delta\Phi_k+R_k-N_k&=0,
\end{aligned}\tag{2.2}$$

where $\tilde{H}:=\tilde{h}'(n_0+\eta n_1)$ and for $i=1,2$,

$$\|r_k^i\|_{H^{s-k}}\lesssim_{K,L}1.$$

Proof. The proof is straightforward once we apply Lemmas 6.1 and 6.2 to obtain the estimate of r_k^i. □

Lemma 2.3. *Define*

$$\mathcal{E}(t):=\sum_{k\le s}\|\,(\eta\partial_t)^k\,(n_1,v_1,\nabla\phi_1)(t)\|_{L^2}^2,$$

we have the following energy estimates:

$$\mathcal{E}(t)\le C_0+(t+\eta)C(K)$$

as long as $K\lesssim 1$.

Proof. This is direct for the case $k=0$. For $k\ge 1$, we use energy estimates, starting from some estimate of Φ_k. The third line of (2.2) gives

$$\|\nabla\Phi_k\|_{L^2}\le C\|(N_k,R_k)\|_{L^2}\lesssim_{K,L}1.\tag{2.3}$$

Multiplying the first line in (2.2) by $\tilde{H}N_k$, the second line in (2.2) by $[n_0+\eta n_1]V_k$ and summing, we obtain:

$$\begin{aligned}
&\partial_t\left[\frac{1}{2}\tilde{H}N_k^2+\frac{1}{2}[n_0+\eta n_1]|V_k|^2\right]-\frac{1}{\eta}[n_0+\eta n_1]V_k\cdot\nabla\Phi_k\\
&\quad+\operatorname{div}\left[\frac{1}{2}\left[\tilde{H}N_k^2+[n_0+\eta n_1]|V_k|^2\right][v_0+v_1]+\frac{1}{\eta}[n_0+\eta n_1]\tilde{H}N_kV_k\right]\\
&=\mathcal{R}_1,
\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}
\mathcal{R}_1:=&\frac{1}{2}\left[\partial_t\tilde{H}\right]N_k^2+\frac{1}{2}\left[\partial_t[n_0+\eta n_1]\right]|V_k|^2+\frac{1}{2}N_k^2\operatorname{div}[\tilde{H}[v_0+\eta v_1]]\\
&+\frac{1}{2}|V_k|^2\operatorname{div}[[n_0+\eta n_1][v_0+v_1]]+r_k^1\tilde{H}N_k+[n_0+\eta n_1]r_k^2\cdot V_k.
\end{aligned}$$

On the other hand, differentiating (1.13e) with respect to t and using (1.13a) and (1.13c) gives

$$\begin{aligned}
-\partial_t\Delta\phi_1=&-[\nu_0+\eta\nu_1]\cdot\nabla\rho_1-[\rho_0+\eta\rho_1]\operatorname{div}[\nu_1]\\
&+v_0\cdot\nabla n_1+\frac{1}{\eta}\operatorname{div}\left[[n_0+\eta n_1]v_1\right]-\sigma_2+\sigma_0.
\end{aligned}\tag{2.5}$$

Applying $(\eta\partial_t)^k$ to this we have

$$\begin{aligned}
-\partial_t\Delta\Phi_k&+[\nu_0+\eta\nu_1]\cdot\nabla R_k+[\rho_0+\eta\rho_1]\operatorname{div}[U_k]\\
&-[v_0+v_1]\cdot\nabla N_k-\frac{1}{\eta}\operatorname{div}\left[[n_0+\eta n_1]V_k\right]=r_k^3,
\end{aligned}\tag{2.6}$$

where

$$\|r_k^3\|_{L^2}\lesssim_{K,L}1.$$

Then we multiply (2.6) by Φ_k to get

$$\begin{aligned}
&\partial_t\left[\frac{1}{2}|\nabla\Phi_k|^2\right]+\frac{1}{\eta}[n_0+\eta n_1]V_k\cdot\nabla\Phi_k\\
&\quad-\operatorname{div}\left[\Phi_k\left[\nabla\partial_t\Phi_k-R_k[\nu_0+\eta\nu_1]-[\rho_0+\eta\rho_1]U_k\right]\right]\\
&\quad-\operatorname{div}\left[\Phi_k\left[N_k[v_0+v_1]+\frac{1}{\eta}[n_0+\eta n_1]V_k\right]\right]\\
&=\mathcal{R}_2,
\end{aligned}\tag{2.7}$$

where

$$\mathcal{R}_2:=R_k\operatorname{div}[\Phi_k[\nu_0+\eta\nu_1]]+U_k\cdot\nabla[[\rho_0+\eta\rho_1]\Phi_k]-N_k\operatorname{div}[\Phi_k[v_0+v_1]]+r_k^3\Phi_k.$$

Adding (2.4) and (2.7) and integrating it over $[0,t]\times X$, we have

$$\begin{aligned}
&\int_X\left[\frac{1}{2}\tilde{H}N_k^2+\frac{1}{2}[n_0+\eta n_1]|V_k|^2+\frac{1}{2}|\nabla\Phi_k|^2\right](t,x)\,dx\\
&=\int_X\left[\frac{1}{2}\tilde{H}N_k^2+\frac{1}{2}[n_0+\eta n_1]|V_k|^2+\frac{1}{2}|\nabla\Phi_k|^2\right](0,x)\,dx\\
&\quad+\int_0^t\int_X\mathcal{R}_1+\mathcal{R}_2\,dxdt\\
&\le C_0+tG(K),
\end{aligned}$$

where we have also used (2.3) and Lemma 2.1 in deriving the last inequality. This completes the proof. □

Now, we obtain control of n_1, v_1, and ϕ_1 in H^s norms from the equation.

Lemma 2.4. *There holds that*

$$\sum_{k=0}^{s} \| (\eta\partial_t)^k (n_1, v_1, \nabla\phi_1)\|_{H^{s-k}} \lesssim C_0 + (t+\eta)C(K).$$

Proof. For $k = 0, \dots, s-1$ and $r = 0, \dots, s-k-1$, it suffices to show

$$\|\nabla\Phi_k\|_{H^{r+1}} \lesssim \begin{cases} \|N_k\|_{H^r} + \eta C(K) & \text{if } k \geq 1, \\ \|N_k\|_{H^r} + tC(K) & \text{if } k = 0, \end{cases} \tag{2.8}$$

$$\|V_k\|_{H^{r+1}} \lesssim \|N_{k+1}\|_{H^r} + \|V_k\|_{H^r} + C_0 + (t+\eta)C(K), \tag{2.9}$$

$$\|N_k\|_{H^{r+1}} \lesssim \|V_{k+1}\|_{H^r} + \|N_k\|_{H^r} + (t+\eta)C(K). \tag{2.10}$$

Indeed, from these three, using Lemma 2.3, we have

$$\begin{aligned} \|(N_k, V_k, \nabla\Phi_k)\|_{H^{s-k}} &\lesssim \sum_{j=0}^{1} \|(N_{k+j}, V_{k+j}, \nabla\Phi_{k+j})\|_{H^{s-k-1}} + C_0 + (t+\eta)C(K) \\ &\lesssim \sum_{j=0}^{s-k} \|(N_{k+j}, V_{k+j}, \nabla\Phi_{k+j})\|_{L^2} + C_0 + (t+\eta)C(K) \\ &\lesssim C_0 + (t+\eta)C(K). \end{aligned}$$

We start with (2.8). Applying elliptic estimates and using the third line of (2.2) and the fact that the average of Φ_k is zero, we have

$$\|\nabla\Phi_k\|_{H^{r+1}} \lesssim \|(N_k, R_k)\|_{H^r}.$$

For the case $k \geq 1$, the right hand side is bounded by $C\|N_k\|_{H^r} + \eta C(K)$ thanks to Lemma 2.1. For the case $k = 0$, we use that

$$\|R_0(t)\|_{H^r} = \|\rho_1(t)\|_{H^r} = \left\|\rho_1(0) + \int_0^t \partial_t\rho_1(\tau)\,d\tau\right\|_{H^r} \leq C_0 + Lt \leq C_0 + tC(K).$$

Next let us show (2.9). Taking the curl of the second equation in (2.2), we obtain that

$$[\partial_t \mathrm{curl}[V_k] + [v_0 + v_1]\cdot\nabla\mathrm{curl}[V_k]] = \mathrm{curl}[r_k^2] - [\mathrm{curl}[v_0+v_1]]\cdot\nabla V_k,$$

from which a direct energy estimate gives

$$\|\mathrm{curl}[V_k]\|_{H^r} \leq C_0 + tC(K).$$

On the other hand, using Lemma 2.1 and the first equation in (2.2), we see that

$$\begin{aligned} \|\mathrm{div}[V_k]\|_{H^r} &= \left\|\frac{1}{n_0+\eta n_1}\left[N_{k+1} + \eta[v_0+v_1]\cdot\nabla N_k + \nabla[n_0+\eta n_1]\cdot V_k - \eta r_k^1\right]\right\|_{H^r} \\ &\lesssim \|N_{k+1}\|_{H^r} + \|V_k\|_{H^r} + \eta C(K). \end{aligned}$$

From these two, we conclude (2.9).

