

MONOPOLES AND LANDAU-GINZBURG MODELS I

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ABSTRACT. The end point of this series of papers is to construct the monopole Floer homology for 3-manifolds with torus boundary. In the first paper, we explain the idea from the standpoint of gauged Landau-Ginzburg models and address a few model problems related to the compactness of moduli spaces, using a Bochner-type formula associated to the gauged Witten equations.

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Part 1. Introduction

1. INTRODUCTION

1.1. Motivations in Floer Homology. The Seiberg-Witten Floer homology of a closed oriented 3-manifold Y is defined by Kronheimer-Mrowka [KM07] and has greatly influenced the study of 3-dimensional topology. The underlying idea is to construct an infinite dimensional Morse theory: solutions to the 3-dimensional Seiberg-Witten equations on Y are critical points of the Chern-Simons-Dirac functional $\mathcal{L}$, and solutions to the 4-dimensional equations on $\mathbb{R}_t \times Y$ are viewed as negative gradient flowlines of $\mathcal{L}$.

The purpose of this series of papers is to generalize their construction and define the Seiberg-Witten Floer homology $HM_*(Y)$ for an oriented 3-manifold $(Y, \partial Y)$ with torus boundary, which has the potential to recover the knot Floer homology of a knot $K \subset S^3$, both the hat-version and the minus-version as special cases. In the first paper, we focus on the geometric aspect and explain its relations with gauged Landau-Ginzburg models, which were first introduced by Witten [Wit93] in his formulation of gauged linear sigma model (see Subsection 1.3 below). This change of standpoints allows us to prove two fundamental results (Theorem 1.2 & 1.3 below) that lead eventually to the compactness theorem of the Seiberg-Witten moduli spaces. The actual construction of the monopole Floer homology of $(Y, \partial Y)$, including the compactness theorem and many other analytic details, will be postponed to the second paper of this series [Wan20].

One reason to develop a relative version of Floer theory for 3-manifolds with boundaries is to give a gluing formula for the absolute version. The second reason is to define invariants for knots and links inside S^3 . These goals are accomplished in the framework of Heegaard Floer Homology, via the construction of bordered Floer homology [LOT08] by Lipshitz-Ozsváth-Thurston and knot Floer homology by Ozsváth-Szabó [OS04] and independently Rasmussen [Ras03]. See [Man16] for a nice survey on their constructions. A long term goal of our program is to interpret their works in the context of gauge theory and hopefully provide new insights for future research.

It has been long believed [Man16, P.1] that the knot Floer homology of (S^3, K) encodes something about the Seiberg-Witten equations on $\mathbb{R}_t$ times the knot complement $S^3 \setminus N(K)$. One may approach this heuristic using the Floer homology constructed in the second paper, which applies to any knot (or link) complements. The **conjectural** relation (see [Wan20] for more details) is as follows

$$\begin{aligned} HM_*(Y) &\rightsquigarrow \text{HFK}_*(S^3, K) && \text{if } Y = S^3 \setminus N(K), \\ HM_*(Y) &\rightsquigarrow \widehat{\text{HFK}}_*(S^3, K) \text{ or } KHM_*(S^3, K) && \text{if } Y = S^3 \setminus N(K \cup m), \end{aligned}$$

where m is a meridian of $K \subset S^3$. In the second case, we will describe a conjectural self-gluing formula of $HM_*(Y)$ in Subsection 2.3 that explains this reduction.

Some constructions of knot Floer homology that uses gauge theory already exist in the literature. It is hoped that the analytic tools introduced in this series of papers can also help extend the existing theory in some directions. Let us give a brief summary:

- In [KM10], Kronheimer-Mrowka defined the monopole knot Floer homology KHM_* using the sutured manifolds technique developed by Juhász [Juh06, Juh08], as the counterpart of $\widehat{HFK}$ in Heegaard Floer homology. In the same paper, they introduced the instanton knot Floer homology whose applications include a new proof of Property P for knots. The rough idea is to close up the boundary of the knot complement $S^3 \setminus N(K)$ (treated as a balanced sutured manifold) and obtain a closed oriented 3-manifold whose monopole Floer homology is already defined. We will come back to this approach and explain its relation with our work in Subsection 2.3;
- Most recently, it is shown that the sutured monopole Floer homology is a functor from the sutured cobordism category [Li18], and Li [Li19] proposed a construction of HFK^- in the Seiberg-Witten theory using a direct system of sutures on the knot complement;
- In [KM11a], Kronheimer-Mrowka introduced the singular instanton Floer homology and proved that the Khovanov homology detects the unknot. See also [KM11b]. The idea is to treat (S^3, K) as a bi-fold, whose metric is singular along the knot K with a cone angle π .

On the other hand, Nguyen [Ngu12, Ngu18] studied the monopole equations on the manifold with boundary $(Y, \partial Y)$ directly and developed analytic foundations for a Floer theory with the Lagrangian boundary condition on ∂Y . We will work instead with a complete Riemannian manifold, as we explain in the next subsection.

1.2. Summary of Results. To give the statement of our main theorems, let us now describe the setup. Given a compact oriented 3-manifold Y with torus boundary $\partial Y = \Sigma$, let g_Y be a Riemannian metric that is cylindrical near Σ and $g_\Sigma = g_Y|_\Sigma$. Then $\Sigma = \coprod_{1 \leq i \leq m} \Sigma_i$ is a disjoint union of 2-tori. Instead of Y , we look at the monopole equations on the complete Riemannian manifold:

$$\hat{Y} = Y \coprod_{\Sigma} [0, \infty)_s \times \Sigma$$

For most results in this paper, there is no need to assume that g_Σ is flat. At this point, a suitable perturbation along the cylindrical end $[0, \infty)_s \times \Sigma$ is required so that Lemma 1.1 below is valid, which is crucial for a Floer theory. It relies on a pair (λ, μ) where

- $\lambda \in \Omega_h^1(\Sigma, i\mathbb{R})$ is an imaginary valued harmonic 1-form on Σ ;
- $\mu \in \Omega_h^2(\Sigma, i\mathbb{R})$ is an imaginary valued harmonic 2-form on Σ , i.e. $\mu = \sum_{j=1}^m \delta_j d\text{vol}_{\Sigma_i}$ for some constants $\delta_j \in i\mathbb{R}$.

Finally, we use the 2-form $\omega := \mu + ds \wedge \lambda$ to perturb the Seiberg-Witten equations on $[0, \infty)_s \times \Sigma$. See (10.1) for the explicit formulae on $\mathbb{R}_s \times \Sigma$ and (7.1) on the 4-manifold $\mathbb{C} \times \Sigma$. As we will focus on the cylindrical end of $\hat{Y}$ in this paper, it does no harm to assume that Σ is connected from now on. We are only interested in $spin^c$ structures with $c_1(S^+)[\Sigma] = 0$. The construction is inspired by the following lemma by Meng-Taubes:

Lemma 1.1 ([MT96]). *For any metric g_Σ on Σ and any pair $(\lambda, \mu) \neq (0, 0)$, there exists a unique $\mathbb{R}_s$ -translation invariant solution (up to gauge) to the perturbed Seiberg-Witten equations (10.1) on the 3-manifold $\mathbb{R}_s \times \Sigma$. Moreover, this solution is irreducible.*

Any finite energy solutions on $\hat{Y}$ will approximate this unique solution along the cylindrical end $[0, \infty)_s \times \Sigma$. As critical points of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega$, they become non-degenerate after further perturbations and form a compact moduli space of dimension 0.

Now consider the moduli space of finite energy monopoles on the 4-manifold $\mathbb{R}_t \times \hat{Y}$ which possesses a planar end $\mathbb{H}_+^2 \times \Sigma$. The upper half plane

$$\mathbb{H}_+^2 := \mathbb{R}_t \times [0, \infty)_s$$

is furnished with the Euclidean metric. The convention here is to use t for the time coordinate and s for the spatial coordinate on cylindrical the end of $\hat{Y}$. To define the Floer differential ∂ and prove that $\partial^2 = 0$ on the Floer complex, the moduli space must have the right compactification. The failure may arise from the cylindrical end of $\hat{Y}$. For a sequence of solutions on $\mathbb{R}_t \times \hat{Y}$, some amount of energy might slide off along the cylindrical ends of $\hat{Y}$ and give rise to finite energy solutions on $\mathbb{C} \times \Sigma$.

This is the first problem that we address. This phenomenon is precluded by the next theorem; the 4-manifold $\mathbb{R}_t \times \hat{Y}$ is non-compact in two directions, but the energy can slide off only in the time direction.

Theorem 1.2. *For any metric g_Σ and any $\lambda \neq 0$, any finite energy solution to the perturbed Seiberg-Witten equations (7.1) on $\mathbb{C} \times \Sigma$, the so-called point-like solution, is irreducible and gauge equivalent to the unique $\mathbb{C}$ -translation invariant solution whose energy is zero.*

In this sense, we say that point-like solutions on $\mathbb{C} \times \Sigma$ are trivial. This result contrasts immensely with the case of the unperturbed equations. Let us give a brief summary:

- For a higher genus surface $\tilde{\Sigma}$ with $g(\tilde{\Sigma}) > 1$, point-like solutions on $\mathbb{C} \times \tilde{\Sigma}$ can be non-trivial. They are classified completely in terms of some algebraic data, see [Wan18];
- When $(\lambda, \mu) = (0, 0)$, point-like solutions on $\mathbb{C} \times \Sigma$ are reducible and correspond to flat $U(1)$ -connections on Σ ;
- When $\lambda = 0$ and $\mu \neq 0$, the moduli space is isomorphic to $\coprod_{d \geq 0} \text{Sym}^d \mathbb{C}$, i.e. the vortex moduli space on $\mathbb{C}$ in the sense of [Tau80]. It is not regular unless $d = 0$. Indeed, the expected dimension is always 0 for any connected component, cf. Example 5.9.

The second problem we address is the exponential decay in the spatial direction on $\mathbb{R}_t \times \hat{Y}$, and we state the result for the planar end $\mathbb{H}_+^2 \times \Sigma$. For any $n \in \mathbb{Z}$ and $R \in [1, \infty)$, define $\Omega_{n,R} := [n-1, n+1] \times [R-1, R+1] \subset \mathbb{H}_+^2$ and let

$$(1.1) \quad \mathcal{E}_{an}(\gamma; \Omega_{n,R})$$

be the analytic energy of the configuration γ on the 4-manifold $\Omega_{n,R} \times \Sigma$, called the local energy functional of γ . See Definition 8.2 for the precise expression. This non-negative

quantity is gauge invariant, and bounds the L_1^2 -norm of γ (up to gauge) on $\Omega_{n,R} \times \Sigma$ and also the L_k^2 -norms in the interior of $\Omega_{n,R} \times \Sigma$ for any $k \geq 1$ when γ is a solution.

Theorem 1.3. *If $\lambda \neq 0 \in \Omega_h^1(\Sigma, i\mathbb{R})$, then there exist constants $\epsilon, \zeta > 0$ depending only on (g_Σ, λ, μ) with the following significance. Suppose γ solves the perturbed Seiberg-Witten equations on $\mathbb{H}_+^2 \times \Sigma$ and $\mathcal{E}_{an}(\gamma, \Omega_{n,R}) < \epsilon$ for any $n \in \mathbb{Z}$ and $R \geq 1$, then*

$$\mathcal{E}_{an}(\gamma; \Omega_{n,R}) < e^{-\zeta R}.$$

From Theorem 1.3, one can easily deduce the decay of L_k^2 -norms for a solution γ . Note that the spatial direction ($s \rightarrow \infty$) is not the direction of downward gradient flowlines of the functional $\mathcal{L}_\omega$, so Theorem 1.3 is not a consequence of the standard theory, e.g. [KM07, Section 13].

The proofs of Theorem 1.2 and 1.3 rely on Proposition 1.4 below which relates the Seiberg-Witten equations on $\mathbb{C} \times \Sigma$ with the gauged Witten equations of an infinite dimensional gauged Landau-Ginzburg model, as we explain in the next subsection.

1.3. Gauged Landau-Ginzburg Models. The gauged Witten equations were first introduced by Witten in his formulation of gauged linear sigma model [Wit93] as a physics theory that explains the so-called Landau-Ginzburg/Calabi-Yau correspondence. Its mathematical foundation is recently developed by Tian-Xu in a series of papers [TX18a, TX18b], in which case the domain is a compact Riemann surface with punctures. We would like to refer readers to their nice introduction for necessary background. Since our focus is slightly different, we give a short discussion below with emphasize on the Picard-Lefschetz theory.

When the dimension is finite and the structure group $G = \{e\}$ is trivial, a Landau-Ginzburg model is a pair (M, W) where

- M is a complete non-compact Kähler manifold, and
- $W = L + iH : M \rightarrow \mathbb{C}$ is a holomorphic function, called the superpotential.

The Landau-Ginzburg Model (M, W) is called *Morse* if $L := \operatorname{Re} W$ is a Morse function on M , so (M, W) defines a Lefschetz fibration. In the viewpoint of symplectic topology, one may define its Fukaya-Seidel category $\mathcal{A}$ in the sense of [Sei08] using Lagrangian Floer cohomology. Each compact Lagrangian submanifold $\mathcal{L}_0 \subset M$ should be assigned a A_∞ -module over $\mathcal{A}$. The construction is based on the Floer equation

$$(1.2) \quad \partial_t P + J \partial_s P + \nabla H = 0,$$

where $H := \operatorname{Im} W$ and $P : \mathbb{R}_t \times [0, 1]_s \rightarrow M$ is a J -holomorphic curve subject to Lagrangian boundary conditions. We wish to generalize this picture in two directions. Here is the first one:

- (1) the structure group G is abelian, but non-trivial.

In this case, the action of G on M is Hamiltonian with a moment map $\mu : M \rightarrow \mathfrak{g}$ and the superpotential W is G -invariant. Readers are referred to Definition 3.1 for other requirements on a gauged Landau-Ginzburg model. The right replacement of (1.2) is the

gauged Witten equations:

$$(1.3) \quad \begin{cases} - *_2 F_A + \mu &= \vec{\delta}, \\ \nabla_{\partial_t}^A P + J \nabla_{\partial_s}^A P + \nabla H &= 0, \end{cases}$$

where A is a connection on the trivial G -bundle Q over $\mathbb{R}_t \times [0, 1]_s$ and $\vec{\delta} \in \mathfrak{g}$. The map P is now regarded as a section of the trivial bundle $Q \times_G M$. The right replacement of the Morse condition is a notion of stability, cf. Definition 3.6. The local energy functional (1.1) in this context is defined as

$$(1.4) \quad \mathcal{E}_{an}(A, P; \Omega_{n,R}) = \int_{\Omega_{n,R}} |F_A|^2 + |\mu - \vec{\delta}|^2 + |\nabla^A P|^2 + |\nabla H|^2.$$

for any $\Omega_{n,R} \subset \mathbb{H}_+^2$. In particular, $\mathcal{E}_{an}(A, P; \Omega_{n,R}) = 0$ implies that up to gauge, $A = d$ and P is a constant map defined on $\Omega_{n,R}$ taking values in $\mu^{-1}(\vec{\delta}) \cap \text{Crit}(H)$.

Here is the second generalization that we make:

- (2) the gauged Landau-Ginzburg model (M, W, G) can be infinite dimensional.

The proofs of Theorem 1.2 and 1.3 start with their counterparts for finite dimensional Landau-Ginzburg models (as toy problems) and are concluded by the following observation.

Proposition 1.4 (Proposition 7.4 & 7.5). *There is an infinite dimensional gauged Landau-Ginzburg model $(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma))$ associated to $(\Sigma, g_\Sigma, \lambda)$ whose gauged Witten equations on $\mathbb{C}$ recover the Seiberg-Witten equations on $\mathbb{C} \times \Sigma$. When $\lambda \neq 0$, this Landau-Ginzburg model is stable in the sense of Definition 3.6, and the superpotential W_λ has infinitely many critical values, which form a lattice inside $\mathbb{C}$.*

Remark 1.5. The perturbations λ and μ play very different roles from the standpoint of the gauged Witten equations (1.3): λ is used to perturb the superpotential W_λ , while μ is used to perturb the moment map equation in (1.3) by changing $\vec{\delta} \in \mathfrak{g}$. $\diamond$

The first clause of Proposition 1.4 does apply to a high genus surface $\tilde{\Sigma}$, but only when $g(\tilde{\Sigma}) = 1$ and $\lambda \neq 0$, the gauged Landau-Ginzburg model we obtain is stable. Roughly speaking, Theorem 1.2 and 1.3 hold in general for any gauged Landau-Ginzburg model with a stable superpotential. The only difference in the infinite dimensional case is that the metric of M depends on the Sobolev completions, and we have to specify the correct norms involved in our estimates. The plot-line of proofs are summarized in the table below:

dim < ∞		dim = ∞
$G = \{e\}$	$G \neq \{e\}$	the Seiberg-Witten equations on $\mathbb{C} \times \Sigma$ or $\mathbb{H}_+^2 \times \Sigma$
Lemma 2.7	Theorem 5.1	Theorem 8.1 (Theorem 1.2)
Lemma 2.5	Theorem 6.1	Theorem 9.1 (Theorem 1.3)

For instance, when $M = \mathbb{C}, G = S^1$ and $W \equiv 0$, the gauged Witten equations (1.3) defined on $\mathbb{C} = \mathbb{R}_t \times \mathbb{R}_s$ come down to the vortex equations in the sense of [Tau80] (with

$\vec{\delta} = i/2$):

$$(1.5) \quad \begin{cases} *_2 iF_A + \frac{1}{2}|P|^2 &= \frac{1}{2}, \\ \partial_A P &= 0, \end{cases}$$

where $A = d + a$ is a $U(1)$ -connection on $\mathbb{C}$ and $P : \mathbb{C} \rightarrow \mathbb{C}$ is a complex valued function. This example is not stable in the sense of Definition 3.6, and Theorem 5.1 fails, by [Tau80, Theorem 1]. Nevertheless, Theorem 6.1 still holds, which states that the local energy functional $\mathcal{E}_{an}(A, P; \Omega_{n,R})$ defined by (1.4) has exponential decay as $R \rightarrow \infty$. This decay is also point-wise, so it recovers a theorem of Jaffe-Taubes:

Theorem 1.6 ([JT80], P.59, Theorem 1.4). *Let (A, P) be a smooth finite energy solution to the vortex equations (1.5). Given any $\epsilon > 0$, there exists $C = C(\epsilon, A, P) < \infty$ such that*

$$0 \leq *_2 iF_A = \frac{1}{2}(1 - |P|^2) < Ce^{-(1-\epsilon)\sqrt{t^2+s^2}}.$$

The proof of Theorem 6.1 uses the maximum principle and a Bochner-type formula (Lemma 6.4) for the energy density functional of (A, P) , which was first proved by Taubes in the special case of the vortex equations in [JT80, Proposition 6.1]. See Remark B.10 for more details.

The example above is more or less degenerate, since the superpotential W is identically zero. Usually one can take any Hamiltonian function to perturb the Floer equation (1.2), but when it is the imaginary part of a holomorphic function W , both (1.2) and (1.3) possess richer structures. Analytically this is encoded in the Bochner-type formula in Lemma 6.4. Geometrically this is related to Fukaya-Seidel categories, as we discuss in the next subsection and Section 2.

