The Modeling Space: An Analytical Tool for Documenting Students' Modeling Activities

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One of the most challenging aspects of doing research in the mathematical modeling genre has been finding an appropriate characterization for the complex interaction of knowledge and cognitive acts that result in coordination of situational referents and mathematical inscriptions. To this end, we introduce the modeling space and illustrate its descriptive and analytic utility.

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To date, the most common characterization of modeling is a "translation" from the real world to the mathematical world. Typically, the "translation" is envisioned as leveraging a one-to-one correspondence between words (or real-world objects) with mathematical counterparts. This characterization is not only empirically inaccurate, but consequently impedes progress in theorizing the teaching and learning of modeling. First, translation between natural languages is typically unbalanced; it is rare that an individual is equally knowledgeable in her first and subsequent languages. Second, the characterization is reductionist since "reliance on translation cues…is more characteristic of students who possess only algorithmic knowledge of the target task and who circumvent the interpretive process of mathematical modeling" (Martin & Bassok, 2005, p. 479). We do not imply that language translation is trivial work, only that referring to mathematization of a real-world situation into a mathematical problem as translation ignores the complexity of modeling which has, in turn, limited the field's ability to systematically investigate it as an idiosyncratic activity. Presently, the field lacks a methodologically cogent approach to studying modeling that reflects the full range of mathematical or real-world meanings individuals might ascribe to a single representation. It also faces an abundance of partially explanatory theories focusing on just one aspect of modeling (e.g., translation). The field is thus ill-equipped to account for how those meanings shift during the process of modeling or to develop means for studying how instructors may impact the process. The broader goal of the research project is a theoretically and methodologically coherent way to trace the evolution of a student's mathematical model, including the representations she produces and her meanings for these representations. Our contribution is twofold: a theoretical construct robust enough to account for students' idiosyncratic and shifting meanings during mathematical modeling and an accompanying descriptive mathematical model capable of tracing the evolving student model.

Relevant Theoretical Constructs

To address the larger question of how to document model evolution, we first acknowledge that the extant theories of mathematical modeling collectively posit both internal (mental) and external (representational) aspects of a mathematical model which must both be accounted for. To date, no one theory does so comprehensively. When multiple theories each offer partial explanations for a phenomenon, such as model evolution, The Networking Theories Group (2014) advocates strategies for coordinating those theories. We adopted the strategies they recommended such as combining and coordinating in order to generate "deeper insights into an empirical phenomenon" (p. 120). We first focused on the aspects of relevant theories that foregrounded conventional mathematical representations (Czocher & Hardison, 2019). However, tracing only the evolution of representations was insufficient for capturing the breadth of ways that a mathematical model could change could change. We next sought to incorporate the theory of quantitative reasoning (Thompson, 2011) by considering the quantities students projected into a situation to be modeled and the interplay between quantities and representations (Czocher & Hardison, under review).

Modeling can be seen as a process of unification among a sign, a referent (the object the sign stands for), and an interpretant (Kehle & Lester, 2003). In mathematics, a sign can be part of an inscription in mathematical notation. In our view, the referent could stand for a real-world object, for a quantity, or for another conceptual entity. Then the interpretant can be characterized as the mathematical conception of how that quantity relates to other quantities. Sherin's (2001) theory of symbolic forms provides one interpretation for how meaning can be read into equations, which are composed of signs. A symbolic form consists of a template and a conceptual meaning (the idea to be expressed in the equation). For example, _ + _ = _ expresses a "parts-of-a-whole" relationship. The blanks can be filled with a single symbol or a group of symbols representing quantities or combinations of quantities (perhaps related via other symbolic forms). Familiarity with symbolic forms helps individuals "know" to use certain operators (e.g., + or ×) or relationships and to know where to place the symbols of quantities in an equation. Symbolic forms are building blocks of equation generation.

Mathematizing a situationinvolves generating mathematical representations and assigning semantic meanings compatible with the modeler's conception of the situation at hand. That is, the modeler must identify relevant quantities and describe how they vary together. Thompson's (2011) theory of quantitative reasoning offers relevant insights. First, Thompson takes the strong position that quantities are mental constructs, not characteristics of objects in the world. It immediately follows that a *quantification* process is carried out by an individual in order to conceive of quantities. Quantification is "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (p. 37). One can conceive of various instantiations of the object, with each instantiation manifesting different extents of the relevant attribute, and coordinate these instantiations with a value. We operationalize quantification as the set the of operations an individual can enact on a particular attribute (Hardison, 2019). Observing a phenomenon and conceptualizing that there are quantities and that they can vary (or may be constant) is foundational to formulating a meaningful mathematical model. For example, quantities like distance and velocity may be more readily available for high school students than torque, electrical current, or GDP.