Using Lemma 2.1, the second equation in (2.2) and (2.8), we see that

$$\begin{aligned} \|\nabla N_k\|_{H^r} &= \left\|\tilde{H}^{-1}\left[V_{k+1} + \eta\,[v_0+v_1]\cdot\nabla V_k + N_k\nabla\tilde{H} - \nabla\Phi_k - \eta r_k^2\right]\right\|_{H^r} \\ &\lesssim \|V_{k+1}\|_{H^r} + \|N_k\|_{H^r} + (t+\eta)C(K), \end{aligned}$$

which gives (2.10) and completes the proof. □

2.4. **Second step: energy estimates for the ion variables.** Proceeding as above, there is little that can be inferred from control of $(\eta\partial_t)^k(\rho_1, u_1)$. However, we can easily recover them from simpler energy estimates.

Lemma 2.5. *There holds that*

$$\|(\rho_1, \nu_1)\|_{H^s} \lesssim C_0 + tC(K).$$

Proof. Simply taking α derivatives of the last three equations in (1.13), we obtain

$$\begin{aligned}\partial_t R^\alpha + [\nu_0 + \eta\nu_1]\,\nabla R^\alpha + [\rho_0 + \eta\rho_1]\,\mathrm{div} U^\alpha &= \varepsilon_1^\alpha,\\ \partial_t U^\alpha + [\nu_0 + \eta\nu_1]\cdot\nabla U^\alpha + \gamma'(\rho_0 + \eta\rho_1)\nabla R^\alpha + \nabla\Phi^\alpha &= \varepsilon_2^\alpha,\end{aligned}$$

where

$$\begin{aligned}(R^\alpha, U^\alpha, \Phi^\alpha) &= \partial_x^\alpha(\rho_1, \nu_1, \phi),\\ \|\varepsilon_i^\alpha\|_{H^{s-|\alpha|}} &\lesssim C(K), \quad i = 1, 2.\end{aligned}$$

Consequently, multiplying the first line by $R^\alpha/(\rho_0 + \eta\rho_1)$, the second line by $U^\alpha/\tilde{\gamma}'(\rho_0 + \eta\rho_1)$ and summing, we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\left[\frac{1}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)}|U^\alpha|^2 + \frac{1}{\rho_0 + \eta\rho_1}|R^\alpha|^2\right]\\ &\quad + \mathrm{div}\left[R^\alpha U^\alpha + \frac{\nu_0 + \eta\nu_1}{\rho_0 + \eta\rho_1}\frac{|R^\alpha|^2}{2} + \frac{\nu_0 + \eta\nu_1}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)}\frac{|U^\alpha|^2}{2}\right]\\ &= \frac{|R^\alpha|^2}{2}\mathrm{div}\left[\frac{\nu_0 + \eta\nu_1}{\rho_0 + \eta\rho_1}\right] + \frac{|U^\alpha|^2}{2}\mathrm{div}\left[\frac{\nu_0 + \eta\nu_1}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)}\right] - \frac{U^\alpha\cdot\nabla\Phi^\alpha}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)}\\ &\quad + \frac{\varepsilon_1^\alpha R^\alpha}{\rho_0 + \eta\rho_1} + \frac{\varepsilon_2^\alpha\cdot U^\alpha}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)} + \frac{|U^\alpha|^2}{2}\partial_t\frac{1}{\tilde{\gamma}'(\rho_0 + \eta\rho_1)} + \frac{|R^\alpha|^2}{2}\partial_t\frac{1}{\rho_0 + \eta\rho_1}.\end{aligned}$$

Using (2.8) with $k = 0$ and $r = s - 1$, we get energy estimates that allow to finish the proof. □

Now we can obtain our main estimates. We just have to let $K' = 2C_0'$, where C_0' is the sum of all the constants C_0 that appear in the previous estimates (in particular, it only depends on the initial data). Then, we see that there exists $\eta(K') > 0$ and $T(K')$ such that we have an energy estimates on $[0, T(K')]$ for all $\eta \le \eta(K')$. This finishes the proof of Theorem 1.1.

2.5. **Adaptation in the case $X = \mathbb{R}^3$.** The problem is that Δ cannot be properly inverted on L^2. We replace (1.13e) by a time-evolution equation for $\nabla\phi_1$, by applying $\nabla(-\Delta)^{-1}$ to (2.5),

$$\eta\partial_t\nabla\phi_1 = \nabla(-\Delta)^{-1}\mathrm{div}\left[-\eta[\rho_1\nu_0 + [\rho_0 + \eta\rho_1]\nu_1] + \eta[n_1 v_0] + [[n_0 + \eta n_1]v_1]\right] \tag{2.11}$$

with initial data

$$\nabla\phi_1(0) = \nabla(-\Delta)^{-1}\left[\rho_1(0) - n_1(0)\right] \in L^2 \tag{2.12}$$

The equation and initial data are well-defined thanks to the facts $\nabla(-\Delta)^{-1}\mathrm{div} : L^2 \to L^2$, $\nabla(-\Delta)^{-1} : \dot{H}^{-1} \to L^2$, and

$$\|\nabla(-\Delta)^{-1}\mathrm{div}(u)\|_{L^2} \lesssim \|u\|_{L^2}. \tag{2.13}$$

By applying div to (2.11) and integrating it over $[0, t]$, one sees that (2.11) with (2.12) implies (1.13e).

We just present the adjustments needed to adapt the proofs of Lemmas 2.1–2.5. Let us define

$$K_{\mathbb{R}^3} := \sup_{t\in[0,T]} \|(n_1, v_1, \rho_1, \nu_1, \nabla\phi_1)\|_{H^s}$$

and prove by induction that

Lemma 2.6. *There holds that*

$$L_{\mathbb{R}^3} := \sup_{t\in[0,T]} \sum_{k=1}^{s} \left[\| (\eta\partial_t)^k (n_1, v_1, \nabla\phi_1)\|_{H^{s-k}} + \eta^{-1}\| (\eta\partial_t)^k (\rho_1, \nu_1)\|_{H^{s-k}}\right] \lesssim_{K_{\mathbb{R}^3}} 1.$$

Adaptation of the proof of Lemma 2.1 to proving Lemma 2.6. It suffices to adjust the estimate of $(\eta\partial_t)^k\nabla\phi_1$ in the induction part. Using (2.11) and (2.13), we simply obtain $\|(\eta\partial_t)^k\nabla\phi_1\|_{H^{s-k}} \lesssim_{K_{\mathbb{R}^3}} 1$. □

For Lemma 2.2, the last equation in (2.2) need to be changed.

Lemma 2.7. *For $k = 0, \dots s$, there holds that*

$$\partial_t\nabla\Phi_k = \frac{1}{\eta}\nabla(-\Delta)^{-1}\mathrm{div}\,[[n_0 + \eta n_1]V_k] + r_k^4, \tag{2.14}$$

where

$$\|r_k^4\|_{H^{s-k}} \lesssim_{K_{\mathbb{R}^3}} 1. \tag{2.15}$$

Proof. The estimate (2.15) follows from (2.13) and Lemma 2.6. □

Now, we show how to adapt Lemma 2.3.

Lemma 2.8. *Define*

$$\mathcal{E}(t) := \sum_{j=0}^{s} \| (\eta\partial_t)^j (n_1, v_1, \nabla\phi_1)(t)\|_{L^2}^2.$$

We have the following energy estimates:

$$\mathcal{E}(t) \le C_0 + tC(K_{\mathbb{R}^3}) \tag{2.16}$$

as long as $K_{\mathbb{R}^3} \lesssim 1$.

Adaptation of the Proof of Lemma 2.3. We only need to replace equality (2.7) by

$$\partial_t\left[\frac{1}{2}|\nabla\Phi_k|^2\right] - \frac{1}{\eta}\nabla(-\Delta)^{-1}\left[\mathrm{div}\,[[n_0 + \eta n_1]V_k]\right]\cdot\nabla\Phi_k = r_k^4\cdot\nabla\Phi_k, \tag{2.17}$$

which can be obtained by multiplying (2.14) by $\frac{1}{\eta}\nabla\Phi_k$.