1.4. Fukaya-Seidel Categories and Floer Homology. As explained in the beginning of Subsection 1.3, each Landau-Ginzburg model (under additional conditions) should be assigned an A_∞ -category. To generalize this Picard-Lefschetz theory to the infinite dimensional gauged Landau-Ginzburg model in Proposition 1.4, one may work with infinite dimensional Lagrangian submanifolds, but there is another alternative: *can we define Lagrangian Floer cohomology without actually mentioning Lagrangian submanifolds?*

When $G = \{e\}$ and $\dim M < \infty$, this idea can be partly realized when the Lagrangian submanifold $\mathcal{L}_0$ is a Lefschetz thimble, i.e. the stable (or unstable) submanifold of a critical point $q \in \text{Crit}(\text{Re } W)$. Instead of a stripe $\mathbb{R}_t \times [0, 1]_s$, we look at J -holomorphic curves defined on the upper (or lower) half plane:

$$P : \mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s \rightarrow M$$

subject to the Floer equation (1.2) and Lagrangian boundary condition on $\mathbb{R}_t \times \{0\}$. The study of Fukaya-Seidel category of Landau-Ginzburg models via this approach has been pioneered by Haydys [Hay15] and Gaiotto-Moore-Witten [GMW15]. See also [FJY18, GMW17, KKS16]. We will give a brief sketch of their proposal in Section 2. The primary application in their cases is when

$$\begin{aligned} M &= SL(2, \mathbb{C}) \text{ connections on a closed 3-manifold } Y, \\ W &= \text{the complex valued Chern-Simons functional,} \end{aligned}$$

so the gauged Witten equations go over to the Haydys-Witten equations on the 5-manifold $\mathbb{C} \times Y$. This idea even goes back to the seminal paper [DT98] by Donaldson and Thomas, in which case

$$\begin{aligned} M &= \text{the space of } \bar{\partial}\text{-operators on a complex vector bundle } E \rightarrow \mathcal{Y}, \\ W &= \text{the holomorphic Chern-Simons functional,} \end{aligned}$$

for a compact Calabi-Yau 3-fold $\mathcal{Y}$, and one recovers the Spin(7)-instanton equation on $\mathbb{C} \times \mathcal{Y}$. The Seiberg-Witten equations should serve as the field test for their programs in higher dimensions, as the compactness does not cause an issue here. Although it is not pursued in the present work, it would be an interesting future direction to develop a bordered monopole Floer theory along this line.

Since we will **not** come back to Landau-Ginzburg models in the second paper of this series, we will explain in Subsection 2.3, how the monopole Floer homology of $(Y, \partial Y)$ fits into this general picture.

1.5. Organization. Section 2 is a continuation of this introduction, in which we explain a variant of Lagrangian Floer cohomology, following the work of Haydys [Hay15] and Gaiotto-Moore-Witten [GMW15]. This variant will allow generalization to the infinite dimensional setting. Instead of holomorphic strips, we look at “holomorphic upper half planes” with boundary condition only on the one side. It is intended to be a general overview; no proofs will be presented.

In Part 2, we study gauged Landau-Ginzburg models on a finite dimensional Kähler manifold. The focus is on the geometric insights that motivate definitions and proofs in the infinite dimensional setting. In Section 3 and 4, we define gauged Landau-Ginzburg models and study the gauged Witten equations on the upper half plane $\mathbb{H}_+^2$. Point-like solutions are solutions on the complex plane $\mathbb{C}$ with finite analytic energy. It is shown in Section 5 that point-like solutions are trivial provided that W is a stable superpotential. In Section 6, we prove an exponential decay result using a Bochner-type formula for the energy density functional.

In the last part of this paper, we introduce the Fundamental Landau-Ginzburg Model associated to a 2-torus Σ and prove Theorem 1.2 and 1.3 by generalizing Theorem 5.1 and 6.1 from Part 2.

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2. A GENERAL OVERVIEW

In this section, we explain a variant of Lagrangian Floer cohomology defined using holomorphic upper half planes, following [Hay15] and [GMW15]. To make it more consistent with the literature, we will work with **cohomology** instead of homology in this section. This variant serves as a toy model for the monopole Floer homology of a 3-manifold

$(Y, \partial Y)$ with torus boundary, as we sketched in the introduction. More details will be given in Subsection 2.3. This analogy is only used as an inspiration or a guideline for future research; it is not our intention to relate these two theories in a precise way.

Analytically, the perturbed Seiberg-Witten equations on $\mathbb{H}_+^2 \times \Sigma$ and the Floer equation (2.5) on $\mathbb{H}_+^2$ share many common features. Lemma 2.7 and Lemma 2.5 below are the counterparts of Theorem 1.2 and 1.3 in this toy model. Their proofs are postponed to the next part, where the corresponding results (Theorem 5.1 and 6.1) are stated and proved for the gauged Witten equations.

2.1. A Variant of Lagrangian Floer Cohomology. Recall that a Landau-Ginzburg Model is a pair (M, W) where

- (M, ω, J, g) is a non-compact complete Kähler manifold with complex structure J and Kähler metric $h := g - i\omega$. The underlying Riemannian metric is g , while ω is the symplectic form.
- $W : M \rightarrow \mathbb{C}$ is a holomorphic function, called the superpotential.

Since M is Kähler, J is parallel. Write $W = L + iH$ with $L = \operatorname{Re} W$ and $H = \operatorname{Im} W$. Then the Cauchy-Riemann equation $(dW)^{0,1} = 0$ comes down to

$$(2.1) \quad \nabla L + J\nabla H = 0,$$

i.e. the gradient ∇L is the Hamiltonian vector field of H .

A Landau-Ginzburg model (M, W) is said to be *Morse* if all critical points of L are non-generate. We always assume (M, W) is Morse in this section. Let $\operatorname{Crit}(L)$ be the set of critical points of L . Taking the covariant derivative of (2.1) yields:

$$(2.2) \quad \operatorname{Hess} L + J \circ \operatorname{Hess} H = 0.$$

Since $\operatorname{Hess} H$ is a symmetric operator and J is skew-symmetric, (2.2) implies that

$$(2.3) \quad J \circ \operatorname{Hess} L + \operatorname{Hess} L \circ J = 0.$$

For any $q \in \operatorname{Crit}(L)$, let $H_q^\pm \subset T_q M$ be the positive (negative) spectral subspace of $\operatorname{Hess}_q L$. Then (2.3) implies $J(H_q^\pm) = H_q^\mp$. In particular, the index of q is (n, n) if $\dim_{\mathbb{R}} M = 2n$. Let U_q and S_q be the unstable and stable submanifolds of q , i.e.

$$U_q = \{x \in M : \exists p : (-\infty, 0]_s \rightarrow M, \partial_s p + \nabla L = 0, p(0) = x, \lim_{s \rightarrow -\infty} p = q\},$$

$$S_q = \{x \in M : \exists p : [0, \infty)_s \rightarrow M, \partial_s p + \nabla L = 0, p(0) = x, \lim_{s \rightarrow \infty} p = q\}.$$

U_q and S_q are called Lefschetz thimbles of (W, q) .

Lemma 2.1. *U_q and S_q are Lagrangian submanifolds of (M, ω) .*

Lemma 2.1 may have been well-known for a long time, but its significance was only realized after the works of Donaldson [Don99] and Seidel [Sei03]; see [Sei08, Remark 16.10].

Assumption 2.2. To simplify our exposition in this section, we make the following assumptions.

- (M, ω) is an exact symplectic manifold, i.e. $\omega = d\theta_M$ is an exact 2-form; the primitive $\theta_M \in \Omega^1(M)$ is a smooth 1-form;
- L has a **unique** critical point on M ; let $\text{Crit}(L) = \{q\}$;
- The superpotential $W : M \rightarrow \mathbb{C}$ is not assumed to be proper. Instead, we assume that $|\nabla W|^2 : M \rightarrow [0, \infty)$ is a proper map. $\diamond$

Example 2.3. Let $M = (\mathbb{C}^n, z_1, \dots, z_n)$ and $W = z_1^2 + \dots + z_n^2$. The unique critical point q is the origin. $\diamond$

Take a pair of **compact** Lagrangian submanifolds $(\mathcal{L}_0, \mathcal{L}_1) \subset M$; assume they intersect transversely. Let $\text{CF}^*(\mathcal{L}_0, \mathcal{L}_1)$ be the $\mathbb{F}_2$ -vector space freely generated by the intersection $\mathcal{L}_0 \cap \mathcal{L}_1$:

$$\text{CF}^*(\mathcal{L}_0, \mathcal{L}_1) = \bigoplus_{y \in \mathcal{L}_0 \cap \mathcal{L}_1} \mathbb{F}_2 \cdot y.$$

The differential ∂ on $\text{CF}^*(\mathcal{L}_0, \mathcal{L}_1)$ is defined by counting J -holomorphic strips of Maslov index 1, subject to Lagrangian boundary conditions. They are smooth maps

$$P : \mathbb{R}_t \times [0, 1]_s \rightarrow M$$

satisfying the equation

$$(2.4) \quad \partial_t P + J \partial_s P = 0,$$

along with the boundary conditions $P(\cdot, 0) \in \mathcal{L}_0$ and $P(\cdot, 1) \in \mathcal{L}_1$. At this point, some assumptions on M and $(\mathcal{L}_0, \mathcal{L}_1)$ are required to ensure that ∂ is well defined after suitable perturbations, but let us skip these technical steps here.

Equation (2.4) can be perturbed by a Hamiltonian function. In our case, we use the imaginary part of W :

$$(2.5) \quad \partial_t P + J \partial_s P + \nabla H = 0.$$

The co-chain complex $\text{CF}^*(\mathcal{L}_0, \mathcal{L}_1)$ is then generated by Hamiltonian chords, which are smooth maps $p : [0, 1]_s \rightarrow M$ satisfying relations

$$p(0) \in \mathcal{L}_0, p(1) \in \mathcal{L}_1, J \partial_s p + \nabla H = 0.$$

Using the Cauchy-Riemann equation (2.1), the last condition is equivalent to

$$(2.6) \quad 0 = J(\partial_s p + \nabla L), s \in [0, 1]_s.$$

i.e. p is a downward gradient flowline of L .

One obtains the Lagrangian Floer cohomology $\text{HF}^*(\mathcal{L}_0, \mathcal{L}_1)$ by taking the cohomology of $(\text{CF}^*(\mathcal{L}_0, \mathcal{L}_1), \partial)$. The underlying idea is an infinite dimensional Morse theory. The configuration space is the loop space

$$\mathcal{C}^\infty([0, 1], M; \mathcal{L}_0, \mathcal{L}_1) := \{p : [0, 1]_s \rightarrow M : p \text{ smooth}, p(0) \in \mathcal{L}_0, p(1) \in \mathcal{L}_1\},$$

and the Morse function defined on $\mathcal{C}^\infty([0, 1], M; \mathcal{L}_0, \mathcal{L}_1)$ is the perturbed symplectic action functional:

$$\mathcal{A}_H(p) = \mathcal{A}(p) + \int_{[0, 1]_s} H \circ p(s) ds.$$

A path p is a critical point of $\mathcal{A}_H$ if and only if p is a Hamiltonian chord. For an $\mathbb{R}_t$ -family of paths $\{p_t\}_{t \in \mathbb{R}} \subset C^\infty([0, 1], M; L_0, L_1)$, it forms a down-ward gradient flowline of $\mathcal{A}_H$ precisely when $P(t, s) = p_t(s)$ solves the equation (2.5) on $\mathbb{R}_t \times [0, 1]_s$.

We would like to generalize this setup for certain non-compact Lagrangian submanifolds, in particular for the unstable and stable submanifolds $U = U_q$ and $S = S_q$. This can not be done in general; some asymptotic behaviors of $\mathcal{L}_i$ at infinity are required to make the cohomology group well-defined. We do not intend to make these conditions precise; instead, we give a few incomplete axiomatic properties:

- (1) There are two classes of non-compact Lagrangian submanifolds: the unstable type and the stable type. Denote them by $\mathcal{C}_{un}$ and $\mathcal{C}_{st}$ respectively.
- (2) $U_q \in \mathcal{C}_{un}$ and $S_q \in \mathcal{C}_{st}$, where U_q and S_q are Lefschetz thimbles of q .
- (3) For any $\mathcal{L}_U \in \mathcal{C}_{un}$ and $\mathcal{L}_S \in \mathcal{C}_{st}$, $L = \text{Re } W$ is bounded above on $\mathcal{L}_U$ and below on $\mathcal{L}_S$. $\mathcal{L}_U$ only intersect $\mathcal{L}_S$ within a compact region of M .
- (4) The Lagrangian Floer cohomology $\text{HF}^*(\mathcal{L}_U, \mathcal{L}_S)$ is well-defined, assuming transversality, by counting Hamiltonian chords and solutions of (2.5).

Our goal now is to give an alternative construction of $\text{HF}^*(\mathcal{L}_U, S_q)$ and $\text{HF}^*(S_q, \mathcal{L}_U)$. We focus on the first case. Suppose $\mathcal{L}_U$ is exact, so the primitive $\theta_M|_{\mathcal{L}_U} = dh$ for some real valued function $h : \mathcal{L}_U \rightarrow \mathbb{R}$. The chain group $\text{CF}^*(\mathcal{L}_U, S_q)$ is generated by the finite set $\mathcal{L}_U \cap S_q$. Each $x \in \mathcal{L}_U \cap S_q$ corresponds to a path $p : [0, \infty)_s \rightarrow M$ such that

$$\partial_s p + \nabla L = 0, \quad p(0) = x, \quad \lim_{s \rightarrow \infty} p(s) = q.$$

Comparing with (2.6), this motives us to look at the space

$$\mathcal{C}^\infty([0, \infty), M; \mathcal{L}_U) := \{p : [0, \infty)_s \rightarrow M : p \text{ smooth}, p(0) \in \mathcal{L}_U, \lim_{s \rightarrow \infty} p(s) = q\},$$

and consider the perturbed action functional:

$$\mathcal{A}_H(p) = -h(p(0)) + \int_{[0, \infty)_s} -p^* \theta_M + H \circ p(s) ds.$$

The differential $\widehat{\partial}$ is defined by counting solutions to (2.5) on the upper half plane $\mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s$ with the boundary condition:

$$(2.7) \quad P(\cdot, 0) \in \mathcal{L}_U, \quad \lim_{s \rightarrow \infty} P(t, s) = q.$$

To prove $\widehat{\partial}^2 = 0$ in this context, it is important to know a compactness result. Since we have omitted some assumptions, we state the result as a property instead of a proposition:

Property 2.4. *For a fixed unstable type Lagrangian submanifold $\mathcal{L}_U$, there exists a function $\eta : [0, \infty)_s \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim_{s \rightarrow \infty} \eta(s) = 0$, and for any solution $P : \mathbb{H}_+^2 \rightarrow M$ of (2.5) subject to the boundary condition (2.7), we have*

$$\sup_{t \in \mathbb{R}} d(P(t, s), q) \leq \eta(s),$$

where d is the distance function of the Riemannian metric g .

The upshot is that the convergence in the boundary condition (2.7) is also uniform for all possible solutions P . In fact, this decay is exponential. The next lemma is the toy model of Theorem 1.3 when $G = \{e\}$ and $\dim M < \infty$.

Lemma 2.5. *There exist constants $\epsilon(M, W), \zeta(M, W) > 0$ with following significance. For any solution $P_1 : \mathbb{H}_+^2 \rightarrow M$ such that $d(P_1(t, s), q) < \epsilon, \forall (t, s) \in \mathbb{H}_+^2$, we have*

$$d(P_1(t, s), q) < e^{-\zeta s}, \forall s \geq 0.$$

The exponent $\zeta(M, W)$ is determined by the first positive eigenvalue of $\text{Hess}_q L$. To derive the exponential decay from Lemma 2.5, set $P_1(t, s) = P(t, s + R)$ for some $R \gg 0$ in Property 2.4.

Remark 2.6. Apparently, Lemma 2.5 holds when P is time-independent, since in this case $\{P(t, s)\}_{s \in [0, \infty)_s}$ is a down-ward gradient flowline of $L = \text{Re} W$ for any fixed $t \in \mathbb{R}_t$ and L is a Morse function. It is not clear to the author whether $\{P(\cdot, s)\}_{s \in [0, \infty)}$ forms a down-ward gradient flowline (in the spatial direction) of some functional in general. $\diamond$

The proof of Property 2.4 relies on the following fact:

Lemma 2.7. *Let $P : \mathbb{C} \rightarrow M$ be a solution of (2.5) on the complex plane. If $P(t, s) \rightarrow q$ as $(t, s) \rightarrow \infty$, then $P \equiv q$.*

Remark 2.8. As we will see later, for a gauged Landau-Ginzburg model, Lemma 2.7 is not true in general. An appropriate condition that ensures Lemma 2.7 is a notion of stability, cf. Definition 3.6. $\diamond$

Remark 2.9. We will prove Lemma 2.7 and Lemma 2.5 in the context of gauged Landau-Ginzburg models in Part 2, cf. Theorem 5.1 and Theorem 6.1.

Property 2.4 is not true in general if $L = \text{Re} W$ has multiple critical points on M . We will address this issue for the Seiberg-Witten equations in the second paper, cf. [Wan20, Theorem 5.3]. $\diamond$

2.2. A Gluing Formula. In general, when $L = \text{Re} W$ is allowed to have multiple critical points (still finite), one may define the Fukaya-Seidel category of (M, W) , an A_∞ -algebra $\mathcal{A}$ generated by Lefschetz thimbles $\{U_q\}_{q \in \text{Crit}(L)}$ or $\{S_q\}_{q \in \text{Crit}(L)}$. Moreover, for each $\mathcal{L}_U \in \mathcal{C}_{un}$ and $\mathcal{L}_S \in \mathcal{C}_{st}$, we assign:

$$\begin{aligned} \mathcal{L}_U &\rightsquigarrow \text{an } A_\infty\text{-right module over } \mathcal{A}, \\ \mathcal{L}_S &\rightsquigarrow \text{an } A_\infty\text{-left module over } \mathcal{A}. \end{aligned}$$

Their underlying co-chain complexes are given respectively by

$$(2.8) \quad \bigoplus_{q \in \text{Crit}(L)} \text{CF}^*(\mathcal{L}_U, S_q) \text{ and } \bigoplus_{q \in \text{Crit}(L)} \text{CF}^*(U_q, \mathcal{L}_S).$$

A theorem of Seidel [Sei08, Corollary 18.27] then suggests a spectral sequence whose E_1 -page is

$$(2.9) \quad \bigoplus_{q \in \text{Crit}(L)} \text{HF}^*(\mathcal{L}_U, S_q) \otimes \text{HF}^*(U_q, \mathcal{L}_S),$$

abutting to $\mathrm{HF}^*(\mathcal{L}_U, \mathcal{L}_S)$ in the E_∞ -page. The underlying geometric picture was probably observed first by Donaldson and elaborated later in the monograph [GMW15] by Gaiotto-Moore-Witten. The Lagrangian Floer cohomology $\mathrm{HF}^*(\mathcal{L}_U, \mathcal{L}_S)$ is defined by counting holomorphic strips of width 1 with boundary conditions, but one can instead work with strips of width R for arbitrarily large $R > 0$ and let $R \rightarrow \infty$.

This neck-stretching picture makes more sense with the perturbed equation (2.5), in place of (2.4). In the limit $R \rightarrow \infty$, a holomorphic strip boils down to a few simpler pieces which can be analyzed by the A_∞ -structures of (2.8). For instance, a differential on the E_0 -page comes from a holomorphic upper half plane in $\mathrm{CF}^*(\mathcal{L}_U, S_q)$ and an intersection point in $U_q \cap \mathcal{L}_S$, or the other way around. This explains why the E_1 -page is a tensor product. Higher multiplications in the A_∞ -structure give rise to higher differentials.

If the A_∞ -algebra $\mathcal{A}$ turns out to be trivial, then the spectral sequence collapses after the E_1 -page. There is one simple geometric condition that yields this triviality:

Lemma 2.10. *If for any $e^{i\theta} \in S^1$, the down-ward gradient flowline equation*

$$\partial_s p(s) + \nabla(\mathrm{Re}(e^{i\theta} W)) = 0, \quad p : \mathbb{R}_s \rightarrow M$$

can only have constant solutions, then the Fukaya-Seidel category of (M, W) is trivial.

