COMMUN. MATH. SCI.
Vol. 18, No. 2, pp. 293–320

RIGOROUS DERIVATION OF A MEAN FIELD MODEL FOR THE OSTWALD RIPENING OF THIN FILMS*

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Abstract. In the late stage of thin liquid films, liquid droplets are connected by an ultra thin residual film. Experimental studies and numerical simulations show that the size distributions of liquid droplets approach a self-similar form. However, theoretical study of the size distributions is lacking because it has been a challenge to retrieve statistical information from the mathematical PDE model of thin films. To facilitate the study of the statistical information, we rigorously derive a mean field model for the Ostwald ripening of thin liquid films through homogenization. This model corresponds to the dilute limit when the droplets are far away from each other and occupy a very small part of the thin film. Our analysis captures the screening effect of the droplets and shows that the mean field spatially varies in a length scale proportional to the screening length.

Keywords. thin film; coarsening; mean field model; homogenization.

Mathematics Subject Classification: 35B27; 76A20; 35K25; 35K55.

1. Introduction

Thin liquid films on a solid substrate can be driven by the dewetting effect into complex nonlinear patterns which are localized quasi-stationary liquid droplets connected by an ultra-thin residual film. The dewetting effect is a result of the balance between surface tension and intermolecular forces. When there is no evaporation, the total mass of droplets is conserved since the mass of the ultra-thin residual film can be ignored. The total number of droplets is observed to decrease and the typical size of droplets increases. Such a phenomenon is called coarsening. There are two mechanisms for the coarsening to occur. One mechanism is the migration and collision of droplets. The other is the exchange of mass between droplets through a diffusive field in the ultra-thin residual film. Bigger droplets grow while smaller ones shrink and disappear. The second mechanism is called Ostwald ripening, as it is similar to what happens in the late stage of phase transitions (see, e.g., [13]).

Experimental studies and numerical simulations show that generically the distribution of droplet sizes approaches a self-similar form and there is a simple relation called the coarsening rate between time and the typical size of droplets (see [7] and references therein). Such a spatio-temporal relation can also be deduced heuristically by asymptotic analysis [3–7] under the assumption that the size distribution is self-similar. But a rigorous justification of the existence of self-similar distributions is lacking. Part of the reason is, it is a challenge to retrieve statistical information from the mathematical model of thin films, which is the lubrication theory [11]. Suppose the thin film is on a substrate $\Omega \subset \mathbb{R}^2$ and $h(x,t) \geq 0$ is the thickness of the thin film. Then h satisfies the following thin film equation

$$h_t = \nabla \cdot (m(h) \nabla(-\Delta h + U'(h))). \tag{1.1}$$

Here U is a van der Waals attraction-repulsion potential. The diffusion mobility $m(h) = h^q$ depends on the boundary condition for the fluid velocity at the substrate. Different types of boundary conditions give different $q > 0$ values. See for example [3, 4, 6] for

*Received: December 13, 2018; Accepted (in revised form): September 9, 2019. Communicated by Andrea Bertozzi.

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discussions about the effect of $m(h)$ on the coarsening mechanisms. There are also estimates of the coarsening rate by directly studying the thin film equation [12]. This approach gives rigorous proofs on the universal upper bound of the coarsening rate, which is true independent of the statistical information.

In two dimensional thin films, the shapes of liquid droplets are paraboloids [3, 4]. The contact angle, which is the angle between the surface of the droplets and the uniform ultra thin residual film, is solely determined by the intermolecular potential and independent of the sizes of the droplets. Because of this feature, the droplets are totally determined by their circular bases. The dynamics of the thin film is hence reduced to the dynamics of the radii of the bases. Suppose at time t, there are $N(t)$ droplets on a square substrate $\Omega=[0,\mathcal{L}]^2$. Let $B_i:=\{x\in\Omega:|x-x_i|\le R_i(t)\}$ be the basis with center x_i and radius $R_i(t)$, $i=1,\dots,N(t)$. Suppose further that the droplets are well-separated. The motion of the centers x_i are affected by the form of the mobility $m(h)$. Under the no-slip boundary condition for the fluid at the substrate, we have $m(h)=h^3$. In this case the motion of the centers x_i are minor effects [4]. Thus we may assume they are fixed in space. Using asymptotic analysis it is derived in [4] that, under a proper rescaling, the evolution of the droplets is determined by the following equations.

$$-\Delta u(x,t)=0 \quad \text{if } x\in\Omega\setminus\bigcup_{i=1}^{N(t)}\bar{B}_i, \tag{1.2}$$

$$u=\frac{1}{R_i} \quad \text{if } x\in\bar{B}_i, \tag{1.3}$$

$$\dot{R}_i=\frac{1}{R_i^2}\int_{\Gamma_i}[\nabla u\cdot n]\,ds \quad \text{on } \Gamma_i:=\partial B_i. \tag{1.4}$$

For simplicity we take periodic boundary condition on $\partial\Omega$. Here u is the rescaled pressure field, $\Gamma_i:=\partial B_i$ is the boundary of the base of the i^{th} droplet, n is the outer normal vector of Γ_i, $[\nabla u\cdot n]$ is the jump of the normal gradient of u across the boundary. Equations (1.2)–(1.4) is a quasi-stationary model for the evolution of droplets. Being quasi-stationary means that even though the pressure field u depends on time, at each moment the evolution of u does not explicitly depend on t, rather it is determined by the boundary condition (1.3) on the boundary of droplets and the boundary condition on $\partial\Omega$. The motion of the droplet boundary is then determined by (1.4), which guarantees that the total volume of droplets is preserved during the ripening process. Because of the parabolic shapes, volumes of droplets are proportional to the cube of the radii of their bases. We assume droplets are far away from each other so that collisions do not occur. Using the same technique as in [8], it can be shown that for any given initial configuration $\{R_i(0)\}$, there exists $T>0$ such that (1.2)–(1.4) has a unique solution in $[0,T]$, where R_i are smooth and $u\in L^2(0,T;H^1(\Omega))$. The solution can be extended to some time t_1 when some droplets disappear (radii become 0). After this moment, we can remove those droplets with radii 0 and restart the evolution.

The first observation is that the total volume of droplets is preserved since

$$\sum_{i=1}^{N}R_i^2\dot{R}_i=\sum_{i=1}^{N}\int_{\Gamma_i}[\nabla u\cdot n]\,ds=-\int_{\Omega\setminus\cup_i\bar{B}(x_i,R_i)}\Delta u\,dx=0. \tag{1.5}$$

In addition, the surface energy, which is proportional to the total surface area of droplets and hence to $\sum_{i=1}^{N}R_i^2$, is decreasing. This can be seen by the following estimate.

Multiplying Equation (1.2) by u and integrating over Ω, we have

$$\begin{aligned} 0 &= \int_\Omega -u\Delta u\, dx = \int_\Omega |\nabla u|^2\, dx + \sum_{i=1}^N \int_{\Gamma_i} u[\nabla u\cdot n]\, ds \\ &= \int_\Omega |\nabla u|^2\, dxdt + \sum_{i=1}^N R_i \dot{R}_i. \end{aligned} \tag{1.6}$$

Integrating over $t_1 < t_2$, we get

$$\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\, dxdt + \frac{1}{2}\sum_{i=1}^N R_i(t_2)^2 = \frac{1}{2}\sum_{i=1}^N R_i(t_1)^2. \tag{1.7}$$

Immediately we have

$$\sum_{i=1}^N R_i^2(t_2) \leq \sum_{i=1}^N R_i^2(t_1) \quad \text{for all } t_1 < t_2. \tag{1.8}$$

To facilitate the study of size distributions, recently mean field descriptions of the Ostwald ripening of thin films were proposed for the simplified case of one dimensional thin films [7] (see also [1]) and the physically realistic two dimensional case [2]. These mean field models were derived using heuristic arguments. Although information about the delicate relations between droplet size, distance between droplets, and the domain size were not accurately captured, these mean field models prove to be useful in predicting the existence and describing the properties of self-similar distributions.

The purpose of this paper is to give a rigorous derivation of a mean field model for thin films on two dimensional solid substrates, and capture the delicate information that is missing in those heuristic mean field models. The study of our new model will provide us more accurate information about the coarsening dynamics of thin liquid films. To make the paper accessible for a wider readership, in Subsection 1.1 we introduce our main result in an informal form, and in Subsection 1.2 we give a heuristic derivation of the mean field model. Also in Subsection 1.2 we state the similarity and difference between our problem and the two-dimensional Mullins-Sekerka model considered in [9]. Starting from Subsection 1.3 we introduce our result in a rigorous form. In Subsection 1.3 we describe the spatial and temporal rescaling under which we can study the homogenization, and also state the main result in a rigorous form. In Section 2 we describe the details about our time scalings. In Section 3 we consider the homogenization limit and derive the necessary estimates. In Section 4 we prove our main theorems. Sections 3 and 4 closely follow the structure of [9], with necessary adaptation for the derivation of our main result. We also provided more details that make the arguments easier for readers to follow. One difference from [9] is that in Subsection 3.4 we use a different approach to prove two lemmas (Lemmas 3.4, and 3.5) that can fulfill the same role as Lemma 3.6, which is proved in [9]. Finally in Section 5 we give some discussion about further investigations.

1.1. Main result in an informal form. Since the total volume of all droplets is conserved, we define $\mathcal{R}$ to be the radius of the droplet whose volume is the average volume of all droplets. Suppose initially there are N_0 droplets with radii $R_i(0)$. Then

$$N_0\mathcal{R}_0^3 = \sum_{i=1}^{N_0} R_i(0)^3, \qquad \text{or} \qquad \mathcal{R}_0 = \left(\frac{\sum_{i=1}^{N_0} R_i(0)^3}{N_0}\right)^{1/3}.$$

Taking $\mathcal{R}_0$ as a characteristic droplet size, we nondimensionalize the radii $R_i(t)$ into $\hat{R}_i(t) := R_i(t)/\mathcal{R}_0$. The space locations of the droplets are as important as their sizes. Let $f(t,x,\hat{R})$ be the joint distribution of the locations and the nondimensionalized sizes of droplets. Our main result is that $f(t,x,\hat{R})$ satisfies the following system of equations in the weak sense.

$$-\Delta \hat{u}_* + 2\pi\delta \int_0^\infty \left(\hat{u}_* - \frac{1}{\hat{R}}\right) f(t,x,\hat{R})\, d\hat{R} = 0, \tag{1.9}$$

$$\frac{\partial}{\partial t} f(t,x,\hat{R}) + \frac{\partial}{\partial \hat{R}} \left(-\frac{2\pi}{\hat{R}^2 \ln \phi^{-1/2}} \left(\frac{1}{\hat{R}} - \hat{u}_*\right) f(t,x,\hat{R}) \right) = 0. \tag{1.10}$$

Here ϕ is the fraction of the substrate covered by the droplet bases, which is decreasing in time, and $\delta > 0$ is a scaling parameter describing the ratio between the domain size $\mathcal{L}$ and the screening length, denoted by $\mathcal{L}_s$. For a specific droplet, its neighbor droplets have a screening effect that blocks the influence from droplets that are far away. A droplet can only receive influence from droplets within the distance of the screening length. For discussion about screening length, we refer to [10] for 3D systems and [9] for 2D systems. Roughly speaking, the screening length is determined by the size of droplets and the distance between droplets. Equation (1.18) gives an approximate formula of the screening length in the scenario that all droplets are of similar size and the distances between droplets are roughly the same. The scaling parameter δ is then defined by

$$\delta := \frac{\mathcal{L}^2}{\mathcal{L}_s^2}.$$

So δ connects the domain size, the typical droplet radius, and the typical distance between droplets. In Subsection 1.2 we will give a heuristic description of the mean field model, and in Subsection 1.3 we will rigorously describe the setting and the mathematical meaning of a weak solution for (1.9)–(1.10).

