

TYPE ALTERNATIVE FOR FROSTMAN MEASURES

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ABSTRACT. For a finite positive Borel measure μ on $\mathbb{R}$ its exponential type, T_μ , is defined as the infimum of $a > 0$ such that finite linear combinations of complex exponentials with frequencies between 0 and a are dense in $L^2(\mu)$. The definition can be easily extended from finite to broader classes of measures. In this paper we prove a new formula for T_μ and use it to study growth and additivity properties of measures with finite positive type. As one of the applications, we show that Frostman measures on $\mathbb{R}$ may only have type zero or infinity.

1. INTRODUCTION

For $a > 0$ denote by $\mathcal{E}_a$ the family of complex exponential functions on $\mathbb{R}$ with frequencies between 0 and a :

$$\mathcal{E}_a = \{e^{isx} \mid s \in [0, a]\}.$$

If μ is a positive finite measure on $\mathbb{R}$ its exponential type T_μ is defined as

$$T_\mu = \inf\{a \mid \mathcal{E}_a \text{ is complete in } L^2(\mu)\},$$

if the set on the right-hand side is non-empty and as infinity otherwise. Recall that a family of vectors in a Banach space is complete in the space if finite linear combinations of vectors from the family are dense in the space. The type problem, the problem of finding T_μ in terms of μ , has many connections in Fourier analysis, spectral analysis of differential operators and related fields, see for instance [4, 5, 6, 7, 8, 9, 16, 17]. A formula for T_μ was recently obtained in [16] and further developed in [17]. One of the implications of these results is that L^2 in the above definition of T_μ may be replaced with any L^p , $p > 1$, without changing the value of T_μ , i.e., that the p -type of μ for any $p > 1$ is equal to the 2-type.

Although the type formula from [16] has improved existing results and gave new examples in the area of the type problem, some of the natural questions remained open. This note originated from one of such questions. Consider

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the Poisson measure Π on the real line,

$$d\Pi(x) = \frac{dx}{1+x^2}.$$

Its type T_Π can be easily shown to be equal to infinity. On the opposite end of the scale, if one considers any measure with 'long' gaps in its support (see Section 3), the type is equal to zero, as follows from another classical result, Beurling's gap theorem. The simplest examples of measures of zero type are measures with semi-bounded (or bounded) support, for instance, a restriction of Π to a half-line. The sharpness of Beurling's theorem was demonstrated by M. Benedicks in [2] where further examples of absolutely continuous measures of zero and infinite type were constructed. Examples from [2] focused on restrictions of the Poisson measure to unions of unit intervals.

To obtain an example of an absolutely continuous measure with finite positive type one needs to utilize further results on the type problem. Define the density f to be equal to $e^{|n|}/(1+n^2)$ on each interval $[n, n+e^{-|n|})$, $n \in \mathbb{Z}$, and to zero elsewhere. Then the type of μ , $d\mu(x) = f(x)dx$, is equal to 2π , as follows from the results of Borichev and Sodin [4] or the results of [16].

Notice that in the last example the density is extremely unbounded, while in the examples with more regular densities the type always comes out to be zero or infinity. This pattern persists over all known examples giving raise to the following natural question: Can an absolutely continuous measure with bounded density have finite positive type?

In this note we give a negative answer to this question extending the result to a slightly wider class of measures. A positive measure μ on $\mathbb{R}$ is Poisson-finite if

$$\int \frac{d\mu(x)}{1+x^2} < \infty.$$

The definition of type given above can be easily extended from finite to Poisson-finite and even wider classes of measures, see Section 2.

A positive measure μ on $\mathbb{R}$ is a Frostman measure if there exist positive constants C and α such that for any interval $I \subset \mathbb{R}$,

$$\mu(I) < C|I|^\alpha, \tag{1.1}$$

where $|I|$ denotes the length of I . It is not difficult to see that every Frostman measure is Poisson-finite. Any absolutely continuous measure μ , $d\mu(x) = f(x)dx$ with bounded density is a Frostman measure. Moreover, if $f \in L^p(\mathbb{R})$, $1 < p < \infty$ then the measure is Frostman since

$$\mu(I) \leq \|f\|_p |I|^{1/p}.$$

A measure μ is doubling if there exists a constant $C > 0$ such that

$$\mu((x - 2r, x + 2r)) < C\mu((x - r, x + r))$$

for any $x \in \mathbb{R}$ and any $r > 0$. Under an additional restriction that $\mu(I) < D$ for all intervals I of fixed (unit) length and some $D > 0$, a doubling measure is a Frostman measure. Among singular measures, a standard Cantor measure on $[0, 1]$ periodically extended to the rest of $\mathbb{R}$ is a relevant example of a singular Frostman measure.

As we show in Theorem 4, Section 6, the type of any Frostman measure can only be equal to zero or infinity. Our result relates local properties of a measure to its type, whereas previously known results only explored the relations between the type and asymptotic properties of the measure near infinity.

To prove Theorem 4 we first obtain a new version of the type formula in Section 5, Theorem 3. Unlike the previous versions, the formula does not include a μ -finite weight inherited from Bernstein's version of the type problem, see Section 4. Such an improvement may be useful in applications, such as the problems discussed here.