As we shall see in Corollary 10.4, this condition can be verified for the Seiberg-Witten equations when the metric g_Σ is flat and $\delta \neq 0$.

Although this geometric intuition is enlightening to keep in mind, the analytic foundation of the web-based formalism [GMW15] is still missing. To implement their proposal for the Seiberg-Witten equations remains a challenging problem.

2.3. Relations with Gauge Theory. As noted in Subsection 1.4, our goal is to define Lagrangian Floer cohomology without using boundary conditions, and we have achieved this goal partly by considering holomorphic upper half planes. To deal with the other boundary component, let us explain the origin of $\mathcal{L}_U$ and $\mathcal{L}_S$ in our primary applications.

Suppose we have a closed oriented 3-manifold Y separated by a homological essential 2-torus Σ :

$$Y = Y_L \#_\Sigma Y_R \text{ and } \mathrm{Im}(H^1(\Sigma; \mathbb{R}) \rightarrow H^1(Y; \mathbb{R})) \neq 0.$$

Let $M(\Sigma)$ be the infinite dimensional Kähler manifold associated to Σ in Proposition 1.4. The solution space of 3-dimensional Seiberg-Witten equations on Y_L , by the work of Nyugen [Ngu12], is infinite-dimensional, whose boundary values on Σ form an infinite dimensional Lagrangian submanifold of $M(\Sigma)$. Denote it by $\mathcal{L}_U$. One may construct the other piece $\mathcal{L}_S$ from the 3-manifold Y_R . It becomes clear that the monopole Floer cohomology $HM^*(Y_L)$ of Y_L , as we sketched in Subsection 1.2, is the analogue of

$$\bigoplus_{q \in \mathrm{Crit} L} \mathrm{HF}^*(\mathcal{L}_U, S_q)$$

defined using holomorphic upper half planes. By working with the extended 3-manifold

$$\hat{Y}_L = Y_L \coprod_{\Sigma} [0, \infty)_s \times \Sigma$$

and the 4-manifold $\mathbb{R}_t \times \hat{Y}_L$, we get rid of boundary conditions completely.

To see the relation with the knot Floer cohomology, recall the construction from [KM10]. For any knot $K \subset S^3$, take a meridian $m \subset S^3 \setminus K$. The link complement

$$Y_K := S^3 \setminus N(m \cup K).$$

is a 3-manifold with boundary $\partial Y_K = \Sigma_1 \cup \Sigma_2$. Using any orientation reversing diffeomorphism $\varphi : \Sigma_1 \rightarrow \Sigma_2$, we close up the boundaries of Y_K and obtain a closed 3-manifold Y'_K . Then define

$$KHM^*(S^3, K) := HM^*(Y'_K).$$

The latter group $HM^*(Y'_K)$ is the monopole Floer cohomology of Y'_K defined using a suitable non-exact perturbation. It is shown in [KM10] that $KHM^*(S^3, K)$ is independent of the isotopy class of φ up to isomorphisms.

On the other hand, we can take $HM^*(Y_K, \partial Y_K)$ as a candidate of knot Floer homology, which is defined by attaching cylindrical ends to Y_K . To see its relation with $KHM^*(S^3, K)$, regard φ as gluing two pieces:

$$Y_K \text{ and } [-R, R]_s \times \Sigma_1.$$

As $R \rightarrow \infty$, we stretch the metric in a neighborhood of Σ_1 in Y'_K , in analogy of the neck stretching picture involved in the spectral sequence (2.9). In fact, Lemma 2.10 applies in this case, so one may recover $KHM^*(S^3, K)$ from $HM^*(Y_K, \partial Y_K)$ by a self-gluing formula.

As an ending remark for this expository section, the monopole Floer homology of 3-manifolds with torus boundary to be defined in the second paper [Wan20] only gives the underlying co-chain complexes (2.8). The construction of A_∞ -structures is left as an interesting future project. The discussion of this subsection is mostly inspirational and philosophical. It will require substantial new ideas to fully realize this picture.

Part 2. Gauged Landau-Ginzburg Models

In this part, we generalize the setup from the previous section by allowing an abelian group G act on the Kähler manifold M . In this case, we obtain the gauged Witten equations (4.1) as the replacement of the Floer equation (2.5). Theorem 5.1 and Theorem 6.1 are analogue of Theorem 1.2 and Theorem 1.3 in the finite dimensional case; their proofs are presented in Section 5 and Section 6 respectively.

3. DEFINITIONS AND EXAMPLES

We generalize the setup from the previous section and introduced the notion of gauged Landau-Ginzburg Models.

Definition 3.1. The quadruple (M, W, G, ρ) is called an (abelian) gauged Landau-Ginzburg model if

- (1) (M, ω, J, g) is a complete non-compact Kähler manifold with complex structure J and Kähler metric $h := g - i\omega$; g is the underlying Riemannian metric, and ω the symplectic form.
- (2) (G, ρ) is a compact abelian Lie group acting on M holomorphically and isometrically, i.e. for any $g \in G$, the action $\rho(g) : M \rightarrow M$ is a holomorphic isometry;
- (3) (G, ρ) is an Hamiltonian group action, and it admits a moment map:

$$\mu : M \rightarrow \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G . Since G is abelian, μ is G -invariant;

- (4) The action of (G, ρ) extends to an action of the complex group $(G_{\mathbb{C}}, \rho_{\mathbb{C}})$. $\rho_{\mathbb{C}} : G_{\mathbb{C}} \times M \rightarrow M$ is holomorphic. $\rho_{\mathbb{C}}$ does not preserve the Riemannian metric g in general.
- (5) $W : M \rightarrow \mathbb{C}$ is a $G_{\mathbb{C}}$ -invariant holomorphic function called the superpotential. Write $W = L + iH$ with $L = \operatorname{Re} W$ and $H = \operatorname{Im} W$. $\diamond$

Again, we assume (M, ω) is an exact symplectic manifold, i.e. $\omega = d\theta_M$ for some $\theta_M \in \Omega^1(M)$. For any $\xi \in \mathfrak{g}$, let $\tilde{\xi}$ be the vector field on M induced from the group action (G, ρ) :

$$\tilde{\xi}(p) = \left. \frac{d}{dt} \rho(e^{t\xi})p \right|_{t=0}.$$

We adopt a non-standard (sign) convention of the moment map in this paper:

$$(3.1) \quad \iota(\tilde{\xi})\omega = -d\langle \mu, \xi \rangle_{\mathfrak{g}}.$$

Since $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, (3.1) is equivalent to

$$(3.2) \quad \langle \nabla \mu, \xi \rangle_{\mathfrak{g}} = \nabla \langle \mu, \xi \rangle_{\mathfrak{g}} = -J\tilde{\xi},$$

where $\nabla \mu \in \Gamma(M, TM \otimes \mathfrak{g})$ is a $\mathfrak{g}$ -valued vector field on M and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denotes a bi-invariant metric of $\mathfrak{g}$.

Example 3.2. Let $G = S^1, G_{\mathbb{C}} = \mathbb{C}^*, M = \mathbb{C}$ and $W \equiv 0$. The group action is the standard complex multiplication. Using our sign convention (3.2), the moment map is $\mu(z) = \frac{i}{2}|z|^2$ for $z \in \mathbb{C}$. $\diamond$

Example 3.3. Let $G = S^1$, $G_{\mathbb{C}} = \mathbb{C}^*$, $M = S^2$ and $W \equiv 0$. Identify M with $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The action $\rho_{\mathbb{C}}$ is the same as in Example 3.2. $\{0, \infty\}$ is the fixed point set of $\rho_{\mathbb{C}}$. $\diamond$

Example 3.4. Let $G = S^1$, $G_{\mathbb{C}} = \mathbb{C}^*$, $M = (\mathbb{C}^2, x, y)$ and $W(x, y) = xy$. W becomes $G_{\mathbb{C}}$ -invariant if we set the action $\rho_{\mathbb{C}}$ as

$$\rho_{\mathbb{C}}(u)(x, y) = (ux, u^{-1}y),$$

for any $u \in \mathbb{C}^*$. The moment map is $\mu(x, y) = \frac{i}{2}(|x|^2 - |y|^2)$. $\diamond$

Just as Assumption 2.2, we wish (W, μ) to satisfy some good properties. The replacement of the Morse condition for gauged Landau-Ginzburg models is a notion of stability. There are two possible candidates; the second one turns out to be more useful.

Definition 3.5. A regular value $\vec{\delta} \in \mathfrak{g}$ of the moment map μ is called W -stable if the restriction of L on

$$\mu^{-1}(\vec{\delta})/G$$

has a unique critical point and it is non-degenerate. Thus, before taking the quotient, there exists a unique critical orbit O_* of L such that $O_* \subset \mu^{-1}(\vec{\delta})$. $\diamond$

$\vec{\delta}$ is a regular value of the moment map μ if and only if the infinitesimal action of G on the fiber $\mu^{-1}(\vec{\delta})$ is free, so the quotient space is a genuine manifold (or orbifold in general).

However, from the viewpoint of Remark 2.8, having a W -stable regular value is not good enough (see Example 5.8). Note that the critical set $\text{Crit}(L) = \{x \in M : \nabla L(x) = 0\}$ is closed and $G_{\mathbb{C}}$ -invariant.

Definition 3.6. The superpotential W is called stable if $\text{Crit}(L)$ contains a unique free $G_{\mathbb{C}}$ -orbit and L is Morse-Bott, i.e. for any $x \in \text{Crit}(L)$, $\ker \text{Hess}_x L$ is precisely $T_x(G_{\mathbb{C}} \cdot x)$. $\diamond$

In fact, any regular value $\vec{\delta} \in \text{Im } \mu \subset \mathfrak{g}$ is a W -stable if W is stable.

In Example 3.2, any $\delta \in i[0, \infty)$ is W -stable, but W itself is not a stable superpotential. Indeed, $\text{Crit}(L) = M$, and it contains two $\mathbb{C}^*$ -orbits.

In Example 3.4, L has a unique critical point $q = (0, 0) \in \mathbb{C}^2$. It is not W -stable. If instead we let $G = \{e\}$, W is stable, and $\vec{\delta} = 0$ is W -stable.

Let us provide a more interesting example.

Example 3.7 (The Fundamental Toy Model). Let $G = S^1$, $G_{\mathbb{C}} = \mathbb{C}^*$, $M = (\mathbb{C}^3, x, y, b)$ and $W_{\lambda}(x, y, b) = (xy - \lambda)b$, where $\lambda \in \mathbb{C}$ is a fixed parameter. $\rho_{\mathbb{C}}$ is defined by

$$\rho_{\mathbb{C}}(u)(x, y, b) = (ux, u^{-1}y, b).$$

for any $u \in \mathbb{C}^*$. The moment map is $\mu(x, y, b) = \frac{i}{2}(|x|^2 - |y|^2)$ and $\nabla L = (\bar{y}\bar{b}, \bar{x}\bar{b}, \bar{x}\bar{y} - \bar{\lambda})$.

If $\lambda \neq 0$, then $\text{Crit}(L) = \{b = 0, xy = \lambda\}$ containing a unique $\mathbb{C}^*$ -orbit, and the superpotential W is stable.

If $\lambda = 0$, then $\text{Crit}(L) = A_{xy} \cup A_{xb} \cup A_{yb}$ where

$$A_{xy} = \{x = 0, y = 0\}, \text{ etc.}$$

So W is not stable. $\vec{\delta} \in i\mathbb{R}$ is W -stable if and only if $\vec{\delta} \neq 0$. For instance, take $\vec{\delta} \in i \cdot (0, \infty)$. If $(x, y, b) \in \mu^{-1}(\vec{\delta})$, then $x \neq 0$, so

$$\text{Crit}(L) \cap \mu^{-1}(\vec{\delta}) = \{(x, 0, 0) : \frac{i}{2}|x|^2 = \vec{\delta}\} \subset A_{yb},$$

which contains a single free G -orbit. Moreover, we compute $\text{Hess } L$ at $(x, y, b) \in M$:

$$\text{Hess } L \begin{pmatrix} x' \\ y' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \bar{b} & \bar{y} \\ \bar{b} & 0 & \bar{x} \\ \bar{y} & \bar{x} & 0 \end{pmatrix} \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{b}' \end{pmatrix},$$

so L is Morse-Bott away from the origin. Note that the $\mathbb{C}^*$ -orbit of $(x, 0, 0)$ is not closed. Its closure contains the origin. $\diamond$

4. THE GAUGED WITTEN EQUATIONS

In this section, we introduce gauged Witten equations, the notion of analytic energy and explain its relation with down-ward gradient flow of the gauged action functional $\mathcal{A}_H$. This serves as a toy model for the Floer theory to be studied in the second paper [Wan20] of this series. Some lemmas are stated and proved only for inspirations; they are not quite related to the proof of Theorem 1.2 and 1.3 in the end.

4.1. The Gauged Action Functional. Let $\vec{\delta} \in \mathfrak{g}$ be a W -stable regular value (in the sense of Definition 3.5) of the moment map $\mu : M \rightarrow \mathfrak{g}$ and $\mathcal{L}_U \subset \mu^{-1}(\vec{\delta})$ be an G -invariant unstable-type Lagrangian sub-manifold of M . Since $\vec{\delta}$ is W -stable, $\mu^{-1}(\vec{\delta}) \cap \text{Crit}(L)$ contains a unique G -orbit O_* . Choose a reference point $q \in O_*$.

Assumption 4.1. Let us first summarize the assumptions we make in this section in order to set up a Floer theory formally :

- the Kähler form ω is exact, i.e. $\omega = d\theta_M$ for some $\theta_M \in \Omega^1(M)$;
- since $\omega|_{\mathcal{L}_U} = 0$, the primitive 1-form θ_M is closed on $\mathcal{L}_U$. For convenience, assume $\theta_M = dh$ is exact on $\mathcal{L}_U$;
- $\vec{\delta} \in \mathfrak{g}$ is a W -stable regular value; $\diamond$

Let $Y = [0, \infty)_s$ and $X = \mathbb{R}_t \times Y = \mathbb{H}_+^2$. Consider a smooth map $P : X \rightarrow M$ and a connection $A = d + a$ of the trivial principal G -bundle Q over X :

$$Q = X \times G.$$

Write the connection 1-form a as $a_t dt + a_s ds$ with $a_t, a_s \in \Gamma(\mathbb{H}_+^2, \mathfrak{g})$. The smooth map P can be differentiated co-variantly with respect to A :

$$\nabla_V^A P := V \cdot P + \tilde{a}(V)$$

for any tangent vector $V \in TX$. Here, $\tilde{a}(V)$ is the induced tangent vector of $a(V) \in \mathfrak{g}$.

We are interested in the gauged Witten equations on $X = \mathbb{H}_+^2$ with boundary values in $\mathcal{L}_U$:

$$(4.1) \quad \begin{cases} - * _2 F_A + \mu &= \vec{\delta}, \\ \nabla_{\partial_t}^A P + J \nabla_{\partial_s}^A P + \nabla H &= 0, \\ P(t, 0) &\in \mathcal{L}_U. \end{cases}$$

The first equation is a moment map constraint. The second one is the J -holomorphic curve equation perturbed by the Hamiltonian $H = \text{Im } W$. When $H \equiv 0$, this reduces to the symplectic vortex equation. The gauged Witten equations (4.1) can be viewed as a down-ward gradient flowline equation in an infinite dimensional space, as we explain now.

For either $Z = Y$ or X , let $\mathcal{A}(Z) = d + \Gamma_0(Z, iT^*Z \otimes \mathfrak{g})$ be the space of smooth connections with decay in the spatial direction. Here,

$$(4.2) \quad \Gamma_0(Z, iT^*Z \otimes \mathfrak{g}) = \{a \in \mathcal{C}^\infty(Z, iT^*Z \otimes \mathfrak{g}) : \lim_{s \rightarrow \infty} a = 0 \text{ and } \langle a, ds \rangle = 0 \text{ at } s = 0\}.$$

A smooth map $p : Z \rightarrow M$ can be viewed as a section of the trivial M bundle over Z :

$$\widetilde{M} = Z \times M = (Z \times G) \times_G M.$$

Consider the space of smooth sections of $\widetilde{M} \rightarrow Z$ subject to the Lagrangian boundary condition and a decay condition at infinity:

$$\Gamma_0(Z, \widetilde{M}; \mathcal{L}_U) = \{p : Z \rightarrow \widetilde{M} : p(0) \in \mathcal{L}_U, \lim_{s \rightarrow \infty} p(s) = q\}.$$

A gauge transformation must converge to the identity element e of G as $s \rightarrow \infty$:

$$\mathcal{G}(Z) := \text{Map}_0(Z, G) = \{u : Y \rightarrow G : \lim_{s \rightarrow \infty} u = e \in G\}.$$

The configuration space is $\mathcal{C}(Z) = \mathcal{A}(Z) \times \Gamma_0(Z, \widetilde{M}; \mathcal{L}_U)$ with $\mathcal{G}(Z)$ acting on by the formula:

$$u(A, p) = (A - u^{-1}du, u \cdot p).$$

Definition 4.2. The gauged action functional $\mathcal{A}_H$ is defined over $\mathcal{C}(Y)$ with $Y = [0, +\infty)_s$ as:

$$(4.3) \quad \mathcal{A}_H(d + a, p) = -h(p(0)) - \int_Y p^* \theta_M + \int_Y H \circ p(s) ds + \langle a, \vec{\delta} - \mu \circ p \rangle_{\mathfrak{g}}.$$

where $\langle a, \vec{\delta} - \mu \circ p \rangle_{\mathfrak{g}} = \langle a_s, \vec{\delta} - \mu \circ p \rangle_{\mathfrak{g}} ds$ is understood as an 1-form on Y and $\theta_M = dh$ on $\mathcal{L}_U$. $\diamond$

For any $\gamma = (A, p) \in \mathcal{C}(Y)$, a tangent vector $(\delta a, \delta p)$ in $T_\gamma \mathcal{C}(Y)$ consists of a smooth form $\delta a \in \Gamma_0(Y, iT^*Y \otimes \mathfrak{g})$ and a vector field δp along the image $p(Y)$:

$$\delta p \in \Gamma_0(Y, p^*TM; \mathcal{L}_U).$$

The tangent space $T_\gamma \mathcal{C}(Y)$ inherits a $\mathcal{G}$ -invariant L^2 -inner product from the Riemannian metric g of M , with respect to which we compute the formal-gradient of $\mathcal{A}_H$:

Proposition 4.3. $\text{grad } \mathcal{A}_H(d + a, p) = (\vec{\delta} - \mu \circ p, J \nabla_{\partial s}^A p + \nabla H).$

Proof. Let $P : [0, 1]_t \times Y \rightarrow M$ be a smooth map such that $P(0, s) = p(s)$, $\partial_t P(0, s) = \delta p(s)$ and $\lim_{s \rightarrow \infty} P(\cdot, s) = q$. Then $\gamma_t = (d + a + t\delta a, P(t, \cdot))$ is a smooth variation of $\gamma_0 = \gamma$. Note that

$$(4.4) \quad \int_{[0, t] \times Y} P^* \omega = \int_{[0, t] \times Y} dP^* \theta_M = h(P(t, 0)) - h(P(0, 0)) + \int_{\{t\} \times Y} P^* \theta_M - \int_{\{0\} \times Y} P^* \theta_M,$$

but also

$$\int_{[0,t] \times Y} P^* \omega = \int_{[0,t] \times Y} \omega(\partial_t P, \partial_s P) dt' ds = - \int_{[0,t]} dt' \int_Y g(\partial_t P, J \partial_s P) ds.$$

Now consider the first variation of (4.3) along a tangent vector $(\delta p, \delta a)$:

$$\begin{aligned} \frac{d}{dt} \mathcal{A}_H(\gamma_t)|_{t=0} &= \int_Y g(\delta p, J \partial_s P + \nabla H) ds + \langle \delta a, \vec{\delta} - \mu \circ p \rangle_{\mathfrak{g}} - \langle \nabla \mu, \delta p \otimes a \rangle_{\mathfrak{g}}. \\ &= \int_Y g(\delta p, J \nabla_{\partial_s}^A P + \nabla H) ds + \langle \delta a, \vec{\delta} - \mu \circ p \rangle_{\mathfrak{g}}, \end{aligned}$$

where we used the relation $\nabla_{\partial_s}^A P = \partial_s P + J \langle \nabla \mu, a_s \rangle_{\mathfrak{g}}$. $\square$

Proposition 4.4. $\mathcal{A}_H$ is $\mathcal{G}(Y)$ -invariant.