Adding (2.4) and (2.17), integrating it over $[0, t] \times X$, and reminding that

$$-\frac{1}{\eta}\int_X \nabla(-\Delta)^{-1}\left[\mathrm{div}\,[[n_0 + \eta n_1]V_k]\right]\cdot\nabla\Phi_k = \frac{1}{\eta}\int_X [n_0 + \eta n_1]V_k\cdot\nabla\Phi_k, \tag{2.18}$$

we have

$$\begin{aligned}
&\int_X \left[\frac{1}{2}\tilde{H}N_k^2 + \frac{1}{2}[n_0 + \eta n_1]|V_k|^2 + \frac{1}{2}|\nabla\Phi_k|^2\right](t, x)\,dx \\
&= \int_X \left[\frac{1}{2}\tilde{H}N_k^2 + \frac{1}{2}[n_0 + \eta n_1]|V_k|^2 + \frac{1}{2}|\nabla\Phi_k|^2\right](0, x)\,dx \\
&\quad + \int_0^t \int_X \mathcal{R}_1 + r_k^4\cdot\nabla\Phi_k\,dxdt \\
&\le C_0 + tG(K_{\mathbb{R}^3}),
\end{aligned}$$

where we have also used Lemma 2.6 in deriving the last inequality. □

Lemma 2.9. *There holds that*

$$\sum_{k=0}^{s} \| (\eta\partial_t)^k (n_1, v_1, \nabla\phi_1)\|_{H^{s-k}} \lesssim C_0 + (t+\eta)C(K_{\mathbb{R}^3}).$$

Adaptation of the Proof of Lemma 2.4. We only need to replace (2.8) by

$$\|\nabla\Phi_k\|_{H^{r+1}} \lesssim \begin{cases} \|N_k\|_{H^r} + C_0 + (\eta+t)C(K_{\mathbb{R}^3}) & \text{if } k \geq 1, \\ \|N_k\|_{H^r} + C_0 + tC(K_{\mathbb{R}^3}) & \text{if } k = 0. \end{cases} \tag{2.19}$$

Indeed, since (2.11) with (2.12) implies (1.13e) and thus we have the third line of (2.2). Then applying standard elliptic theory, we see that

$$\begin{aligned} \|\nabla\Phi_k\|_{H^{r+1}} &\leq C\|(N_k, R_k)\|_{H^r} + C\|\nabla\Phi_k\|_{L^2} \\ &\leq C\|(N_k, R_k)\|_{H^r} + C_0 + (\eta+t)C(K_{\mathbb{R}^3}), \end{aligned}$$

where we have used (2.16). One then gets (2.19) as in the proof of Lemma 2.4. □

Lemma 2.10. *There holds that*

$$\|(\rho_1, \nu_1)\|_{H^s} \lesssim C_0 + tC(K_{\mathbb{R}^3}).$$

Adaptation of the Proof of Lemma 2.5. We use (2.19) instead of (2.8). □

3. Localization of limit solutions

In this section, we discuss the localization of limit solutions when the underlying space is $X = \mathbb{R}^3$. This property is essential to control the initial layer and justify the derivation of the electron equations (1.9) and (1.10). We consider the limit equations (1.8) for the ions and show the following lemma, where $E_0(T)$ is the linearized energy:

$$E_0(T) := \sup_{[0,T]} \|(\rho_0 - \bar{\rho}, \nu_0, \phi_0)(t)\|_{H^3}^2.$$

Lemma 3.1. *Let $k = 1, 2, 3$. Assume that (ρ_0, ν_0, ϕ_0) is a solution of (1.8) as $(\rho_0 - \bar{\rho}, \nu_0, \nabla\phi_0) \in C^1([0,T] : H^3)$ and, in addition, the initial data $(\rho_0(0), \nu_0(0))$ satisfies*

$$\|\langle x\rangle^k [\rho_0 - \bar{\rho}](0)\|_{H^3}^2 + \|\langle x\rangle^k \nu_0(0)\|_{H^3}^2 \leq \delta$$

for some $\delta > 0$. Then there holds that

$$\sup_{[0,T]} \|(\langle x\rangle^k [\rho_0 - \bar{\rho}], \langle x\rangle^k \nu_0, \langle x\rangle^k \phi_0)(t)\|_{H^3}^2 \leq C(T, E_0(T))\delta. \tag{3.1}$$

Furthermore, for $n_0 = \tilde{n}(\phi_0)$, it holds that

$$\sup_{[0,T]} \|\langle x\rangle^k [n_0 - \bar{n}](t)\|_{H^3}^2 \leq C(T, E_0(T))\delta. \tag{3.2}$$

For this proof, we first introduce the weighted unknown functions

$$\rho_{R,\beta,k} := \langle \beta x\rangle^k \varphi_R [\rho_0 - \bar{\rho}], \quad \nu_{R,\beta,k} := \langle \beta x\rangle^k \varphi_R \nu_0, \quad \phi_{R,\beta,k} := \langle \beta x\rangle^k \varphi_R \phi_0$$

for $R > 0$, $k = 1, 2, 3$, and $\beta \in (0, 1]$, where $\varphi_R := \varphi_0^2(x/R)$ for a radially non-increasing function φ_0 such that $\varphi_0(x) = 1$ when $|x| \leq 1$, $\varphi_0(x) = 0$ if $|x| \geq$

2. Furthermore, we rewrite equations (1.8) into those for the weighted functions $(\rho_{R,\beta,k}, \nu_{R,\beta,k}, \phi_{R,\beta,k})$ by using the facts $\bar{\rho} = \bar{n}$ and $\tilde{n}(0) = \bar{n}$.

$$
\begin{aligned}
&\partial_t \rho_{R,\beta,k} + \nu_0 \cdot \nabla \rho_{R,\beta,k} + \rho_0 \mathrm{div}[\nu_{R,\beta,k}] \\
&\qquad = [\nu_0 \cdot w^1_{\beta,k}]\rho_{R,\beta,k} + \rho_0 w^1_{\beta,k} \cdot \nu_{R,\beta,k} + \tau^1_{R,\beta,k}, \\
&\partial_t \nu_{R,\beta,k} + \nu_0 \cdot \nabla \nu_{R,\beta,k} + \tilde{\gamma}'(\rho_0)\nabla \rho_{R,\beta,k} + \nabla \phi_{R,\beta,k} \\
&\qquad = [\nu_0 \cdot w^1_{\beta,k}]\nu_{R,\beta,k} + \tilde{\gamma}'(\rho_0)\rho_{R,\beta,k} w^1_{\beta,k} + \phi_{R,\beta,k} w^1_{\beta,k} + \tau^2_{R,\beta,k}, \\
&-\Delta \phi_{R,\beta,k} + \tilde{N}\phi_{R,\beta,k} \\
&\qquad = -2w^1_{\beta,k} \cdot \nabla \phi_{R,\beta,k} - [w^2_{\beta,k} - 2|w^1_{\beta,k}|^2]\phi_{R,\beta,k} + \rho_{R,\beta,k} + \tau^3_{R,\beta,k},
\end{aligned}
\tag{3.3}
$$

where

$$
\tilde{N} := \int_0^1 \tilde{n}'(\theta\phi_0)\, d\theta \geq \tilde{c}_0 > 0, \qquad w^1_{\beta,k} := \frac{\nabla\langle \beta x\rangle^k}{\langle \beta x\rangle^k}, \quad w^2_{\beta,k} := \frac{\Delta\langle \beta x\rangle^k}{\langle \beta x\rangle^k}.
$$

Furthermore, we also see that

$$
\begin{aligned}
&\int_0^T \|(\tau^1_{R,\beta,k}, \tau^2_{R,\beta,k}, \tau^3_{R,\beta,k})(t)\|^2_{H^3}\, dt \\
&\lesssim_{E_0(T)} \beta^{-1}(\beta R)^{k-1} \int_0^T \|(\rho_0 - \bar{\rho}, \nu_0, \phi_0, \nabla\phi_0)(t)\|^2_{H^3(R \leq |x| \leq 2R)}\, dt.
\end{aligned}
\tag{3.4}
$$

We now show the localization.