If a measure with bounded density cannot have a finite positive type, it is natural to ask how fast the density of a measure of such a type must grow. We give an answer to this question in Section 8, Theorem 6. In Section 7 we investigate additivity properties of type.

2. PRELIMINARIES

If f is a function from $L^1(\mathbb{R})$ we denote by $\hat{f}$ its Fourier transform

$$\hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-izt} dt. \quad (2.1)$$

Let M be a set of all finite Borel complex measures on the real line. Similarly, for $\mu \in M$ we define

$$\hat{\mu}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-izt} d\mu(t).$$

Via Parseval's theorem, the Fourier transform may be extended to be a unitary operator from $L^2(\mathbb{R})$ onto itself and can be defined for even broader classes of distributions.

If one extends all functions from $L^2(-a, a)$ as 0 to the rest of the line, one can apply the Fourier transform to all such functions and obtain the Paley-Wiener space of entire functions

$$\mathcal{PW}_a = \{\hat{f} \mid f \in L^2(-a, a)\}.$$

The definition of type T_μ , given in the introduction for finite measures μ , can be naturally extended to broader classes of measures on $\mathbb{R}$:

$$T_\mu = 2 \inf\{a \mid \mathcal{PW}_a \cap L^2(\mu) \text{ is dense in } L^2(\mu)\},$$

if the set of such a is non-empty and as infinity otherwise. Via this definition one can consider the type problem in the class of polynomially growing measures, i.e., measures ν such that $\nu = (1 + |x|^n)\mu$ for some $n > 0$ and some finite measure μ . Poisson-finite measures are polynomially growing measures with $n = 2$. We will denote the set of all polynomially growing Borel complex measures on the real line by M_{poly} . The notation M_{poly}^+ will be used for the subset of positive measures.

Note that by duality, a family F of functions is not dense in $L^q(\mu)$, $q > 1$, iff there exists

$$f \in L^p(\mu), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

annihilating the family, i.e., such that

$$\int f \bar{g} d\mu = 0$$

for every g from the family. In such cases we write $f \perp F$. We denote by T_μ^p the p -type of μ defined as

$$T_\mu^p = 2 \inf\{a \mid \exists f \in L^p(\mu), f \not\equiv 0, f \perp (\mathcal{PW}_a \cap L^q(\mu))\},$$

Note that $T_\mu = T_\mu^2$. Cases $p \neq 2$ were considered in several papers, see for instance articles by Koosis [10] or Levin [11] for the case $p = \infty$.

It is well-known that if $\nu = (1 + |x|^n)\mu$ then $T_\nu^p = T_\mu^p$, which reduces the problem for polynomially growing measures to its original settings of finite measures. The equality of T_μ^p , $p > 1$, to T_μ remains true for polynomially growing measures.

Suppose that $T_\mu^p > a$ for some finite measure μ , $p > 1$ and $a > 0$. By our definitions this means that there exists a function

$$f \in L^q(\mu), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

which annihilates $\mathcal{E}_a$, i.e., such that

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int e^{-isx} f(x) dx = 0 \text{ for all } s \in [0, a].$$

An alternative way to extend the definition of type is to say that $T_\mu \geq a$ if $\widehat{f\mu}$, understood in the sense of distributions for infinite measures, has no support on $[0, a]$ for some $f \in L^p(\mu)$, $f \not\equiv 0$.

Hence the type problem becomes a version of the gap problem, which studies measures whose Fourier transform has a non-trivial gap in its support. As we can see, to find T_μ^p is the same as to find what non-zero density $f \in L^q(\mu)$ gives the largest spectral gap for the measure $f\mu$. It turns out that to approach the type problem it is beneficial to first solve the gap problem in the case $q = 1$, which is no longer a dual version of a p -type problem.

3. GAP FORMULA

We say that a polynomially growing measure $\mu \in M_{poly}$ annihilates a Paley-Wiener space $\mathcal{PW}_a$, and write $\mu \perp \mathcal{PW}_a$ if for all functions $f \in \mathcal{PW}_a \cap L^1(|\mu|)$,

$$\int f d\mu = 0.$$

Note that $\mathcal{PW}_a$ contains a dense set of fast-decreasing functions belonging to $L^1(|\mu|)$ for any polynomially growing μ . If X is a closed subset of the real line we denote by $M_{poly}(X)$ the set of polynomially growing measures supported on X . We denote by $\mathbf{G}_X$ the gap characteristic of X defined as

$$\mathbf{G}_X = 2 \sup\{a \mid \exists \mu \in M_{poly}(X), \mu \neq 0, \text{ such that } \mu \perp \mathcal{PW}_a\}. \quad (3.1)$$

when the set is non-empty and zero otherwise. Alternatively, one could define $\mathbf{G}_X$ as the supremum of the size of the gap in the support of $\hat{\mu}$, taken over all non-zero finite measures μ supported on X (which explains the name).

The problem of finding $\mathbf{G}_X$ in terms of X has many connections and applications, see for instance [13, 17] for results and further references. It was recently solved in [15]. This version of the gap problem is related to the version mentioned in the last section via the following observation.