Proof. Since elements of $\mathcal{G}(Y)$ are subject to the boundary condition $\lim_{s \rightarrow \infty} u = e$, $\mathcal{G}(Y)$ is contractible. It suffices to consider the infinitesimal action. The Lie algebra of $\mathcal{G}$ is

$$\text{Lie}(\mathcal{G}) = \Gamma_0(Y, \mathfrak{g}) = \{\xi : Y \rightarrow \mathfrak{g} : \lim_{s \rightarrow \infty} \xi(s) = 0\}$$

For $\xi \in \text{Lie}(\mathcal{G})$, the tangent vector generated at $\gamma \in \mathcal{C}(Y)$ is

$$(4.5) \quad \mathbf{d}_\gamma(\xi) := (-\partial_s \xi, \tilde{\xi}) = (-\partial_s \xi, J \langle \nabla \mu, \xi \rangle).$$

It suffices to verify this vector is L^2 -orthogonal to $\text{grad } \mathcal{A}_H$. For any path $p \in \Gamma_0(Y, \widetilde{M}; \mathcal{L}_U)$, $\vec{\delta} - \mu \circ p(s) = 0$ for $s = 0$ and ∞ . Hence, the boundary terms involved in the integration by parts below vanish:

$$\int_Y \langle \vec{\delta} - \mu \circ p, -\partial_s \xi \rangle = - \int_Y \langle \partial_s(\mu \circ p), \xi \rangle = - \int_Y \langle \nabla \mu, \partial_s p \otimes \xi \rangle,$$

On the other hand, we use Lemma B.1 (5)(6) to compute:

$$\int_Y \langle J \nabla_{\partial_s}^A p + \nabla H, J \langle \nabla \mu, \xi \rangle \rangle = \int_Y \langle \nabla \mu, \partial_s p \otimes \xi \rangle. \quad \square$$

Remark 4.5. In the expression (4.3), the first two terms come from the usual action functional, motivated by the integration by parts (4.4). The third part comes from the Hamiltonian perturbation. The last one is added by requiring $\mathcal{A}_H$ to be gauge-invariant. $\diamond$

Hence, the gauged Witten equations (4.1) can be formally written as

$$\partial_t \gamma_t + \text{grad } \mathcal{A}_H(\gamma_t) = 0$$

if $a_t \equiv 0$ and $\gamma_t = (d + a_s(t, \cdot) ds, P(t, \cdot)) \in \mathcal{C}(Y)$. There is a classical notion of analytic energy associated to any down-ward gradient flow equation:

$$\begin{aligned} \infty > \mathcal{E}_{an}(\{\gamma_t\}) &= - \lim_{t \rightarrow \infty} \mathcal{A}_H(\gamma_t) + \lim_{t \rightarrow -\infty} \mathcal{A}_H(\gamma_t) = \int_{\mathbb{R}_t} \langle -\partial_t \gamma_t, \text{grad } \mathcal{A}_H(\gamma_t) \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}_t} |\partial_t \gamma_t|^2 + |\text{grad } \mathcal{A}_H(\gamma_t)|^2 \geq 0. \end{aligned}$$

This formula is only valid when A is in the temporal gauge, i.e. when $a_t \equiv 0$. On the contrary, the gauged Witten equations (4.1) are invariant under the larger gauge group $\mathcal{G}(X)$ (recall that $X = \mathbb{R}_t \times Y$). In fact, the left hand side of (4.1) defines a $\mathcal{G}(X)$ -equivariant map:

$$\mathfrak{F} : \mathcal{C}(X) \rightarrow \Gamma_0(X, Q \times_G (TM \oplus \mathfrak{g})),$$

called the gauged Witten map.

Definition 4.6. Let $X = \mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s$. For any $(A, P) \in \mathcal{C}(X)$, let $T = \nabla_{\partial_t}^A P$, $S = \nabla_{\partial_s}^A P$ and $F = - *_2 F_A \circ P \in \Gamma(X, \mathfrak{g})$. The analytic energy of (A, P) is defined as

$$(4.6) \quad \mathcal{E}_{an}(A, P) = \int_X |T|^2 + |JS + \nabla H|^2 + |F|^2 + |\vec{\delta} - \mu|^2. \quad \diamond$$

With this convention of notations understood, the equation (4.1) takes a more compact form:

$$(4.7a) \quad F + \mu = \vec{\delta},$$

$$(4.7b) \quad T + JS + \nabla H = 0,$$

$$(4.7c) \quad P(t, 0) \in \mathcal{L}_U.$$

We are interested in the moduli space of solutions of (4.1) with finite analytic energy. One may impose a gauge-fixing condition, produce an elliptic theory and finally construct a Morse complex in this context. However, we will only carry out the proof in the infinite dimensional setting for the Seiberg-Witten equations.

4.2. The Extended Hessian. Although we will only get into linear analysis in the second paper [Wan20] of this series, it is enlightening to first work out the extended Hessian of the gauged action functional $\mathcal{A}_H$ here. The discussion below will be used in [Wan20, Section 11] where we compute the essential spectrum of the extended Hessian for the perturbed Chern-Simons-Dirac functional on $\hat{Y}$.

At any $\gamma = (A, p) \in \mathcal{C}(Y)$, the linearized gauge action

$$\mathbf{d}_\gamma : \text{Lie}(\mathcal{G}) = \Gamma_0(Y, \mathfrak{g}) \rightarrow T_\gamma \mathcal{C}(Y)$$

defined by the formula (4.5) has a formal adjoint:

$$\begin{aligned} \mathbf{d}_\gamma^* : T_\gamma \mathcal{C}(Y) &\rightarrow \Gamma_0(Y, \mathfrak{g}) \\ ((\delta a_s)ds, \delta p) &\mapsto \partial_s(\delta a_s) + \langle J\nabla\mu, \delta p \rangle. \end{aligned}$$

By formally linearizing the expression in Proposition 4.3, we obtain the Hessian of $\mathcal{A}_H$ at γ :

$$\begin{aligned} \mathcal{D}_\gamma \mathcal{A}_H : T_\gamma \mathcal{C}(Y) &\rightarrow T_\gamma \mathcal{C}(Y) \\ ((\delta a_s)ds, \delta p) &\mapsto (-\langle \nabla\mu, \delta p \rangle, J(\partial_s \delta p) - \langle \nabla\mu, a_s \rangle_{\mathfrak{g}} + \text{Hess } H(\delta p)). \end{aligned}$$

The upshot is that these operators can be combined to form a larger operator, the extended Hessian of $\mathcal{A}_H$, which is essential self-adjoint:

$$\widehat{\text{Hess}}_\gamma = \begin{pmatrix} 0 & \mathbf{d}_\gamma^* \\ \mathbf{d}_\gamma & \mathcal{D}_\gamma \mathcal{A}_H \end{pmatrix} : L_1^2(Y, \mathfrak{g} \oplus (T^*Y \otimes \mathfrak{g}) \oplus p^*TM) \rightarrow L^2(\mathbb{R}, \mathfrak{g} \oplus (T^*Y \otimes \mathfrak{g}) \oplus p^*TM),$$

and it is cast into the form $\sigma(\partial_s + \hat{D}_{p(s)})$ with

$$(4.8) \quad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & J \end{pmatrix} \text{ and } \hat{D}_{p(s)} = \begin{pmatrix} 0 & 0 & \langle \nabla \mu, \cdot \rangle_{T_{p(s)}M} \\ 0 & 0 & \langle J \nabla \mu, \cdot \rangle_{T_{p(s)}M} \\ \langle \nabla \mu, \cdot \rangle_{\mathfrak{g}} & \langle J \nabla \mu, \cdot \rangle_{\mathfrak{g}} & \text{Hess } L \end{pmatrix}.$$

where we identify $T^*Y \otimes \mathfrak{g}$ with $\mathfrak{g}$ by omitting the form ds . The operator $\hat{D}$ is a self-adjoint bundle endomorphism over the vector bundle

$$\widehat{TM} := \mathfrak{g} \oplus \mathfrak{g} \oplus TM \rightarrow M.$$

Moreover, σ acts on $\widehat{TM}$ as an almost complex structure and anti-commutes with $\hat{D}$, i.e.

$$\sigma^2 = -\text{Id}, \quad \sigma \hat{D} + \hat{D} \sigma = 0.$$

The operator $\hat{D}$ is tied to the stability of W by the following observation:

Lemma 4.7. *The super-potential W is stable in the sense of Definition 3.6 if and only if $\hat{D}_q$ is invertible for any critical point $q \in \text{Crit}(L)$ and $\text{Crit}(L)$ contains a unique free $G_{\mathbb{C}}$ -orbit.*

These structures of the extended Hessians form the basis of linear analysis in [Wan20, Section 11]. As a preview, the essential spectrum of $\widehat{\text{Hess}}_{\gamma}$ will be

$$(-\infty, -\lambda_1] \cup [\lambda_1, +\infty)$$

where λ_1 is the first non-negative eigenvalue of $\hat{D}_q$. In particular, $\widehat{\text{Hess}}_{\gamma}$ is Fredholm if and only if $\lambda_1 > 0$.

Finally, we end this section by a remark on the domain of $\widehat{\text{Hess}}_{\gamma}$. To make it self-adjoint, a section $(f, (\delta a_s)ds, \delta p)$ in the domain must satisfy the boundary condition:

$$(f(0), \delta a_s(0), \delta p(0)) \in \mathfrak{g} \oplus \{0\} \oplus p^* T\mathcal{L}_U \text{ at } s = 0,$$

which is a Lagrangian subspace with respect to σ . This is the reason why we have imposed the boundary condition

$$\langle a, ds \rangle = 0 \text{ at } s = 0,$$

in the definition (4.2) of $\mathcal{A}(Z)$. Otherwise, $\mathbf{d}_{\gamma}^*$ is not the formal adjoint of $\mathbf{d}_{\gamma}$.

5. POINT-LIKE SOLUTIONS

In this section, we study finite energy solutions of (4.1) on the complex plane $\mathbb{C}$, the so-called point-like solutions in terms of [GMW15, Section 14.1]. Assuming W is a stable superpotential, we will prove that all point-like solutions are trivial, i.e. they are gauge equivalent to the constant solutions. Interesting solutions may occur if W is not stable, cf. Example 5.8.

Let $P : \mathbb{C} \rightarrow M$ be a smooth map and A be a smooth connection in the trivial principal G -bundle $\mathbb{C} \times G \rightarrow \mathbb{C}$. We shall frequently use the abbreviations from Definition 4.6. The main result of this section is the following:

Theorem 5.1. *Suppose (M, W, G, ρ) is a gauged Landau-Ginzburg model and W is stable in the sense of Definition 3.6. Take any $\vec{\delta} \in \text{Im } \mu \subset \mathfrak{g}$. Suppose (A, P) is a solution of the gauged Witten equations*

$$(5.1) \quad \begin{cases} - * F_A + \mu = \vec{\delta}, \\ \nabla_{\partial_t}^A P + J \nabla_{\partial_s}^A P + \nabla H = 0. \end{cases}$$

on $\mathbb{C}$ with $\mathcal{E}_{an}(A, P) < \infty$, and (A, P) is subject to the boundary condition

$$(5.2) \quad \lim_{(t,s) \rightarrow \infty} P(t + is) \rightarrow q,$$

then (A, P) is gauge equivalent to the constant solution $(A_0 = d, P \equiv q)$. Here $q \in O_* = \mu^{-1}(\vec{\delta}) \cap \text{Crit}(L)$ is an arbitrary base point.

The proof is based on an interesting observation. Since W is holomorphic and P is “ J -holomorphic”, it is reasonable to ask if the composition:

$$W \circ P : \mathbb{C} \xrightarrow{P} M \xrightarrow{W} \mathbb{C}.$$

is still holomorphic. In fact, we have

Lemma 5.2. *If (A, P) is a solution to the gauged Witten equations (5.1) on $\mathbb{C}$, then*

$$\bar{\partial}(W \circ P) := (\partial_t + i\partial_s)(W \circ P) = -i|\nabla H|^2,$$

Proof. By the Cauchy-Riemann equation $\nabla L = -J\nabla H$ and (5.1), we have

$$\bar{\partial}(W \circ P) = \langle \nabla L + i\nabla H, \nabla_{\partial_t}^A P + J \nabla_{\partial_s}^A P \rangle = -i|\nabla H|^2. \quad \square$$

Remark 5.3. When $A = d$ is the trivial connection and $\nabla_{\partial_t}^A P \equiv 0$, $P(t, \cdot)$ is a downward gradient flowline of L . In this case, this lemma recovers the usual identity:

$$\partial_s(L \circ P) = -|\nabla L|^2.$$

$P(t, \cdot)$ is also a Hamiltonian flow, so $\partial_s(H \circ P) = 0$. $\diamond$

We also need a more useful notion of energy:

Lemma 5.4. *Under the conditions of Theorem 5.1, define*

$$(5.3) \quad \mathcal{E}_{an}(A, P; \mathbb{C}) = \int_{\mathbb{C}} |\nabla_A P|^2 + |\nabla H|^2 + |F|^2 + |\vec{\delta} - \mu|^2.$$

Then $\mathcal{E}_{an}(A, P, \mathbb{C}) = \mathcal{E}_{an}(A, P) < \infty$.

Proof. Using the Cauchy-Riemann equation (2.1), we have

$$\begin{aligned} \int_{\mathbb{C}} |JS + \nabla H|^2 &= \int_{\mathbb{C}} |S + \nabla L|^2 = \int_{\mathbb{C}} |S|^2 + |\nabla L|^2 + \lim_{t' \rightarrow \infty} \lim_{s' \rightarrow \infty} \int_{[-t', t'] \times [-s', s']} 2\langle S, \nabla L \rangle \\ &= \int_{\mathbb{C}} |S|^2 + |\nabla L|^2 + \lim_{t' \rightarrow \infty} \lim_{s' \rightarrow \infty} \int_{[-t', t']} 2(L \circ P(t', s') - L \circ P(t', -s')). \end{aligned}$$

By (5.2), the boundary term tends to zero as $s' \rightarrow \infty$, so $\mathcal{E}_{an}(A, P; \mathbb{C}) = \mathcal{E}_{an}(A, P)$. $\square$

Lemma 5.5. *Under the assumption of Theorem 5.1, $\nabla L \equiv 0$, so $P(z) \in \text{Crit}(L)$ for any $z = t + is \in \mathbb{C}$ and $W \circ P$ is a constant function on $\mathbb{C}$.*

Proof. As W is Morse-Bott, for some G -invariant neighborhood Ω of $O_* \subset M$ and $C > 0$, the estimate

$$(5.4) \quad |W(x) - W(q)| \leq C|\nabla H(x)|^2.$$

holds for any $x \in \Omega$. By the boundary condition (5.2), for a large constant $R(\Omega) > 0$, $P(z) \in \Omega$ if $|z| > R$. As a result,

$$(5.5) \quad |W \circ P(z) - W(q)| \leq C|\nabla H(P(z))|^2.$$

when $|z| > R$. Write $(W \circ P)(z) - W(q) = U + iV$ with U, V real. Then Lemma 5.2 implies that

$$\partial_t U - \partial_s V = 0, \partial_t V + \partial_s U = -|\nabla H|^2 \leq 0.$$

Set $K(z) = \int_0^z V dt + U ds$. By the first equation above, this integral is independent of the path we choose. Therefore,

$$U = \partial_s K, V = \partial_t K \text{ and } \Delta_{\mathbb{C}} K = (-\partial_s^2 - \partial_t^2)K = |\nabla H|^2 \geq 0.$$

Then the Morse-Bott inequality (5.5) is equivalent to $|\nabla K| = |W \circ P - W(q)| \leq C|\Delta K|$.

Our goal is to show $K \equiv 0$. Let $Z(r) := \int_{\partial B(0,r)} \Delta K \geq 0$. Take $r > R(\Omega)$ and integrate by parts:

$$\begin{aligned} 0 \leq E(r) &:= \int_0^r Z(r') dr' = \int_{B(0,r)} \Delta K = \left| \int_{\partial B(0,r)} \vec{n} \cdot \nabla K \right| \\ &\leq C \left(\int_{\partial B(0,r)} \Delta K \right) \leq CE(r)'. \end{aligned}$$

Therefore, for any $r > r_0 > R(\Omega)$,

$$(5.6) \quad 0 \leq E(r_0) \leq E(r) e^{\frac{r_0 - r}{C}}.$$

Let $r \rightarrow \infty$. Note that $\lim_{r \rightarrow \infty} E(r) = \int_{\mathbb{C}} |\nabla H|^2 \leq \mathcal{E}'_{an}(A, P) < \infty$. Hence, $E(r_0) \equiv 0$, and

$$\Delta K = |\nabla H|^2 \equiv 0 \Rightarrow W \circ P(z) \equiv W(q). \quad \square$$

Remark 5.6. The proof of Lemma 5.5 does not require W to be stable. It suffices to assume that W is Morse-Bott near O_* . $\diamond$

Proof of Theorem 5.1. Since W is stable, the multiplication $g \mapsto g \cdot q$ defines a closed embedding ι of $G_{\mathbb{C}}$ into M . Let $\tilde{\mu} : G_{\mathbb{C}} \rightarrow \mathfrak{g}$ be the composition $\mu \circ \iota$.

By Lemma 5.5, $\text{Im } P \subset \text{Im } \iota$, so $P(z) = g(z) \cdot q$ for a unique element $g(z) \in G_{\mathbb{C}}$.

We first deal with the case when $G = S^1$ and $G_{\mathbb{C}} = \mathbb{C}^*$. Since we are interested in solutions modulo gauge, $g(z)$ may be assumed to be real. Suppose $g(z) = e^{\alpha(z)}$ for some $\alpha : \mathbb{C} \rightarrow \mathbb{R}$. The boundary condition (5.2) implies

$$\lim_{z \rightarrow \infty} \alpha(z) = 0.$$

Moreover, the first equation of (5.1) implies $A = d + i *_2 d\alpha$. Plugging this into the second equation of (5.1), we obtain that

$$(5.7) \quad i\Delta_{\mathbb{C}}\alpha + (\mu(e^{\alpha(z)} \cdot q) - \mu(q)) = 0.$$

Suppose $|\alpha|$ attains its maximum at $z_0 \in \mathbb{C}$ and $\beta := \alpha(z_0) \neq 0$. Then

$$\langle \mu(e^{\beta} \cdot q) - \mu(q), i\beta \rangle_{\mathfrak{g}} = -\langle \Delta_{\mathbb{C}}\alpha(z_0), \beta \rangle \leq 0.$$

We claim that for any $\beta \neq 0$, the inner product $\langle \mu(e^{\beta} \cdot q) - \mu(q), i\beta \rangle_{\mathfrak{g}} \geq 0$. Indeed,

$$\langle \mu(e^{\beta} \cdot q) - \mu(q), i\beta \rangle_{\mathfrak{g}} = \int_0^1 \langle \partial_t \mu(e^{t\beta} \cdot q), i\beta \rangle_{\mathfrak{g}} dt = \int_0^1 |\langle \nabla \mu(e^{t\beta} \cdot q), i\beta \rangle_{\mathfrak{g}}|^2 dt.$$

Since the base point q generates a free $G_{\mathbb{C}}$ -orbit, the integrand is strictly positive. So $\beta \equiv 0$. The general case is dealt with in a similar manner. $\square$

Let us end this section with a few examples.