Quantitative reasoning more generally entails conceiving of quantities and relationships among quantities. It involves conceiving of relationships among quantities whose values may vary independently. These constructs allow modeling to viewed as conceiving and representing relationships among the quantities involved. Coordinating quantities and attending to relationships among quantities is *covariational reasoning* (Carlson, et al., 2002). It involves identifying ways to combine quantities through operations and trace their changes, rates of changes, and intensities of changes whether they are directly measurable or not (e.g., Johnson, 2015). Relationships can be identified through observation, a priori reasoning, or through knowledge of principles rooted in physical theory.

When quantities and relationships being modeled are expressed externally in mathematical notation, they become the mathematical representation of a physical model. The mathematical representation is intended to convey an individual's mathematical model (mathematical concepts, objects, and structures) and the relationships among the constituent inscriptions' situational quantitative referents. In this way, the theory of covariation of quantities elaborates an important aspect of how the conceptual counterparts to mathematical models are formalized into mathematical representations and expressed in symbolic forms. In the next section, we introduce the modeling space, a theoretical construct we use to communicate the quantities present in a student's mathematical model, to document changes in this model over time, and to provide some predictive utility for modifications a student might or might not make to her model.

The Modeling Space: Construct and Representational Tool

We preface our discussion of the modeling space by noting that we distinguish between the quantities an individual projects onto a situation, operations (quantitative or numerical) enacted on quantities or their values, and the representations (inscriptions as well as utterances) she uses. We refer to the set of mathematical models a student might generate within a given modeling task as *the modeling space.* We intentionally draw an analogy to a Cartesian product space. Each quantity corresponds to a dimension of the product space. We view the modeling space as the set of mathematical relationships that act via composition on the situationally relevant quantities available to the student. In this section we build to a mathematical description of an individual's modeling space. For example, suppose that in the course of addressing a modeling task about a falling body under the influence of gravity only, we are able to infer that Janet has introduced the quantities *initial height above ground*, *time elapsed*, *current height above ground*, *mass of the object,* and *initial velocity*. We represent her available quantities, organized by type, as the sequence $(\mathrm{TIME}, \mathrm{HT_I}, \mathrm{HT}, \mathrm{MASS}, \mathrm{V_I})$. Her modeling space would be all of the meaningful (to her) mathematical combinations of those quantities. For example, one element in the modeling space Janet may represent is *current height above ground*, $h = h_0 - v_0 \cdot t$ (Eqn 1), where the symbol names correspond to experts' conventions.

We can formalize the modeling space by structuring it with a descriptive mathematical model. Suppose during modeling, a student projects a set of N quantities, $Q = \{q_1, q_2, \ldots q_N\}$ onto the referent situation. At interview time τ, we assign one of three values from the set $S = \{-,0,1\}$ to each quantity (– means the potential quantity q_i has not yet been projected by the individual, 0 means q_i has been projected by the individual at some $t < \tau$ but is not referred during a time period of analytic interest, 1 means there is evidence that q_i is referred during the vignette). The set S has a natural order $- < 0 < 1$. We can impose a commutative binary operation, the *tropical addition* defined by $a * b = \max(a, b)$. S is then a monoid[1] under $*$ with identity $-$. We then form a Cartesian product from the set S over the N dimensions supplied by Q:

$$M = \prod_{i=1}^{N} S_i = \underbrace{\{-,0,1\} \times \{-,0,1\} \times \cdots \times \{-,0,1\}}_{N\ times}$$

The product M is again a commutative monoid under the operation $+$, coordinate-wise addition using the operation $*$.

Elements of M, written as $(s_1, s_2, \ldots, s_N)$ are assigned to a vignette of student work on a modeling task according to whether *during that vignette* each quantity q_i is used during the vignette (value 1), has been projected prior to the vignette but is not used during the vignette (value 0), or has not yet been projected onto the situation (value $-$). The individual's mathematical model evolves over time and is captured by changing values of s_k for each q_k. Thus, we model the start of the interview with *M* as expressed as $(-, -, \cdots, -)$. As quantities are

[1] A monoid is a set that is closed under an associative binary operation and contains an identity. It is different from a group because its elements lack inverses.

introduced, the $-$'s are replaced with $0'$s and $1'$s. For example, if $N = 4$, the notation $(-,0,1,1,0)$ would indicate that the individual has available four quantities and is using quantities q_2 and q_3, but not q_1 and q_4. The quantity q_1 has not yet been projected. At the end of the interview, M is expressed as a sequence of $0's$ and $1's$. We can use elements of M to represent the accumulated modeling space (all quantities we infer a student has projected up to at a given moment during the interview) or, by acknowledging that q_i is a function of time and treating elements of M as finite states, we could trace change over designated periods of interview time. It is beyond the scope of this paper to fully explore the distinctions; instead we wish to exemplify the construct and its utility to support further theorizing by treating accumulated quantities.