Lemma 1 ([15, 17]). *For $\mu \in M_{poly}^+$,*

$$T_\mu^1 = \mathbf{G}_{\text{supp } \mu}.$$

As we discussed in the last section, for $p > 1$ the type $T_\mu^p = T_\mu$ is different from T_μ^1 . Further formulas for the type will be discussed in the next two sections.

To give the formula for $\mathbf{G}_X$ [15] we will need the following definitions.

A sequence of disjoint intervals $\{I_n\}$ on the real line is called long (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1 + \text{dist}^2(0, I_n)} = \infty \quad (3.2)$$

where $|I_n|$ stands for the length of I_n . If the sum is finite we call $\{I_n\}$ short.

Let

$$\dots < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < \dots$$

be a two-sided sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \rightarrow \infty$ as $n \rightarrow \pm\infty$ and the sequence $\{I_n\}$ is short, i.e. the sum in (3.2) is finite.

Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a finite set of points on $\mathbb{R}$. Consider the quantity

$$E(\Lambda) = \sum_{\lambda_k, \lambda_l \in \Lambda, \lambda_k \neq \lambda_l} \log |\lambda_k - \lambda_l|. \quad (3.3)$$

As usual, a sequence of points $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ is called discrete if it has no finite accumulation points.

Now we are ready to give the definition of d -uniform sequences used in our type formulas.

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of distinct real points and let d be a positive number. We say that Λ is a d -uniform sequence if there exists a short partition $\{I_n\}$ such that

$$\Delta_n = \#(\Lambda \cap I_n) = d|I_n| + o(|I_n|) \text{ as } n \rightarrow \pm\infty \text{ (density condition)} \quad (3.4)$$

and

$$\sum_n \frac{\Delta_n^2 \log |I_n| - E_n}{1 + \text{dist}^2(0, I_n)} < \infty \quad (\text{energy/work condition}), \quad (3.5)$$

where

$$E_n = E(\Lambda \cap I_n).$$

The following formula for the gap characteristic of a closed set was obtained in [15], see also [17].

Theorem 1.

$$\mathbf{G}_X = 2\pi \sup\{d \mid X \text{ contains a } d\text{-uniform sequence}\},$$

if the set on the right is non-empty and $\mathbf{G}_X = 0$ otherwise.

4. TYPE FORMULA IN BERNSTEIN'S SETTINGS

We first approach the type problem in the settings of Bernstein's weighted uniform approximation.

Consider a weight W , i.e. a lower semicontinuous function $W : \mathbb{R} \rightarrow [1, \infty]$ that tends to ∞ as $x \rightarrow \pm\infty$. The space C_W is the space of all continuous functions on $\mathbb{R}$ satisfying

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{W(x)} = 0.$$

We define the semi-norm in C_W as

$$\|f\|_W = \|fW^{-1}\|_\infty.$$

Finding conditions on W ensuring completeness of polynomials or exponentials in C_W is a classical problem, see for instance [3, 14, 12, 17] for further discussion and references.

For a weight W we define

$$T_W = \inf\{a \mid \mathcal{E}_a \text{ is complete in } C_W\}.$$

We put $T_W = 0$ if the last set is empty.

Theorem 2 ([17]).

$$T_W = 2\pi \sup \left\{ d \mid \sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} < \infty \text{ for some } d\text{-uniform sequence } \Lambda \right\},$$

if the set is non-empty, and 0 otherwise.

As was shown by A. Bakan [1], $\mathcal{E}_a$ is complete in $L^p(\mu)$, $1 \leq p \leq \infty$, iff there exists a weight $W \in L^p(\mu)$ such that $\mathcal{E}_a$ is complete in C_W . This result yields the following corollary from the last theorem. For $\mu \in M_{poly}^+$, we call a weight W a μ -weight if

$$\int W d\mu < \infty.$$

Corollary 1 ([16]). *Let μ be a finite positive measure on the line. Let $1 < p \leq \infty$ and $d > 0$ be constants.*

Then $T_\mu^p \geq 2\pi d$ if and only if for any μ -weight W there exists a d -uniform sequence $\Lambda = \{\lambda_n\} \subset \text{supp } \mu$ such that

$$\sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} < \infty. \quad (4.1)$$

Note that this statement implies $T_\mu^p = T_\mu$ for all $p > 1$, the property mentioned in previous sections. We will need these statements to obtain a new version of the type formula in the next section.

5. TYPE FORMULA IN L^p -SETTINGS

Definition 1. If $\Lambda = \{\lambda_n\} \subset \mathbb{R}$ is a discrete sequence of distinct points we denote by $\Lambda^* = \{\lambda_n^*\}$ the sequence of closed intervals such that each λ_n^* is centered at λ_n and has the length equal to one-third of the distance from λ_n to the rest of Λ . Note that then the intervals λ_n^* are pairwise disjoint.

Definition 2. If Λ is a discrete sequence we will write that $D_1(\Lambda) = d$ if there exists a short partition on which Λ satisfies (3.4).

Note that if Λ satisfies (3.4) with some d on some short partition then the asymptotic density of Λ is d and therefore Λ cannot satisfy (3.4) with any other d on a different short partition, which implies correctness of the last definition.