Example 5.7. In our Fundamental Example 3.7, suppose $\lambda = r_+ r_-$, $q = (r_+, r_-, 0)$ and $\vec{\delta} = \frac{i}{2}(r_+^2 - r_-^2)$. In this case, the equation (5.7) becomes

$$\Delta_{\mathbb{C}}\alpha + \frac{1}{2}(r_+^2(e^{2\alpha} - 1) + r_-^2(1 - e^{-2\alpha})) = 0. \quad \diamond$$

Example 5.8. For Example 3.2, the gauged Witten equations come down to the vortex equation on $\mathbb{C}$ (with $\delta = \frac{i}{2}$):

$$(5.8) \quad \begin{cases} \bar{\partial}_A P = 0, \\ i * F_A + \frac{1}{2}(|P|^2 - 1) = 0. \end{cases}$$

By [Tau80], the moduli space $\mathcal{M}_n$ with $\mathcal{E}_{an} = 2\pi n$ is $\text{Sym}^n \mathbb{C}$ for any $n \geq 1$, so Theorem 5.1 fails. W is not stable in this case, even though $\delta = \frac{i}{2}$ is W -stable. Note that $\mathcal{M}_n$ is regular; its dimension agrees with the prediction of the index formula. $\diamond$

Example 5.9. In Example 3.7, let $\lambda = 0$ and $\vec{\delta} = \frac{i}{2}$. For a solution (A, P) of (5.1), write $P(z) = (x(z), y(z), b(z))$. Lemma 5.5 and Remark 5.6 implies $y(z) = b(z) \equiv 0$. The equations are reduced to the previous example. However, in this case, the moduli space $\mathcal{M}'_n$ is not regular. Its formal dimensions are always zero for any $n \geq 0$. $\diamond$

6. EXPONENTIAL DECAY IN THE SPATIAL DIRECTION

In this section, we generalize Lemma 2.5 in the context of gauged Landau-Ginzburg models, which is also the analogue of Theorem 1.3 in the finite dimensional case. We state and prove the theorem for the energy density function.

Theorem 6.1. *For any stable gauged Landau-Ginzburg model (M, W, G, ρ) , there exist $\epsilon(M, W), \zeta(M, W) > 0$ with following significance. Given a solution $\gamma = (A, P) \in \mathcal{C}(X)$ to the gauged Witten equations (4.1) on the upper half plane $X = \mathbb{R}_t \times [0, \infty)_s$, suppose the point-wise estimate*

$$U_{\gamma}(t, s) := |\nabla^A P|^2 + |\nabla H|^2 + |F|^2 + |\vec{\delta} - \mu|^2 < \epsilon$$

holds and any $(t, s) \in X$. Then

$$U_\gamma(t, s) < e^{-\zeta s}, \quad \forall s \geq 0.$$

The function $U_\gamma : X \rightarrow [0, \infty)$ is called the energy density function.

Fix a base point $q \in O_* = \mu^{-1}(\vec{\delta}) \cap \text{Crit}(L)$. Then any configuration $(A, P) \in \mathcal{C}(X)$ is subject to the boundary condition

$$\lim_{s \rightarrow \infty} P(\cdot, s) \rightarrow q.$$

In this case, the energy density U_γ provides an upper bound for the distance:

$$U_\gamma \geq |\nabla H|^2 + |\vec{\delta} - \mu|^2 \geq \epsilon \cdot [d(P(t, s), O_*)]^2,$$

so Theorem 6.1 implies Lemma 2.5 when $G = \{e\}$ is trivial. U_γ is only a bounded function on X . Its integral is not finite and does not yield the analytic energy $\mathcal{E}_{an}$ in Definition 4.6. In fact, it is more relevant with the variant $\mathcal{E}_{an}(A, P; \mathbb{C})$ defined in Lemma 5.4.

Remark 6.2. The analogue of Property 2.4 (the uniform L^∞ decay) continues to hold for the gauged Witten equation, which can be improved into an exponential decay using Theorem 6.1. However, if the superpotential W is allowed to have multiple critical $\mathcal{G}_{\mathbb{C}}$ -orbits, Property 2.4 is not automatic. We will come back to this issue in the second paper. $\diamond$

Proof of Theorem 6.1. By the gauged Witten equations (4.1), it suffices to show the exponential decay for the quantity

$$u(t, s) := |\nabla^A P|^2 + |F|_{\mathfrak{g}}^2.$$

We use a lemma from Appendix A and verify its conditions:

Lemma 6.3 (Corollary A.2). *Take $\zeta > 0$. Suppose $u : \mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s \rightarrow \mathbb{R}$ is a bounded $\mathcal{C}^2$ -function on the upper half plane $\mathbb{H}_+^2$ such that*

$$(U1) \quad (\Delta_{\mathbb{H}_+^2} + \zeta^2)u \leq 0, \text{ and}$$

$$(U2) \quad u(t, 0) \leq K \text{ for some } K > 0 \text{ and any } t \in \mathbb{R}_t.$$

Then $u(t, s) \leq Ke^{-\zeta s}$ for any $(t, s) \in \mathbb{H}_+^2$.

(U2) follows from the assumption that $u(t, s) \leq U_\gamma(t, s) < \epsilon$. To verify (U1), we find an explicit formula of $\Delta_{\mathbb{H}_+^2} u(s, t)$. It is convenient to define a bundle map:

$$D : TM \rightarrow TM \oplus \mathfrak{g} \oplus \mathfrak{g}$$

$$(p, v) \mapsto (\text{Hess}_p H(v), \langle \nabla \mu, v \rangle, \langle J \nabla \mu, v \rangle), \quad \forall p \in M, v \in T_p M.$$

Lemma 6.4 (Corollary B.8). *We have the following Bochner-type formula for $\Delta_{\mathbb{H}_+^2} u(t, s)$:*

$$0 = \frac{1}{2} \Delta_{\mathbb{H}_+^2} (|\nabla^A P|^2 + |F|_{\mathfrak{g}}^2) + I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$\begin{aligned} I_1 &= |\text{Hess}_A P|^2 + |\nabla F|_{\mathfrak{g}}^2, & I_2 &= |D(\nabla^A P)|^2 + |\langle \nabla \mu, F \rangle_{\mathfrak{g}}|^2, & I_3 &= 2\langle R(S, T)S, T \rangle, \\ I_4 &= \langle (\nabla_T \text{Hess } H)(\nabla H), T \rangle + \langle (\nabla_S \text{Hess } H)(\nabla H), S \rangle, \\ I_5 &= 6\langle \text{Hess } \mu(JS), T \otimes F \rangle - \langle \text{Hess } \mu(T), T \otimes F \rangle - \langle \text{Hess } \mu(S), S \otimes F \rangle. \end{aligned}$$

and R is the Riemannian curvature tensor of M .

Remark 6.5. This identity was first proved by Taubes in [JT80, Proposition 6.1] for the vortex equation on $\mathbb{C}$, in which case $M = \mathbb{C}$ is furnished with the flat metric, $W \equiv 0$ and $\mu = \frac{i}{2}|z|^2$, cf. Example 3.2. For more details, see Remark B.10. $\diamond$

Let us digest the consequence of Lemma 6.4. $I_1 \geq 0$. I_4 and I_5 involve only trilinear tensors:

$$\begin{aligned} \langle \nabla \cdot \text{Hess } H(\cdot), \cdot \rangle : TM \otimes TM \otimes TM &\rightarrow \mathbb{R}, \\ \langle \text{Hess } \mu(\cdot), \cdot \otimes \cdot \rangle : TM \otimes TM \otimes \mathfrak{g} &\rightarrow \mathbb{R}. \end{aligned}$$

Within a G -invariant neighborhood Ω of O_* with compact closure, we may assume these trilinear maps have uniformly bounded operator norms. Hence, whenever $P(t, s) \in \Omega$,

$$|I_4| + |I_5| \leq Cu^{3/2}.$$

for some $C > 0$. The same estimate holds for I_3 with a different exponent of u :

$$|I_3| \leq Cu^2.$$

Since the critical orbit O_* is free,

$$|\langle \nabla \mu, F \rangle_{\mathfrak{g}}|^2 > \zeta_1^2 |F|^2$$

for a small number $\zeta_1 > 0$ for any $p \in O_*$ and $F \in \mathfrak{g}$. The same estimate also holds for any $p \in \Omega$ by possibly shrinking the open neighborhood Ω .

Finally, since W is a stable superpotential, D_p is injective for any $p \in O_*$ (by the Morse-Bott condition); so

$$|D_p(v)|^2 > \zeta_2^2 |v|^2$$

for any $v \in T_p M$. The same estimate also holds for $p \in \Omega$ (by possibly shrinking Ω). Hence, for $\zeta = \min\{\zeta_1, \zeta_2\}$,

$$|I_2| \geq \zeta^2 u$$

whenever $P(s, t) \in \Omega$. By taking $\epsilon \ll 1$ such that $\epsilon + \epsilon^{1/2} < \zeta^2/2C$. Lemma 6.4 then implies

$$0 \geq \frac{1}{2} \Delta_{\mathbb{H}_+^2} u + \zeta^2 u - C(u^2 + u^{3/2}) \geq \frac{1}{2} (\Delta_{\mathbb{H}_+^2} + \zeta^2) u.$$

Now apply Lemma 6.3 with $K = \epsilon$. $\square$

Remark 6.6. The bundle maps D and $\langle \nabla \mu, \cdot \rangle$ involved in I_2 are components of the operator $\hat{D}$. The invertibility of $\hat{D}$ is essential to this proof. $\diamond$

Remark 6.7. Let us add a remark to explain the mysterious Bochner-type formula in Lemma 6.4. The baby case is Example 2.3 in which the structure group $G = \{e\}$ is trivial. In that case, $\text{Hess } H$ is a constant self-adjoint $\mathbb{R}$ -linear operator on $\mathbb{C}^n$, so

$$(\nabla H)_x = \text{Hess } H(x), \quad x \in \mathbb{C}^n.$$

Applying the operator $(\partial_t - J\partial_s)$ to (4.7b), we obtain that

$$0 = \partial_t T + \partial_s S + \text{Hess } H(T + JS) = -\Delta P - (\text{Hess } H)^2(P),$$

from which one can easily prove that the map $P : \mathbb{H}_+^2 \rightarrow \mathbb{C}^n$ along with its all higher derivatives has exponential decay as $s \rightarrow \infty$. Example 3.2 is the other extreme where $W \equiv 0$ and μ is quadratic. The proof of Lemma 7.7 is a tedious exercise in Riemannian geometry, and is deferred to Appendix B. $\diamond$

Part 3. The Seiberg-Witten Equations on $\mathbb{C} \times \mathbb{T}^2$

In the third part of this paper, we study an infinite dimensional Landau-Ginzburg model associated to a 2-torus $\Sigma = (\mathbb{T}^2, g_\Sigma)$ whose gauged Witten equations recover the Seiberg-Witten equations on $X = \mathbb{C} \times \Sigma$ or $\mathbb{H}_+^2 \times \Sigma$. We generalize Theorem 5.1 and 6.1 from the previous part to this infinite dimensional case. The main difference is that the topology of M depends on a Sobolev completion of smooth sections, and we need to specify the correct norms used in the estimates.

One main obstacle in defining a Floer homology for a 3-manifold $\hat{Y}$ with cylindrical ends is a compactness issue, and its resolution relies on three key ingredients:

- (K1) A uniform upper bound on the analytic energy;
- (K2) Point-like solutions are trivial on $\mathbb{C} \times \Sigma$, namely, they have to be $\mathbb{C}$ -translation invariant up to gauge.
- (K3) Finite energy solutions on $\mathbb{R}_s \times \Sigma$ are trivial, namely, they have to be $\mathbb{R}_s$ -translation invariant up to gauge.

In order to achieve these properties, a suitable perturbation of the Seiberg-Witten equations on either $\mathbb{C} \times \Sigma$ or $\mathbb{R}_s \times \Sigma$ has to be perturbed by a 2-form $\omega = \mu + ds \wedge \lambda$ where

- $\lambda \in \Omega_h^1(\Sigma, i\mathbb{R})$ is a harmonic 1-form on Σ ;
- $\mu \in \Omega_h^2(\Sigma, i\mathbb{R})$ is a harmonic 2-form on Σ .

However, most results in this part, except Section 10, do not require μ to be harmonic. One can take $\mu = \vec{\delta} \cdot d\text{vol}_\Sigma$ to be any smooth 2-form on Σ instead. While λ is used to perturb the superpotential W_λ , μ is used to perturb the moment map equation in (4.1).

The first property (K1) will be postponed to the second paper [Wan20] where we set up the cobordism category properly. (K2) is achieved by Theorem 1.2. The proof uses ideas from Section 5 and is accomplished in Section 8. As for (K3), we invoke a theorem of Taubes, which will be explained in Section 10.

7. THE FUNDAMENTAL LANDAU-GINZBURG MODEL

In this section, we explain the construction of the fundamental Landau-Ginzburg model $(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma))$ associated to a Riemannian 2-torus (Σ, g_Σ) . When $\lambda \neq 0$, we will verify the superpotential W_λ is stable in the sense of Definition 3.6 and any $\vec{\delta} \in \mathfrak{g}$ is W -stable in the sense of Definition 3.5, cf. Proposition 7.5.

7.1. Review. Recall that a $spin^c$ structure $\mathfrak{s}$ on a smooth 4-manifold X is a pair (S_X, ρ_4) where $S_X = S^+ \oplus S^-$ is the spin bundle, and the bundle map $\rho_4 : T^*X \rightarrow \text{Hom}(S_X, S_X)$ defines the Clifford multiplication. A configuration $\gamma = (A, \Phi) \in \mathcal{C}(X, \mathfrak{s})$ consists of a smooth $spin^c$ connection A and a smooth section Φ of S^+ . Let A^t be the induced connection on $\bigwedge^2 S^+$. Let ω be a closed 2-form on X and ω^+ denote its self-dual part. The Seiberg-Witten equations perturbed by ω are defined on $\mathcal{C}(X, \mathfrak{s})$ by the formula:

$$(7.1) \quad \begin{cases} \frac{1}{2}\rho_4(F_{A^t}^+) - (\Phi\Phi^*)_0 - \rho_4(\omega^+) = 0, \\ D_A^+\Phi = 0, \end{cases}$$

where $D_A^+ : \Gamma(S^+) \rightarrow \Gamma(S^-)$ is the Dirac operator and $(\Phi\Phi^*)_0 = \Phi\Phi^* - \frac{1}{2}|\Phi|^2 \otimes \text{Id}_{S^+}$ denotes the traceless part of the endomorphism $\Phi\Phi^* : S^+ \rightarrow S^+$.

The gauge group $\mathcal{G}(X) = \text{Map}(X, S^1)$ acts naturally on $\mathcal{C}(X, \mathfrak{s})$ by the formula:

$$\mathcal{G}(x) \ni u : \mathcal{C}(X, \mathfrak{s}) \rightarrow \mathcal{C}(X, \mathfrak{s}), (A, \Phi) \mapsto (A - u^{-1}du, u\Phi).$$

The monopole equations (7.1) is invariant under gauge transformations.

Let $\Sigma = (\mathbb{T}^2, g_\Sigma)$ be the 2-torus with a Riemannian metric g_Σ , which is **not** necessarily flat. In the special case when $X = \mathbb{C} \times \Sigma$ is a product of complex manifolds, the equations (7.1) can be understood more explicitly. In what follows, the 4-manifold X is equipped with the product metric and the complex orientation.

Let $dvol_{\mathbb{C}}$ and $dvol_\Sigma$ denote volume forms on $\mathbb{C}$ and Σ respectively. Then the 2-form $\omega_{sym} = dvol_{\mathbb{C}} + dvol_\Sigma$ makes X into a symplectic manifold. The spin bundle S^+ splits as $L^+ \oplus L^-$: they are $\mp 2i$ eigenspace of $\rho_4(\omega_{sym}) : S^+ \rightarrow S^+$. The spin section Φ decomposes as (Φ_+, Φ_-) with $\Phi_\pm \in \Gamma(X, L^\pm)$. We are only interested in the $spin^c$ structure on $\mathbb{C} \times \Sigma$ with

$$c_1(S^+)[\Sigma] = 0,$$

so both L^+ and L^- are topologically trivial.

Let $z = t + is$ be the coordinate function on $\mathbb{C}$. The Clifford multiplication $\rho = \rho_4 : T^*X \rightarrow \text{Hom}(S, S)$ can be constructed by setting:

$$\rho_4(dt) = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix}, \quad \rho_4(ds) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} : S^+ \oplus S^- \rightarrow S^+ \oplus S^-,$$

where $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : S^+ = L^+ \oplus L^- \rightarrow L^+ \oplus L^-$ is the first Pauli matrix.

If we identify $L^+ \cong \mathbb{C}$ and $L^- \cong \bigwedge^{0,1} \Sigma$, then

$$\rho_3(w) := \rho_4(dt)^{-1} \cdot \rho_4(w) = \begin{pmatrix} 0 & -\iota(\sqrt{2}w^{0,1}) \cdot \\ \sqrt{2}w^{0,1} \otimes \cdot & 0 \end{pmatrix} : S^+ \rightarrow S^+,$$

for any $x \in \Sigma$ and $w \in T_x \Sigma$.

Remark 7.1. We will frequently work with Clifford multiplications in dimension 2, 3 and 4, denoted by ρ_2 , ρ_3 and ρ_4 respectively. Identify $\mathbb{C}$ as $\mathbb{R}_t \times \mathbb{R}_s$, then they are related by

$$\rho_3(w) = \rho_4(dt)^{-1} \cdot \rho_4(w), \quad \rho_2(v) = \rho_3(ds)^{-1} \cdot \rho_3(v) : S^+ \rightarrow S^+.$$

for any $w \in T^*(\mathbb{R}_s \times \Sigma)$ and $v \in T^*\Sigma$. ◇

The symplectic form ω_{sym} is parallel, so is the decomposition $S^+ = L^+ \oplus L^-$. Thus, any $spin^c$ connection A must split as

$$\nabla_A = \begin{pmatrix} \nabla_{A_+} & 0 \\ 0 & \nabla_{A_-} \end{pmatrix}.$$

We regard L^+ and L^- as bundles over Σ , and they pull back to spin bundles over X via the projection map $X \rightarrow \Sigma$. Let $\tilde{B}_0 = (d, \nabla^{LC})$ be the reference connection on

$\mathbb{C} \otimes \bigwedge^{0,1} \mathbb{T}^2 \rightarrow \mathbb{T}^2$. We obtain a reference connection A_0 on S^+ by setting

$$\nabla_{A_0} = \nabla_{\check{B}_0} + \frac{d}{dt} + \frac{d}{ds}$$

One can easily check that A_0 is a *spin*^c connection. Any other *spin*^c connection A differs from A_0 by an imaginary valued 1-form $a \in \Gamma(X, iT^*X)$. Their curvature tensors are related by

$$F_A = F_{A_0} + da \otimes \text{Id}_S.$$

Using the product structure on X , the connection $\nabla_A = (\nabla_A^{\mathbb{C}}, \nabla_A^{\Sigma})$ is decomposed into $\mathbb{C}$ -direction part and Σ -direction part. The curvature tensor F_A is decomposed accordingly as:

$$F_A = F_A^{\Sigma} + F_A^{\mathbb{C}} + F_A^m$$

where $F_A^m \in \Gamma(X, iT^*X \otimes \text{End}(S))$ is the mixed term. Similar decomposition applies to the induced curvature form F_{A^t} on $\bigwedge^2 S^+ = L^+ \otimes L^-$:

$$(7.2) \quad F_{A^t} = F_{A^t}^{\Sigma} d\text{vol}_{\Sigma} + F_{A^t}^{\mathbb{C}} d\text{vol}_{\mathbb{C}} + F_{A^t}^m,$$

where $F_{A^t}^m \in \Gamma(X, i\Omega^1(\mathbb{C}) \wedge \Omega^1(\Sigma))$. Our description of F_A then shows that

$$(7.3) \quad F_A^m = \frac{1}{2} F_{A^t}^m \otimes \text{Id}.$$

and

$$(7.4) \quad F_A^{\Sigma} = \begin{pmatrix} F_{A^+}^{\Sigma} & 0 \\ 0 & F_{A^-}^{\Sigma} \end{pmatrix} d\text{vol}_{\Sigma} = \begin{pmatrix} \frac{1}{2} F_{A^t}^{\Sigma} + \frac{i}{2} K & 0 \\ 0 & \frac{1}{2} F_{A^t}^{\Sigma} - \frac{i}{2} K \end{pmatrix} d\text{vol}_{\Sigma}.$$

where K is the Gaussian curvature of $(\mathbb{T}^2, g_{\Sigma})$.