1.2. Heuristic description of the mean field model. Now we give some heuristic argument about the behavior of u. Assume the distance between droplets is much bigger than the size of droplets. Then at a given time, near the droplets, the pressure u should be similar to the fundamental solution to the 2D Laplacian operator. Away from droplets, u should be approaching a slowly-varying mean field u_* which is the average of the hydrodynamic pressure of neighbor droplets. Consequently u_* depends on the spatial distribution of the droplets. This indicates a "cutoff" of the fundamental solution. Assume d is the cutoff distance, that is, d is a number bigger than droplet sizes but smaller than, or at most comparable to, the distance between droplets so that u becomes the mean field u_* at points x when $|x - x_i| \geqslant d$ for all i. Then because of boundary condition (1.3), u should have the following form near the i^{th} droplet,

$$u(x) \approx \frac{\left(\frac{1}{R_i} - u_*\right)}{\ln \frac{R_i}{d}} \ln \frac{|x - x_i|}{d} + u_* \qquad \text{for } |x - x_i| \leq d. \tag{1.11}$$

From the boundary condition (1.4), we obtain an evolution law for R_i.

$$\dot{R}_i \approx \frac{2\pi}{R_i^2} \frac{1}{\ln \frac{R_i}{d}} \left(\frac{1}{R_i} - u_*\right). \tag{1.12}$$

Note that the cutoff distance d is artificial. We assume that d is comparable to the average distance between the droplets. So heuristically we may take d to be the average distance, that is, $d=\mathcal{L}/\sqrt{N}$, or $Nd^2=\mathcal{L}^2$. In addition, without changing the leading order behavior of the model, we may replace R_i by the characteristic radius $\mathcal{R}$ of all droplets. Define $\mathcal{R}=\mathcal{R}(t)$ to be the radius of the droplet whose volume is the average volume of all droplets at time t. That is,

$$\mathcal{R}(t)=\left(\frac{\sum_{i=1}^{N(t)} R_i(t)^3}{N(t)}\right)^{1/3}.$$

Then roughly speaking, $N\mathcal{R}^2\sim\sum_{i=1}^{N} R_i^2$ and the coefficient $(\ln\frac{R_i}{d})^{-1}$ is replaced by

$$\left(\ln\frac{\mathcal{R}}{d}\right)^{-1}=\left(\ln\frac{\sqrt{N}\mathcal{R}}{\sqrt{N}d}\right)^{-1}\sim\left(\frac{1}{2}\ln\frac{\sum_{i=1}^{N} R_i^2}{\mathcal{L}^2}\right)^{-1}=-\frac{1}{\ln\phi^{-1/2}} \tag{1.13}$$

where

$$\phi(t):=\frac{\sum_{i=1}^{N} R_i^2}{\mathcal{L}^2} \tag{1.14}$$

is the fraction of the substrate covered by the bases of all droplets. After the above simplifications, we obtain the following mean field model.

$$\dot{R}_i=-\frac{2\pi}{R_i^2\ln\phi(t)^{-1/2}}\left(\frac{1}{R_i}-u_*\right)\qquad\text{for all } i=1,\cdots,N. \tag{1.15}$$

Next we consider the evolution of the distribution of radii. Since the evolution of droplets depends on their spatial positions, the radius distribution is not independent of the spatial distribution of their centers. Let $f(t,x,R)$ be the joint distribution of centers and radii at time t. Given the evolution law for radii (1.15), the transport equation for f is

$$\frac{\partial}{\partial t}f(t,x,R)+\frac{\partial}{\partial R}\left(-\frac{2\pi}{R^2\ln\phi^{-1/2}}\left(\frac{1}{R}-u_*\right)f(t,x,R)\right)=0. \tag{1.16}$$

Here

$$\phi=\frac{\int_\Omega\int_0^\infty R^2 f(t,x,R)\,dR\,dx}{\mathcal{L}^2}. \tag{1.17}$$

The minimum requirement on $f(t,x,R)$ is $\int_\Omega\int_0^\infty R^2 f(t,x,R)\,dR\,dx<<\mathcal{L}^2$ since the droplets can not occupy all the substrate. In addition, the conservation of total volume of droplets translates into $\int_\Omega\int_0^\infty R^3 f(t,x,R)\,dR\,dx=\text{const}$.

The mean field u_* "slowly varies" on the scale of screening length, denoted by $\mathcal{L}_s$. According to [9], if $\mathcal{R}$ is the typical radius of droplets and d is the average distance between droplets, then

$$\mathcal{L}_s^2\approx d^2\ln(d/\mathcal{R}). \tag{1.18}$$

So if $\mathcal{R}$ is much smaller than d, $\mathcal{L}_s$ is much bigger than d and it can even be much bigger than the system size $\mathcal{L}$. When $\mathcal{L}_s>>\mathcal{L}$, since u_* varies on a scale $\mathcal{L}_s$, it is approximately constant in Ω.

In [2], u_* was taken to be a spatial constant and it corresponds to the case $\mathcal{L}_s >> \mathcal{L}$. In this case, the spatial positions of droplets do not matter and the joint distribution $f(t,x,R)$ is simplified into just a distribution of radii, $f(t,R)$. The value of u_* is then determined by the conservation of total volume of droplets. Hence

$$u_*(t) = \frac{\sum_{i=1}^{N} R_i^{-1}}{N} = \frac{\int_0^\infty R^{-1} f(t,R)\,dR}{\int_0^\infty f(t,R)\,dR}. \tag{1.19}$$

In the case when $\mathcal{L}_s$ is comparable with $\mathcal{L}$, u_* is no longer a spatial constant. Furthermore, the evolution of droplets also depends on their spatial locations. In [9], Niethammer and Otto studied a similar problem of the two-dimensional Mullins-Sekerka model for phase transitions of a two-phase mixture. In the dilute limit, the minor phase consists of disjoint circular islands. They considered the effect of screening length and derived a transport equation for the joint distribution of location and size of the circular particles. Our thin liquid film problem has some similarity to the problem in [9] since the substrate is two-dimensional. Indeed this paper is motivated by [9], and our approach closely follows the framework of [9]. However there are some differences that require extra care and different treatment.

(i) The liquid droplets are three dimensional objects while the substrate is two dimensional, hence our problem bears a feature of a mixture of dimensions.

(ii) In [9], the total area covered by the 2D islands is preserved, resulting in a constant ϕ. In our situation, the total volume of the 3D liquid droplets is preserved, while the total area covered by the 3D liquid droplets is decreasing. As a result, our $\phi(t)$ defined by (1.14) is decreasing in time.

(iii) In [9], the authors consider the case when the screening length is comparable with the system size. Under the simplified assumption that $\mathcal{L}_s = \mathcal{L}$ and the initial characteristic radius is one, they derived a limiting model. In our problem, to signify the importance of various situations, we describe in details how the system should be rescaled based on the relations between the characteristic radius, the distance between droplets, the domain size, and the screening length. Specifically we introduce and keep track of a parameter δ that describes the relation between the typical radius of droplets, the average distance between droplets, and the domain size. δ can also be considered as a measure of the system size relative to the screening length.

1.3. Homogenization and the main result. Mathematically, a mean field model corresponds to the dilute limit of the evolution equations (1.2)–(1.4). The dilute limit is the limiting behavior of the system when the fraction of the substrate covered by the droplets is disappearing, while the total number of droplets is increasing toward infinity. To study the dilute limit, we need to consider a sequence of domains with bigger and bigger average droplet distances. Let $\Omega_k := (0,\mathcal{L}_k)^2$ be a sequence of domains in which initially there are $N_k(0)$ droplets of radii $\{R_{k,i}, i=1,\ldots,N_k(0)\}$, whose centers of bases are on the lattice of spacing d_k. Thus $\mathcal{L}_k^2 = N_k(0)d_k^2$. We require $N_k(0)\to\infty$ and $d_k\to\infty$, while $\phi_k(0) := \frac{\sum_{i=1}^{N_k(0)} R_{k,i}^2}{\mathcal{L}_k^2} \to 0$ as $k\to\infty$. Also we take $\mathcal{R}_k$ as the characteristic radius of the initial distribution of droplets, defined by

$$\mathcal{R}_k = \left(\frac{\sum_{i=1}^{N_k(0)} R_{k,i}^3}{N_k(0)}\right)^{1/3}. \tag{1.20}$$

By (1.18), the screening length is defined by

$$\mathcal{L}_{k,s}^2 := d_k^2 \ln\left(\frac{d_k}{\mathcal{R}_k}\right). \tag{1.21}$$

In such a system, there are three dimensionless ratios,

$$\varepsilon := \frac{d_k}{\mathcal{L}_k}, \qquad a_\varepsilon := \frac{\mathcal{R}_k}{\mathcal{L}_k}, \qquad \delta := \frac{\mathcal{L}_k^2}{\mathcal{L}_{k,s}^2}.$$

Only two of these ratios are independent. Indeed,

$$\delta^{-1} = \frac{d_k^2}{\mathcal{L}_k^2} \ln\left(\frac{d_k}{\mathcal{R}_k}\right) = \varepsilon^2 \ln\left(\frac{\varepsilon}{a_\varepsilon}\right). \tag{1.22}$$

The homogenization limit corresponds to

$$\varepsilon = \frac{d_k}{\mathcal{L}_k} = \frac{1}{\sqrt{N_k(0)}} \to 0 \quad \text{as } k \to \infty.$$

Since we want to capture the effect of screening length, we keep $\delta > 0$ as a positive constant. Thus

$$a_\varepsilon = \varepsilon e^{-1/(\delta\varepsilon^2)} \tag{1.23}$$

decays to zero exponentially as $\varepsilon \to 0^+$.

We rescale the sequence of domains Ω_k into the reference domain

$$\Omega_0 := (0,1)^2$$

by taking $x = y/\mathcal{L}_k$ for $y \in \Omega_k$ and $x \in \Omega_0$. Then the radii are rescaled into

$$\frac{R_{k,i}}{\mathcal{L}_k} = \frac{\mathcal{R}_k}{\mathcal{L}_k}\frac{R_{k,i}}{\mathcal{R}_k} = a_\varepsilon \frac{R_{k,i}}{\mathcal{R}_k}. \tag{1.24}$$

Equation (1.24) indicates that it is the normalized radii

$$\hat{R}_{k,i} := \frac{R_{k,i}}{\mathcal{R}_k} \tag{1.25}$$

that we should study the distribution of. By (1.20) we see that

$$\frac{\sum_{i=1}^{N_k(0)} \hat{R}_{k,i}^3(0)}{N_k(0)} = 1. \tag{1.26}$$

In summary, after the nondimensionalization and normalization, we get a domain $\Omega_0 = (0,1)^2$, in which we have a lattice grid of size ε. At each lattice point x_i there is a circle

$$B_i^\varepsilon(t) := B(x_i, a_\varepsilon \hat{R}_{k,i}(t)),$$

which is the base of a liquid droplet.

Next we consider the rescaling in time. According to (1.15), the logarithmic factor

$$\frac{1}{\ln \phi(t)^{-1/2}}$$

determines the time scale on which the radius $R_{k,i}$ varies. Since ϕ is decreasing in time, and is disappearing as $\varepsilon \to 0$ (see Section 2 for a proof), it is natural to consider the following nonlinear rescaling of time

$$\tilde{t} := \int_0^t \frac{1}{\ln \phi^{-1/2}(s)} ds. \tag{1.27}$$

In Section 2 it is shown that

$$\frac{\tilde{t}}{\delta\varepsilon^2 t} \to 1 \qquad \text{uniformly for } t \in (0,\infty) \text{ as } \varepsilon \to 0.$$

Thus it suffices to consider the following linear rescaling of time

$$\hat{t} = \mathcal{R}_k^{-4} \delta\varepsilon^2 t. \tag{1.28}$$

The appearance of $\mathcal{R}_k^{-4}$ not only simplifies notations but also is consistent with the spatio-temporal relation derived in [2].