Theorem 3. Let $\mu \in M_{poly}^+$. Suppose that $T_\mu < \infty$. Then

$$T_\mu = 2\pi \max\{d \mid \exists d\text{-uniform } \Lambda = \{\lambda_n\} \text{ such that } \sum \frac{\log \mu(\lambda_n^*)}{1 + n^2} > -\infty\}$$

if the set of such d is non-empty and $T_\mu = 0$ otherwise.

Note that the maximum on the right-hand side exists whenever the set is non-empty. A maximal sequence can always be chosen inside the support of the measure, which will be useful for us in applications.

The statement covers the case of finite type T_μ , which is needed for Theorem 4 in the next section. We would like to leave it as an open problem whether the formula can be extended to the infinite case, i.e., if it is true that the equation (with supremum in place of the maximum) holds when $T_\mu = \infty$. It is not difficult to see that if the supremum of the set on the right is infinite then $T_\mu = \infty$. It therefore remains to check the opposite implication.

Proof. As follows from our discussion in Section 2, it is enough to prove the theorem for finite μ .

Let $T_\mu = 2\pi d < \infty$. One can show that then there exists a μ -weight W such that (4.1) is satisfied for some d -uniform sequence Λ but is not satisfied for any $(d + \varepsilon)$ -uniform sequence. Define

$$s = \max\{d \mid \exists d\text{-uniform } \{\lambda_{n_k}\} \subset \Lambda, \sum \frac{\log \mu(\lambda_{n_k}^*)}{1 + n_k^2} > -\infty\}$$

(note that $\max$ can be used in the last formula instead of $\sup$). Suppose that $s < d$. Let Γ be a maximal subsequence in the last formula.

Then the subsequence $\Lambda_1 = \{\lambda_{n_k}\} = \Lambda \setminus \Gamma$ satisfies $D_1(\Lambda_1) = \varepsilon > 0$ and the corresponding sequence of intervals $\Lambda_1^* = \{\lambda_{n_k}^*\}$ has the property that for any subsequence $\Lambda_2 = \{\lambda_{n_{k_l}}\} \subset \Lambda_1$, $D_1(\Lambda_2) > 0$,

$$\sum_l \frac{\log \mu(\lambda_{n_{k_l}}^*)}{1 + n_{k_l}^2} = -\infty.$$

Indeed, if there existed

$$\Lambda_2 \subset \Lambda_1, \quad D_1(\Lambda_2) > 0$$

for which the last sum were finite, then by Lemma 3 from Section , $\Gamma \cup \Lambda_2$ would have an $(s + \varepsilon)$ -uniform subsequence with finite sum, which would contradict maximality of Γ .

Now one can obtain a contradiction in the following way. Define a new weight W_1 to be equal to

$$\frac{\max(W(\lambda_{n_k}), 1/\mu(\lambda_{n_k}^*))}{1 + n_k^2}$$

on each interval $\lambda_{n_k}^*$ from Λ_1^* and equal to W elsewhere. Then W_1 is a μ -weight and therefore (4.1) must be satisfied with some d -uniform sequence Φ . Notice that then Φ intersects the intervals from Λ_1^* only for a subsequence of density zero, i.e., there exists a subsequence $\Theta = \{\theta_n\} \subset \Lambda_1$ such that $D_1(\Theta) = D_1(\Lambda_1) = \varepsilon$ and

$$\Phi \setminus \cup_{\lambda_{n_k} \in \Theta} \lambda_{n_k}^* = \emptyset.$$

Since Λ_1 is an ε -uniform sequence, Θ is an ε -uniform sequence. By Lemma 4 from Section , $\Phi \cup \Theta$ is a $(d + \varepsilon)$ -uniform sequence on which the original weight W satisfies (4.1), which contradicts our choice of W . Hence, $s \geq d$.

In the opposite direction, suppose that there exists a d -uniform sequence Λ such that

$$\sum \frac{\log \mu(\lambda_n^*)}{1 + n^2} > -\infty.$$

Note that every μ -weight W satisfies

$$\min_{\lambda_n^*} W(x) < \frac{C}{\mu(\lambda_n^*)}$$

on every λ_n^* , with

$$C = \int W d\mu.$$

The sequence of points where the minima occur will give us a d -uniform sequence on which W satisfies (4.1). Thus $T_\mu \geq 2\pi d$. $\square$

6. TYPE OF FROSTMAN MEASURES

In this section we solve the problem of type for measures with bounded densities discussed in the introduction. As it turns out, our result can be formulated for a broader class of Frostman measures, which seems to be the right class for such a statement due to the elementary property that $\log |I|^\alpha = \alpha \log |I|$.

Recall that a positive measure μ on $\mathbb{R}$ is a Frostman measure if there exist positive constants α and C such that

$$\mu((x - \epsilon, x + \epsilon)) < C\epsilon^\alpha \quad (6.1)$$

for all $\epsilon > 0, x \in \mathbb{R}$. It easily follows that Frostman measures are Poisson-finite.