7.2. The Fundamental Landau-Ginzburg Model. We now provide another perspective on the Seiberg-Witten equations on $\mathbb{C} \times \Sigma$ in the language of Landau-Ginzburg models and the gauged Witten equations. The fundamental Landau-Ginzburg model $(M(\Sigma), W_{\lambda}, \mathcal{G}(\Sigma))$ to be defined below will allow us to apply results from Part 2 to the Seiberg-Witten equations on either $\mathbb{C} \times \Sigma$ or $\mathbb{H}_+^2 \times \Sigma$.

The Kähler manifold $M(\Sigma) = \mathcal{C}(\Sigma)$ is also the configuration space on Σ :

$$(\check{B}_0, 0) + \Omega^1(\Sigma, i\mathbb{R}) \oplus \Gamma(\Sigma, L^+ \oplus L^-),$$

where $\check{B}_0 = (d, \nabla^{LC})$ is the reference *spin*^c connection on $S^+ \rightarrow \Sigma$. A configuration $\kappa \in M$ is a triple $(\check{b}, \check{\Psi}_+, \check{\Psi}_-)$, where the sum $\check{B}_0 + \check{b}$ is viewed as a *spin*^c connection on Σ and $\check{\Psi} = (\check{\Psi}_+, \check{\Psi}_-) \in \Gamma(\Sigma, S^+)$ is a spin section.

The complex structure of M is defined by the bundle map

$$J = (*_{\Sigma}, \rho_3(ds)) = (*_{\Sigma}, \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}),$$

while the Riemannian metric g_M of M is the flat L^2 metric:

$$\langle (\check{b}_1, \check{\Psi}_1), (\check{b}_2, \check{\Psi}_2) \rangle = \int_{\Sigma} \langle \check{b}_1, \check{b}_2 \rangle + \text{Re} \langle \check{\Psi}_1, \check{\Psi}_2 \rangle.$$

Let h_M be the Hermitian metric on M induced from J and g_M .

The abelian group acting on M is $\mathcal{G}(\Sigma) = \text{Map}(\mathbb{T}^2, S^1)$ with the usual action:

$$u(\check{b}, \check{\Psi}) = (\check{b} - u^{-1}du, u\check{\Psi}).$$

Occasionally, we will use a smaller group $\mathcal{G}^e$, the identity component of $\mathcal{G}$, to develop the theory. They fits into a short exact sequence:

$$0 \rightarrow \mathcal{G}^e \rightarrow \mathcal{G} \xrightarrow{\pi} H^1(\mathbb{T}^2, \mathbb{Z}) \rightarrow 0, \quad \pi(u) = [\frac{u^{-1}du}{2\pi i}].$$

The Lie algebra of $\mathcal{G}(\Sigma)$ or $\mathcal{G}^e$ is $\text{Lie}(\mathcal{G}) = \Gamma(\Sigma, i\mathbb{R})$. Since J acts on $\Gamma(\Sigma, L^-)$ by the conjugate of the standard complex structure, we say that $\mathcal{G}(\Sigma)$ acts on $\Omega^1(\Sigma, i\mathbb{R}), \Gamma(\Sigma, L^+)$ and $\Gamma(\Sigma, L^-)$ by weights $(0, 1, -1)$.

The moment map μ is given by

$$\mu(\check{b}, \check{\Psi}) = - *_{\Sigma} d\check{b} + \frac{i}{2}(|\check{\Psi}_+|^2 - |\check{\Psi}_-|^2) + \frac{i}{2}K.$$

where the Gaussian curvature K is added for conventions. If $v = (\delta\check{b}, \delta\check{\Psi})$ is a tangent vector at $(\delta\check{b}, \delta\check{\Psi})$, then we have

$$(7.5) \quad \begin{aligned} \langle \nabla\mu, v \rangle &= - *_{\Sigma} d_{\Sigma}\delta\check{b} + i \text{Re}\langle i\delta\check{\Psi}, \rho_3(ds)\check{\Psi} \rangle \in \text{Lie}(\mathcal{G}), \\ \langle \nabla\mu, Jv \rangle &= d_{\Sigma}^*\delta\check{b} + i \text{Re}\langle i\delta\check{\Psi}, \check{\Psi} \rangle \in \text{Lie}(\mathcal{G}). \end{aligned}$$

The super-potential W is the Dirac functional. The Clifford multiplication on Σ :

$$\rho_2 : T^*\Sigma \rightarrow \text{Hom}(S^+, S^+),$$

defines a Dirac operator for each spin^c connection $\check{B}$ on $S^+ \rightarrow \Sigma$:

$$D_{\check{B}}^{\Sigma} : \Gamma(\Sigma, S^+) \xrightarrow{\nabla_{\check{B}}} \Gamma(\Sigma, T^*\Sigma \otimes S^+) \xrightarrow{\rho_2} \Gamma(\Sigma, S^+).$$

This operator is self-adjoint and switches the parity, i.e.

$$D_{\check{B}}^{\Sigma} = \begin{pmatrix} 0 & D_{\check{B}}^- \\ D_{\check{B}}^+ & 0 \end{pmatrix} : \Gamma(\Sigma, L^+ \oplus L^-) \rightarrow \Gamma(\Sigma, L^+ \oplus L^-).$$

The super-potential W_0 is then defined as

$$W_0(\check{b}, \check{\Psi}_+, \check{\Psi}_-) = \int_{\Sigma} \langle D_{\check{B}}^+ \check{\Psi}_+, \check{\Psi}_- \rangle = \int_{\Sigma} \langle \check{\Psi}_+, D_{\check{B}}^- \check{\Psi}_- \rangle.$$

The perturbation that we consider takes the form

$$W_{\lambda}(\check{b}, \check{\Psi}) = W_0 - \langle \check{b}, \lambda \rangle_{h_M},$$

where $\lambda \in \Omega^1(\Sigma, i\mathbb{R})$ and h_M is the Hermitian inner product.

The complex gauge group $\mathcal{G}_{\mathbb{C}} = \text{Map}(\Sigma, \mathbb{C}^*)$ acts on M by the formula:

$$e^{\alpha}u(\check{b}, \check{\Psi}) = (\check{b} + i *_{\Sigma} d_{\Sigma}\alpha - u^{-1}du, e^{\alpha}u\check{\Psi}_+, e^{-\alpha}u\check{\Psi}_-),$$

where $u \in \mathcal{G}(\Sigma)$ and $\alpha : \Sigma \rightarrow \mathbb{R}$ is real.

Lemma 7.2. *The super potential $W_0 : M(\Sigma) \rightarrow \mathbb{C}$ is invariant under $\mathcal{G}_{\mathbb{C}}$.*

Proof of Lemma. It suffices to verify that $e^{-\alpha} D_{\check{B}'}^+(e^\alpha \cdot) = D_{\check{B}}^+(\cdot)$ if $\check{B}' = e^\alpha \cdot \check{B}$, or equivalently

$$\rho_2(d\alpha) + \rho_2(i *_{\Sigma} d\alpha) = 0 : \Gamma(L^+) \rightarrow \Gamma(L^-).$$

If one restricts instead to sections of $\Gamma(L^-)$, then

$$\rho_2(d\alpha) - \rho_2(i *_{\Sigma} d\alpha) = 0 : \Gamma(L^-) \rightarrow \Gamma(L^+). \quad \square$$

As for the perturbed super-potential W_λ ,

- for W_λ to be invariant under $\mathcal{G}^e$, λ has to be co-closed;
- for W_λ to be invariant under the identity component $\mathcal{G}_{\mathbb{C}}^e$, λ has to be harmonic;
- for W_λ to be invariant under $\mathcal{G}_{\mathbb{C}}$, λ has to be zero.

Assumption 7.3. We choose $\lambda \neq 0 \in \Omega_h^1(\Sigma, i\mathbb{R})$ to be a harmonic form, so W_λ is only invariant under $\mathcal{G}_{\mathbb{C}}^e$. $\diamond$

Write $W_\lambda = L + iH$. Then

$$(7.6) \quad \nabla L(\check{b}, \check{\Psi}) = (\rho_2^{-1}(\check{\Psi}\check{\Psi}^*)_{\Pi} - \lambda, D_{\check{B}}^-\check{\Psi}_-, D_{\check{B}}^+\check{\Psi}_+).$$

The equation $\nabla L = 0$ has solutions if and only if λ is a harmonic 1-form. When it is the case, $\text{Crit}(L)$ contains a unique $\mathcal{G}_{\mathbb{C}}$ orbit if and only if $\lambda \neq 0$ (see Proposition 7.4 below). This is the second reason why we insist Assumption 7.3. When $\lambda = 0$, $\text{Crit}(L)$ consists of three classes of orbits corresponding respectively to

$$(\check{\Psi}_+ \neq 0, \check{\Psi}_- \equiv 0), (\check{\Psi}_+ \equiv 0, \check{\Psi}_- \neq 0), (\check{\Psi}_+ \equiv 0, \check{\Psi}_- \equiv 0).$$

Consider the trivial principal bundle $\mathbb{C} \times \mathcal{G}(\Sigma) \rightarrow \mathbb{C}$. A connection $\bar{A}$ is expressed as

$$\bar{A} = d + a_t(z)dt + a_s(z)ds$$

with $a_t, a_s \in \Gamma(\mathbb{C}, \text{Lie}(\mathcal{G}(\Sigma))) = \Gamma(\mathbb{C} \times \Sigma, i\mathbb{R})$.

Proposition 7.4. *With the gauged Landau-Ginzburg model $(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma))$ defined as above, the associated gauged Witten equations over $\mathbb{C}$ is equivalent to the Seiberg-Witten equations (7.1) on $\mathbb{C} \times \Sigma$ with $\omega = ds \wedge \lambda - \check{\delta} \text{vol}_{\Sigma}$. Let*

$$\begin{aligned} P : \mathbb{C} &\rightarrow M(\Sigma) \\ z &\mapsto (\check{b}(z), \check{\Psi}(z)) \end{aligned}$$

be a smooth map defined on $\mathbb{C}$. Then the identification $(A, \Phi) \leftrightarrow (\bar{A}, P)$ is made by taking

$$\begin{aligned} A - A_0 &= (\bar{A} - d) + (\check{B}(z) - \check{B}_0) = a_t(z)dt + a_s(z)ds + \check{b}(z), \\ \Phi &= \check{\Psi}(z) \text{ on } \{z\} \times \Sigma. \end{aligned}$$

Proof. The J -holomorphic equation in (4.1) in our case becomes

$$\nabla_{\partial_t}^{\bar{A}} \begin{pmatrix} \check{b} \\ \check{\Psi} \end{pmatrix} + \begin{pmatrix} *_{\Sigma} & 0 \\ 0 & \rho_3(ds) \end{pmatrix} \left(\nabla_{\partial_s}^{\bar{A}} \begin{pmatrix} \check{b} \\ \check{\Psi} \end{pmatrix} + \nabla L \right).$$

More concretely, it is

$$(7.7) \quad \begin{aligned} (\partial_t \check{b} - d_{\Sigma} a_t) + *_{\Sigma}(\partial_s \check{b} - d_{\Sigma} a_s + \rho_2^{-1}(\check{\Psi}\check{\Psi}^*)_{\Pi} - \lambda) &= 0, \\ (\partial_t \check{\Psi} + a_t \check{\Psi}) + \rho_3(ds)(\partial_s \check{\Psi} + a_s \check{\Psi} + D_{\check{B}}^{\Sigma} \check{\Psi}) &= 0. \end{aligned}$$

The second equation gives rise to the Dirac operator $D_A^+ \Phi = 0$, while the first equation gives the off-diagonal part of the curvature equation:

$$\frac{1}{2} \rho_4(F_{A^t}^+)_{\Pi} - (\Phi \Phi^*)_{\Pi} - \rho_4(\omega^+) = 0$$

with $\omega = ds \wedge \lambda - \vec{\delta} d\text{vol}_{\Sigma}$. The diagonal part comes from the moment map equation in (4.1):

$$- *_C d_C(a_t dt + a_s ds) - *_\Sigma d_\Sigma \check{b} + \frac{i}{2} (|\check{\Psi}_+|^2 - |\check{\Psi}_-|^2) + \frac{i}{2} K = \vec{\delta}.$$

Indeed, $\frac{1}{2} F_{A^t}^\Sigma = d_\Sigma \check{b} - i/2 \cdot K d\text{vol}_{\Sigma}$ and $\frac{1}{2} F_{A^t}^{\mathbb{C}} = d_C(a_t dt + a_s ds)$ in terms of the decomposition (7.2). $\square$

7.3. Stability. Now we examine the stability of the superpotential W_λ . Even though W_λ is not $\mathcal{G}_{\mathbb{C}}$ -invariant, $\nabla L = 0$ is a $\mathcal{G}_{\mathbb{C}}$ -invariant equation on M .

Proposition 7.5. *Suppose $\lambda \neq 0 \in \Omega_h^1(\Sigma, i\mathbb{R})$. Then $\text{Crit}(L)$ contains a unique free $\mathcal{G}_{\mathbb{C}}$ -orbit. For any $\vec{\delta} \in \text{Lie}(\mathcal{G})$, $\mu^{-1}(\vec{\delta}) \cap \text{Crit}(L)$ consists of a unique $\mathcal{G}(\Sigma)$ -orbit. Moreover, W_λ is a Morse-Bott function. In this sense, we say that W_λ is stable and any $\vec{\delta} \in \text{Lie}(\mathcal{G})$ is W_λ -stable.*

Proof. The verification that W_λ is Morse-Bott is postponed to Proposition 7.9. It concerns only the linearized operator at the critical $\mathcal{G}_{\mathbb{C}}$ -orbit. We focus on the other statements. The equation $\nabla L = 0$ implies:

$$(7.8) \quad (\rho_2^{-1}(\check{\Psi} \check{\Psi}^*)_{\Pi} - \lambda, D_{\check{B}}^- \check{\Psi}_-, D_{\check{B}}^+ \check{\Psi}_+) = 0.$$

In terms of the Hodge decomposition of $\Omega^1(\Sigma, i\mathbb{R})$, write

$$(7.9) \quad \check{b} = \check{b}^1 + \check{b}^h + \check{b}^2.$$

with $\check{b}^1$ exact, $\check{b}^h$ harmonic and $\check{b}^2$ co-exact. The equation (7.8) is invariant under the action of $\mathcal{G}_{\mathbb{C}}$, so we may kill $\check{b}^1$ and $\check{b}^2$ by an element $e^{\alpha+i\beta} \in \mathcal{G}_{\mathbb{C}}$ with

$$id_{\Sigma} \beta = \check{b}^1 \text{ and } id_{\Sigma} \alpha = *_\Sigma \check{b}^2.$$

Since $\lambda \neq 0$, $|\lambda|$ is non-vanishing everywhere, so are $\check{\Psi}_+$ and $\check{\Psi}_-$ by the first equation of (7.8). Hence, $\check{b}^h$ can be killed by a harmonic gauge transformation as well (the trivial connection is the only flat connection on Σ that supports non-trivial holomorphic global sections), and $\check{\Psi}_+$ has to be a constant function. Using the identification $L^+ \cong \mathbb{C}$ and $L^- \cong \bigwedge^{0,1} \Sigma$, a representative in the orbit can be written as

$$(b, \check{\Psi}_+, \check{\Psi}_-) = (0, 1, \sqrt{2}\lambda^{0,1}).$$

In general, a solution of the equation (7.2) is cast into the form:

$$e^\alpha \cdot (0, 1, \sqrt{2}\lambda^{0,1}) = (i *_\Sigma d_\Sigma \alpha, e^\alpha, e^{-\alpha} \sqrt{2}\lambda^{0,1}).$$

up to the gauge action of $\mathcal{G}(\Sigma)$. Take any smooth function $\vec{\delta} \in \Gamma(\Sigma, i\mathbb{R})$. To show $\vec{\delta}$ is W_λ -stable, we have to find solutions of (7.2) in the fiber $\mu^{-1}(\vec{\delta})$, or equivalently, to solve

the equation

$$(7.10) \quad i(\Delta_\Sigma \alpha + \frac{1}{2}(e^{2\alpha} - e^{-2\alpha}|\lambda|^2) + \frac{1}{2}K) = \vec{\delta}.$$

The strategy is to use the variational principle and show that the non-linear map:

$$\begin{aligned} \eta : L_k^2(\Sigma, \mathbb{R}) &\rightarrow L_{k-2}^2(\Sigma, \mathbb{R}), \\ \alpha &\mapsto \eta(\alpha) = \Delta_\Sigma \alpha + \frac{1}{2}(e^{2\alpha} - e^{-2\alpha}|\lambda|^2) \end{aligned}$$

is a bijection for any $k \geq 2$. It suffices to prove the special case for $k = 2$; the rest will follow from elliptic regularity. For any $g \in L^2(\Sigma, \mathbb{R})$, define an energy functional as

$$\mathcal{E}_g(\alpha) = \|\eta(\alpha) - g\|_2^2$$

If α_0 achieves the infimum $\inf_{\alpha \in L_2^2} \mathcal{E}_g(\alpha)$, let $f = \eta(\alpha_0) - g \in L^2(\Sigma)$. Then, for any tangent vector $v \in L_2^2(\Sigma, \mathbb{R})$ at the minimizer α_0 , we have

$$0 = \partial_t \mathcal{E}_g(\alpha_0 + tv) = \langle f, \Delta_\Sigma v + (e^{2\alpha_0} + e^{-2\alpha_0}|\lambda|^2)v \rangle.$$

Since the linearized operator $\Delta_\Sigma + (e^{2\alpha_0} + e^{-2\alpha_0}|\lambda|^2)$ is positive and self-adjoint on L_2^2 , $f = 0$. To find such a minimizer α_0 , let $\{\alpha_n\} \subset L_2^2$ be a sequence that minimizes $\mathcal{E}_g(\alpha)$, i.e.

$$\lim_{n \rightarrow \infty} \mathcal{E}_g(\alpha_n) = \inf_{\alpha \in L_2^2} \mathcal{E}_g(\alpha).$$

To show that $\{\alpha_n\}$ contains a weakly convergent subsequence, we need an a priori estimate:

Lemma 7.6. *For any fixed $g \in L^2(\Sigma)$, there exists function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $\alpha \in L_2^2$, $\mathcal{E}_g(\alpha) < C$ implies $\|\alpha\|_{L_2^2} \leq \varphi(C)$.*