The pressure field $u(t,y)$, $y \in \Omega_k$, correspondingly is rescaled into

$$u(t,y) = \frac{1}{\mathcal{R}_k} \hat{u}_\varepsilon(\hat{t},x), \qquad \text{where } x = y/\mathcal{L}_k \text{ and } \hat{t} = \delta\varepsilon^2 t.$$

Then

$$\nabla_y u(t,y) = \frac{1}{\mathcal{R}_k \mathcal{L}_k} \nabla_x \hat{u}_\varepsilon(\hat{t},x) \tag{1.29}$$

and

$$\begin{aligned} \frac{d\hat{R}_{k,i}}{d\hat{t}} &= \frac{\mathcal{R}_k^3}{\delta\varepsilon^2} \frac{dR_{k,i}}{dt} = \frac{\mathcal{R}_k^3}{\delta\varepsilon^2} \frac{1}{R_{k,i}^2} \int_{\Gamma_i} \nabla_y u(t,y) \cdot n \, ds(y) \\ &= \frac{1}{\delta\varepsilon^2} \frac{1}{\hat{R}_i^2} \int_{\Gamma_i^\varepsilon} \nabla_x \hat{u}_\varepsilon(\hat{t},x) \cdot n \, ds(x). \end{aligned} \tag{1.30}$$

Here $\Gamma_i^\varepsilon = \partial B_i^\varepsilon$. According to (1.2)–(1.4), $\hat{u}_\varepsilon$ satisfies

$$-\Delta \hat{u}_\varepsilon(\hat{t},x) = 0 \qquad \text{in } \Omega_0 \setminus \cup_i \bar{B}_i^\varepsilon, \tag{1.31}$$

$$\hat{u}_\varepsilon = \frac{1}{\hat{R}_{k,i}} \qquad \text{in } \bar{B}_i^\varepsilon, \tag{1.32}$$

$$\frac{d\hat{R}_{k,i}}{d\hat{t}} = \frac{1}{\delta\varepsilon^2 \hat{R}_{k,i}^2} \int_{\Gamma_i^\varepsilon} [\nabla \hat{u}_\varepsilon \cdot n] \, ds \qquad \text{on } \Gamma_i^\varepsilon. \tag{1.33}$$

Our main result is the following theorem.

THEOREM 1.1. *Consider the system (1.31)–(1.33). Suppose $\delta > 0$ is a fixed constant. Then for any $0 < \lambda \le 1/2$, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the droplets $B(x_i, a_\varepsilon \hat{R}_{k,i}(\hat{t}))$ do not collide. More precisely, we have*

$$B(x_i, a_\varepsilon \hat{R}_{k,i}(\hat{t})) \subset B(x_i, \lambda\varepsilon) \qquad \textit{for all } \; \hat{t} \in [0,\infty). \tag{1.34}$$

Furthermore, if we assume that initially the total volume is not concentrated on the few biggest droplets, that is

$$\frac{\sum_{i:\hat{R}_{k,i}(0) \geqslant R} \hat{R}_{k,i}^3(0)}{\sum_{i=1}^{N_0} \hat{R}_{k,i}^3(0)} \to 0 \qquad \textit{uniformly in } \varepsilon \textit{ as } R \to \infty, \tag{1.35}$$

then the diffusion field $\hat{u}_\varepsilon$ *converges as* $\varepsilon \to 0$ *to a mean field* $\hat{u}_*$*, which satisfies the following system of equations together with the joint distribution* $d\nu_{\hat{t}}(x,r)$*:*

$$\int_0^\infty \int_{\Omega_0} \psi(t)\nabla\zeta(x)\cdot\nabla\hat{u}_*\,dx\,d\hat{t} \\ + 2\pi\delta\int_0^\infty\int_{\Omega_0\times(0,\infty)} \psi(t)\zeta(x)\left(\hat{u}_* - \frac{1}{r}\right)d\nu_{\hat{t}}(x,r)\,d\hat{t} = 0, \tag{1.36}$$

$$\int_0^\infty \int_{\Omega_0\times(0,\infty)} \left(\psi'(t)\zeta - 2\pi\psi(t)\frac{\partial\zeta}{\partial r}\frac{1}{r^2}\left(\frac{1}{r} - \hat{u}_*\right)\right)d\nu_{\hat{t}}(x,r)\,d\hat{t} = 0, \tag{1.37}$$

for all $\psi \in C_0^\infty(0,\infty)$ *and* $\zeta \in C_p^0$*. The function space* C_p^0 *is defined by (3.9).*

REMARK 1.1. If $d\nu_{\hat{t}}(x,r)$ has a density $f(\hat{t},x,r)$, then $\hat{u}_*$ and $f(\hat{t},x,r)$ are weak solutions for

$$-\Delta\hat{u}_* + 2\pi\delta\int_0^\infty \left(\hat{u}_* - \frac{1}{r}\right) f(\hat{t},x,r)\,dr = 0, \tag{1.38}$$

and

$$\frac{\partial}{\partial\hat{t}} f(\hat{t},x,r) + \frac{\partial}{\partial r}\left(-\frac{2\pi}{r^2}\left(\frac{1}{r} - \hat{u}_*\right) f(\hat{t},x,r)\right) = 0. \tag{1.39}$$

The topology in which $\hat{u}_\varepsilon$ converges to $\hat{u}_*$ will be made clear in Section 4. Apparently (1.38) is a generalization of (1.19). In addition, in the regime when $\mathcal{L} << \mathcal{L}_s$, $\delta \approx 0$ and heuristically (1.38) reduces to $\Delta\hat{u}_* \approx 0$. Combined with the periodic boundary condition, u_* is a constant in Ω_0 and (1.38) heuristically implies (1.19). To make this argument rigorous, we may consider the situation that δ depends on ε and $\delta \to 0$ as $\varepsilon \to 0$. It will be studied in further explorations. Equation (1.39) is of the same form as (1.16), except that (1.39) is in the $\hat{t} = \mathcal{R}_k^{-4}\delta\varepsilon^2 t \approx \mathcal{R}_k^{-4}\int_0^t \frac{1}{\ln\phi^{-1/2}(s)}\,ds$ time scale.

As was pointed out in [9], it is not necessary for x_i to be the grid points. It suffices to assume the minimum distance between droplets is a fixed portion of the average distance. In that sense, we need only to find $0 < \lambda < 1/2$ such that $B(x_i,\lambda\varepsilon) \cap B(x_j,\lambda\varepsilon) = \emptyset$ if $i \neq j$. For Assumption (1.35), if initially all droplets are bounded uniformly in ε, then this assumption is automatically true.

2. The time scalings

In this section we explain the relation between the two time scalings: the nonlinear rescaling $\tilde{t} := \int_0^t \frac{1}{\ln\phi^{-1/2}(s)}\,ds$ and the linear rescaling $\hat{t} = \mathcal{R}_k^{-4}\delta\varepsilon^2 t$. For $\tilde{t}$, by (1.27),

$$\frac{d\tilde{t}}{dt} = \frac{1}{\ln\phi^{-1/2}(t)}. \tag{2.1}$$

Let's estimate $d\tilde{t}/dt$.

$$\ln\phi(t) = \ln\left(\frac{\sum_{i=1}^N R_{k,i}^2}{\mathcal{L}_k^2}\right) = \ln\left(\frac{\mathcal{R}_k^2}{\mathcal{L}_k^2}\sum_{i=1}^N \hat{R}_{k,i}^2\right) = \ln\left(a_\varepsilon^2\sum_{i=1}^N \hat{R}_{k,i}^2\right) \\ = \ln\frac{a_\varepsilon^2}{\varepsilon^2} + \ln\left(\varepsilon^2\sum_{i=1}^N \hat{R}_{k,i}^2\right)$$

$$= \left(\ln \frac{a_\varepsilon^2}{\varepsilon^2}\right)\left(1 + \frac{\ln\left(\varepsilon^2 \sum_{i=1}^N \hat{R}_{k,i}^2\right)}{\ln \frac{a_\varepsilon^2}{\varepsilon^2}}\right)$$

$$= -2\left(\ln \frac{\varepsilon}{a_\varepsilon}\right)\left(1 - \frac{\ln\left(\varepsilon^2 \sum_{i=1}^N \hat{R}_{k,i}^2\right)}{2\ln \frac{\varepsilon}{a_\varepsilon}}\right),$$

$$\frac{1}{\ln \phi(t)^{-1/2}} = \left(\ln \frac{\varepsilon}{a_\varepsilon}\right)^{-1}\left(1 - \frac{\ln\left(\varepsilon^2 \sum_{i=1}^N \hat{R}_{k,i}^2\right)}{2\ln \frac{\varepsilon}{a_\varepsilon}}\right)^{-1}$$

$$= \delta\varepsilon^2 \left(1 - \frac{1}{2}\delta\varepsilon^2 \ln\left(\varepsilon^2 \sum_{i=1}^N \hat{R}_{k,i}^2\right)\right)^{-1} \qquad \text{by (1.23)}$$

$$= \delta\varepsilon^2 \beta_\varepsilon(t). \tag{2.2}$$

Here

$$\beta_\varepsilon(t) := \left(1 - \frac{1}{2}\delta\varepsilon^2 \ln\left(\varepsilon^2 \sum_{i=1}^N \hat{R}_{k,i}^2\right)\right)^{-1}. \tag{2.3}$$

So

$$\tilde{t} = \delta\varepsilon^2 \int_0^t \beta_\varepsilon(s)\, ds, \tag{2.4}$$

$$\phi(t) = e^{-2/(\delta\varepsilon^2 \beta_\varepsilon(t))}. \tag{2.5}$$

Lemma 2.1. *$\beta_\varepsilon(t)$ uniformly converges to 1 for all $t \in (0,\infty)$ as $\varepsilon \to 0$. Consequently*

$$\phi(t) \to 0 \quad \textit{and} \quad \frac{\tilde{t}}{\delta\varepsilon^2 t} \to 1 \quad \textit{uniformly in } t \in (0,\infty) \textit{ as } \varepsilon \to 0. \tag{2.6}$$

Proof. By (1.26) we have

$$\sum_{i=1}^{N_k(0)} \hat{R}_{k,i}^3(0) = N_k(0) =: N_0 = \varepsilon^{-2}.$$

Then by Hölder's inequality, the conservation of mass, and the fact that $N_k(t) \le N_k(0)$

$$\varepsilon^2 \sum_{i=1}^{N_k(t)} \hat{R}_{k,i}^2(t) \le \varepsilon^2 N_k(t)^{1/3} \left(\sum_{i=1}^{N_k(t)} \hat{R}_{k,i}^3(t)\right)^{2/3} \le \varepsilon^2 N_0^{1/3} \left(\sum_{i=1}^{N_k(0)} \hat{R}_{k,i}^3(0)\right)^{2/3}$$

$$= \varepsilon^2 N_0 \le 1. \tag{2.7}$$

On the other hand, since the surface area $\sum_{i=1}^{N(t)} \hat{R}_{k,i}^2$ is decreasing, it is bigger than or equal to the surface area of the eventual stable equilibrium state when there remains just one droplet, whose volume equals the preserved total volume of droplets, and thus the radius is $\hat{R}_{k,f} := \left(\sum_{i=1}^{N_k(0)} \hat{R}_{k,i}^3(0)\right)^{1/3} = \varepsilon^{-2/3}$. So

$$\varepsilon^2 \sum_{i=1}^{N(t)} \hat{R}_{k,i}^2 \geqslant \varepsilon^2 \hat{R}_{k,f}^2 = \varepsilon^{2/3}. \tag{2.8}$$

Combining (2.7) and (2.8), we have

$$1\le 1-\frac{1}{2}\delta\varepsilon^2\ln\left(\varepsilon^2\sum_{i=1}^{N}\hat{R}_{k,i}^2\right)\le 1-\frac{1}{2}\delta\varepsilon^2\ln\varepsilon^{2/3}=1-\frac{1}{3}\delta\varepsilon^2\ln\varepsilon. \tag{2.9}$$

By (2.3)

$$\frac{1}{1-\frac{1}{3}\delta\varepsilon^2\ln\varepsilon}\le\beta_\varepsilon(t)\le 1. \tag{2.10}$$

Letting $\varepsilon\to 0$, we obtain $\beta_\varepsilon(t)\to 1$ uniformly for all $t\in[0,\infty)$. Hence $\phi(t)\to 0$ uniformly for all $t\in[0,\infty)$ by (2.5). In addition,

$$\frac{\tilde{t}}{\delta\varepsilon^2 t}=\frac{1}{t}\int_0^t\beta_\varepsilon(s)\,ds\to 1 \qquad \text{uniformly in } t\in(0,\infty) \text{ as } \varepsilon\to 0. \tag{2.11}$$

□

Because of Lemma 2.1, and since we only consider the limiting behavior when $\varepsilon\to 0$, it suffices to consider the following linear rescaling of time

$$\hat{t}=\mathcal{R}_k^{-4}\delta\varepsilon^2 t. \tag{2.12}$$

The following a priori estimate is similar to (1.7).

$$\int_0^{\hat{t}}\int_{\Omega_0}|\nabla\hat{u}_\varepsilon|^2\,dx d\hat{t}+\frac{1}{2}\delta\varepsilon^2\sum_{i=1}^{N(\hat{t})}\hat{R}_{k,i}^2(\hat{t})=\frac{1}{2}\delta\varepsilon^2\sum_{i=1}^{N_0}\hat{R}_{k,i}^2(0). \tag{2.13}$$

So according to (2.13) and (2.7),

$$\int_0^{\infty}\int_{\Omega_0}|\nabla\hat{u}_\varepsilon|^2\,dx d\hat{t}\le\frac{1}{2}\delta\varepsilon^2\sum_{i=1}^{N_0}R_{k,i}^2(0)\le\frac{1}{2}\delta. \tag{2.14}$$

By (2.14), if $\delta\approx 0$, then

$$\int_0^{\infty}\int_{\Omega_0}|\nabla\hat{u}_\varepsilon|^2\,dx d\hat{t}\approx 0 \tag{2.15}$$

and $\hat{u}_\varepsilon$ is approximately a spatial constant. This is consistent with the screening length argument given in Section 1. If δ is a nonzero constant, then we do not expect $\int_0^\infty\int_{\Omega_0}|\nabla\hat{u}_\varepsilon|^2\,dx d\hat{t}$ to be small and u_ε will not be a spatial constant.