Theorem 4. *If μ is a Frostman measure then T_μ equals either 0 or ∞ .*

Proof. If $T_\mu = 2\pi d$, $0 < d < \infty$, then there exists a d -uniform sequence Λ such that Λ^* satisfies

$$\sum \frac{\log \Delta_n}{1 + n^2} > -\infty, \quad (6.2)$$

where $\Delta_n = \mu(\lambda_n^*)$. WLOG μ satisfies (6.1) with $C = 1$ and some α , $0 < \alpha \leq 1$. Divide the interval λ_n^* into M_n equal subintervals, so that

$$\frac{1}{2}[\Delta_n/6]^{1/\alpha} \leq |\lambda_n^*|/M_n \leq [\Delta_n/6]^{1/\alpha}.$$

Then the mass of each subinterval is at most $\Delta_n/6$ (and hence $M_n \geq 6$). Since the total mass is Δ_n , there exist at least 3 subintervals of mass at least

$$\Delta_n/2M_n \asymp \Delta_n^{1+\frac{1}{\alpha}}/|\lambda_n^*|. \quad (6.3)$$

Let l_1^n and l_2^n be two of these intervals, not adjacent to each other. Then the distance d_n between the intervals is greater or equal to the length of each interval and its logarithm can be estimated from below by

$$\log d_n \geq \log |\lambda_n^*|/M_n \geq C_1 \log \Delta_n + C_2,$$

which together with (6.2) implies

$$\sum \frac{\log d_n}{1 + n^2} > -\infty. \quad (6.4)$$

Consider the sequence $\Gamma = \{\gamma_m\}$ composed of the centers of all such intervals l_1^n, l_2^n for all n . Since the sequence Λ was d -uniform and because of (6.4), Γ is a $2d$ -uniform sequence. Our construction implies that each of the intervals γ_m^* contains one of the intervals l_1^n, l_2^n , whose mass is at least the expression in (6.3). Since $\Delta_n = \mu(\lambda_n^*)$ satisfy (6.2), the sequence of intervals $\{\gamma_m^*\}$ satisfies

$$\sum \frac{\log \mu(\gamma_m^*)}{1+n^2} \gtrsim \sum \frac{\log \Delta_n}{1+n^2} + \text{const} > -\infty$$

which implies $T_\mu \geq 4\pi d$ contradicting our initial assumption. $\square$

As was discussed in the introduction, the last theorem has the following corollaries.

Corollary 2. *Let $\mu = fm$, where $f \in L^p(\mathbb{R})$, $p > 1$. Then T_μ is either 0 or infinity.*

Corollary 3. *If μ is a doubling measure such that*

$$\sup_{x \in \mathbb{R}} \mu((x, x+1)) < \infty$$

then T_μ is either 0 or infinity.

7. ADDITIVITY PROPERTIES OF TYPE

It is natural to ask if the type of a measure satisfies any additivity conditions, in general or in special cases. Our first observation in this direction is that the inequality $T_{\mu+\nu} \lesssim T_\mu + T_\nu$ fails in general. Indeed, let μ and ν be the restrictions of Lebesgue measure m to $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. Then $T_\mu = T_\nu = 0$, because if $f \in L^2(\nu)$ ($\in L^2(\mu)$) then $\widehat{f\nu}$ ($\widehat{f\mu}$) belongs to the Hardy space H^2 ($\bar{H}^2$) and cannot vanish on a set of positive measure, unless $f \equiv 0$. On the other hand $T_{\mu+\nu} = T_m = \infty$.

Further examples of this type, without semi-bounded supports, can be obtained using Beurling's theorem and choosing μ to be the restriction of m to the union of odd dyadic intervals $[2^{2n+1}, 2^{2n+2}]$, $n \in \mathbb{Z}$ and ν as the restriction to even intervals. Then once again $T_\mu = T_\nu = 0$ but $T_{\mu+\nu} = T_m = \infty$.

In the opposite direction, we obviously have $T_{\mu+\nu} \geq \max(T_\mu, T_\nu)$. However, the inequality $T_{\mu+\nu} \geq T_\mu + T_\nu$ does not hold in general. To construct an example, consider

$$\mu = \sum_{n=-\infty}^{\infty} \delta_n \text{ and } \nu = \sum_{n=-\infty}^{\infty} \delta_{n+e^{-|n|}}.$$

Since both measures are supported by 1-uniform sequences, $T_\mu = T_\nu = 2\pi$ by Theorem 3 (in this simple case the type can be calculated directly without

any advanced results). Since the support of $\mu + \nu$ does not contain any d -uniform sequences with $d > 1$, $T_{\mu+\nu} = 2\pi$ by Theorem 1 or 3.

Nonetheless, the following 'splitting' property holds for measures of finite positive type.

Theorem 5. *Let $\mu \in M_{poly}^+$, $T_\mu = 2\pi d < \infty$. Let c_1, c_2 be non-negative constants such that $c_1 + c_2 = d$. Then there exists a closed set $X \subset \mathbb{R}$ such that the measures $\mu_1 = \mu|_X$, $\mu_2 = \mu - \mu_1$ satisfy*

$$T_{\mu_1} = 2\pi c_1, \quad T_{\mu_2} = 2\pi c_2.$$

Note that the analog of the above statement for measures of infinite type is false. If μ is a measure with bounded density such that $T_\mu = \infty$, c_1 is finite positive and c_2 is infinite then any restriction of μ will again be a measure with bounded density and therefore, by Theorem 4, will not have its type equal to $2\pi c_1$.