Proof of Lemma. It suffices to prove the lemma for a particular $g \in L^2(\Sigma)$. We do this for

$$g = g_0 := \frac{1}{2}(1 - |\lambda|^2).$$

However, in the computation below, we take $g = g_0 - g_1$ where g_1 is an auxiliary function to be fixed later. By direct computation, we have

$$\begin{aligned} (7.11) \quad \mathcal{E}_g(\alpha) &= \|\eta(\alpha) - g\|^2 = \int_\Sigma |\Delta_\Sigma \alpha + g_1 + \frac{1}{2}(e^{2\alpha} - 1) + \frac{1}{2}(1 - e^{-2\alpha})|\lambda|^2|^2 \\ &= \int_\Sigma |\Delta_\Sigma \alpha + g_1|^2 + \frac{1}{4}|e^\alpha - e^{-\alpha}|^2|e^\alpha + e^{-\alpha}|\lambda|^2|^2 + 2|e^\alpha d\alpha|^2 + 2|e^{-\alpha} d\alpha \otimes \lambda|^2 \\ &\quad + \int_\Sigma \operatorname{Re} \langle g_1, (e^\alpha - e^{-\alpha})(e^\alpha + e^{-\alpha}|\lambda|^2) \rangle + \operatorname{Re} \langle d\alpha, (1 - e^{-2\alpha})d|\lambda|^2 \rangle. \end{aligned}$$

The last term is bounded below by

$$(7.12) \quad 2 \operatorname{Re} \langle e^{-\alpha} d\alpha \otimes \lambda, (e^\alpha - e^{-\alpha}) \nabla \lambda \rangle \geq -|e^{-\alpha} d\alpha \otimes \lambda|^2 - |(e^\alpha - e^{-\alpha}) \nabla \lambda|^2.$$

If we set $g_1 = (e^\alpha - e^{-\alpha})|\nabla\lambda|^2/(e^\alpha + e^{-\alpha}|\lambda|^2)$ in (7.11), then (7.11) and (7.12) imply that

$$\begin{aligned}\mathcal{E}_g(\alpha) &\geq \int_{\Sigma} |\Delta_{\Sigma}\alpha + g_1|^2 + \frac{1}{4}|e^\alpha - e^{-\alpha}|^2|e^\alpha + e^{-\alpha}|\lambda|^2|^2 \\ &\geq \int_{\Sigma} \frac{1}{2}|\Delta_{\Sigma}\alpha|^2 - |g_1|^2 + 4|\alpha|^2|\lambda|^2.\end{aligned}$$

using the fact that $|e^\alpha - e^{-\alpha}| \geq 2|\alpha|$ and $e^\alpha + e^{-\alpha}|\lambda|^2 \geq 2|\lambda|$. Technically, g_1 can not be α -dependent. However, for this particular choice, there is a point-wise estimate

$$|g_1| \leq |\nabla\lambda|^2 \max\{1, |\lambda|^{-2}\},$$

from which we deduce that

$$\mathcal{E}_{g_0}(\alpha) + C \geq c \cdot \|\alpha\|_{L_2^2}^2.$$

for some $C \gg 1$ and $1 \gg c > 0$, since $|\lambda|$ is non-vanishing everywhere and

$$2\mathcal{E}_{g_0}(\alpha) \geq \mathcal{E}_g(\alpha) - 2\|g_1\|_2^2. \quad \square$$

Remark 7.7. It is essential for this proof to have $|\lambda|$ non-vanishing. One can also prove Lemma 7.6 using the Weitzenböck formula and the fact that λ is harmonic. The equation $\eta(\alpha) = g$ is a variant of the Kazdan-Warner equation. The third approach is then to apply their general theory. $\diamond$

Back to the proof of Proposition 7.5. Lemma 7.6 allows us to find a weakly convergent subsequence among $\{\alpha_n\}$. Denote the limit as α_0 . Then

$$\mathcal{E}_g(\alpha_0) \leq \lim_{n \rightarrow \infty} \mathcal{E}_g(\alpha_n) = \inf \mathcal{E}_g(\alpha),$$

so the infimum is attained at $\alpha = \alpha_0$. This proves $\eta : L_2^2 \rightarrow L^2$ is surjective. If $\eta(\alpha_1 + \delta\alpha) = \eta(\alpha_1)$, then

$$\Delta_{\Sigma}\delta\alpha + \frac{1}{2}(e^{2\delta\alpha} - 1)e^{2\alpha_1} + \frac{1}{2}(1 - e^{-2\delta\alpha})e^{-2\alpha_1}|\lambda|^2 = 0.$$

By the maximum principle, $\delta\alpha \equiv 0$ on Σ , so η is injective. $\square$

Remark 7.8. By Proposition 7.5, $\mu^{-1}(\vec{\delta})$ contains many $\mathcal{G}^e$ -orbits. They are parametrized by $\mathcal{G}/\mathcal{G}^e = \pi_0(\mathcal{G}) \cong H^1(\Sigma, \mathbb{Z})$. $\diamond$

7.4. The Morse-Bott Condition. In this subsection, we take up the task to verify that $\text{Re}(W_\lambda)$ is a Morse-Bott function on M , which completes the proof of Proposition 7.5.

Proposition 7.9. *For the fundamental Landau-Ginzburg model $(M, W_\lambda, \mathcal{G}(\Sigma))$, $L = \text{Re}(W_\lambda)$ is Morse-Bott if $\lambda \neq 0 \in \Omega_h^1(\Sigma, i\mathbb{R})$. In particular, the Morse-Bott estimate (5.4) continues to hold in our case.*

Proof. Since M is a complex linear space, the tangent space at any $\kappa = (\check{B}, \check{\Psi}) \in M$ is identified with

$$\mathcal{H} := \Omega^1(\Sigma, i\mathbb{R}) \oplus \Gamma(\Sigma, L^+ \oplus L^-),$$

and let $\mathcal{H}_k$ be the completion of $\mathcal{H}$ with respect to the $L^2_{k,\check{B}}$ norm:

$$\|(\delta\check{b}, \delta\check{\Psi})\|_{L^2_{k,\check{B}}}^2 = \sum_{0 \leq j \leq k} \int_{\Sigma} |\nabla^j \delta\check{b}|^2 + |\nabla^j_{\check{B}} \delta\check{\Psi}|^2.$$

This family of norms on the tangent bundle of $M(\Sigma)$ is equivariant under the gauge action of $\mathcal{G}(\Sigma)$. The Lie algebra $\mathfrak{g}$ of $\mathcal{G}(\Sigma)$ is $\Gamma(\Sigma, i\mathbb{R})$ and let $\mathfrak{g}_k = L^2_k(\Sigma, i\mathbb{R})$ be its L^2_k -completions for $k = 0, 1$. By Lemma 4.7, it suffices to prove the extended operator

$$\hat{D}_\kappa = \begin{pmatrix} 0 & 0 & \langle \nabla \mu, \cdot \rangle_{T_q M} \\ 0 & 0 & \langle J \nabla \mu, \cdot \rangle_{T_q M} \\ \langle \nabla \mu, \cdot \rangle_{\mathfrak{g}} & \langle J \nabla \mu, \cdot \rangle_{\mathfrak{g}} & \text{Hess } L \end{pmatrix} : \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathcal{H}_1 \rightarrow \mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathcal{H}_0,$$

is invertible for any $\kappa = (\check{B}, \check{\Psi}) \in \text{Crit}(L)$. Since $\hat{D}_\kappa$ is self-adjoint and Fredholm by the standard elliptic theory, it suffices to verify that $\hat{D}_\kappa$ is injective. Notice that the images of

$$\langle \nabla \mu, \cdot \rangle_{\mathfrak{g}}, \langle J \nabla \mu, \cdot \rangle_{\mathfrak{g}}, D_\kappa := (\langle \nabla \mu, \cdot \rangle_{\mathfrak{g}}, \langle J \nabla \mu, \cdot \rangle_{\mathfrak{g}}, \text{Hess } L)$$

are pairwise orthogonal in $\mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathcal{H}_0$. The first two are injective, because the $\mathcal{G}_\mathbb{C}$ -orbit of κ is free. We focus the last operator

$$D_\kappa : \mathcal{H}_1 \rightarrow \mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathcal{H}_0.$$

Suppose $v = (\delta\check{b}, \delta\check{\Psi}) \in \ker D_\kappa$, then the tangent vector v solves the following equations by (7.5) and (7.8):

$$(7.13) \quad - *_{\Sigma} d_{\Sigma} \delta\check{b} + i \text{Re} \langle i \delta\check{\Psi}, \rho_3(ds) \check{\Psi} \rangle = 0,$$

$$(7.14) \quad d_{\Sigma}^* \delta\check{b} + i \text{Re} \langle i \delta\check{\Psi}, \check{\Psi} \rangle = 0,$$

$$(7.15) \quad (\delta\check{\Psi} \check{\Psi}^* + \check{\Psi} \delta\check{\Psi}^*)_{\Pi} = 0,$$

$$(7.16) \quad D_{\check{B}}^{\Sigma} \delta\check{\Psi} + \rho_2(\delta\check{b}) \check{\Psi} = 0.$$

The key observation is that the third equation (7.15) imposes an algebraic constraint on the spinor $\delta\check{\Psi}$. Recall that entries of $\check{\Psi} = (\check{\Psi}_+, \check{\Psi}_-)$ are non-vanishing everywhere by the proof of Proposition 7.5, so (7.15) implies that

$$(7.17) \quad \delta\check{\Psi} = (h\check{\Psi}_+, -\bar{h}\check{\Psi}_-) = i\beta\check{\Psi} + (\alpha\check{\Psi}_+, -\alpha\check{\Psi}_-),$$

for a complex valued function $h = \alpha + i\beta : \Sigma \rightarrow \mathbb{C}$. By (7.16)(7.17) and the fact that $D_{\check{B}}^{\Sigma} \check{\Psi} = 0$, we have

$$\rho_2(dh + \delta\check{b})\check{\Psi}_+ = 0, \rho_2(-d\bar{h} + \delta\check{b})\check{\Psi}_- = 0.$$

Again, by the non-vanishing property of $\check{\Psi}$,

$$\delta\check{b} = i(*_{\Sigma} d\alpha - d\beta).$$

In other words, $(\delta\check{b}, \delta\check{\Psi})$ is generated by the linearized action by $\mathcal{G}_\mathbb{C}$ at κ . By the gauge fixing condition (7.14), $\beta \equiv 0$. By (7.13), $\alpha \equiv 0$. $\square$

8. POINT-LIKE SOLUTIONS ARE TRIVIAL

With all machineries developed so far, we are now ready to study the monopole equations on $\mathbb{C} \times \mathbb{T}^2$. By [Wan18], for a higher genus Riemann surface $\tilde{\Sigma}$, finite energy solutions to the unperturbed equations on $\mathbb{C} \times \tilde{\Sigma}$ are not trivial in general and can be classified in terms of some algebraic data. In our case, we show the other extreme:

Theorem 8.1. *Take any smooth function $\vec{\delta} \in \text{Lie}(\mathcal{G}) = \Gamma(\Sigma, i\mathbb{R})$ and $\lambda \neq 0 \in \Omega_h^1(\Sigma, i\mathbb{R})$. Consider the fundamental Landau-Ginzburg model $(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma))$ and the gauged Witten equations on the complex plane $\mathbb{C}$:*

$$(8.1) \quad \begin{cases} - * F_{\bar{A}} + \mu = \vec{\delta}, \\ \nabla_{\partial_t}^{\bar{A}} P + J \nabla_{\partial_s}^{\bar{A}} P + \nabla H = 0. \end{cases}$$

where $P : \mathbb{C} \rightarrow M(\Sigma)$ is a smooth map and $\bar{A}$ is a smooth connection in the trivial $\mathcal{G}(\Sigma)$ -bundle. Suppose the analytic energy

$$(8.2) \quad \mathcal{E}_{an}(\bar{A}, P; \mathbb{C}) = \int_{\mathbb{C}} |\nabla_{\bar{A}} P|^2 + |\nabla H|^2 + |F|^2 + |\vec{\delta} - \mu|^2$$

is finite, then $(\bar{A}, P)$ is gauge equivalent to a constant configuration, so $\mathcal{E}_{an}(\bar{A}, P; \mathbb{C}) = 0$.

Proof. We follow the proof of Theorem 5.1. Let $\kappa_* = (\check{b}^0, \check{\Psi}_+^0, \check{\Psi}_-^0)$ be a representative in the critical $\mathcal{G}$ -orbit in $\mu^{-1}(\vec{\delta})$, so

$$(8.3) \quad - *_{\Sigma} d_{\Sigma} \check{b}^0 + \frac{i}{2} (|\check{\Psi}_+^0|^2 - |\check{\Psi}_-^0|^2) + \frac{i}{2} K = \vec{\delta}.$$

Define a family of metrics on the quotient configuration space $M(\Sigma)/\mathcal{G}(\Sigma)$ using L_k^2 -Sobolev norms:

$$d_k([\kappa_1], [\kappa_2]) = \inf_{g \in \mathcal{G}} \|\kappa_1 - g \cdot \kappa_2\|_{L_k^2}.$$

We first verify the condition of Theorem 5.1 by showing that

$$(8.4) \quad d_k([P(t, s)], [\kappa_*]) \rightarrow 0$$

as $z = t + is \rightarrow \infty$ for any $k \geq 0$. Note that $\mathcal{E}_{an}(\bar{A}, P; \mathbb{C})$ coincides with the more classical notion of analytic energy for the monopole equations, cf. [Wan18, Lemma 2.1]:

$$\mathcal{E}_{an}(\bar{A}, P; \mathbb{C}) = \int_{\mathbb{C}} \int_{\Sigma} \frac{1}{4} |F_{At}|^2 + |\nabla_A \Phi|^2 + |(\Phi \Phi^*)_0 + \rho_4(\omega^+)|^2 + \frac{K}{2} |\Phi|^2 + \text{Re} \langle F_{At}^{\Sigma}, \vec{\delta} \rangle,$$

with $\omega = ds \wedge \lambda - \vec{\delta} d\text{vol}_{\Sigma}$. The equation even holds before integrating over $\mathbb{C}$. Let $n = (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \subset \mathbb{C}$ and define

$$(A_n, \Phi_n)(z, x) = (A, \Phi)(z + n, x), \forall z \in \Omega := \overline{B(0, 10)} \text{ and } x \in \Sigma.$$

Then $\{(A_n, \Phi_n)\}$ is a family of solutions on $\Omega \times \Sigma$ with $\mathcal{E}_{an}(A_n, \Phi_n; \Omega) \rightarrow 0$ as $n \rightarrow \infty$, where $\mathcal{E}_{an}(A, \Phi; \Omega)$ is given by Definition 8.2. By the standard compactness theorem [KM07, Theorem 5.1.1], up to gauge transformations, any subsequence of $\{(A_n, \Phi_n)\}$ contains a further subsequence converging in C^∞ -topology in the interior. Let (A_∞, Φ_∞) be the limit. Since $\mathcal{E}_{an}(A_\infty, \Phi_\infty; \Omega) = 0$, it is gauge equivalent to a constant family of κ_* with $\bar{A} = d$. This proves (8.4).

By Proposition 7.5 and Lemma 5.5, the superpotential W_λ is stable, and if $(\bar{A}, P)$ solves the gauged Witten equation (8.1), then

$$\nabla L(P(z)) \equiv 0.$$

Because the superpotential W is stable, $P(z)$ lies in the $\mathcal{G}_\mathbb{C}$ -orbit of κ_* . In terms of Hodge decomposition (7.9), we apply the gauge fixing condition $(\check{b}(z) - \check{b}^0)^1 \equiv 0$, i.e. $\check{b}(z) - \check{b}^0$ is co-closed for each $z \in \mathbb{C}$. It follows that

$$P(z) = e^{\alpha(z)} \kappa_* = (\check{b}^0 + i *_\Sigma d_\Sigma \alpha(z), e^{\alpha(z)+i\theta(z)} \check{\Psi}_+^0, e^{-\alpha(z)+i\theta(z)} \check{\Psi}_-^0)$$

for some smooth function $\alpha \in \Gamma(\mathbb{C} \times \Sigma, \mathbb{R})$ and $\theta : \mathbb{C} \rightarrow \mathbb{R}$. One may kill θ by a further gauge transformation, so we set $\theta \equiv 0$ in the sequel. Write $\bar{A} = d + a_t dt + a_s ds$. The first equation of (8.1) then imply (comparing (7.7)):

$$\begin{aligned} (\partial_t \check{b} - d_\Sigma a_t) + *_\Sigma (\partial_s \check{b} - d_\Sigma a_s) &= 0, \\ (\partial_t \check{\Psi} + a_t \check{\Psi}) + \rho_3(ds)(\partial_s \check{\Psi} + a_s \check{\Psi}) &= 0, \end{aligned}$$

so $a_t = -i\partial_s \alpha$, $a_s = i\partial_t \alpha$. Combining with (8.3), the moment map equation in (8.1) then gives

$$(8.5) \quad (\Delta_\mathbb{C} + \Delta_\Sigma)\alpha + \frac{1}{2}(e^{2\alpha} - 1)|\check{\Psi}_+^0|^2 + \frac{1}{2}(1 - e^{-2\alpha})|\check{\Psi}_-^0|^2 = 0.$$

By the boundary condition (8.4), $\|\alpha(z)\|_\infty \rightarrow 0$ as $z \rightarrow \infty$. The maximum principle then implies that $\alpha \equiv 0$, so $(\bar{A}, P)$ is gauge equivalent to the constant configuration $(P \equiv \kappa_*, \bar{A} \equiv d)$. $\square$

Theorem 8.1 will play an important role in the proof of compactness theorem in the second paper. In practice, it is convenient to work with a weaker condition than the finiteness of the total energy:

$$\mathcal{E}_{an}(\bar{A}, P; \mathbb{C}) < \infty.$$

To state the result, let $I_n = [n-2, n+2]_t \subset \mathbb{R}_t$. Choose a compact domain $\Omega_0 \subset I_0 \times [0, \infty)_s$ with a smooth boundary such that

$$(8.6) \quad I_0 \times [1, 3] \subset \Omega_0 \subset I_0 \times [0, 4].$$

Define $\Omega_{n,R}$ to be the translated domain

$$(8.7) \quad \{(t, s) : (t - n, s - R) \in \Omega_0\} \subset I_n \times [0, \infty)_s$$

for any $n \in \mathbb{Z}$ and $R > 0$.

Definition 8.2. For any region $\Omega \subset \mathbb{C}$ and any configuration $\gamma = (\bar{A}, P)$ or (A, Φ) , define the local energy functional of γ over Ω as

$$\begin{aligned} \mathcal{E}_{an}(\bar{A}, P; \Omega) &= \mathcal{E}_{an}(A, \Phi; \Omega) := \int_\Omega |\nabla_{\bar{A}} P|^2 + |\nabla H|^2 + |F|^2 + |\vec{\delta} - \mu|^2, \\ &= \int_\Omega \int_\Sigma \frac{1}{4} |F_{A^t}|^2 + |\nabla_A \Phi|^2 + |(\Phi \Phi^*)_0 + \rho_4(\omega^+)|^2 + \frac{K}{2} |\Phi|^2 + \text{Re} \langle F_{A^t}^\Sigma, \vec{\delta} \rangle. \end{aligned} \quad \diamond$$

Proposition 8.3. *There exists a constant $\epsilon_* > 0$ with the following significance. Under the assumptions of Theorem 8.1, suppose instead that the local energy functional*

$$\mathcal{E}_{an}(\bar{A}, P; \Omega_{n,R}) < \epsilon_*$$

when $|n| + |R| \gg 1$, then $(\bar{A}, P)$ is gauge equivalent to the constant configuration.

Apparently, Proposition 8.3 implies Theorem 8.1.

Proof. There are two ways to proceed. In the first approach, we apply Theorem 1.3 to show the total analytic energy $\mathcal{E}_{an}(\bar{A}, P; \mathbb{C})$ is actually finite, since the local energy functional $\mathcal{E}_{an}(\bar{A}, P; \Omega_{n,R})$ has exponential decay as $|n| + |R| \rightarrow \infty$. Then our proposition follows from Theorem 8.1.