3. Homogenization

Our goal is to consider the homogenization of the system (1.31)–(1.33). To simplify notations, from now on we omit the hat and the subscript k, and rewrite $\hat{t}$ as t, $\hat{R}_{k,i}$ as R_i, and $\hat{u}_\varepsilon$ as u_ε. Also we drop the subscript of Ω_0 and just write $\Omega=(0,1)^2$. So the system we are working on is

$$-\Delta u_\varepsilon(t,x)=0 \quad \text{in } \Omega\setminus\cup_i\bar{B}_i^\varepsilon, \tag{3.1}$$

$$u_\varepsilon=\frac{1}{R_i} \quad \text{in } \bar{B}_i^\varepsilon, \tag{3.2}$$

$$\frac{dR_i}{dt}=\frac{1}{\delta\varepsilon^2R_i^2}\int_{\Gamma_i^\varepsilon}[\nabla u_\varepsilon\cdot n]\,ds \quad \text{on } \Gamma_i^\varepsilon=\partial B_i^\varepsilon. \tag{3.3}$$

The natural space for the pressure u_ε is $L^2(0,\infty;H_p^1)$, where

$$H_p^1 := \{u \in H_{loc}^1(\mathbb{R}^2) : u \text{ is periodic with respect to } \Omega\}. \tag{3.4}$$

Recall that the initial distribution of R_i satisfies

$$\frac{\sum_{i=1}^{N(0)} R_i^3(0)}{N(0)} = 1. \tag{3.5}$$

As is done when estimating (2.8), the biggest possible radius is attained when there remains only one droplet, of radius $R_f = \varepsilon^{-2/3}$. Thus we have the following estimate on the size of droplets.

$$R_i(t) \le \varepsilon^{-2/3} \quad \text{for all } t \ge 0, i = 1,\dots,N(t). \tag{3.6}$$

Then by (1.23),

$$a_\varepsilon R_i(t) \le \varepsilon^{1/3} e^{-1/(\delta\varepsilon^2)} \quad \text{for all } t \ge 0, i = 1,\dots,N(t). \tag{3.7}$$

Since $\delta > 0$ is a fixed constant, it is easily seen that

$$\varepsilon^{-2/3} e^{-1/(\delta\varepsilon^2)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence for any $\lambda > 0$, there exists ε_0 such that

$$\varepsilon^{1/3} e^{-1/(\delta\varepsilon^2)} < \lambda\varepsilon \quad \text{for any } 0 < \varepsilon < \varepsilon_0.$$

Combined with (3.7), we get $a_\varepsilon R_i(t) < \lambda\varepsilon$ for all $t > 0$ and all indices i, which is (1.34), the first conclusion of Theorem 1.1.

Similar to what is done in [9], we introduce the rescaled empirical joint distribution of $\{(x_i, R_i(t)) : i = 1,\dots,N(t)\}$

$$\int_{\Omega\times(0,\infty)} \zeta \, d\nu_t^\varepsilon = \frac{1}{N_0} \sum_{i=1}^{N(t)} \zeta(x_i, R_i(t)) = \varepsilon^2 \sum_{i=1}^{N(t)} \zeta(x_i, R_i(t)) \quad \text{for } \zeta \in C_p^0 \tag{3.8}$$

where

$$C_p^0 := \left\{ \begin{array}{l} \zeta = \zeta(x,r) : \zeta \in C^0(\mathbb{R}^2 \times (0,\infty)), \\ \zeta \text{ is periodic with respect to } \Omega = (0,1)^2 \\ \ \text{and has compact support in } r \in (0,\infty) \end{array} \right\}. \tag{3.9}$$

From here on, to simplify notations we omit the domain $\Omega \times (0,\infty)$ when writing integrals with respect to $d\nu_t^\varepsilon$, unless specified otherwise. That is, $\int \zeta \, d\nu_t^\varepsilon := \int_{\Omega\times(0,\infty)} \zeta \, d\nu_t^\varepsilon$.

The rest of this section is devoted to estimates related to ν_t^ε and u_ε. They are analogous to the estimates in [9]. However, since our problem has the distinctive feature that the droplets are three dimensional while the diffusion field is two dimensional, and since we want to capture the effect of the screening length through the parameter δ, we give detailed proofs for some of the results.

3.1. A priori estimates. By taking ζ smooth and

$$\zeta(x,r)=\begin{cases}1 \text{ if } \min_i\{R_i(t)\}\le r\le \max_i\{R_i(t)\},\\ 0 \text{ if } r\geqslant 2\max_i\{R_i(t)\} \text{ or } r\le \frac{1}{2}\min_i\{R_i(t)\},\end{cases}\tag{3.10}$$

we translate $N(t)\le N_0=\varepsilon^{-2}$ into

$$\int d\nu_t^\varepsilon \le 1 \qquad \text{for all } t\in(0,\infty).\tag{3.11}$$

Similarly by taking ζ smooth and

$$\zeta(x,r)=\begin{cases}r^3 \text{ if } \min_i\{R_i(t)\}\le r\le \max_i\{R_i(t)\},\\ 0 \quad \text{if } r\geqslant 2\max_i\{R_i(t)\} \text{ or } r\le \frac{1}{2}\min_i\{R_i(t)\},\end{cases}\tag{3.12}$$

the conservation of volume is written as

$$\int r^3\, d\nu_t^\varepsilon = 1 \qquad \text{for all } t\in(0,\infty).\tag{3.13}$$

The condition (1.35) on the initial distribution of volumes translates into

$$\int_{r>R} r^3\, d\nu_0^\varepsilon \to 0 \quad \text{uniformly in } \varepsilon \text{ as } R\to\infty.\tag{3.14}$$

In addition, using u_ε as a test function for (3.1)–(3.3), we get

$$\delta\varepsilon^2 \sum_{i=1}^{N(t)} R_i\dot{R}_i = -\int_\Omega |\nabla u_\varepsilon|^2\, dx\tag{3.15}$$

and thus for any $t>0$ we have

$$\int_0^t \int_\Omega |\nabla u_\varepsilon|^2\, dx dt + \frac{\delta}{2}\int r^2\, d\nu_t^\varepsilon = \frac{\delta}{2}\int r^2\, d\nu_0^\varepsilon \le \frac{\delta}{2}\tag{3.16}$$

by (3.11) and (3.13).

3.2. Control of $\dot{R}_i$ and the Hölder continuity of ν_t^ε. Since u_ε satisfies the non-homogeneous Dirichlet condition (3.2), for each i we define the following auxiliary function w_i as the solution for

$$-\Delta w_i = 0 \quad \text{in } B_{\lambda\varepsilon}(x_i)\setminus \bar{B}_i\tag{3.17}$$

$$w_i = 0 \quad \text{on } \partial B_{\lambda\varepsilon}(x_i)\tag{3.18}$$

$$w_i = 1 \quad \text{in } B_i^\varepsilon.\tag{3.19}$$

Then w_i can be extended by zero outside of $B_{\lambda\varepsilon}(x_i)$ into a function $\tilde{w}_i\in H_p^1(\Omega)$. Using $\tilde{w}_i$ as a test function for (3.1)–(3.3), we derive the following lemma concerning the growth rate of the volume of droplets, in terms of $\dot{R}_i$.

Lemma 3.1. *For any $\varepsilon>0$ and $t>0$ we have*

$$\delta\varepsilon^2 \sum_{i=1}^{N(t)} \left\{\left(1-\delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)\right) R_i^4 \dot{R}_i^2\right\} \le 2\pi \int_\Omega |\nabla u_\varepsilon|^2\, dx.\tag{3.20}$$

In addition, there exists $\varepsilon_1 > 0$ depending only on λ and δ such that for all $t > 0$ and all $\varepsilon < \varepsilon_1$, we have

$$\int_0^t \left(\sum_{i=1}^{N(s)} R_i^4(s) \dot{R}_i^2(s) \right) ds \le \frac{2\pi}{\varepsilon^2}. \tag{3.21}$$

Proof. w_i can be solved explicitly

$$\begin{aligned} w_i(x) &= \frac{\ln(\frac{|x-x_i|}{\lambda\varepsilon})}{\ln(\frac{a_\varepsilon R_i}{\lambda\varepsilon})} = \frac{\ln(\frac{|x-x_i|}{\lambda\varepsilon})}{\ln(\frac{a_\varepsilon}{\varepsilon}) + \ln\left(\frac{R_i}{\lambda}\right)} = \frac{\ln(\frac{|x-x_i|}{\lambda\varepsilon})}{-\delta^{-1}\varepsilon^{-2} + \ln\left(\frac{R_i}{\lambda}\right)} \\ &= -\delta\varepsilon^2 \frac{\ln(\frac{|x-x_i|}{\lambda\varepsilon})}{1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)}, \end{aligned} \tag{3.22}$$

$$\nabla w_i = \frac{1}{\ln(\frac{a_\varepsilon R_i}{\lambda\varepsilon})} \frac{x - x_i}{|x - x_i|^2}, \tag{3.23}$$

$$\int_{B_{\lambda\varepsilon}(x_i)} |\nabla w_i|^2 \, dx = \frac{2\pi}{\ln(\frac{\lambda\varepsilon}{a_\varepsilon R_i})} = \frac{2\pi\delta\varepsilon^2}{1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)}. \tag{3.24}$$

By (3.3),

$$\delta\varepsilon^2 R_i^2 \dot{R}_i = \int_{\partial B_i^\varepsilon} \nabla u_\varepsilon \cdot n \, ds = \int_{\partial B_i^\varepsilon} w_i \nabla u_\varepsilon \cdot n \, ds = -\int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \nabla u_\varepsilon \cdot \nabla w_i \, dx. \tag{3.25}$$

Hence

$$\begin{aligned} \delta^2 \varepsilon^4 R_i^4 \dot{R}_i^2 &\le \left(\int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx \right) \left(\int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} |\nabla w_i|^2 \, dx \right) \\ &\le \left(\int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx \right) \frac{2\pi\delta\varepsilon^2}{1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)} \end{aligned} \tag{3.26}$$

and

$$\delta\varepsilon^2 \left(1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right) \right) R_i^4 \dot{R}_i^2 \le 2\pi \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx.$$

Since $B_{\lambda\varepsilon}(x_i)$ are disjoint, summation over all i gives

$$\delta\varepsilon^2 \sum_{i=1}^{N(t)} \left\{ \left(1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right) \right) R_i^4 \dot{R}_i^2 \right\} \le 2\pi \int_\Omega |\nabla u_\varepsilon|^2 \, dx. \tag{3.27}$$

Since $R_i \le \varepsilon^{-2/3}$, we have

$$1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right) \geqslant 1 - \delta\varepsilon^2 \ln\left(\frac{1}{\varepsilon^{2/3}\lambda}\right) = 1 + \delta\varepsilon^2 \ln(\varepsilon^{2/3}\lambda). \tag{3.28}$$