Proof. By Theorem 3 there exists a d -uniform sequence Λ such that the sequence of intervals Λ^* satisfies

$$\sum \frac{\log \mu(\lambda_n^*)}{1 + n^2} > -\infty. \quad (7.1)$$

If $\{I_n\}$ is a short partition corresponding to Λ from the definition of d -uniform sequences, choose a subsequence $\Gamma \subset \Lambda$ satisfying (3.4) on $\{I_n\}$ with c_1 in place of d . Note that the energy condition for Γ will then be satisfied on the same partition and therefore Γ is a c_1 -uniform sequence.

Put $X = \cup_{\lambda_n \in \Gamma} \lambda_n^*$ and $\mu_1 = \mu|_X$. We claim that μ_1 and $\mu_2 = \mu - \mu_1$ are the desired measures. First, notice that $T_{\mu_1} = c_1$. Indeed, since Γ is a c_1 -uniform sequence, $T_{\mu_1} \geq 2\pi c_1$ by Theorem 3. If $T_{\mu_1} = 2\pi p > 2\pi c_1$ then there exists a p -uniform sequence Φ satisfying (7.1) for μ_1 . As was remarked after Theorem 3, we can choose Φ from X . Moreover, we can assume that each interval λ_n^* , $\lambda_n \in \Gamma$ contains at least one point from Φ . Then the intervals from Φ^* do not intersect the intervals λ_n^* , $\lambda_n \in \Lambda \setminus \Gamma$.

Divide each interval λ_n^* , $\lambda_n \in \Lambda \setminus \Gamma$ into two equal subintervals. Note that at least one of them has mass of at least one half of the original interval, with respect to μ . For each n choose the half-interval with the larger mass and denote the centers of these half-intervals by ψ_n , $\psi_n \in \lambda_n^*$. Note that $\Psi = \{\psi_n\}$ is a c_2 -uniform sequence. By Lemma 4 from Section , the sequence $\Phi \cup \Psi$ is a $(p + c_2)$ -uniform sequence. By our construction, the sequence of intervals $(\Phi \cup \Psi)^*$ satisfies (7.1), which implies that $T_\mu \geq 2\pi(p + c_2) > 2\pi d$, a contradiction.

Similarly, $T_{\mu_2} = 2\pi c_2$. □

8. GROWTH OF DENSITY IN THE CASE OF FINITE POSITIVE TYPE

As we saw from Corollary 2, a measure of finite positive type cannot have bounded density. It is natural to ask how fast should its density grow. To this account we prove the following statement.

Theorem 6. *Let $\mu \in M_{poly}^+$, be absolutely continuous, $\mu = fm$. Suppose that $T_\mu = 2\pi d$, $0 < d < \infty$. For $x > 0$ denote*

$$\mathcal{M}_f(x) = \text{ess sup}_{[-x,x]} f(x).$$

Then

$$\int_0^\infty \frac{\log(1 + \mathcal{M}_f(x))}{1 + x^2} dx = \infty.$$

Proof. Suppose that the integral from the statement is finite. Consider the measure $\nu(x) = \mu(x)/(1 + \mathcal{M}_f(|x|))$. This measure has bounded density and therefore has type 0 or infinity. Note that for any ν -weight W , $U = W/(1 + \mathcal{M}_f) + 1$ is a μ -weight. Hence there exists a d -uniform sequence Λ such that U satisfies (4.1). But

$$\log U(\lambda_n) = \log W(\lambda_n) - \log(1 + \mathcal{M}_f(|\lambda_n|))$$

and

$$\sum_n \frac{\log(1 + \mathcal{M}_f(|\lambda_n|))}{1 + \lambda_n^2} \lesssim \int_0^\infty \frac{\log(1 + \mathcal{M}_f(|x|))}{1 + x^2} dx < \infty.$$

Therefore W satisfies (4.1) on the same sequence. Hence, by Corollary 1, $T_\nu \geq 2\pi d$, which implies that $T_\nu = \infty$.

Therefore, for any ν -weight U there exists a $2d$ -uniform sequence Λ which satisfies (4.1). Since any μ -weight W is equal to $U(1 + \mathcal{M}_f(|x|))$ for a ν -weight U , similar to above we conclude that W satisfies (4.1) for a $2d$ -uniform sequence Λ . Thus by Corollary 1, $T_\mu \geq 4\pi d$, a contradiction. $\square$

Let us show that the condition of the last theorem is sharp in its scale. Let $M : \mathbb{R}_+ \rightarrow [1, \infty)$ be an increasing function such that

$$\int_0^\infty \frac{\log M(x)}{1 + x^2} dx = \infty.$$

Then there exists $\mu = fm$, $\mathcal{M}_f(x) \leq M(x)$ such that T_μ is finite positive. Indeed, put f to be equal to $M(n)$ on every interval

$$I_n = \left[n, n + \frac{1}{M(n)} \right]$$

and zero elsewhere. One can show that then every sequence $S \subset \mathbb{Z}$ such that

$$\sum_{n \in S} \frac{\log |I_n|}{1 + n^2} > -\infty$$

has density zero. Hence, $\mathbb{Z}$ is the maximal d -uniform sequence from the statement of Theorem 3 and $T_\mu = 2\pi$.