Suppose Janet considered gravity, but she has not yet explicitly introduced it as a quantity, but does later. We can represent her mathematical model as $m_1 = (1,1,1,0,1,-)$, where – indicates that gravity has yet to be introduced. Note, the symbol 1 denotes only that the quantity is "active" for the individual. The symbol does not signify *which* value, unit, or mathematical symbol the individual associates with the quantity at that moment. Because a given attribute of an object can be measured in different ways, expressing gravitational acceleration in feet per second squared or object-lengths per second per second would map to the same coordinate value. In this way, the sequence $m_1 = (1,1,1,0,1,-)$ could represent any number of mathematical inscriptions whose elements are associated with quantitative referents (e.g., a graph and an equation), mathematical relationships among the quantities, or quantitative operations among the quantities. Said differently, elements of M are an equivalence class of mathematical models that compose quantities in Q. The collection of possible mathematical compositions and expressions of those compositions that can be mapped to $m \in M$ we refer to as the *modeling space of m* and indicate as $\widetilde{m}$. The collection of all possible models she might conceive using the quantities available to her during the interview would be her modeling space, $\widetilde{M}$.

Constructing Merik's Modeling Space

Data were collected as part of a larger study of the characteristics of tasks and facilitator interventions that could elicit specific mathematical modeling competencies among undergraduate STEM majors. Pursuant to this goal, we conducted a series of think-aloud task-based interviews. Participants addressed a variety of modeling and applications tasks requiring participants to make simplifying assumptions about the contextual situation. Tasks included opportunities to use arithmetic, algebra, calculus, and differential equations. In this analysis, we present the work of Merik, an engineering student who had completed linear algebra and calculus 3, because he was especially articulate in describing his mathematical thinking and regularly exhibited a variety of modeling competencies as he engaged in modeling tasks. One problem he addressed was the Monkey Problem: *A wildlife veterinarian is trying to hit a monkey in a tree with a tranquilizing dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquilizing dart to shoot the monkey.*

We chose Merik's work on the Monkey Problem because he introduced many different inscriptions, quantities, and mathematical representations indicating that it would be possible to closely examine changes in his mathematical and contextual knowledge about the situation. Merik interpreted the prompt as an invitation to find a model representing the situation. He was given unlimited time and was assured that his responses were not being judged for "correctness." We provisionally accepted all of his work without actively teaching, leading, or removing ambiguity (Goldin, 2000). A key aspect of the interview protocol and subsequent analysis was to assume Merik's interpretations of his own work differed from our own. Follow-up questions and interventions were intended to either clarify his thinking or to test any conjectures the interviewer had about his thinking in the moment. The audio/video recorded interview session lasted 46 minutes and was subsequently transcribed.

We followed the procedure outlined in Czocher and Hardison (2019, under review), which had three stages to analysis: (1a) identify and catalog all mathematical representations by examining the spatial and temporal organization of inscriptions on Merik's paper (1b) determine whether the representations or their meanings may have changed, (2) identify quantities Merik projected onto the Monkey Problem context, and (3) document whether there was sufficient evidence to infer whether the quantitative situational referent of a given inscription changed during the interview. We elaborate on (2) and then show how the modeling space construct enabled (3).

According to our theoretical frame, a quantity is an individual's conception of a measurable attribute of an object in a situation. We analyzed the interview and identified situational attributes to which Merik attended in the Monkey Problem. By situational attributes, we mean we were able to infer a referent within the Monkey context with a quality that Merik might have quantified (e.g., the tree's height). Instances in which Merik mentioned generic attributes—those for which we were unable to infer situational referents (e.g., distance)—were not considered situational attributes. Additionally, we searched for evidence that suggested Merik might have quantified these situational attributes. In particular, we sought evidence of Merik engaging in mental operations necessary for, or suggestive of, a conceived measurement process for each situational attribute. Through iterative cycles of analysis, we stabilized a set of 8 criteria that we took as evidence of quantification during mathematical modeling. Three independent coders systematically applied those criteria to the video and transcript; disagreements were resolved through consensus seeking. A quantity was included as a potential quantity for Merik if it met at least one inclusion criteria (see Table 1). We recorded the times at which we could infer that situational referents actively served as counterparts to inscriptions and symbols within the representations (or not).