Let $\varepsilon_1 = \varepsilon_1(\lambda, \delta)$ be small enough so that

$$1 + \delta\varepsilon^2 \ln(\varepsilon^{2/3}\lambda) > \frac{1}{2} \quad \text{for all } \varepsilon \le \varepsilon_1.$$

Then by (3.27), for all $\varepsilon < \varepsilon_1$ we have

$$\delta\varepsilon^2 \sum_{i=1}^{N(t)} R_i^4 \dot{R}_i^2 \le 4\pi \int_\Omega |\nabla u_\varepsilon|^2 \, dx,$$

and by (3.16), for any $t > 0$ we have

$$\int_0^t \left(\sum_{i=1}^{N(s)} R_i^4(s) \dot{R}_i^2(s) \right) ds \le \frac{4\pi}{\delta\varepsilon^2} \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \, dx dt \le \frac{2\pi}{\varepsilon^2}.$$

□

LEMMA 3.2. *For any $0 < \varepsilon < \varepsilon_1$ and any $\zeta \in C_p^0$, we have the following Hölder continuity of $\int \zeta \, d\nu_t^\varepsilon$ in t.*

$$\left| \int \zeta \, d\nu_{t_1}^\varepsilon - \int \zeta \, d\nu_{t_2}^\varepsilon \right| \le \sqrt{2\pi} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) |t_1 - t_2|^{1/2}. \tag{3.29}$$

Proof. For any given $\zeta \in C_p^0$, we have $\partial_r \zeta \in C_p^0$ and

$$\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} < \infty.$$

Then

$$\begin{aligned}
\left| \int \zeta \, d\nu_{t_1}^\varepsilon - \int \zeta \, d\nu_{t_2}^\varepsilon \right| &\le |t_1 - t_2|^{1/2} \left(\int_0^\infty \left| \frac{d}{dt} \int \zeta \, d\nu_t^\varepsilon \right|^2 dt \right)^{1/2} \\
&\le |t_1 - t_2|^{1/2} \left(\int_0^\infty \left| \varepsilon^2 \sum_{i=1}^N \partial_r \zeta(R_i) \dot{R}_i \right|^2 dt \right)^{1/2} \\
&\le |t_1 - t_2|^{1/2} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) \left(\int_0^\infty \left| \varepsilon^2 \sum_{i=1}^N R_i^2 \dot{R}_i \right|^2 dt \right)^{1/2} \\
&\le |t_1 - t_2|^{1/2} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) \left(\int_0^\infty \left| \varepsilon^2 \sum_{i=1}^N 1 \right| \left| \varepsilon^2 \sum_{i=1}^N R_i^4 \dot{R}_i^2 \right| dt \right)^{1/2} \\
&\le |t_1 - t_2|^{1/2} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) \left(\int_0^\infty \varepsilon^2 \sum_{i=1}^N R_i^4 \dot{R}_i^2 \, dt \right)^{1/2} \\
&\le \delta^{-1/2} |t_1 - t_2|^{1/2} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) \left(\int_0^\infty \delta\varepsilon^2 \sum_{i=1}^N R_i^4 \dot{R}_i^2 \, dt \right)^{1/2} \\
&\le \sqrt{2\pi} \left(\sup_{x,r} \frac{|\partial_r \zeta|}{r^2} \right) |t_1 - t_2|^{1/2} \quad \text{by (3.21).}
\end{aligned} \tag{3.30}$$

□

3.3. Tightness of ν_t^ε. Now we show that if initially the total volume is not concentrated on the few biggest droplets, neither will it in later times.

LEMMA 3.3. *For all $\eta > 0$ and $T > 0$ there exists $R_1 > 0$ such that*

$$\sup_{t\in(0,T)} \int_{r>R_1} r^3\, d\nu_t^\varepsilon < \eta \quad \textit{uniformly in } \varepsilon. \tag{3.31}$$

Proof. This Lemma mimics Lemma 3.2 in [9]. Let R be a parameter to be determined and let φ be a smooth cutoff function on $(0,\infty)$ such that $\varphi(r)=0$ for $r<R/2$, $\varphi(r)=1$ for $r>R$ and that $0\le \varphi'(r)\le 3/R$. Then for any $t\in(0,T)$ we have

$$\begin{aligned}
&\int_{r>R} r^3\, d\nu_t^\varepsilon \le \int \varphi(r) r^3\, d\nu_t^\varepsilon = \int \varphi(r) r^3\, d\nu_0^\varepsilon + \int_0^t \left(\frac{d}{dt}\int \varphi(r) r^3\, d\nu_t^\varepsilon\right) dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \int_0^t \left(\frac{d}{dt}\sum_{i=1}^N \varepsilon^2 \varphi(R_i) R_i^3\right) dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \int_0^t \left(\varepsilon^2 \sum_{i=1}^N \Big(\varphi'(R_i)R_i + 3\varphi(R_i)\Big) R_i^2 \dot R_i\right) dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \int_0^t \left(\varepsilon^2 \sum_{i=1}^N R_i^4 \dot R_i^2\right)^{1/2} \left(2\varepsilon^2 \sum_{i=1}^N \Big(\varphi'(R_i)^2 R_i^2 + 9\varphi(R_i)^2\Big)\right)^{1/2} dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \int_0^t \left(\varepsilon^2 \sum_{i=1}^N R_i^4 \dot R_i^2\right)^{1/2} \left(2\varepsilon^2 \sum_{i:R_i\geqslant R/2} \Big(\frac{9}{R^2} R_i^2 + 36\frac{R_i^2}{R^2}\Big)\right)^{1/2} dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \frac{\sqrt{90}}{R}\int_0^t \left(\varepsilon^2 \sum_{i=1}^N R_i^4 \dot R_i^2\right)^{1/2} \left(\varepsilon^2 \sum_{i=1}^N R_i^2\right)^{1/2} dt \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \frac{(90t)^{1/2}}{R}\left(\int_0^t \varepsilon^2 \sum_{i=1}^N R_i^4 \dot R_i^2\, dt\right)^{1/2} \\
&\le \int_{r>R/2} r^3\, d\nu_0^\varepsilon + \frac{(180\pi T)^{1/2}}{R} \quad \text{by (3.21)}.
\end{aligned}$$

For any given $\eta > 0$, because of (3.14), we can choose $R_1 > 0$ depending on T and η such that

$$\int_{r>R/2} r^3\, d\nu_0^\varepsilon + \frac{(180\pi T)^{1/2}}{R} < \eta \quad \text{uniformly in } \varepsilon$$

for all $R > R_1$. □

3.4. The regularity of u_ε and ν_t^ε. To obtain the necessary compactness, we need the boundedness of u_ε in $L^2_{\text{loc}}(0,\infty;H^1(\Omega))$. Since we already have the boundedness of ∇u_ε in $L^2(0,\infty;L^2(\Omega))$ by (3.16), to obtain the boundedness of u_ε in $L^2_{\text{loc}}(0,\infty;H^1(\Omega))$, we need to estimate $\int_0^\infty \int_\Omega |u_\varepsilon|^2\, dxdt$. By Poincaré's inequality, we need only to get a bound on $⨍_\Omega u_\varepsilon\, dx := \frac{1}{|\Omega|}\int_\Omega u_\varepsilon\, dx$. In fact we will do such estimates locally and then patch them together.

For each i, consider an auxiliary function ϕ_i that satisfies the following equation for some constant c_i,

$$-\Delta \phi_i = c_i \quad \text{in } B_{\lambda\varepsilon}(x_i)\setminus \bar B_i, \tag{3.32}$$

$$\frac{\partial \phi_i}{\partial n} = 1 \quad \text{on } \partial B_i, \tag{3.33}$$

$$\frac{\partial \phi_i}{\partial n} = 0 \quad \text{on } \partial B_{\lambda\varepsilon}(x_i). \tag{3.34}$$

We will use ϕ_i to help us estimate the average of u_ε in $B_{\lambda\varepsilon} \setminus \bar{B}_i$. The solvability of (3.32)–(3.34) requires

$$c_i = \frac{|\partial B_i|}{|B_{\lambda\varepsilon}(x_i) \setminus \bar{B}_i|} = \frac{2a_\varepsilon R_i}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2}.$$

Due to symmetry, ϕ_i depends only on the radial coordinate $r := |x - x_i|$. Solving it in polar coordinates, we get

$$\phi_i(r) = -\frac{c_i}{4} r^2 + \alpha_i \ln r + \beta_i,$$
$$\phi_i'(r) = -\frac{c_i}{2} r + \frac{\alpha_i}{r},$$

where

$$\alpha_i := \frac{\lambda^2\varepsilon^2 a_\varepsilon R_i}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2} > 0, \quad \beta_i \text{ is an arbitrary constant.} \tag{3.35}$$

So

$$\begin{aligned}
\int_{B_{\lambda\varepsilon}(x_i)\setminus \bar{B}_i} |\nabla \phi_i|^2 \, dx &= 2\pi \int_{a_\varepsilon R_i}^{\lambda\varepsilon} |\phi_i'(r)|^2 r dr \le 2\pi \int_{a_\varepsilon R_i}^{\lambda\varepsilon} \left(\frac{c_i^2}{4} r^3 + \frac{\alpha_i^2}{r} \right) dr \\
&= 2\pi \left(\alpha_i^2 \ln(\frac{\lambda\varepsilon}{\alpha_\varepsilon R_i}) + \frac{c_i^2}{16}(\lambda^4\varepsilon^4 - a_\varepsilon^4 R_i^4) \right) \\
&= 2\pi a_\varepsilon^2 R_i^2 \left(\frac{\lambda^4\varepsilon^4}{(\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2)^2} \ln(\frac{\lambda\varepsilon}{a_\varepsilon R_i}) + \frac{\lambda^2\varepsilon^2 + a_\varepsilon^2 R_i^2}{4(\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2)} \right).
\end{aligned} \tag{3.36}$$

By (1.23), we have

$$\ln(\frac{\lambda\varepsilon}{a_\varepsilon R_i}) = \frac{1}{\delta\varepsilon^2} \left(1 - \delta\varepsilon^2 \ln(\frac{R_i}{\lambda}) \right).$$

In addition, since $R_i \le \varepsilon^{-2/3}$ for all $t > 0$, we have

$$\lim_{\varepsilon \to 0^+} \frac{\lambda^4\varepsilon^4}{(\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2)^2} = 1, \qquad \lim_{\varepsilon \to 0^+} \frac{\lambda^2\varepsilon^2 + a_\varepsilon^2 R_i^2}{4(\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2)} = \frac{1}{4}$$

uniformly in i and $t > 0$. Thus by (3.36), there exists $\varepsilon_2 = \varepsilon_2(\lambda, \delta) > 0$ such that for all $0 < \varepsilon < \varepsilon_2$, we have

$$\begin{aligned}
\int_{B_{\lambda\varepsilon}(x_i)\setminus \bar{B}_i} |\nabla \phi_i|^2 \, dx &\le 3\pi a_\varepsilon^2 R_i^2 \left(\frac{1 - \delta\varepsilon^2 \ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2} + \frac{1}{4} \right) \\
&\le 4\pi a_\varepsilon^2 R_i^2 \left(\frac{1 - \delta\varepsilon^2 \ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2} \right).
\end{aligned} \tag{3.37}$$

Multiplying (3.32) by u_ε and integrating over $B_{\lambda\varepsilon}\setminus\bar{B}_i$, we get

$$\begin{aligned}\frac{2a_\varepsilon R_i}{\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2}\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx&=\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}\nabla u_\varepsilon\cdot\nabla\phi_i\,dx+\int_{\partial B_i}u_\varepsilon\,ds\\&=\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}\nabla u_\varepsilon\cdot\nabla\phi_i\,dx+2\pi a_\varepsilon.\end{aligned}\tag{3.38}$$

Writing

$$⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx=\frac{1}{\pi(\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2)}\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx,$$

we get

$$⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx-\frac{1}{R_i}=\frac{1}{2\pi a_\varepsilon R_i}\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}\nabla u_\varepsilon\cdot\nabla\phi_i\,dx.\tag{3.39}$$

So

$$\begin{aligned}&\left|⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx-\frac{1}{R_i}\right|^2\\&\le\frac{1}{4\pi^2a_\varepsilon^2R_i^2}\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2dx\right)\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla\phi_i|^2dx\right)\\&\le\frac{1}{\pi}\left(\frac{1-\delta\varepsilon^2\ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2}\right)\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2dx\right)\quad\text{by (3.37).}\end{aligned}\tag{3.40}$$

Multiplying both sides by $\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}$ and summing over all i, we obtain the following lemma.