9. ADDITIVITY PROPERTIES OF UNIFORM SEQUENCES

This section contains Lemma 3, used in the proof of Theorem 3, and Lemma 4 used in the proofs of Theorems 3 and 5.

We start with the following statement from [15].

Lemma 2. *Let Λ be a sequence of real points and let $\{I_n\}$ be a short partition such that Λ satisfies*

$$a|I_n| < \#(\Lambda \cap I_n)$$

for all n with some $a > 0$ and the energy condition (3.5) on $\{I_n\}$. Then for any short partition $\{J_n\}$, there exists a subsequence $\Gamma \subset \Lambda$ which satisfies

$$\#(\Lambda \setminus \Gamma) \cap J_n = o(|J_n|)$$

as $n \rightarrow \pm\infty$, and the energy condition (3.5) on $\{J_n\}$.

All d -uniform sequences satisfy the following simple properties.

Lemma 3. *Let Λ be a d -uniform sequence.*

1) If $\Gamma \subset \Lambda$ is a c -uniform sequence then $\Lambda \setminus \Gamma$ contains a $(d - c)$ -uniform subsequence.

2) if $\Gamma_1 \subset \Lambda$ is a c_1 -uniform sequence and $\Gamma_2 \subset \Lambda$ is a c_2 -uniform sequence, $\Gamma_1 \cap \Gamma_2 = \emptyset$, then $\Gamma_1 \cup \Gamma_2$ contains a $(c_1 + c_2)$ -uniform subsequence.

Proof. 1) Let $\{I_n\}$ and $\{J_n\}$ be the short partitions from the definition of d -uniform sequences in Section 3 for the sequences Λ and Γ correspondingly. Let L_n be a third short partition with the property that

$$\begin{aligned} \max\{|I_m| : I_m \cap L_n \neq \emptyset\} + \max\{|J_m| : J_m \cap L_n \neq \emptyset\} = \\ = o(|L_n|) \text{ as } n \rightarrow \infty. \end{aligned} \tag{9.1}$$

By Lemma 2 there is a subsequence $\Lambda' \subset \Lambda$ such that

$$\#((\Lambda \setminus \Lambda') \cap L_n) = o(|L_n|)$$

and Λ' satisfies the energy condition on $\{L_n\}$. Notice that then $\Lambda' \setminus \Gamma$ satisfies the energy condition and the density condition with the constant $d - c$ on $\{L_n\}$.

2) Similarly to the last part, if $\{I_n\}$ and $\{J_n\}$ are short partitions corresponding to Γ_1 and Γ_2 , choose a short partition $\{L_n\}$ to satisfy (9.1).

Choose a subsequence $\Lambda' \subset \Lambda$ like in the last part. Note that since Λ' satisfies the energy condition on L_n , so does $\Lambda' \cap (\Gamma_1 \cup \Gamma_2)$. The last sequence also satisfies the density condition with the constant $c_1 + c_2$ on L_n . $\square$

A union of uniform sequences is a uniform sequence, up to a sequence of density zero, if the original sequences are separated from each other in the following precise sense.

Lemma 4. *Let $\Lambda = \{\lambda_n\}$ be a c -uniform sequence and let $\Gamma = \{\gamma_n\}$ be a d -uniform sequence. Let $\{I_n\}$ be a sequence of disjoint intervals such that I_n is centered at λ_n ,*

$$|I_n| \leq \frac{1}{3} \text{dist}(\lambda_n, \Lambda \setminus \{\lambda_n\})$$

and

$$\sum \frac{\log |I_n|}{1 + n^2} > -\infty.$$

Suppose that $\Gamma \cap (\cup I_n) = \emptyset$. Then $\Lambda \cup \Gamma$ contains a $(c + d)$ -uniform subsequence.

Proof. Denote by p_n the intervals

$$p_n = \frac{1}{2} I_n.$$

Choose the intervals q_n to be centered at γ_n and such that

$$|q_n| = \frac{1}{3} \text{dist}(\gamma_n, \cup I_n \cup \Gamma \setminus \{\gamma_n\}).$$

Note that then the intervals q_n, p_n are disjoint and satisfy

$$\sum \frac{\log |p_n|}{1 + n^2} > -\infty, \quad \sum \frac{\log |q_n|}{1 + n^2} > -\infty. \quad (9.2)$$

As follows from Lemma 2, one can choose a short partition $\{S_n\}$ on which both sequences Λ and Γ have subsequences which satisfy (3.5) and (3.4) with constants c and d correspondingly. WLOG we will assume that the subsequences are equal to Λ and Γ . We will also assume that the endpoints of S_n do not fall into any of the intervals p_n, q_n and that

$$|S_n| = \frac{1}{c} \# \Lambda \cap S_n = \frac{1}{d} \# \Gamma \cap S_n$$

(omitting the $o(|\cdot|)$ term in subsequent formulas).

For each $k \in \mathbb{Z}$ consider the functions u_k and v_k defined as follows. Each function u_k is continuous and piecewise linear. It is zero on $(-\infty, a_k)$, where $S_k = (a_k, b_k)$ and its derivative is zero outside of $(\cup p_n) \cap S_k$. On each $p_n, p_n \subset S_k$, u_k grows linearly by one. It follows that $u_k = c|S_k|$ on (b_k, ∞) . Repeat the same construction for v_k with the intervals q_n in place of p_n .