Proof of Lemma 3.1. We begin by showing that

$$
\sup_{[0,T]} \|(\rho_{R,\beta,k}, \nu_{R,\beta,k})(t)\|^2_{H^3} \lesssim_{E_0(T),T} \delta + \int_0^T \|(\tau^1_{R,\beta,k}, \tau^2_{R,\beta,k}, \tau^3_{R,\beta,k})(t)\|^2_{H^3}\, dt. \tag{3.5}
$$

It is easy to check that $w^1_{\beta,k}$ and $w^2_{\beta,k}$ belong to $W^{5,\infty}$ and satisfy

$$
\|w^1_{\beta,k}\|_{L^\infty} \leq 3\beta \quad \text{for } k = 1, 2, 3.
$$

Now let us fix β by $\min\{1, \sqrt{\tilde{c}_0}/9\}$ for $\tilde{c}_0$ being in the definition of $\tilde{N}$, and apply energy method to the third equality of (3.3) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left[|\nabla \phi_{R,\beta,k}|^2 + \tilde{N}\phi^2_{R,\beta,k} \right] dx \\
&= \int_{\mathbb{R}^3} [|w^1_{\beta,k}|^2 \phi_{R,\beta,k} + \rho_{R,\beta,k} + \tau^3_{R,\beta,k}]\phi_{R,\beta,k}\, dx \\
&\leq \frac{2\tilde{c}_0}{3} \int_X \phi^2_{R,\beta,k}\, dx + C(\tilde{c}_0) \int_{\mathbb{R}^3} [\rho^2_{R,\beta,k} + [\tau^3_{R,\beta,k}]^2]\, dx,
\end{aligned}
$$

so that $\|\phi_{R,\beta,k}(t)\|_{H^1} \lesssim_{E(T_0)} \|(\rho_{R,\beta,k}, \tau^3_{R,\beta,k})\|_{L^2}$. Elliptic estimates then give

$$
\|\phi_{R,\beta,k}(t)\|_{H^5} \lesssim_{E(T_0)} \|(\rho_{R,\beta,k}, \tau^3_{R,\beta,k})\|_{H^3}. \tag{3.6}
$$

Performing the same energy method in as the proof of Lemma 2.5 for the first and second equations in (3.3), and using the above elliptic estimate and the fact $\|(\rho_{R,\beta,k}, \nu_{R,\beta,k})(0)\|^2_{H^3} \leq \delta$, we conclude that

$$
\begin{aligned}
&\|(\rho_{R,\beta,k}, \nu_{R,\beta,k})(t)\|^2_{H^3} \\
&\lesssim_{E_0(T)} \delta + \int_0^t \left[\|(\rho_{R,\beta,k}, \nu_{R,\beta,k})(s)\|^2_{H^3} + \|(\tau^1_{R,\beta,k}, \tau^2_{R,\beta,k}, \tau^3_{R,\beta,k})(s)\|^2_{H^3} \right] ds,
\end{aligned}
$$

which together with the Gronwall lemma leads to (3.5).

From now on we derive (3.1). In the case $k = 1$, we see from (3.4) that the righthand side of (3.5) converges to zero as R tends to infinity and thus have

$$\|(\langle\beta x\rangle[\rho_0 - \bar{\rho}], \langle\beta x\rangle\nu_0)(t)\|_{H^3}^2 \lesssim_{E_0(T),T} \delta.$$

We can then plug this into (3.6) and let $R \to \infty$ to get

$$\|\langle\beta x\rangle\phi_0(t)\|_{H^5}^2 \lesssim_{E_0(T),T} \delta.$$

Induction over k then gives the result.

Finally (3.2) follows from the mean value theorem and the fact that $n_0 = \tilde{n}(\phi_0)$ and $\bar{n} = \tilde{n}(0)$. The proof is complete. □

4. Decay of remainder

In this section, we assume Theorem 1.3 for a moment and finish the proof of Theorem 1.2 by proving the following:

Proposition 4.1. *If the initial data $(n_1(0), v_1(0), \rho_1(0), \nu_1(0)) \in H^s(\mathbb{R}^3)$ satisfies (1.14), (1.18), $\rho_1(0) = 0$, and $\nu_1(0) = 0$, then the remainder $(n_r, v_r, \rho_r, \nu_r, \phi_r)$ converges to 0 in $H^{s-1}(\mathbb{R}^3)$ as $\eta \to 0$.*

The proof is based on the energy method used in Section 2.5. The remainder $(n_r, v_r, \rho_r, \nu_r, \phi_r)$ satisfies the equations

$$\begin{aligned}
&\partial_t n_r + [v_0 + v_\ell] \cdot \nabla n_r + \frac{1}{\eta}\mathrm{div}[[n_0 + \eta n_\ell + \eta n_r]v_r] + n_r \mathrm{div}[v_0 + v_\ell] \\
&\qquad = -\mathrm{div}[n_\ell v_0], && \text{(4.1a)}\\
&\partial_t v_r + [v_0 + v_\ell + v_r] \cdot \nabla v_r + \frac{1}{\eta}\nabla[\tilde{H}_r n_r - \phi_r] + v_r \cdot \nabla[v_0 + v_\ell] \\
&\qquad = -\nabla[v_0 \cdot v_\ell] + v_\ell \times \mathrm{curl}[v_0], && \text{(4.1b)}\\
&\partial_t \rho_r + \nu_0 \cdot \nabla \rho_r + \mathrm{div}[\rho_0 \nu_r] + \rho_r \mathrm{div}[\nu_0] = -\eta \mathrm{div}[\rho_r \nu_r], && \text{(4.1c)}\\
&\partial_t \nu_r + \nu_0 \cdot \nabla \nu_r + \nabla\left[\tilde{G}_r \rho_r + \phi_r\right] + \nu_1 \cdot \nabla \nu_0 + \nabla \phi_\ell = -\eta \nu_r \cdot \nabla \nu_r, && \text{(4.1d)}\\
&-\Delta \phi_r = \rho_r - n_r && \text{(4.1e)}
\end{aligned}$$

with the initial data condition $(n_r(0), v_r(0), \rho_r(0), \nu_r(0)) = (0, 0, 0, 0)$, where

$$\tilde{H}_r := \int_0^1 \tilde{h}'(n_0 + \eta n_\ell + \theta\eta n_r)\, d\theta > 0, \quad \tilde{G}_r := \int_0^1 \tilde{\gamma}'(\rho_0 + \theta\eta\rho_r)\, d\theta > 0.$$

Similarly to Section 2.5, we rewrite (4.1e) into an evolution equation for $\nabla\phi_r$ as

$$\begin{aligned}
\partial_t \nabla\phi_r = \nabla(-\Delta)^{-1}\mathrm{div}&\left[\frac{1}{\eta}[[n_0 + \eta n_l + \eta n_r]v_r] - \rho_r\nu_0 - \rho_0\nu_r + n_r[v_0 + v_\ell]\right] \\
&+ \nabla(-\Delta)^{-1}\mathrm{div}[n_\ell v_0 - \eta\rho_r\nu_r].
\end{aligned} \tag{4.2}$$

We notice from Theorems 1.1 and 1.3 that

$$\sup_{[0,\min\{T_0,T_\ell\}]} [\|(n_r, v_r, \rho_r, \nu_r, \nabla\phi_r)\|_{H^s} + \|(\eta\partial_t n_r, \eta\partial_t v_r, \partial_t\rho_r, \partial_t\nu_r)\|_{H^{s-1}}] \lesssim 1. \tag{4.3}$$

Furthermore, we also see the following convergence.

Lemma 4.1. *For any $T \in [0, \min\{T_0, T_l\}]$, the function $\mathcal{D}_\ell(t) := \|(n_\ell v_0, v_0 \cdot v_\ell, v_\ell \times \mathrm{curl}[v_0])(t)\|_{L^2}$ satisfies*

$$\int_0^T \mathcal{D}_\ell^2(t)\,dt \to 0 \quad \text{as } \eta \to 0. \tag{4.4}$$

Proof. For any $\epsilon > 0$, there exists $R_0 > 0$ such that

$$\int_0^T \int_{|x| \geq R_0} [|v_0|^2 + |\nabla v_0|^2]\,dtdx < \epsilon.$$

Using this, the uniform estimate (4.3), and the local decay estimate (1.21), and then setting η in (1.21) so that $|\log \eta|^{\frac{1}{4}} > R_0$ and $|\log \eta|^{-\frac{1}{2}} < \epsilon$, we conclude that

$$\begin{aligned}\int_0^T \mathcal{D}_\ell^2\,dt &\lesssim \int_0^T \int_{|x| \leq |\log \eta|^{\frac{1}{4}}} [n_\ell^2 + |v_\ell|^2]\,dtdx + \int_0^T \int_{|x| \geq R_0} [|v_0|^2 + |\nabla v_0|^2]\,dtdx \\ &\lesssim \epsilon,\end{aligned}$$

which implies the desired convergence. □

The rest of this section is devoted to the proof of Proposition 4.1. It suffices to show that

$$\partial_t[\mathcal{E}_r(t) + \eta \mathcal{F}_\ell(t)] \lesssim \mathcal{E}_r(t) + \mathcal{D}_\ell(t) + \eta, \tag{4.5}$$

where $\sup_{[0,\min\{T_0,T_l\}]} |\mathcal{F}_\ell(t)| \lesssim 1$ and

$$\mathcal{E}_r(t) := \int_X \left[\tilde{G}_r \rho_r^2 + \rho_0 |\nu_r|^2 + \tilde{H}_r n_r^2 + [n_0 + \eta n_\ell + \eta n_r]|v_r|^2 + |\nabla \phi_r|^2\right] dx.$$

Indeed, Gronwall lemma allows us to conclude that $\sup_{[0,\min\{T_0,T_\ell\}]} \mathcal{E}_r(t) \to 0$ as $\eta \to 0$. This and (4.3) lead to Proposition 4.1 with the aid of the Gagliardo–Nirenberg inequality.