LEMMA 3.4. *There exists a constant C such that for all $\varepsilon>0$ sufficiently small and all $t>0$ we have*

$$\sum_{i=1}^N\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i}-⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx\right|^2\le\frac{1}{\pi}\int_\Omega|\nabla u_\varepsilon|^2\,dx.\tag{3.41}$$

The average over $B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i$ can be replaced by the average over $B_{\lambda\varepsilon}$. Indeed, by the triangle inequality, we have

$$\begin{aligned}&\frac{1}{\pi(\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2)}\left|\int_{B_{\lambda\varepsilon}(x_i)}u_\varepsilon\,dx\right|\\&=\left|⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx-\frac{1}{R_i}+\frac{1}{\pi(\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2)}\int_{B_i}u_\varepsilon\,dx+\frac{1}{R_i}\right|\\&\le\left|⨍_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}u_\varepsilon\,dx-\frac{1}{R_i}\right|+\frac{1}{\pi(\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2)}\int_{B_i}u_\varepsilon\,dx+\frac{1}{R_i}\\&\le\frac{1}{\sqrt{\pi}}\left(\frac{1-\delta\varepsilon^2\ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2}\right)^{1/2}\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2dx\right)^{1/2}+\frac{a_\varepsilon^2R_i}{\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2}+\frac{1}{R_i}.\end{aligned}\tag{3.42}$$

Thus

$$
\begin{aligned}
&\left|\fint_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i} u_\varepsilon\,dx - \fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right| \\
&= \frac{1}{\pi}\left|\left(\frac{1}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2} - \frac{1}{\lambda^2\varepsilon^2}\right)\int_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx - \frac{1}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2}\int_{B_i} u_\varepsilon\,dx\right| \\
&\le \frac{1}{\pi}\frac{a_\varepsilon^2 R_i^2}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2}\frac{1}{\lambda^2\varepsilon^2}\left|\int_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right| + \frac{a_\varepsilon^2 R_i}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2} \\
&\le \frac{a_\varepsilon^2 R_i^2}{\lambda^2\varepsilon^2}\frac{1}{\sqrt{\pi}}\left(\frac{1-\delta\varepsilon^2\ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2}\right)^{1/2}\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2 dx\right)^{1/2} \\
&\quad + \frac{a_\varepsilon^2 R_i^2}{\lambda^2\varepsilon^2}\left(\frac{a_\varepsilon^2 R_i}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2} + \frac{1}{R_i}\right) + \frac{a_\varepsilon^2 R_i}{\lambda^2\varepsilon^2 - a_\varepsilon^2 R_i^2} \\
&\le \varepsilon\left(\frac{1-\delta\varepsilon^2\ln(\frac{R_i}{\lambda})}{\delta\varepsilon^2}\right)^{1/2}\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2 dx + \varepsilon^3\right)^{1/2} \qquad (3.43)
\end{aligned}
$$

for ε sufficiently small. Then

$$
\begin{aligned}
&\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i} - \fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right|^2 \\
&\le \frac{2\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left(\left|\fint_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i} u_\varepsilon\,dx - \frac{1}{R_i}\right|^2 + \left|\fint_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i} u_\varepsilon\,dx - \fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right|^2\right) \\
&\le C\left(\int_{B_{\lambda\varepsilon}(x_i)\setminus\bar{B}_i}|\nabla u_\varepsilon|^2 dx + \varepsilon^3\right). \qquad (3.44)
\end{aligned}
$$

Here $C = C(\delta,\lambda) > 0$ is a generic constant. Summing up all i, since $N(t) \le \varepsilon^{-2}$, we get the following bound.

LEMMA 3.5. *There exists a constant $C = C(\delta,\lambda)$ such that for all $\varepsilon > 0$ sufficiently small and all $t > 0$ we have*

$$
\sum_{i=1}^{N}\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i} - \fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right|^2 \le C\left(\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\right). \qquad (3.45)
$$

Inequality (3.45) can be improved using a different approach as in [9], so that ε is removed from the right-hand side. We write down the result as the following lemma but omit the proof here.

LEMMA 3.6. *There exists a constant $C = C(\delta,\lambda) > 0$ such that for all $\varepsilon > 0$ sufficiently small and all $t > 0$ we have*

$$
\sum_{i=1}^{N}\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i} - \fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right|^2 \le C\int_\Omega |\nabla u_\varepsilon|^2\,dx. \qquad (3.46)
$$

Lemmas 3.4, 3.5, and 3.6 fulfill the same role. We only need one of them to carry out our later analysis. For instance, we will use Lemma 3.4 to prove the following lemma about ν_t^ε, which can also be proved using the other two lemmas.

LEMMA 3.7. *There exists a constant $C=C(\delta,\lambda)>0$ such that for all $\varepsilon>0$ sufficiently small and all $t>0$ we have*

$$\int \frac{\delta}{1-\delta\varepsilon^2\ln(r/\lambda)}\frac{1}{r^2}\,d\nu_t^\varepsilon \le C\left(\frac{\delta}{\lambda^2}\int_\Omega |u_\varepsilon|^2\,dx+\int_\Omega |\nabla u_\varepsilon|^2\,dx\right). \tag{3.47}$$

Proof.

$$\int \frac{\delta}{1-\delta\varepsilon^2\ln(r/\lambda)}\frac{1}{r^2}\,d\nu_t^\varepsilon = \sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\frac{1}{R_i^2}$$

$$\le 2\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i}-\fint_{B_{\lambda\varepsilon}(x_i)\setminus B_i} u_\varepsilon\,dx\right|^2 + 2\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\fint_{B_{\lambda\varepsilon}(x_i)\setminus B_i} u_\varepsilon\,dx\right|^2$$

$$\le C\int_\Omega |\nabla u_\varepsilon|^2\,dx+\frac{2}{\pi}\sum_{i=1}^N \frac{\delta\varepsilon^2}{(1-\delta\varepsilon^2\ln\frac{R_i}{\lambda})}\frac{1}{(\lambda^2\varepsilon^2-a_\varepsilon^2R_i^2)}\int_{B_{\lambda\varepsilon}(x_i)\setminus B_i}|u_\varepsilon|^2\,dx$$

$$\le C\int_\Omega |\nabla u_\varepsilon|^2\,dx+\frac{\delta}{\lambda^2}\max_i\left\{\frac{1}{(1-\delta\varepsilon^2\ln\frac{R_i}{\lambda})}\frac{1}{(1-\lambda^{-2}\varepsilon^{-2}a_\varepsilon^2R_i^2)}\right\}\int_\Omega |u_\varepsilon|^2\,dx$$

$$\le C\left(\int_\Omega |\nabla u_\varepsilon|^2\,dx+\frac{\delta}{\lambda^2}\int_\Omega |u_\varepsilon|^2\,dx\right) \quad \text{when } \varepsilon \text{ is sufficiently small.}$$

□

Now it is time to prove the L^2 bound of u_ε.

LEMMA 3.8. *There exists a constant $C=C(\lambda,\delta)$ such that for all ε sufficiently small and all $t>0$ we have*

$$\int_\Omega |u_\varepsilon|^2\,dx \le C\left\{\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2+\left(\frac{1+(\lambda^{-1}+\delta^{-1/2})\|\nabla u_\varepsilon\|_{L^2(\Omega)}}{\int\frac{r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon}\right)^2\right\}. \tag{3.48}$$

Proof.

$$\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|R_i\fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon - 1\right| = \sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}R_i\left|\fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon - \frac{1}{R_i}\right|$$

$$\le \max_i\left(\frac{\delta}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\right)^{1/2}\left(\sum_{i=1}^N \varepsilon^2R_i^2\right)^{1/2}\cdot\left(\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|\frac{1}{R_i}-\fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\,dx\right|^2\right)^{1/2}$$

$$\le C\delta^{1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega)}. \tag{3.49}$$

$$\sum_{i=1}^N \frac{\delta}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}R_i\left|\int_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\right| = \pi\lambda^2\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}R_i\left|\fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon\right|$$

$$\le \pi\lambda^2\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|R_i\fint_{B_{\lambda\varepsilon}(x_i)} u_\varepsilon - 1\right| + \pi\lambda^2\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}$$

$$\leq C\lambda^2\delta^{1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega)}+C\lambda^2\delta. \tag{3.50}$$

Let $\bar{u}_\varepsilon := \fint_\Omega u_\varepsilon\, dx$. Then

$$\begin{aligned}
&\left(\int \frac{\delta r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon\right)|\bar{u}_\varepsilon| = \sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\left|R_i\fint_{B_{\lambda\varepsilon}(x_i)}\bar{u}_\varepsilon\right| \\
&\leq \sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\frac{R_i}{\pi\lambda^2\varepsilon^2}\left(\int_{B_{\lambda\varepsilon}(x_i)}|u_\varepsilon-\bar{u}_\varepsilon|+\left|\int_{B_{\lambda\varepsilon}(x_i)}u_\varepsilon\right|\right) \\
&\leq \sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\frac{R_i}{\sqrt{\pi}\lambda\varepsilon}\left(\int_{B_{\lambda\varepsilon}(x_i)}|u_\varepsilon-\bar{u}_\varepsilon|^2\right)^{1/2}+\sum_{i=1}^N \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\frac{R_i}{\lambda}}\frac{R_i}{\pi\lambda^2\varepsilon^2}\left|\int_{B_{\lambda\varepsilon}(x_i)}u_\varepsilon\right| \\
&\leq C\frac{\delta}{\lambda}\left(\sum_{i=1}^N\varepsilon^2R_i^2\right)\left(\sum_{i=1}^N\int_{B_{\lambda\varepsilon}(x_i)}|u_\varepsilon-\bar{u}_\varepsilon|^2\right)^{1/2}+C\delta^{1/2}\left(\|\nabla u_\varepsilon\|_{L^2(\Omega)}+\delta^{1/2}\right) \\
&\leq C\frac{\delta}{\lambda}\|\nabla u_\varepsilon\|_{L^2(\Omega)}+C\delta^{1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega)}+C\delta.
\end{aligned} \tag{3.51}$$

So

$$|\bar{u}_\varepsilon|\leq C\left(\int\frac{r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon\right)^{-1}\left(1+(\lambda^{-1}+\delta^{-1/2})\|\nabla u_\varepsilon\|_{L^2(\Omega)}\right),$$

and

$$\begin{aligned}
&\int_\Omega |u_\varepsilon|^2\,dx\leq 2\int_\Omega|u_\varepsilon-\bar{u}_\varepsilon|^2+2\int_\Omega|\bar{u}_\varepsilon|^2 \\
&\leq C\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}+C\left(\int\frac{r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon\right)^{-2}\left(1+(\lambda^{-1}+\delta^{-1/2})\|\nabla u_\varepsilon\|_{L^2(\Omega)}\right)^2.
\end{aligned}$$

□

Corollary 3.1. *For all $T<\infty$ there exists $C_{T,\delta,\lambda}=C(T,\delta,\lambda)<\infty$ such that*

$$\int_0^T\int\frac{\delta}{1-\delta\varepsilon^2\ln(r/\lambda)}\frac{1}{r^2}\,d\nu_t^\varepsilon+\int_0^T\int_\Omega|u_\varepsilon|^2\,dxdt\leq C_{T,\delta,\lambda}. \tag{3.52}$$

Proof. Combining Lemma 3.7 and Lemma 3.8, we only need to prove that for any $T<\infty$, there exists a constant $c(T,\delta,\lambda)$ such that

$$\inf_{t\in(0,T)}\int\frac{r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon\geqslant c(T,\delta,\lambda) \tag{3.53}$$

uniformly as $\varepsilon\to 0$.

$$\begin{aligned}
1=\int r^3\,d\nu_t^\varepsilon&=\int_{r\leq R_0}r^3\,d\nu_t^\varepsilon+\int_{R_0<r\leq R_1}r^3\,d\nu_t^\varepsilon+\int_{r>R_1}r^3\,d\nu_t^\varepsilon \\
&\leq R_0^3+R_1^2\left(1-\delta\varepsilon^2\ln\frac{R_0}{\lambda}\right)\int_{R_0<r\leq R_1}\frac{r}{1-\delta\varepsilon^2\ln\frac{r}{\lambda}}\,d\nu_t^\varepsilon+\int_{r>R_1}r^3\,d\nu_t^\varepsilon.
\end{aligned} \tag{3.54}$$

According to Lemma 3.3, we can choose R_1 big enough so that $\int_{r>R_1}r^3\,d\nu_t^\varepsilon<\frac{1}{4}$ for all ε. In addition, we choose $R_0^3<\frac{1}{4}$. □

REMARK 3.1. Combining Corollary 3.1 and the fact that $\int r^3\,d\nu_t^\varepsilon = 1$, a simple application of Hölder's inequality shows that

$$\int_0^T \int \frac{1}{r}\,d\nu_t^\varepsilon\,dt \le C_{T,\delta,\lambda}. \tag{3.55}$$

This indicates that $\int r^{-1}\,d\nu_t^\varepsilon$ exists almost everywhere in $t\in[0,\infty)$ and it is in $L^1_{loc}([0,\infty))$.