Now define the functions ϕ_k as $u_k(x) - cx$ on S_k and as 0 outside of S_k . Define ψ_k as $v_k(x) - dx$ on S_k and as 0 outside of S_k . Employing some of the techniques used in [15] we notice the following.

Claim 1. *The functions ϕ_k and ψ_k belong to the Dirichlet class $\mathcal{D}$ with*

$$\|\phi_k\|_{\mathcal{D}}^2 = \frac{1}{\pi}(c^2|S_k|^2 \log |S_k| - E_{\Lambda \cap S_k}) - \sum_{p_n \subset S_k} \log |p_n| + O(|S_k|^2),$$

$$\|\psi_k\|_{\mathcal{D}}^2 = \frac{1}{\pi}(d^2|S_k|^2 \log |S_k| - E_{\Gamma \cap S_k}) - \sum_{q_n \subset S_k} \log |q_n| + O(|S_k|^2).$$

and

$$\begin{aligned} \|\phi_k + \psi_k\|_{\mathcal{D}}^2 &= \frac{1}{\pi}((c+d)^2|S_k|^2 \log |S_k| - E_{(\Lambda \cup \Gamma) \cap S_k}) - \\ &\quad \sum_{p_n \subset S_k} \log |p_n| - \sum_{q_n \subset S_k} \log |q_n| + O(|S_k|^2), \end{aligned}$$

as $k \rightarrow \infty$.

Proof of Claim. The function $\phi = \phi_k$ is a function with bounded harmonic conjugate and compactly supported on $S_k = S$ derivative. Hence it belongs to $\mathcal{D}$ and its norm can be calculated as

$$\|\phi\|_{\mathcal{D}}^2 = \int_{\mathbb{R}} \phi d\tilde{\phi} = - \int_{\mathbb{R}} \tilde{\phi} d\phi.$$

Integrating by parts,

$$\begin{aligned} - \int_{\mathbb{R}} \tilde{\phi} d\phi &= \frac{1}{\pi} \int_S \phi' \left[\int_S \frac{\phi(t) dt}{t-x} \right] dx = \\ &= - \frac{1}{\pi} \int_S \phi' \left[\int_S \log |t-x| \phi'(t) dt \right] dx. \end{aligned}$$

Let $p_1, p_2, \dots, p_n$ be the intervals from our construction above inside S . Recall that by our construction of $u = u_k$, $u' = 1/|p_l|$ on each p_l and to zero elsewhere. Since $\phi = u - cx$ we obtain

$$\begin{aligned} - \int_S \phi' \left[\int_S \log |t-x| \phi'(t) dt \right] dx &= -c^2 \int_S \int_S \log |t-x| dt dx \\ &\quad + 2c^2 \int_S \sum_{m=1}^n \frac{1}{|p_m|} \int_{p_m} \log |t-x| dt dx \\ &\quad - \sum_{m=1}^n \sum_{k=1}^m \frac{1}{|p_m||p_k|} \int_{p_m} \int_{p_k} \log |t-x| dt dx = \\ &= -I + II - III. \end{aligned}$$

Recall that the points $\lambda_1, \lambda_2, \dots, \lambda_n$ are the centers of the intervals $p_1, p_2, \dots, p_n$. Because the distance between adjacent intervals p_l, p_{l+1} is at least $\max(|p_l|, |p_{l+1}|)$, for the last term we have

$$III = \sum_{1 \leq m, k \leq n, m \neq k} \log |\lambda_m - \lambda_k| + \sum_{p_n \subset S_k} \log |p_n| + O(|S|^2).$$

Similarly,

$$II = 2c^2 |S|^2 \log |S| + O(|S|^2).$$

Finally, via elementary calculations,

$$I = c^2 |S|^2 \log |S| + O(|S|^2).$$

Combining the last three equations we obtain the statement for $\phi = \phi_k$. The equations for ψ_k and $\phi_k + \psi_k$ can be proved similarly. $\square$

To finish the proof of the lemma, it remains to notice that the claim implies

$$\begin{aligned} & \sum_k \frac{(c+d)^2 |S_k|^2 \log |S_k| - E_{(\Lambda \cup \Gamma) \cap S_k}}{1+k^2} + \text{const} \\ & \lesssim \sum_k \|\phi_k + \psi_k\|_{\mathcal{D}}^2 + \text{const} \lesssim \sum_k (\|\phi_k\|_{\mathcal{D}}^2 + \|\psi_k\|_{\mathcal{D}}^2) + \text{const} \lesssim \\ & \sum_k \frac{(c^2 |S_k|^2 \log |S_k| - E_{\Lambda \cap S_k}) + (d^2 |S_k|^2 \log |S_k| - E_{\Gamma \cap S_k})}{1+k^2} + \text{const}. \end{aligned}$$

Since the sequences Λ and Γ are c - and d -uniform respectively on the partition $\{S_n\}$, the last sum is finite. Therefore the sequence $\Lambda \cup \Gamma$ satisfies the energy condition (3.5) on the partition $\{S_n\}$. Since it also satisfies the density condition (3.5) with $a = c + d$ on $\{S_n\}$, it is a $(c + d)$ -uniform sequence. $\square$

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