Remark 4.1. *We can obtain rates of convergence in* (4.4) *and thus for the remainders if the localization $\|\langle x\rangle^\alpha v_0\|_{L^\infty} < +\infty$ holds for some $\alpha > 0$.*

4.1. **Ion Variables.** We start by energy estimates for the ion variables. Multiply (4.1c) by $\tilde{G}_r \rho_r$ and (4.1d) by $\rho_0 \nu_r$, respectively. Add these two results and integrate it over $\mathbb{R}^3$. Then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \left[\tilde{G}_r \rho_r^2 + \rho_0 |\nu_r|^2\right] dx + \int_{\mathbb{R}^3} \rho_0 \nu_r \cdot \nabla \phi_\ell\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \left[[\partial_t \tilde{G}_r]\rho_r^2 + [\partial_t \rho_0]|\nu_r|^2 + [\mathrm{div}[\tilde{G}_r \nu_0]]\rho_r^2 + [\mathrm{div}[\rho_0 \nu_0]]|\nu_r|^2\right] dx \\ &\quad - \int_{\mathbb{R}^3} \left[[\rho_r \mathrm{div}[\nu_0]]\tilde{G}_r \rho_r + [-\nabla \phi_r + \nu_r \cdot \nabla \nu_0] \cdot [\rho_0 \nu_r]\right] dx \\ &\quad - \eta \int_{\mathbb{R}^3} \left[[\mathrm{div}[\rho_r \nu_r]]\tilde{G}_r \rho_r + [\nu_r \cdot \nabla \nu_r] \cdot [\rho_0 \nu_r]\right] dx \\ &\lesssim \mathcal{E}_r(t) + \eta,\end{aligned} \tag{4.6}$$

where we have also used (4.3) and Theorem 1.3 in deriving the inequality. Thus, we see that the main task is to control the forcing by the fast oscillating electrostatic field $\nabla \phi_\ell$. This is done by a normal form transformation using the fast time oscillations of ϕ_ℓ. In order to use the fast oscillations in time, we write

$$v_\ell = \nabla \Psi_\ell, \quad (\partial_t v_0 + v_0 \cdot \nabla v_0) = \nabla \Upsilon_0$$

with the aid of the facts (1.10) and (1.17c). Then, from (1.17b) and (1.17d), we have the equation

$$-\tilde{h}'(n_0)\Delta\phi_\ell + \phi_\ell = \eta\partial_t\Psi_\ell + \eta\sigma_r,$$

where

$$\sigma_r := \frac{|v_\ell|^2}{2} + \left[\frac{\tilde{h}(n_0+\eta n_\ell) - \tilde{h}(n_0)}{\eta} - \tilde{h}'(n_0)n_\ell\right] + \Upsilon_0.$$

We postpone until Lemma 6.3 in Appendix to show that $[1-\tilde{h}'(n_0)\Delta]^{-1}$ is well-defined. Apply $\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}$ to the above equation and use the time-derivative formula in Lemma 6.3 to obtain

$$\begin{aligned}
\nabla\phi_\ell &= \eta\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}[\partial_t\Psi_\ell + \sigma_r] \\
&= \eta\partial_t\left[\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\Psi_\ell\right] + \eta\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\left[\frac{\partial_t[\tilde{h}'(n_0)]}{\tilde{h}'(n_0)}[\Psi_\ell - \phi_\ell] + \sigma_r\right].
\end{aligned}$$

Using this and letting

$$\mathcal{F}_\ell(t) := \int_{\mathbb{R}^3} \rho_0\nu_r \cdot \nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\Psi_\ell\, dx,$$

we control the fast oscillating electrostatic field $\nabla\phi_\ell$ as

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho_0\nu_r \cdot \nabla\phi_\ell dx &= \partial_t[\eta\mathcal{F}_\ell] - \eta\int_{\mathbb{R}^3} \partial_t[\rho_0\nu_r]\cdot\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\Psi_\ell\, dx \\
&\quad + \eta\int_{\mathbb{R}^3} \rho_0\nu_r\cdot\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\left[\frac{\partial_t[\tilde{h}'(n_0)]}{\tilde{h}'(n_0)}[\Psi_\ell-\phi_\ell]+\sigma_r\right]dx \\
&\leq \partial_t[\eta\mathcal{F}_\ell] + C\eta,
\end{aligned} \tag{4.7}$$

where we also have used (2.13), (4.3), Theorem 1.3, Lemma 6.3, and Sobolev's inequality in deriving the inequality. We now note that $\sup_{[0,\min\{T_0,T_l\}]}|\mathcal{F}_\ell(t)| \lesssim 1$ holds.

4.2. **Electron variables.** Let us estimate electron variables in the same way as in the proof of Lemma 2.8. Multiply (4.1a) by $\tilde{H}_r n_r$, (4.1b) by $[n_0+\eta n_\ell+\eta n_r]v_r$, and (4.2) by $\nabla\phi_r$, respectively. Add these three results and integrate it over $\mathbb{R}^3$.

Then, reminding the computation in (2.18), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left[\tilde{H}_r n_r^2 + [n_0+\eta n_\ell+\eta n_r]|v_r|^2 + |\nabla\phi_r|^2\right]\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}^3}\left[[\partial_t\tilde{H}_r]n_r^2 + [\partial_t[n_0+\eta n_\ell+\eta n_r]]|\nu_r|^2\right]dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^3}\left[[\mathrm{div}[\tilde{H}_r[v_0+v_\ell]]]n_r^2+\mathrm{div}\left[[n_0+\eta n_\ell+\eta n_r][v_0+v_\ell+v_r]\right]|v_r|^2\right]dx\\
&\quad-\int_{\mathbb{R}^3}\left[[n_r\mathrm{div}[v_0+v_\ell]]\tilde{H}_r n_r+[v_r\cdot\nabla[v_0+v_\ell]]\cdot[[n_0+\eta n_\ell+\eta n_r]v_r]\right]dx\\
&\quad-\int_{\mathbb{R}^3}\left[\nabla(-\Delta)^{-1}\mathrm{div}\left[\rho_r\nu_0+\rho_0\nu_r-n_r[v_0+v_\ell]\right]\right]\cdot\nabla\phi_r\,dx\\
&\quad+\int_{\mathbb{R}^3}\left[[v_0\cdot v_\ell]\mathrm{div}[[[n_0+\eta n_\ell+\eta n_r]v_r]+[v_\ell\times\mathrm{curl}[v_0]]\cdot[[n_0+\eta n_\ell+\eta n_r]v_r]\right]dx\\
&\quad+\int_{\mathbb{R}^3}n_\ell v_0\cdot\nabla[\tilde{H}_r n_r]+\left[\nabla(-\Delta)^{-1}\mathrm{div}\left[n_\ell v_0-\eta\rho_r\nu_r\right]\right]\cdot\nabla\phi_r\,dx\\
&\lesssim\mathcal{E}_r(t)+\mathcal{D}_\ell(t)+\eta,
\end{aligned}
\tag{4.8}
$$

where we also have used (2.13), (4.3), and Theorem 1.3. Adding (4.6) and (4.8) and then using (4.7), we conclude (4.5). Therefore, the proof of Proposition 4.1 is complete.

5. Local decay of initial layer

This section is devoted to the proof of Theorem 1.3. One can show the independence of T_ℓ with respect to η and the uniform estimate (1.19) in much the same way as the proof of Theorem 1.1. Thus, we omit the proof and focus only on deriving the local decay estimate (1.21).