4. The limit as $\varepsilon \to 0$

4.1. The existence of the limit of ν_t^ε. As is in [9], according to (3.11) and (3.29), the Arzela-Ascoli theorem guarantees that there exists a family of nonnegative Borel measures $\{\nu_t\}_t$ and a subsequence of ν_t^ε, still denoted by ν_t^ε, such that

$$\int \zeta\,d\nu_t^\varepsilon \to \int \zeta\,d\nu_t \quad \text{locally uniformly in } t\in[0,\infty). \tag{4.1}$$

for any ζ in a countable dense subset of $C_p^0\cap C^1$. Again by the uniform boundedness of ν_t^ε (3.11), we can extend (4.1) to all $\zeta\in C_p^0$.

Since $\int d\nu_t^\varepsilon \le 1$, the weak lower semicontinuity guarantees the uniform boundedness of ν_t,

$$\int d\nu_t \le 1. \tag{4.2}$$

By the tightness of ν_t^ε (Lemma 3.3), we have

$$\int_{r>M} r^3\,d\nu_t^\varepsilon \to 0 \quad \text{uniformly in } \varepsilon \text{ as } M\to\infty. \tag{4.3}$$

On the other hand, (3.11) indicates that

$$\int_{r<\sigma} r^3\,d\nu_t^\varepsilon \to 0 \quad \text{uniformly in } \varepsilon \text{ as } \sigma\to 0. \tag{4.4}$$

So the conservation of volume (3.13) is preserved by ν_t,

$$\int r^3\,d\nu_t = 1 \quad \text{for all } t\in[0,\infty). \tag{4.5}$$

4.2. The limit of u_ε. First we observe that, according to (3.16) and (3.52), for any given T, u_ε is bounded in $L^2(0,T;H^1(\Omega))$. So there exists $u_*\in L^2(0,T;H^1(\Omega))$ and a subsequence of u_ε, still denoted by u_ε, such that $u_\varepsilon \rightharpoonup u_*$ in $L^2(0,T;H^1(\Omega))$. In addition, the trace theorem indicates that u_* is spatially periodic in space with respect to Ω.

Let $C_0(0,T)$ denote continuous functions in t with compact support in $(0,T)$ and $C_p^1(\Omega)$ denote continuously differentiable functions that are periodic with respect to Ω.

THEOREM 4.1. *For any $\psi(t)\in C_0(0,T)$ and any $\zeta\in C_p^1(\Omega)$, we have*

$$\int_0^T \int \psi(t)\nabla\zeta(x)\cdot\nabla u_*\,dx\,dt + 2\pi\delta\int_0^T\int \psi(t)\zeta(x)\left(u_* - \frac{1}{r}\right)d\nu_t\,dt = 0. \tag{4.6}$$

Before proving Theorem 4.1, we state the following lemma, which is a fact used in [9] but we think it deserves to be singled out as a lemma. It is essential when we need to exchange integrals in a ball into an integral on the boundary of that ball.

LEMMA 4.1. *For any $u\in H^1(B_r(0))$, we have*

$$\left|\int_{B_r} u\,dx - \frac{r}{2}\int_{\partial B_r} u\,ds\right| \le \sqrt{2\pi}r^2\|\nabla u\|_{L^2(B_r)}. \tag{4.7}$$

Proof. The idea is to use integration by parts to transform the volume integral into a boundary integral.

Let $\varphi(x) = -\frac{1}{4}|x|^2$. Then

$$\begin{aligned} -\Delta\varphi &= 1 \quad \text{in } B_r(0),\\ \nabla\varphi\cdot n &= -\frac{1}{2}r \quad \text{on } \partial B_r. \end{aligned}$$

$$\begin{aligned} \int_{B_r} u\,dx &= \int_{B_r}(-\Delta\varphi)u\,dx = \int_{B_r}\nabla u\cdot\nabla\varphi - \int_{\partial B_r} u\nabla\varphi\cdot n\\ &= \int_{B_r}\nabla u\cdot\nabla\varphi + \frac{1}{2}r\int_{\partial B_r} u\,ds. \end{aligned}$$

Hence

$$\left|\int_{B_r} u\,dx - \frac{r}{2}\int_{\partial B_r} u\,ds\right| \le \|\nabla u\|_{L^2(B_r)}\|\nabla\varphi\|_{L^2(B_r)} = \sqrt{2\pi}r^2\|\nabla u\|_{L^2(B_r)}. \tag{4.8}$$

□

REMARK 4.1. A special case of Lemma 4.1 is when u is harmonic in B_r. The mean value property says

$$u(0) = \frac{1}{2\pi r}\int_{\partial B_r} u\,ds = \frac{1}{\pi r^2}\int_{B_r} u\,dx. \tag{4.9}$$

Hence

$$\int_{B_r} u\,dx = \frac{r}{2}\int_{\partial B_r} u\,ds \tag{4.10}$$

LEMMA 4.2. *For w_i defined by (3.17)–(3.19), we have*

$$\int_0^T \psi(t)\sum_{i=1}^N \zeta(x_i,R_i)\int_{\partial B_{\lambda_\varepsilon}(x_i)} u_\varepsilon(-\nabla w_i\cdot n)\,dt \to 2\pi\int_0^T\psi(t)\int\zeta u_*\,d\nu_t\,dt. \tag{4.11}$$

Proof. This is exactly Lemma 4.1 in [9]. □

Proof. **(Proof of Theorem 4.1.)** For any $\psi(t)$ and $\zeta(x)$,

$$\begin{aligned} 0 &= \int_0^T\psi(t)\int_\Omega\zeta\cdot(-\Delta u_\varepsilon)\,dxdt\\ &= \int_0^T\psi(t)\int_\Omega\nabla\zeta\cdot\nabla u_\varepsilon - \nabla\cdot(\zeta\nabla u_\varepsilon)\,dxdt \end{aligned}$$

$$= \int_0^T \psi(t) \left(\int_\Omega \nabla\zeta \cdot \nabla u_\varepsilon \, dx dt + \sum_{i=1}^N \int_{\partial B_i^\varepsilon} \zeta \nabla u_\varepsilon \cdot n \, ds \right) dt$$

$$= \int_0^T \psi(t) \int_\Omega \nabla\zeta \cdot \nabla u_\varepsilon \, dx dt + \sum_{i=1}^N \int_0^T \psi(t) \int_{\partial B_i^\varepsilon} \zeta \nabla u_\varepsilon \cdot n \, ds dt. \tag{4.12}$$

Apparently

$$\int_0^T \psi(t) \int_\Omega \nabla\zeta \cdot \nabla u_\varepsilon \, dx dt \to \int_0^T \psi(t) \int_\Omega \nabla\zeta \cdot \nabla u_* \, dx dt \quad \text{as } \varepsilon \to 0. \tag{4.13}$$

We only need to consider the limit of

$$\sum_{i=1}^N \int_0^T \psi(t) \int_{\partial B_i^\varepsilon} \zeta \nabla u_\varepsilon \cdot n \, ds dt \quad \text{as } \varepsilon \to 0.$$

Because of the definition of ν_t^ε, we want to replace ζ by $\zeta(x_i)$ on ∂B_i^ε. Since $\Delta w_i = 0$, $w_i = 1$ on ∂B_i^ε, and $w_i = 0$ on $\partial B_{\lambda\varepsilon}(x_i)$,

$$\begin{aligned}
&\int_{\partial B_i^\varepsilon} \zeta \nabla u_\varepsilon \cdot n \, ds \\
&= \int_{\partial B_i^\varepsilon} \zeta(x_i) w_i \nabla u_\varepsilon \cdot n \, ds + \int_{\partial B_i^\varepsilon} (\zeta - \zeta(x_i)) w_i \nabla u_\varepsilon \cdot n \, ds \\
&= -\zeta(x_i) \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \nabla \cdot (w_i \nabla u_\varepsilon) \, dx - \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \nabla \cdot \Big((\zeta - \zeta(x_i)) w_i \nabla u_\varepsilon \Big) dx \\
&= -\zeta(x_i) \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \nabla \cdot (u_\varepsilon \nabla w_i) \, dx - \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \nabla \cdot \Big((\zeta - \zeta(x_i)) w_i \nabla u_\varepsilon \Big) dx \\
&= \zeta(x_i) \int_{\partial B_i^\varepsilon} u_\varepsilon \nabla w_i \cdot n \, ds - \zeta(x_i) \int_{\partial B_{\lambda\varepsilon}(x_i)} u_\varepsilon \nabla w_i \cdot n \, ds \\
&\quad - \int_{B_{\lambda\varepsilon}(x_i) \setminus B_i^\varepsilon} \Big(w_i \nabla\zeta \cdot \nabla u_\varepsilon + (\zeta - \zeta(x_i)) \nabla w_i \cdot \nabla u_\varepsilon \Big) dx \\
&=: J_1^i + J_2^i + J_3^i.
\end{aligned} \tag{4.14}$$

We analyze the above equation term by term. J_1^i is simple.

$$\begin{aligned}
J_1^i &= \zeta(x_i) \int_{\partial B_i^\varepsilon} u_\varepsilon \nabla w_i \cdot n \, ds = \zeta(x_i) \frac{1}{R_i} \int_{\partial B_i^\varepsilon} \nabla w_i \cdot n \, ds \\
&= -\frac{\delta\varepsilon^2}{1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)} 2\pi \zeta(x_i) \frac{1}{R_i}, \\
\sum_{i=1}^N \int_0^T \psi(t) J_1^i \, dt &= -2\pi\delta \int_0^T \varepsilon^2 \sum_{i=1}^N \psi(t)\zeta(x_i) \frac{1}{R_i} \frac{1}{1 - \delta\varepsilon^2 \ln\left(\frac{R_i}{\lambda}\right)} \\
&= -2\pi\delta \int_0^T \psi(t)\zeta(x) \frac{1}{1 - \delta\varepsilon^2 \ln\frac{r}{\lambda}} \frac{1}{r} \, d\nu_t^\varepsilon \, dt \\
&\to -2\pi\delta \int_0^T \psi(t) \left(\int \zeta(x) \frac{1}{r} \, d\nu_t \right) dt \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{4.15}$$

For J_2^i and J_3^i, we have the following lemma, which completes the proof of Theorem 4.1. □

LEMMA 4.3.

$$\sum_{i=1}^{N}\int_0^T \psi(t)J_2^i\,dt \to 2\pi\delta\int_0^T\psi(t)\left(\int\zeta(x)u_*(x)\,d\nu_t\right)dt, \tag{4.16}$$

$$\sum_{i=1}^{N}\int_0^T \psi(t)J_3^i\,dt \to 0. \tag{4.17}$$

Proof. For J_2^i, we further split it into two terms using Lemma 4.1.

$$J_2^i = -\zeta(x_i)\int_{\partial B_{\lambda\varepsilon}(x_i)} u_\varepsilon\nabla w_i\cdot n\,ds = \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\frac{1}{\lambda\varepsilon}\int_{\partial B_{\lambda\varepsilon}(x_i)}u_\varepsilon\,ds$$
$$= \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\frac{2}{\lambda^2\varepsilon^2}\int_{B_{\lambda\varepsilon}(x_i)}u_\varepsilon\,dx + I_1^i \quad \text{by (4.7)}.$$
$$|I_1^i| \le \left|\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\right| 2\sqrt{2\pi}\|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))}. \tag{4.18}$$