Table 1 Potential quantities projected onto the Monkey Problem context, chronological order

Quantity	Type	Time	Description
ANG_{STR}	Angle	2:08	Measure of angle gun is aimed relative to the horizontal, for straight path
$\text{DIST}_{\text{VET/TREE}}$	Length	2:09	Horizontal distance from vet to the tree/under the monkey.
$\text{HT}_{\text{MKY/GUN}}$	Length	2:10	Height of the monkey relative to the vet's gun.
$\text{VVEL}_{\text{DART-I}}$	Rate	2:47	Initial vertical velocity of the dart
ACC_{DART}	Rate	3:20	(Vertical) acceleration of dart
$\text{HT}_{\text{GUN/GRD}}$		3:35	Height of gun (or vet) relative to ground.
$\text{HT}_{\text{TREE/GRD}}$	Length	4:13	Height of the tree
$\text{DIST}_{\text{VET/MKY}}$	Length	4:36	Length of the straight path from the vet's gun to the monkey.
ANG_{PAR}	Angle	6:04	Measure of angle gun is aimed relative to the horizontal, for parabolic path
$\text{IVEL}_{\text{DART}}$	Rate	11:37	Initial linear velocity of the dart.
HT_{DART}	Length	15:35	Height of the dart
TIME	Time	16:08	Elapsed time
$\text{ANG}_{\text{VET/3D}}$	Angle	24:38	Measure of the plane angle formed by a designated axis and the line through the tree & veterinarian in 3-space.
$\text{HVEL}_{\text{DART-I}}$	Rate	25.42	Initial horizontal velocity of the dart

In total, we identified 14 potential quantities that Merik cumulatively introduced to structure the Monkey Problem. Thus $Q = (\Big\{\text{DIST}_{\frac{\text{VET}}{\text{TREE}}}, \text{HT}_{\frac{\text{MKY}}{\text{GUN}}}, \text{HT}_{\frac{\text{GUN}}{\text{GRD}}}, \text{DIST}_{\frac{\text{VET}}{\text{MKY}}}, \text{HT}_{\frac{\text{TREE}}{\text{GRD}}}\ \text{HT}_{\text{DART}}, \Big\},$

$\{ANG_{STR}, ANG_{PAR}, ANG_{VET,3D}\}, , \{VVEL_{DART,I}, IVEL_{DART}, HVEL_{DART}, ACC_{DART}\}, \{TIME\}$)
At interview time τ, we represent the active equivalence class within his modeling space via the tuple Q with appropriate substitutions from S made for each quantity.

Illustrations

Merik initially imposed a right triangle and considered the angle to fire the dart such that the hypotenuse would pass through the vet and the monkey. However, after introducing ACC_{DART}, Merik stated that he was seeking a quadratic equation because "that is the path the bullet is going to follow." At this moment, there were no inscriptions resembling a quadratic equation, so we interpreted his stated goal to produce an equation as indicating an implicit symbolic form relevant to him. To elicit the form from Merik as well as to gain insight into the situationally specific meanings Merik might have for it, the interviewer asked, "What variables and parameters would be present in your equation?" Merik immediately inscribed $f(x) = Ax^2 + Bx + C$. At this point we were unable to infer that Merik had projected meanings specific to the task at hand. Moments later, Merik explained, "I know that my A is negative 10," which indicated he was attending to gravity based on his earlier activities. As Merik continued, he indicated that B "would be whatever the initial velocity is, which I don't have." Merik went on to explain, "the image of 30 feet which is, in this particular case, 40 feet." He also explained that the "image of 0 is 0." These specific values were references to Merik's earlier simplification of the task wherein he considered a specific scenario: the vet was 30 feet from the tree and the monkey was 40 feet high. Although Merik substituted 0 for C, we were unable to infer whether Merik had any situationally specific quantitative referent for C at this point in the interview. For this portion of the interview (roughly 9:30-11:40), the equivalence class for models he could generate was $(\overbrace{1,1,0,0,0,0}^{\text{lengths}}, \overbrace{0,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}}, \overbrace{-}^{\text{time}})$, which corresponds to his quadratic.