The proof will rely on estimate on the local energy decay. In order to do this, we compute variants of (1.23) and (1.24) adapted to our case. Letting

$$
G(n_0,\alpha)=\int_{s=0}^{\alpha}s\tilde{h}'(n_0+\eta s)ds,\qquad H(n_0,\alpha)=\int_{s=0}^{\alpha}\tilde{h}'(n_0+\eta s)ds,
$$

then

$$
\eta\partial_t[n_\ell v_{\ell j}]+\mathrm{div}L_j=-e^{(1)}\partial_j n_0-\eta n_\ell\pi_j,\tag{5.1}
$$

for $j=1,2,3$, where $L_j:=(L_{j1},L_{j2},L_{j3})$ and, for $k=1,2,3$,

$$
L_{jk}:=\left[(n_0+\eta n_\ell)v_{\ell j}v_{\ell k}-\delta_{jk}n_0\frac{|v_\ell|^2}{2}\right]-\left[\partial_j\phi_\ell\partial_k\phi_\ell-\delta_{jk}\frac{|\nabla\phi_\ell|^2}{2}\right]+\delta_{jk}G(n_0,n_\ell)
$$

and

$$
\pi=\partial_t v_0+v_0\cdot\nabla v_0,\qquad e^{(1)}=\frac{|v_\ell|^2}{2}+n_\ell\int_{s=0}^{n_\ell}h''(n_0+\eta s)ds,
$$

which satisfy

$$
\|\pi\|_{L^2}\lesssim 1,\qquad |e^{(1)}|\lesssim|v_\ell|^2+|n_\ell|^2.\tag{5.2}
$$

The equirepartition of energy becomes

$$\begin{aligned}&\eta\partial_t[v_\ell\cdot\nabla\phi_\ell]+\mathrm{div}\left[[H(n_0,n_\ell)+\eta\frac{|v_\ell|^2}{2}]\nabla\phi_\ell\right]\\&=n_\ell H(n_0,n_\ell)+|\nabla\phi_\ell|^2-(n_0+\frac{1}{2}\eta n_\ell)|v_\ell|^2\\&\quad+v_\ell\cdot[(n_0+\eta n_\ell),\nabla\Delta^{-1}\mathrm{div}]v_\ell-\eta\nabla\phi_\ell\cdot\pi.\end{aligned}\tag{5.3}$$

Next we introduce some localization operator. For a cut-off scale R, whose precise value will be chosen later, define

$$\varphi_R(x):=\varphi_0^2(R^{-1}x)$$

for a radially non-increasing function φ_0 such that $\varphi_0(x)=1$ when $|x|\le 1$, $\varphi_0(x)=0$ if $|x|\ge 2$. It is easy to check that for α with $|\alpha|\ge 1$,

$$|\partial^\alpha\varphi_R|\lesssim |x|^{-|\alpha|}\mathbf{1}_{\{R\le|x|\le 2R\}}.\tag{5.4}$$

Let us also introduce the linearized energy $E(T_\ell)$ which is bounded uniformly in η:

$$E(T_\ell):=\sup_{[0,T_\ell]}\|(n_\ell,v_\ell,\nabla\phi_\ell)(t)\|_{H^2}^2.$$

The commutator in (5.3) plays an important role and we record a few important properties. It is skew-symmetric

$$\int_{\mathbb{R}^3}v\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]v\,dx=0,\tag{5.5}$$

and has a simpler structure on gradients $v=\nabla\Psi$,:

$$\begin{aligned}{}[n_0,\nabla\Delta^{-1}\mathrm{div}]\nabla\Psi&=-\Psi\nabla n_0+\nabla\Delta^{-1}\mathrm{div}[\Psi\nabla n_0],\\ [n_0,\nabla\Delta^{-1}\mathrm{div}]v&=\Delta^{-1}\mathrm{div}[v\otimes\nabla n_0-\nabla n_0\otimes v]\end{aligned}\tag{5.6}$$

or, in coordinates,

$$[n_0,\nabla\Delta^{-1}\mathrm{div}]\partial_j\Psi=\Delta^{-1}\partial_k[\partial_j\Psi\partial_k n_0-\partial_k\Psi\partial_j n_0].$$

The above controls are complemented by the localization properties of n_0 derived in (3.2).

5.1. **Control of velocity and electric field by density.** This subsection provides several inequalities for the velocity and electric field. Those will be used to control the density n_ℓ later.

Lemma 5.1. *Under the same assumptions as in Theorem 1.3, it holds that*

$$\int_{\mathbb{R}^3}\varphi_R\frac{|\nabla\phi_\ell|^2}{\langle x\rangle^2}\,dx\lesssim\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dx+R^{-1}E(T_\ell)\quad\textit{for } R>0.\tag{5.7}$$

Proof. By Hardy's inequality and (5.4), we observe that

$$\int_{\mathbb{R}^3}\varphi_R\frac{|\nabla\phi_\ell|^2}{\langle x\rangle^2}\,dx\le\int_{\mathbb{R}^3}\varphi_R\frac{|\nabla\phi_\ell|^2}{|x|^2}\,dx\lesssim\|\varphi_R^{1/2}\nabla^2\phi_\ell\|_{L^2}^2+R^{-1}E(T_\ell).\tag{5.8}$$

Next multiply $\partial_i\Delta\phi_\ell=\partial_i n_\ell$ by $\varphi_R\partial_i\phi_\ell$ and integrate the result to obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\varphi_R|\nabla\partial_i\phi_\ell|^2\,dx&=\int_{\mathbb{R}^3}\left[\varphi_R n_\ell^2+(\partial_i\varphi_R)(n_\ell\partial_i\phi_\ell)-\partial_i\phi_\ell[\nabla\varphi_R\cdot\nabla\partial_i\phi_\ell]\right]\,dx\\&\le\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dx+CR^{-1}E(T_\ell),\end{aligned}$$

and using (5.8), we complete the proof. □

Lemma 5.2. *Assume that*

$$\|\langle x\rangle\nabla n_0\|_{L_t^\infty L_x^3}\lesssim\delta, \tag{5.9}$$

and assume that v_ℓ satisfies (5.3)*, then there hold that*

$$\left|\int_{\mathbb{R}^3}\varphi_R v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]v_\ell\,dx\right|\lesssim R^{-1}E(T_\ell), \tag{5.10}$$

$$\int_0^T\int_{\mathbb{R}^3}\varphi_R\frac{|v_\ell|^2}{\langle x\rangle^2}\,dxdt\lesssim\int_0^T\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dxdt+C(E(T_\ell))(R^{-2}+\eta)(1+T) \tag{5.11}$$

for $R>0$ and $T\in[0,\min\{T_0,T_\ell\}]$.

Proof. Let us first show (5.10). Owing to the skew-symmetry (5.5), we find that

$$\int_{\mathbb{R}^3}\varphi_R v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]v_\ell\,dx=\int_{\mathbb{R}^3}\varphi_R v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]([1-\varphi_R]v_\ell)\,dx.$$

To rewrite the right hand side, we first use $v_\ell=\nabla\Psi_\ell$, the product rule, (5.5) and the first line in (5.6), to get

$$\begin{aligned}&\int_{\mathbb{R}^3}\varphi_R v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]([1-\varphi_R]v_\ell)\,dx\\
&=\int_{\mathbb{R}^3}\varphi_R\nabla\Psi_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]\nabla[(1-\varphi_R)\Psi_\ell]\,dx\\
&\quad+\int_{\mathbb{R}^3}\nabla[\varphi_R\Psi_\ell]\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}](\Psi_\ell\nabla\varphi_R)\,dx\\
&=\mathcal{I}_{\ell 1}+\mathcal{I}_{\ell 2}-\mathcal{I}_{\ell 3},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{\ell 1}&:=\int_{\mathbb{R}^3}\varphi_R v_\ell\cdot\left[-(1-\varphi_R)\Psi_\ell\nabla n_0+\nabla\Delta^{-1}\mathrm{div}[(1-\varphi_R)\Psi_\ell\nabla n_0]\right]\,dx,\\
\mathcal{I}_{\ell 2}&:=\int_{\mathbb{R}^3}\varphi_R\Psi_\ell^2[\nabla n_0\cdot\nabla\varphi_R]\,dx,\\
\mathcal{I}_{\ell 3}&:=\int_{\mathbb{R}^3}\nabla\Delta^{-1}\mathrm{div}[\varphi_R\Psi_\ell\nabla n_0]\cdot\Psi_\ell\nabla\varphi_R\,dx.\end{aligned}$$

For the completion of the derivation of (5.10), it suffices to estimate $\mathcal{I}_{\ell 1}$, $\mathcal{I}_{\ell 2}$, and $\mathcal{I}_{\ell 3}$ from above by $R^{-1}E(T_\ell)$.