So

$$\left|\sum_{i=1}^{N}\int_0^T\psi(t)I_1^i\,dt\right| \le C(\sup|\zeta|)\delta\varepsilon^2\int_0^T|\psi(t)|\sum_{i=1}^{N}\|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))}\,dt$$
$$\le C(\sup|\zeta|)\delta\varepsilon^2\int_0^T|\psi(t)|\left(\sum_{i=1}^{N}1\right)^{1/2}\left(\sum_{i=1}^{N}\|\nabla u_\varepsilon\|^2_{L^2(B_{\lambda\varepsilon}(x_i))}\right)^{1/2}dt$$
$$\le C(\sup|\zeta|)\delta\varepsilon\int_0^T|\psi(t)|\left(\varepsilon^2\sum_{i=1}^{N}1\right)^{1/2}\left(\int_\Omega|\nabla u_\varepsilon|^2\,dx\right)^{1/2}dt$$
$$\le C(\sup|\zeta|)\delta\varepsilon\|\psi\|_{L^2(0,T)}\left(\int_0^T\int_\Omega|\nabla u_\varepsilon|^2\,dx\right)^{1/2}$$
$$\le C(\sup|\zeta|)\delta^{3/2}\varepsilon\|\psi\|_{L^2(0,T)} \quad \text{by (3.16)}$$
$$\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\int_0^T\psi(t)J_2^i\,dt$$
$$= \lim_{\varepsilon\to 0}\int_0^T\psi(t)\left(\sum_{i=1}^{N}\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\frac{2}{\lambda^2\varepsilon^2}\int_{B_{\lambda\varepsilon}(x_i)}u_\varepsilon\,dx\right)dt. \tag{4.19}$$

Let

$$\varphi_\varepsilon(x) = \begin{cases} \frac{\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\frac{1}{\pi\lambda^2\varepsilon^2}, & \text{if } |x-x_i|\le\lambda\varepsilon,\\ 0, & \text{otherwise.}\end{cases} \tag{4.20}$$

Then for any $v\in C_0^\infty((0,T)\times\Omega)$,

$$\begin{aligned}\int_0^T\int_\Omega \varphi_\varepsilon(x)v(t,x)\,dxdt &= \int_0^T\left(\sum_{i=1}^N \frac{\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)\frac{1}{\pi\lambda^2\varepsilon^2}\int_{B_{\lambda\varepsilon}(x_i)} v(x)\,dx\right)dt\\ &= \int_0^T\left(\sum_{i=1}^N \frac{\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\zeta(x_i)v(x_i)\right)dt + I_\varepsilon\\ &= \int_0^T\int \zeta(x)v(t,x)\,d\nu_t^\varepsilon\,dt + I_\varepsilon\\ &\to \int_0^T\int \zeta(x)v(t,x)\,d\nu_t\,dt. \end{aligned}\tag{4.21}$$

Here we used the fact that $I_\varepsilon\to 0$ as $\varepsilon\to 0$.

So by the strong convergence of $u_\varepsilon\to u_*$ in $L^2(0,T;L^2(\Omega))$ and the boundedness of φ_ε in $L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{i=1}^N\int_0^T \psi(t)J_2^i\,dt &= \lim_{\varepsilon\to 0}2\pi\delta\int_0^T\int_\Omega \psi(t)u_\varepsilon(x)\varphi_\varepsilon(x)\,dxdt\\ &= 2\pi\delta\int_0^T\left(\int \psi(t)\zeta(x)u_*(x)\,d\nu_t\right)dt. \end{aligned}\tag{4.22}$$

Now we consider J_3^i. For $x\in B_{\lambda\varepsilon}(x_i)$, since

$$|\zeta(x)-\zeta(x_i)| = \left|\int_0^1 \nabla\zeta(x_i+t(x-x_i))\cdot(x-x_i)\,dt\right| \le \lambda\varepsilon(\sup|\nabla\zeta|),$$

we have

$$\begin{aligned}|J_3^i| &= \left|-\int_{B_{\lambda\varepsilon}(x_i)\setminus B_i^\varepsilon}(w_i\nabla\zeta\cdot\nabla u_\varepsilon + (\zeta-\zeta(x_i))\nabla w_i\cdot\nabla u_\varepsilon)\,dx\right|\\ &\le (\sup|\nabla\zeta|)\left(\|w_i\|_{L^2(B_{\lambda\varepsilon}(x_i)\setminus B_i^\varepsilon)} + \lambda\varepsilon\|\nabla w_i\|_{L^2(B_{\lambda\varepsilon}(x_i))}\right)\|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))},\\ &\le (\sup|\nabla\zeta|)\lambda\varepsilon\left(\sqrt{\frac{\pi}{2}}\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)} + \left(2\pi\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)}\right)^{1/2}\right)\\ &\quad\cdot\|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))}\\ &\le C(\sup|\nabla\zeta|)\delta^{1/2}\varepsilon^2(\delta^{1/2}\varepsilon+1)\|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))}. \end{aligned}\tag{4.23}$$

Here we used Equation (3.24) and the following estimate

$$\begin{aligned}\|w_i\|^2_{L^2(B_{\lambda\varepsilon}(x_i)\setminus B_i^\varepsilon)} &= 2\pi\frac{1}{\ln^2\left(\frac{a_\varepsilon R_i}{\lambda\varepsilon}\right)}\int_{a_\varepsilon R_i}^{\lambda\varepsilon} r\left(\ln\left(\frac{r}{\lambda\varepsilon}\right)\right)^2 dr\\ &= 2\pi\frac{1}{\ln^2\left(\frac{a_\varepsilon R_i}{\lambda\varepsilon}\right)}\left\{-\frac{(a_\varepsilon R_i)^2}{2}\left(\ln\left(\frac{a_\varepsilon R_i}{\lambda\varepsilon}\right)\right)^2 - \int_{a_\varepsilon R_i}^{\lambda\varepsilon} r\ln\left(\frac{r}{\lambda\varepsilon}\right)dr\right\}\\ &\le 2\pi\frac{1}{\ln^2\left(\frac{a_\varepsilon R_i}{\lambda\varepsilon}\right)}\left\{-\int_{a_\varepsilon R_i}^{\lambda\varepsilon} r\ln\left(\frac{r}{\lambda\varepsilon}\right)dr\right\} \end{aligned}$$

$$= 2\pi \frac{1}{\ln^2(\frac{a_\varepsilon R_i}{\lambda\varepsilon})} \left\{ \frac{(a_\varepsilon R_i)^2}{2} \ln(\frac{a_\varepsilon R_i}{\lambda\varepsilon}) + \frac{\lambda^2\varepsilon^2}{4} - \frac{(a_\varepsilon R_i)^2}{4} \right\}$$

$$\le \frac{\pi\lambda^2\varepsilon^2}{2} \left(\frac{\delta\varepsilon^2}{1-\delta\varepsilon^2 \ln(\frac{R_i}{\lambda})} \right)^2. \tag{4.24}$$

So

$$\left| \sum_{i=1}^N \int_0^T \psi(t) J_3^i \, dt \right|$$

$$\le C(\sup|\nabla\zeta|) \|\psi\|_{L^2(0,T)} \delta^{1/2}\varepsilon^2(\delta^{1/2}\varepsilon+1) \left(\int_0^T \left(\sum_{i=1}^N \|\nabla u_\varepsilon\|_{L^2(B_{\lambda\varepsilon}(x_i))} \right)^2 \right)^{1/2}$$

$$\le C(\sup|\nabla\zeta|) \|\psi\|_{L^2(0,T)} \delta^{1/2}\varepsilon(\delta^{1/2}\varepsilon+1) \cdot \left(\int_0^T \left(\varepsilon^2 \sum_{i=1}^N 1 \right) \left(\sum_{i=1}^N \|\nabla u_\varepsilon\|^2_{L^2(B_{\lambda\varepsilon}(x_i))} \right) \right)^{1/2}$$

$$\le C(\sup|\nabla\zeta|) \|\psi\|_{L^2(0,T)} \delta^{1/2}\varepsilon(\delta^{1/2}\varepsilon+1) \left(\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \, dx dt \right)^{1/2}$$

$$\le C(\sup|\nabla\zeta|) \|\psi\|_{L^2(0,T)} \delta\varepsilon(\delta^{1/2}\varepsilon+1) \quad \text{by (3.16)}$$

$$\to 0 \quad \text{as } \varepsilon \to 0. \tag{4.25}$$

□

4.3. The limit equation for the droplet distribution. Finally we state the following theorem about the droplet distribution.

THEOREM 4.2. *For all $\psi \in C_0^\infty(0,\infty)$ and $\zeta \in C_p^0$ with $\partial\zeta/\partial r \in C_p^0$, we have*

$$\int_0^T \int \left(\psi'(t)\zeta + 2\pi\psi(t) \frac{\partial\zeta}{\partial r} \frac{1}{r^3}(ru_* - 1) \right) d\nu_t \, dt = 0. \tag{4.26}$$

Proof. For a given ψ, let $T > 0$ be big enough so that $\operatorname{supp}\psi \subset (0,T)$. Then

$$0 = \int_0^T \frac{d}{dt} \varepsilon^2 \sum_{i=1}^N \psi(t)\zeta(x_i,R_i) \, dt$$

$$= \int_0^T \varepsilon^2 \sum_{i=1}^N \left\{ \psi'(t)\zeta(x_i,R_i) + \psi(t)\frac{\partial\zeta}{\partial r}(x_i,R_i)\dot{R}_i \right\} dt$$

$$= \int_0^T \varepsilon^2 \sum_{i=1}^N \left\{ \psi'(t)\zeta(x_i,R_i) + \psi(t)\frac{\partial\zeta}{\partial r}(x_i,R_i)\frac{1}{\delta\varepsilon^2 R_i^2} \int_{\partial B_i^\varepsilon} \nabla u_\varepsilon \cdot n \, ds \right\} dt$$

$$= \int_0^T \varepsilon^2 \sum_{i=1}^N \left\{ \psi'(t)\zeta(x_i,R_i) + \psi(t)\frac{\partial\zeta}{\partial r}(x_i,R_i)\frac{1}{\delta\varepsilon^2 R_i^2} \left(-2\pi \frac{\delta\varepsilon^2}{1-\delta\varepsilon^2\ln\left(\frac{R_i}{\lambda}\right)} \frac{1}{R_i} - \int_{\partial B_{\lambda\varepsilon}(x_i)} u_\varepsilon \nabla w_i \cdot n \, ds \right) \right\} dt$$

$$\to \int_0^T \int \left(\psi'(t)\zeta + 2\pi\psi(t)\frac{\partial\zeta}{\partial r}\frac{1}{r^3}(-1+ru_*) \right) d\nu_t \, dt. \tag{4.27}$$

Here we used Lemma 4.2. □

5. Discussions and conclusions

We have derived a mean field model for the Ostwald ripening of thin liquid films. In this model, the system size $\mathcal{L}$ and the screening length $\mathcal{L}_s$ induced by the droplets are involved through a parameter $\delta \approx \mathcal{L}^2/\mathcal{L}_s^2$, which we assume to be a positive constant. This model is a generalization of the mean field model proposed in [2], which heuristically corresponds to the case when $\mathcal{L} << \mathcal{L}_s$, or $\delta \approx 0$. The result in this paper promotes the understanding of the delicate relations between the domain size, the distance between droplets, and the characteristic size of the droplets. In the case when the system size $\mathcal{L}$ is much bigger than $\mathcal{L}_s$, i.e., when $\delta \approx \infty$, little is known about a mean field approach and we are currently investigating this possibility. On the other hand, both the mean field model in this paper and that in [2] (and that for 1D thin films [1,7]) only consider the Ostwald ripening mechanism in thin liquid films. The migration and collision of liquid droplets are ignored. We expect to derive a model that includes the migration and collisions. Such a model will likely have some terms similar to those in the Smoluchowski coagulation models.

Acknowledgments. The author thanks the anonymous referees for their careful reading and valuable comments that improved the quality of the paper. The author acknowledges support by US NSF through grants DMS-1802863 and DMS-1815746.

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