One of Merik's chief difficulties in constructing a model to his satisfaction lay in the fact that there were competing meanings attached to the quadratic template $\square = \square \cdot \square^2 + \square \cdot \square + \square$. At times the symbol x represented the horizontal position of the dart (implicitly at a given moment in time), while at others it implicitly represented time elapsed in the situation. The three meanings in play were $f(x)$ as predicting vertical position in terms of elapsed time, $f(x)$ as predicting the vertical position in terms of horizontal position, and $f(x)$ as an alternative representation of the mathematical object parabola. The shift itself was not consciously realized by Merik. Indeed, he abandoned the representation at 11:45, as he referred a previously identified quantity, the angle the veterinarian should fire at ANG_{PAR} prompting a new inscription. Based on the quantities we could infer were *active* for Merik from 9:30-15:40 The equivalence class for models he could generate was: $(\overbrace{1,1,0,0,1,1}^{\text{lengths}}, \overbrace{1,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}}, \overbrace{-}^{\text{time}})$ He did not resolve the competing schema until 15:16, when in response to an interviewer prompt to provide explicit meanings for the symbols Merik realized that plugging in 30 for x and 40 for y (distances) was not compatible with the parabola which recycled the symbol x for time. Thus, it was not until after 16 minutes into his work on the problem that Merik referenced time in a way that we could infer he had projected the quantity onto the situation. The representation for quantities available to Merik for composition became $\left(\overbrace{1,1,0,0,1,-}^{\text{lengths}}, \overbrace{1,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}} \overbrace{1}^{\text{time}}\right)$.

Due to space constraints, we provide one more example from later in the interview. At this point, Merik's stated goal was to find ANG_{PAR} and we symbolize the *active* equivalence class within his modeling space as $\left(\overbrace{1,1,0,0,0,0}^{\text{lengths}}, \overbrace{1,1,0}^{\text{angles}}, \overbrace{0,1,0,1}^{\text{rates}} \overbrace{1}^{\text{time}}\right)$. The interviewer intended to direct Merik's attention to the angle between the straight-line path and the angle that would produce the parabolic path, asking "How do you anticipate the two angles will compare?" Merik responded that ANGPAR would be "larger not by a wide margin but I think that because the way that it's traveling more like [[draws arc'ed curve between the veterinarian and the monkey]] then you have to aim up more to increase the angle." Thus, Merik was able to consider variation in ANG_{PAR} in relation to ANG_{STR}. However, this was not sufficient for quantifying the difference between the two angles, even after the interviewer prompted him to think about finding an angle measure between two curves and he responded that he could use tangent lines to do so. He introduced the signs $\underline{u}$ and $\underline{v}$ and the inscription $\underline{u} \cdot \underline{v} = \frac{\cos\theta}{|\underline{u}||\underline{v}|}$. We lacked observable evidence to support the claim that θ corresponded to a situational referent. We suggest that Merik did not (in that moment) apply his formula because, for him, the angle between $\underline{u}$ and $\underline{v}$ was not salient as a quantity. His reasonable, sensible options for modeling the situation were constrained by the quantities that he had available. This perspective explains not only why the prompt or his further mathematical conceptual work did not help him to make progress, but also why it could not help him – without a quantity, the formula had no situational meaning.

Value and Future Work

Our theoretical and methodological considerations have resulted in an examination of Merik's mathematical modeling activity as a process of composition of quantities via mathematical operations. Our approach separated the acts of quantification from the acts of introducing variables from the acts of generating inscriptions from the acts of ascribing meaning to mathematical inscriptions. We found evidence that each of these acts can be carried out independently or to varying degrees of alignment. These aspects are overlooked when viewing modeling as translation. The modeling space enables a precise description of this finding. As a theoretical tool, (1) it predicts at any given moment, a student's modeling process will be constrained by the elements in her modeling space at that time and (2) we can trace how the modeling space expands and supports (or excludes) formation of mathematical relations over time. The modeling space (at least partially) predicts, and simultaneously constrains, the mathematical models the student might produce. Ultimately, the research community's goal is to articulate opportunities for effective pedagogical intervention. In contrast to the majority of research in the modeling genre, which tends to be representations-forward, our theoretical and methodological approach put quantities in the fore. Because the modeling space is focused on documenting and tracing meanings as well as inscriptions, it may be able to support models of pedagogy as well. Our analyses offered explanations for two key moments for pedagogical intervention: one successful intervention (asking the student to be explicit about his meanings for symbols) and one failed intervention (introducing a strategy based in quantities Merik had not projected) and the outcomes of the interviewers' moves are reflected as amendments to the modeling space. As a representational tool, the modeling space approach affords overviews of an individual's work as a time series, could facilitate comparison of individuals' productions, be used to evaluate potential task prompts and indicate potential sites for interventions as well as predict whether those interventions are likely to be taken up by the student. Finally, the methodology moves the field one step closer to being able to trace changes in a mathematical model: how they are precipitated, ways they change, and how students respond to interventions.

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