Let us estimate those terms separately. The term $\mathcal{I}_{\ell 1}$ can be controlled by using (2.13), the localization (3.2), and Sobolev inequality as

$$\begin{aligned}|\mathcal{I}_{\ell 1}|&\lesssim\|v_\ell\|_{L^2}\|\Psi_\ell\|_{L^6}\|(1-\varphi_R)\nabla n_0\|_{L^3}\\
&\lesssim R^{-1}\|v_\ell\|_{L^2}^2\|\langle x\rangle\nabla n_0\|_{L^3}\lesssim R^{-1}E(T_\ell).\end{aligned}$$

The term $\mathcal{I}_{\ell 2}$ is estimated by (5.4) as

$$|\mathcal{I}_{\ell 2}|\lesssim\|\Psi_\ell\|_{L^6}^2\|\nabla\varphi_R\cdot\nabla n_0\|_{L^{3/2}}\lesssim R^{-1}\|v_\ell\|_{L^2}^2\|\langle x\rangle\nabla n_0\|_{L^3}\lesssim R^{-1}E(T_\ell).$$

We also estimate $\mathcal{I}_{\ell 3}$ as

$$\begin{aligned}|\mathcal{I}_{\ell 3}|&\lesssim R^{-1}\|\Psi_\ell\|_{L^6}\|\nabla\Delta^{-1}\mathrm{div}[\varphi_R\Psi_\ell\nabla n_0]\|_{L^{6/5}}\\
&\lesssim R^{-1}\|\Psi_\ell\|_{L^6}\|\Psi_\ell\nabla n_0\|_{L^{6/5}}\\
&\lesssim R^{-1}\|\Psi_\ell\|_{L^6}\|\Psi_\ell/\langle x\rangle\|_{L^2}\|\langle x\rangle\nabla n_0\|_{L^3}\lesssim R^{-1}E(T_\ell).\end{aligned}$$

Hence, the proof of (5.10) is complete.

Now we turn to (5.11). Multiply (5.3) by $\varphi_R/\langle x\rangle^2$ to obtain

$$\int_0^T\int_{\mathbb{R}^3} \varphi_R(n_0+\frac{1}{2}\eta n_\ell)\frac{|v_\ell|^2}{\langle x\rangle^2}\,dxdt = \mathcal{I}_{\ell 4},$$

where

$$\begin{aligned}\mathcal{I}_{\ell 4} :=& \int_0^T\int_{\mathbb{R}^3} \varphi_R\frac{n_\ell H(n_0,n_\ell)+|\nabla\phi_\ell|^2}{\langle x\rangle^2}\,dxdt - \eta\int_0^T\int_{\mathbb{R}^3}\frac{\varphi_R}{\langle x\rangle^2}\nabla\phi_\ell\cdot\pi\,dxdt\\ &+\int_0^T\int_{\mathbb{R}^3}\frac{\varphi_R}{\langle x\rangle^2}v_\ell\cdot[n_0+\eta n_\ell,\nabla\Delta^{-1}\mathrm{div}]v_\ell\,dxdt - \eta\left[\int_{\mathbb{R}^3}\frac{\varphi_R}{\langle x\rangle^2}v_\ell\cdot\nabla\phi_\ell\,dx\right]_0^T\\ &+\int_0^T\int_{\mathbb{R}^3}[H(n_0,n_\ell)+\eta\frac{|v_\ell|^2}{2}]\nabla\phi_\ell\cdot\nabla\left[\frac{\varphi_R}{\langle x\rangle^2}\right]\,dxdt.\end{aligned}$$

Using the second line in (5.6), we first compute that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\frac{\varphi_R}{\langle x\rangle^2}v_\ell\cdot[n_0,\nabla\Delta^{-1}\mathrm{div}]v_\ell\,dx\right| &\lesssim \left|\int_{\mathbb{R}^3}\frac{\varphi_R}{\langle x\rangle^2}v_\ell\cdot\Delta^{-1}\mathrm{div}\left\{\nabla n_0\otimes v_\ell - v_\ell\otimes\nabla n_0\right\}\,dx\right|\\ &\lesssim \|\frac{v_\ell}{\langle x\rangle^2}\|_{L^2}\|v_\ell\otimes\nabla n_0\|_{L^{\frac{6}{5}}}\\ &\lesssim \|\frac{v_\ell}{\langle x\rangle}\|_{L^2}^2\|\langle x\rangle\nabla n_0\|_{L^3}\\ &\lesssim \|\langle x\rangle\nabla n_0\|_{L^3}\int_{\mathbb{R}^3}\varphi_R n_0\frac{|v_\ell|^2}{\langle x\rangle^2}\,dx + R^{-2}E(T_\ell),\end{aligned}$$

and (5.7) and (5.9), allows to estimate

$$\begin{aligned}\mathcal{I}_{\ell 4} &\lesssim \|\langle x\rangle\nabla n_0\|_{L^\infty_t L^3}\int_0^T\int_{\mathbb{R}^3}\varphi_R n_0\frac{|v_\ell|^2}{\langle x\rangle^2}\,dxdt + \int_0^T\int_{\mathbb{R}^3}\varphi_R\frac{n_\ell^2+|\nabla\phi_\ell|^2}{\langle x\rangle^2}\,dxdt\\ &\quad + C(E(T_\ell))(\eta+R^{-2})(1+T)\\ &\lesssim \delta\int_0^T\int_{\mathbb{R}^3}\varphi_R n_0\frac{|v_\ell|^2}{\langle x\rangle^2}\,dxdt + \int_0^T\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dxdt + C(E(T_\ell))(\eta+R^{-2})(1+T).\end{aligned}$$

Therefore, we conclude (5.11). □

5.2. **Control of the density.** We are now in a position to show the local decay in Theorem 1.3. Let us define

$$A_R(T) := \int_0^T\int_{\{R\le|x|\le 2R\}}\left[n_\ell^2+|v_\ell|^2+|\nabla\phi_\ell|^2\right]\,dxdt$$

and choose R so that

$$\eta^{-1/2}\le R\le\eta^{-3/4},\quad A_R(T)\lesssim|\log\eta|^{-1}TE(T_\ell).$$

We can find the above R since $\sum_{\eta^{-1/2}\le R\le\eta^{-3/4}}A_R(T)\lesssim TE(T_\ell)$. From now on we show the local decay of the density n_ℓ.

Proposition 5.1. *Under the same assumptions as in Theorem 1.3, there holds that*

$$\int_0^T\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dxdt \lesssim_{E(T_\ell)} |\log\eta|^{-1}(1+T)\quad \textit{for } T\in[0,\min\{T_0,T_\ell\}].$$

Proof. Multiply (5.1) by $x_j\varphi_R$ and sum up them for $j=1,2,3$. Multiply (5.3) by $\varphi_R/2$. Then, adding both results, we have

$$\begin{aligned}&\varphi_R\left\{\sum_{j=1}^{3}L_{jj}+\frac{1}{2}\left[(n_0+\frac{1}{2}\eta n_\ell)|v_\ell|^2-n_\ell H(n_0,n_\ell)-|\nabla\phi_\ell|^2\right]\right\}\\&=\operatorname{div}\left[\varphi_R\left\{x_jL_j-\frac{1}{2}(H(n_0,n_\ell)+\eta\frac{|v_\ell|^2}{2})\nabla\phi_\ell\right\}\right]+\eta\partial_t\mathcal{R}_{\ell 0}+\mathcal{R}_{\ell 1}+\eta\mathcal{R}_{\ell 2},\end{aligned}\tag{5.12}$$

where

$$\begin{aligned}\mathcal{R}_{\ell 0}&:=\varphi_R n_\ell[x\cdot v_\ell]-\frac{1}{2}\varphi_R[v_\ell\cdot\nabla\phi_\ell],\\ \mathcal{R}_{\ell 1}&:=-\sum_{j=1}^{3}x_j[L_j\cdot\nabla\varphi_R]+\frac{1}{2}(H(n_0,n_\ell)+\eta\frac{|v_\ell|^2}{2})\nabla\phi_\ell\cdot\nabla\varphi_R,\\&\quad+\frac{1}{2}\varphi_Re^{(1)}[x\cdot\nabla n_0]-\varphi_Rv_\ell\cdot[n_0+\eta n_\ell,\nabla\Delta^{-1}\mathrm{div}]v_\ell,\\ \mathcal{R}_{\ell 2}&:=\varphi_R\pi\cdot\left[n_\ell x-\frac{1}{2}\nabla\phi_\ell\right].\end{aligned}$$

The key observation on the left hand side of (5.12) is that

$$\begin{aligned}&\sum_{j=1}^{3}L_{jj}+\frac{1}{2}\left[(n_0+\frac{1}{2}\eta n_\ell)|v_\ell|^2-n_\ell H(n_0,n_\ell)-|\nabla\phi_\ell|^2\right]\\=&\tilde{h}_0'(n_0)n_\ell^2+\eta C(E(T_\ell)).\end{aligned}\tag{5.13}$$

Therefore, it remains to control the integrations of all terms on the right hand side of (5.12).

Let us estimate those integrations one by one. We can control the first term by the linearized energy

$$\eta\left|\left[\int_{\mathbb{R}^3}\mathcal{R}_{\ell 0}\,dx\right]_0^T\right|\lesssim\eta RE(T_\ell).$$

Using (5.2), we see that

$$\eta\left|\int_0^T\int_{\mathbb{R}^3}\mathcal{R}_{\ell 2}\,dxdt\right|\lesssim\eta RT(1+E(T_\ell)).$$

The decaying property (5.4) enables us to estimate the first line of $\mathcal{R}_{\ell 1}$ by $A_R(T)$:

$$\int_0^T\int_{\mathbb{R}^3}\left|\sum_{j=1}^{3}x_j[L_j\cdot\nabla\varphi_R]+\frac{1}{2}(H(n_0,n_\ell)+\eta\frac{|v_\ell|^2}{2})\nabla\phi_\ell\cdot\nabla\varphi_R\right|dxdt\lesssim A_R(T).$$

Using (5.2) and Lemma 5.2, we see that[1]

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\varphi_Re^{(1)}[x\cdot\nabla n_0]\,dxdt&\lesssim\delta\int_0^T\int_{\mathbb{R}^3}\frac{|v_\ell|^2+|n_\ell|^2}{1+|x|^2}\,dxdt\\&\lesssim\delta\int_0^T\int_{\mathbb{R}^3}\varphi_Rn_\ell^2\,dxdt+(\eta+R^{-2})(1+T)C(E(T_\ell)).\end{aligned}$$

[1] It is to control $|v_\ell|^2$ by Lemma 5.2 that we require the strong localization condition for n_0.

Using Lemma 5.2 for the last term and adding all the estimates above, we get

$$\int_0^T\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dxdt \lesssim \delta\int_0^T\int_{\mathbb{R}^3}\varphi_R n_\ell^2\,dxdt + C(E(T_\ell))(\eta R+R^{-2})(1+T)+A_R(T)$$

and if $\delta>0$ is small enough, recalling that $\eta^{-\frac{1}{2}}\le R\le \eta^{-\frac{3}{4}}$, we obtain the result. $\square$

Combining Lemma 5.1, Lemma 5.2 and Proposition 5.1, we have the following corollary which completes Theorem 1.3.

Corollary 5.1. *For any $\mu\in(0,1)$ and $T\in[0,\min\{T_0,T_\ell\}]$, it holds that*

$$\int_0^T\int_{B(0,|\log\eta|^{(1-\mu)/2})}\left\{|v_\ell|^2+|\nabla\phi_\ell|^2\right\}\,dxdt \lesssim_{E(T_\ell)} |\log\eta|^{-\mu}(1+T).$$

6. Appendix

We introduce several basic estimates.

Lemma 6.1. *Suppose that $k,l\ge 0$, $s>3/2$ and $k+l\le s$, then*

$$\|uv\|_{H^{s-(k+l)}}\lesssim \|u\|_{H^{s-k}}\|v\|_{H^{s-l}}.$$

and if $k\ge 1$, $s>5/2$, then

$$\|u\nabla v\|_{H^{s-(k+l)}}\lesssim \|u\|_{H^{s-k}}\|v\|_{H^{s-l}}.$$

Proof. These can be shown by the same way as in the proofs of inequalities (B1)–(B3) in [14]. $\square$

For the notational convenience, we use

$$\mathfrak{X}_s(T):=\bigcap_{k=0}^{s} C^j([0,T];H^{s-j}(\Omega)) \quad \text{for s} = 0,1,2,\dots,$$

$$|||u(t)|||_{s,k}^2 := \sum_{j=0}^{k}\|\partial_t^j u(t)\|_{s-k}^2.$$

Furthermore, for a nonnegative integer $s\ge 0$, $\mathcal{B}^s(X)$ denotes the space of the functions whose derivatives up to s-th order are continuous and bounded over X.

Lemma 6.2. *Let $s>3/2$, $k=0,\dots,s$ and $\mathcal{O}\subset\mathbb{R}^d$, an open subset containing the origin. Suppose that $A\in\mathcal{B}^{s+1}([0,T]\times X\times\mathcal{O})$ and $A(t,x,0)=0$. If $f\in\mathfrak{X}_s(T)$, then $A(\cdot,\cdot,f)$ belongs to $\mathfrak{X}_s(T)$ and the following inequality holds.*

$$|||A(t,\cdot,f(t))|||_{s,k}\lesssim (1+|||f(t)|||_{s,k})^{s-1}|||f(t)|||_{s,k}. \tag{6.1}$$

For the commutator $[\partial_t^k,\cdot]$ with $k\le s$, and for $f\in\mathfrak{X}_{s+1}(T)$ and $g\in\mathfrak{X}_s(T)$, it holds that

$$\|[\partial_t^k,f(t)]g(t)\|_{H^{s+1-k}}\lesssim |||f(t)|||_{s+1,s+1}|||g(t)|||_{s,s}. \tag{6.2}$$

Proof. Inequality (6.1) is shown by the same way as in the proof of inequality (B4) in [14]. Since the commutator in (6.2) is a linear combination of terms of the form $\partial_t^{p+1}f\cdot\partial_t^{k-p-1}g$, for $0\le p\le k-1$, the estimate is direct. $\square$

Lemma 6.3. *Consider the following problem for an elliptic equation*

$$[1-\tilde{h}'(n_0)\Delta]u = f \quad in\ \mathbb{R}^3,$$

where $\tilde{h}$ *and* n_0 *are defined in* (1.5) *and* (1.12). *Assume that*

$$\begin{aligned} h'(n_0) > \kappa > 0, \\ \|(h'(n_0))^{-\frac{1}{2}}\nabla(h'(n_0))\|_{L^\infty} \le 1/10. \end{aligned} \tag{6.3}$$

Then the inverse operator $[1-\tilde{h}(n_0)\Delta]^{-1} : H^1 \to H^1$ *is well-defined and satisfies*

$$\|[1-\tilde{h}'(n_0)\Delta]^{-1}f\|_{H^1} \lesssim_\kappa \|f\|_{L^2}, \tag{6.4}$$

$$\|\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}f\|_{L^2} \le \|\nabla f\|_{L^2}. \tag{6.5}$$

Furthermore, $\nabla[1-\tilde{h}'(n_0)\Delta]^{-1} : H^1 \to L^2$ *can be extended to a bounded operator from* $\dot{H}^1$ *to* L^2*. For any* $f \in C^1([0,T];\dot{H}^1)$*, it also holds*

$$\begin{aligned} &\partial_t\left[\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}f\right] \\ &= \nabla[1-\tilde{h}'(n_0)\Delta]^{-1}\left[\partial_t f + \frac{\partial_t[\tilde{h}'(n_0)]}{\tilde{h}'(n_0)}[[1-\tilde{h}(n_0)\Delta]^{-1}f - f]\right]. \end{aligned}$$

Proof. Once estimates (6.4) and (6.5) are obtained, we can show easily the well-posedness of $[1-\tilde{h}(n_0)\Delta]^{-1}$, extension of $\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}$, and time-differentiability of $\nabla[1-\tilde{h}(n_0)\Delta]^{-1}f$ by a standard method with the Lax–Milgram theorem. Furthermore, differentiating the elliptic equation in t leads to the formula of the time-derivative. Indeed,

$$[1-\tilde{h}'(n_0)\Delta]\partial_t u = \partial_t f + \Delta u \cdot \partial_t[\tilde{h}'(n_0)] = \partial_t f + \frac{\partial_t[\tilde{h}'(n_0)]}{\tilde{h}'(n_0)}[u-f].$$

Applying $\nabla[1-\tilde{h}'(n_0)\Delta]^{-1}$ to this, we have the formula of the time-derivative. Therefore, it is sufficient to show estimates (6.4) and (6.5).

Multiplying the elliptic equation by $u/h'(n_0)$ and integrating it over $\mathbb{R}^3$, we have

$$\int\left[\frac{u^2}{h'(n_0)} + |\nabla u|^2\right]dx = \int \frac{fu}{h'(n_0)}\,dx$$

which gives (6.4) using Cauchy Schwartz inequality and the first assumption in (6.3). Differentiate the elliptic equation in x_j multiply the result by $\partial_j u$, and integrate them over $\mathbb{R}^3$. Then sum up the results for $j = 1,2,3$ to obtain

$$\begin{aligned} &\int\left[|\nabla u|^2 + \tilde{h}'(n_0)\sum_{j=1}^{3}|\nabla\partial_j u|^2\right]dx \\ &= \sum_{j=1}^{3}\int \partial_j f \partial_j u\,dx + \sum_{j=1}^{3}\int \tilde{h}''(n_0)[\partial_j n_0 \Delta u - \nabla n_0 \cdot \nabla\partial_j u]\partial_j u\,dx. \end{aligned}$$

This together with the second line in (6.3) and Schwarz's inequality gives (6.5). □

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