In the second approach, we adapt the proof of Theorem 8.1 to our situation. There are three major modifications:

Step 1. If ϵ_* is small enough, then the Morse-Bott estimate (5.4) in the proof of Lemma 5.5 still holds for any $P(z)$ when $|z| \gg 1$. This step requires the compactness theorem [KM07, Theorem 5.1.1].

Step 2. In the proof of Lemma 5.5, we concluded from (5.6) that if $E(r_0) > 0$, then

$$E(r) = \int_{B(0,r)} |\nabla H|^2$$

blows up exponentially as $r \rightarrow \infty$. In our case, since $\mathcal{E}_{an}(\bar{A}, P; \Omega_{n,R})$ is uniformly bounded for all $n \in \mathbb{Z}$ and $R \in \mathbb{R}$, $E(r)$ can grow at most in the rate r^2 . We still arrive at a contradiction, so $\nabla H \equiv 0$.

Step 3. Finally, using the stability of the superpotential W , we have to show the equation (8.5) can only have the trivial solution $\alpha \equiv 0$. At this point, we only know α is uniformly bounded on $\mathbb{C} \times \Sigma$ and we argue as follows. If $\alpha : \mathbb{C} \times \Sigma \rightarrow \mathbb{R}$ is a solution of (8.5), then

$$\begin{aligned} \frac{1}{2}(\Delta_{\mathbb{C}} + \Delta_{\Sigma})\alpha^2 &\leq \langle (\Delta_{\mathbb{C}} + \Delta_{\Sigma})\alpha, \alpha \rangle \\ &= -\frac{1}{2}\alpha(e^{2\alpha} - 1)|\check{\Psi}_+^0|^2 - \frac{1}{2}\alpha(1 - e^{-2\alpha})|\check{\Psi}_-^0|^2 \leq -c^2|\alpha|^2 \end{aligned}$$

for some $c > 0$. Let $V(z) = \int_{\{z\} \times \Sigma} \alpha^2$, then $V(z)$ is a bounded subharmonic function on $\mathbb{C}$:

$$(8.8) \quad \Delta_{\mathbb{C}} V(z) \leq -2c^2 V(z) \leq 0,$$

so $V(z)$ is constant. (8.8) implies that $V(z) \equiv 0$. □

9. PROOF OF THEOREM 1.3

Now we present the proof of Theorem 1.3 by generalizing ideas from Section 6. In fact, we do not need any assumptions on $\tilde{\delta} \in \mathfrak{g}$: it can be any smooth function on Σ .

Theorem 9.1. *There exists constants $\epsilon, \zeta > 0$ depending only the metric $g_\Sigma, \lambda \neq 0$ and $\vec{\delta} \in \Gamma(\Sigma, i\mathbb{R})$ with the following significance. Suppose a configuration $\gamma = (\bar{A}, P)$ solves the gauged Witten equations (8.1) on $\mathbb{H}_+^2$ and $\mathcal{E}_{an}(\gamma; \Omega_{n,R}) < \epsilon$ for any $n \in \mathbb{Z}$ and $r \geq 0$, then*

$$\mathcal{E}_{an}(\gamma; \Omega_{n,R}) < e^{-\zeta R}.$$

Here the subset $\Omega_{n,R} \subset \mathbb{H}_+^2$ is defined by (8.7).

Proof. We adapt the proof of Theorem 6.1 and follow the notations from Proposition 7.9. Let $(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma))$ be the fundamental Landau-Ginzburg model introduced in Subsection 7.2. For any $\kappa = (\check{B}, \check{\Psi}) \in M$, recall that $\mathcal{H}_k$ is the completion of the tangent space $T_\kappa M$ with respect to the $L_{k,\check{B}}^2$ norm for any $k \geq 0$:

$$\|(\delta\check{b}, \delta\check{\Psi})\|_{L_{k,\check{B}}^2}^2 = \sum_{0 \leq j \leq k} \int_\Sigma |\nabla^k \delta\check{b}|^2 + |\nabla_{\check{B}}^k \delta\check{\Psi}|^2.$$

We claim that the trilinear tensors defined in the proof of Theorem 6.1:

$$(9.1) \quad \begin{aligned} \langle \nabla \cdot \text{Hess } H(\cdot), \cdot \rangle : \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 &\rightarrow \mathbb{R}, \\ \langle \text{Hess } \mu(\cdot), \cdot \otimes \cdot \rangle : \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathfrak{g}_1 &\rightarrow \mathbb{R}, \end{aligned}$$

are bounded operators. Indeed, take tangent vectors $v_i = (\delta\check{b}_i, \delta\check{\Psi}_i) \in T_\kappa M$ for $i = 1, 2$. Using (7.5) and (7.6), we compute that:

$$\begin{aligned} \text{Hess } L(v_1) &= (\rho_2^{-1}(\check{\Psi}\delta\check{\Psi}_1^* + \delta\check{\Psi}_1\check{\Psi}^*)_0, D_{\check{B}}^\Sigma \delta\check{\Psi}_1 + \rho_2(\delta\check{b}_1)\check{\Psi}), \\ (\nabla_{v_2} \text{Hess } L)(v_1) &= (\rho_2^{-1}(\delta\check{\Psi}_2\delta\check{\Psi}_1^* + \delta\check{\Psi}_1\delta\check{\Psi}_2^*)_0, \rho_2(\delta\check{b}_2)\delta\check{\Psi}_1 + \rho_2(\delta\check{b}_1)\delta\check{\Psi}_2), \\ \langle \text{Hess } \mu(v_1), v_2 \rangle &= i \text{Re} \langle i\delta\check{\Psi}_1, \rho_3(ds)\delta\check{\Psi}_2 \rangle. \end{aligned}$$

Hence, tensors in (9.1) are independent of $\gamma \in M$ and involve only point-wise multiplications of sections. Since $L_{1,\check{B}}^2 \hookrightarrow L^3$ in dimension 2 (with a uniform norm independent of $\check{B}$), and the multiplication map $L^3 \times L^3 \times L^3 \rightarrow L^1$ is bounded, our claim follows.

Now we come to analyze the differential operators

$$\begin{aligned} D_\kappa : \mathcal{H}_1 &\rightarrow \mathcal{H}_0 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_0, \\ v = (\delta\check{b}, \delta\check{\Psi}) &\mapsto (\text{Hess } H(v), \langle \nabla \mu, v \rangle, \langle J\nabla \mu, v \rangle), \end{aligned}$$

and $J\langle \nabla \mu, \cdot \rangle : \mathfrak{g}_1 \rightarrow \mathcal{H}_0$, $\xi \mapsto (-d\xi, \xi\check{\Psi})$.

Lemma 9.2. *Suppose $\kappa_* = (\check{B}_*, \check{\Psi}_*) \in M$ is a reference point in the critical orbit $O_* = \mu^{-1}(\vec{\delta}) \cap \text{Crit}(L)$. Then for any $\epsilon > 0$, we can find an $L^{2+\epsilon_1}$ neighborhood $\Omega(\epsilon_1)$ of κ_* (or a $\mathcal{G}$ -invariant neighborhood of O_*) such that for any $\kappa = (\check{B}, \check{\Psi}) \in \Omega$, $v \in T_\kappa M$ and $\xi \in \mathfrak{g}$, we have*

$$(9.2) \quad \|D_\kappa(v)\|_{L^2(\Sigma)} \geq c\|v\|_{L_{1,\check{B}}^2} \quad \text{and} \quad \|J\langle \nabla \mu, \xi \rangle\|_2 \geq c\|\xi\|_{L_1^2}$$

for some $c > 0$.

Proof of Lemma. If $\kappa = \kappa_* = (\check{B}_*, \check{\Psi}_*)$, then estimates (9.2) follow from the injectivity of the extended operator $\hat{D}_\kappa$ in the proof of Proposition 7.9. In general, let $w = \kappa - \kappa_*$. Then $\hat{D}_\kappa(v) = \hat{D}_{\kappa_*}v + I(w, v)$ for a bilinear operator $I(\cdot, \cdot)$ involving only point-wise multiplication, so

$$\|I(w, v)\|_2 \leq \|w\|_p \|v\|_q \leq C(p) \|w\|_p \|v\|_{L^2_{1, \check{B}}}$$

for any positive (p, q) with $1/p + 1/q = 1/2$. The constant $C(p)$ arises from the Sobolev embedding $L^2_{1, \check{B}} \hookrightarrow L^q$ where $2 < p, q < \infty$. Similarly, we have

$$\|v\|_{L^2_{1, \check{B}_*}} \geq \|v\|_{L^2_{1, \check{B}}} - C(p) \|w\|_p \|v\|_{L^2_{1, \check{B}}}.$$

Thus, estimate (9.2) hold when $\|w\|_p \ll 1$. $\square$

Back to the proof of Theorem 1.3. Now the proof of Theorem 6.1 can proceed with no difficulty. Following the notations therein, define

$$u(z) = \|\nabla^A P\|_{L^2(\Sigma)}^2 + \|F\|_{L^2(\Sigma)}^2 \text{ and } w(z) = \|\nabla^A P\|_{\mathcal{H}_1}^2 + \|F\|_{L^2_1(\Sigma)}^2.$$

For any number $\eta > 0$, by the compactness theorem [KM07, Theorem 5.2.1], there exists a constant $\epsilon(\eta) > 0$ such that for any configuration $\gamma = (\bar{A}, P)$ with

$$\mathcal{E}_{an}(\gamma, \Omega_0) = \int_{\Omega} u(z) dz < \epsilon(\eta),$$

we have the point-wise estimate

$$0 \leq u(z) \leq w(z) \leq \eta, \forall z \in \Omega'_0$$

for a smaller domain $\Omega'_0 \subset \Omega_0$. By taking $\eta \ll 1$, we ensures that $P(z) \in \Omega(4)$, where $\Omega(4)$ is the neighborhood obtained in Lemma 9.2 with $\epsilon_1 = 4$.

Now replace Ω_0 by $\Omega_{n,R}$ for any $n \in \mathbb{Z}$ and $R > 1$. The Bochner-type formual in Lemma 6.4 then implies that

$$0 \geq \frac{1}{2} \Delta_{\mathbb{H}_+^2} u + \zeta^2 w - C w^{3/2} \geq \frac{1}{2} \Delta_{\mathbb{H}_+^2} u + \frac{\zeta^2}{2} w \geq \frac{1}{2} (\Delta_{\mathbb{H}_+^2} + \zeta^2) u,$$

for some $\zeta > 0$, and we use Lemma 6.3 to conclude. $\square$

10. FINITE ENERGY SOLUTIONS ON $\mathbb{R}_s \times \mathbb{T}^2$

In this section, we study the 3-dimensional Seiberg-Witten equations:

$$(10.1) \quad \begin{cases} \frac{1}{2} \rho_3(F_{B^t}) - (\Psi \Psi^*)_0 - \rho_3(\omega) = 0, \\ D_B \Psi = 0. \end{cases}$$

on the cylinder $\mathbb{R}_s \times \Sigma$ with $\omega = ds \wedge \lambda - \vec{\delta} \text{vol}_\Sigma$. Here, $\lambda \neq 0 \in \Omega_h^1(\Sigma)$ is a harmonic 1-form on Σ and $\vec{\delta} \in \text{Lie}(\mathcal{G}) = \Gamma(\Sigma, i\mathbb{R})$ is an imaginary valued function. In terms of Landau-Ginzburg models, the equations (10.1) are equivalent to the downward gradient flow equation of $L = \text{Re } W_\lambda$:

$$(10.2) \quad \frac{d}{ds} p(s) + \nabla L \circ p = 0.$$

where $p(s) = (\check{b}(s), \check{\Psi}(s)) : \mathbb{R} \rightarrow M(\Sigma)$ is a smooth path in the Kähler manifold $M(\Sigma)$. Its relation with (10.1) can be seen by setting

$$B = \frac{d}{ds} + \check{B}_0 + \check{b}(s) \text{ and } \Psi = \check{\Psi}(s) \text{ on } \{s\} \times \Sigma.$$

We require the path p to have finite analytic energy:

$$(10.3) \quad \mathcal{E}_{an}(p) := \int_{\mathbb{R}_s} \left| \frac{d}{ds} p \right|^2 + |\nabla L|^2 < \infty.$$

Under (10.3), it is easily seen that the path p has a finite length and

$$q_{\pm} = \lim_{s \rightarrow \pm\infty} p(s)$$

lies in $\text{Crit}(L)$. By the Cauchy-Riemann equation (2.1), (10.2) is also the Hamiltonian flow equation for $H = \text{Im } W_{\lambda}$. Hence, a flowline connecting q_- and q_+ can exist only if

$$(10.4) \quad L(q_-) \geq L(q_+) \text{ and } H(q_-) = H(q_+).$$

Proposition 10.1. *When $\lambda \in \Omega_h^1(\Sigma) \cong H^1(\Sigma, i\mathbb{R})$ is not a multiple of any integral classes, any finite energy solution of (10.1) has to be $\mathbb{R}_s$ -translation invariant, i.e $p(s) \equiv q_- = q_+$.*

Proof. By Proposition 7.5, q_- and q_+ lie in the same $\mathcal{G}(\Sigma)$ -orbit, so $q_+ = u \cdot q_-$ for some $u : \Sigma \rightarrow S^1$. Hence,

$$W_{\lambda}(q_-) - W_{\lambda}(q_+) = - \int_{\Sigma} \langle u^{-1} du, \lambda \rangle_{h_M}.$$

In particular, $H(q_-) - H(q_+) = 4\pi^2 ([\frac{u^{-1} du}{2\pi i}] \cup [\frac{\lambda}{2\pi i}])[\Sigma]$. If λ is not a multiple of an integral class, this pairing can not be zero unless $[\frac{u^{-1} du}{2\pi i}] = 0 \in H^1(\Sigma, \mathbb{Z})$. This implies that

$$\mathcal{E}_{an}(p) = 2(L(q_-) - L(q_+)) = 0,$$

so the path p has to be $\mathbb{R}_s$ -translation invariant. $\square$

Remark 10.2. A solution of (10.1) can be viewed as an S^1 -invariant solution of the 4-dimensional equations (7.1) on $S^1 \times \mathbb{R}_s \times \mathbb{T}^2$. When g_{Σ} is flat, Proposition 10.1 follows from a theorem of Taubes, see [Tau01, Proposition 4.4]. $\diamond$

Taubes' theorems provide another simple condition that precludes non-trivial solutions.

Proposition 10.3. [Tau01, Proposition 4.7] *Suppose g_{Σ} is flat and $\vec{\delta} \in \text{Lie}(\mathcal{G})$ is a constant function on Σ , i.e. $\vec{\delta} \equiv \delta$ for some $\delta \in i\mathbb{R}$. If $\delta \neq 0$, then any finite energy solution of (10.1) has to be $\mathbb{R}_s$ -translation invariant, i.e $p(s) \equiv q_- = q_+$.*

Proof. The proof is adapted from [Tau01, P. 486-487]. The closed 2-form $\omega = ds \wedge \lambda - \delta d\text{vol}_{\Sigma}$ that is used to perturb the equations (10.1) is a parallel 2-form. Thus, the spin bundle S^+ splits as

$$(10.5) \quad L_{\omega}^+ \oplus L_{\omega}^-$$

with $\rho_3(\omega)$ acting on by a diagonal matrix

$$m \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $m = \sqrt{|\delta|^2 + |\lambda|^2}$ is a positive number. The splitting (10.5) is parallel. Let $p(t) = (b(t), \Phi(t))$ be a solution of (10.2) on $\mathbb{R}_s \times \Sigma$. Write $\Phi(t) = \sqrt{2m}(\alpha(t), \beta(t))$ with respect to the decomposition (10.5). By Witten's vanishing spinor argument [Wit94], $\beta \equiv 0$. The first equation of (10.1) becomes

$$\frac{1}{2}F_{B^t} = (1 - |\alpha|^2)\omega.$$

The curvature form F_{B^t} is closed, so $d(1 - |\alpha|^2) \wedge \omega = 0$. Alternatively, it says that $\langle d|\alpha|^2, *_3\omega \rangle = 0$. The dual tangent vector of $i *_3\omega$ generates a flow on $\mathbb{R}_s \times \Sigma$ along which $|\alpha|^2$ stays constant. Since $\delta \neq 0$, this flow translates the spatial coordinate s as time varies. Since $|\alpha| \rightarrow 1$ as $s \rightarrow \pm\infty$, $|\alpha| \equiv 1$. This completes the proof. $\square$

When g_Σ is flat, $\delta = 0$ and λ is a multiple of an integral class, there is a non-trivial moduli space of flowlines for any pair (q_-, q_+) subject to (10.4). They are pulled back from vortices on the cylinder $\mathbb{R}_s \times S^1$. These moduli spaces are not regular; their expected dimensions are always zero from the index computation. For more details, see [Tau01, Section 4(d)(e)]. Here is an immediate corollary of Proposition 10.3.

Corollary 10.4. *Suppose g_Σ is flat and $\vec{\delta} \equiv \delta \in i\mathbb{R}$ is a constant function on Σ . If $\delta \neq 0$, then for any $e^{i\theta} \in S^1$, any down-ward gradient flowline of the functional $\text{Re}(e^{i\theta}W_\lambda)$:*

$$\partial_s p(s) + \nabla \text{Re}(e^{i\theta}W_\lambda) = 0,$$

has to be a constant path.

Proof. This corollary follows from Proposition 10.3 by noting that

$$e^{i\theta}W_\lambda(\check{b}, \check{\Psi}_+, \check{\Psi}_-) = W_0(\check{b}, \check{\Psi}_+, e^{-i\theta}\check{\Psi}_-) + \langle \check{b}, e^{-i\theta}\lambda \rangle_{h_M}. \quad \square$$

APPENDIX A. THE MAXIMUM PRINCIPLE

This appendix is meant to state a version of maximum principle, from which one can easily deduce exponential decay. The author is greatly indebted to Ao Sun for teaching me this elementary and enlightening proof (of Lemma A.1). Laplacian operators are assumed to have positive spectra. In particular, over the complex plane,

$$\Delta_{\mathbb{C}} := -\partial_t^2 - \partial_s^2.$$

This sign convention is adopted throughout this paper.

Proposition A.1. *Take $\zeta > 0$. Suppose $u : \mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s \rightarrow \mathbb{R}$ is a bounded $\mathcal{C}^2$ -function on the upper half plane such that*

- (1) $(\Delta_{\mathbb{C}} + \zeta^2)u \leq 0$, and
- (2) $u(t, 0) \leq 0$ for any $t \in \mathbb{R}_t$.

Then $u(t, s) \leq 0$ for any $(t, s) \in \mathbb{H}_+^2$.

Proof. Choose a smooth cut-off function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

- $\psi \equiv 1$ on $[0, 1]$,
- $\psi \equiv 0$ on $[2, \infty)$ and
- $0 \leq \psi \leq 1$.

Let $\phi_R : \mathbb{C} \rightarrow \mathbb{R}$ such that $\phi_R(z) = \psi(|z|/R)$. Then

- $\phi_R \equiv 1$, $\nabla \phi_R = 0$ and $\Delta_{\mathbb{C}} \phi_R \equiv 0$ for $|z| < R$;
- $\phi_R \equiv 0$, $\nabla \phi_R = 0$ and $\Delta_{\mathbb{C}} \phi_R \equiv 0$ for $|z| > 2R$;
- For some $L > 0$, $|\nabla \phi_R| < \frac{L}{R}$ and $|\Delta_{\mathbb{C}} \phi_R| < \frac{L}{R^2}$.

Only the last property requires some explanation. In general, we have

$$\begin{aligned} \nabla \phi_R &= \frac{1}{R} \phi' \left(\frac{|z|}{R} \right), \\ \Delta_{\mathbb{C}} \phi_R &= -(\partial_r^2 \phi_R + \frac{1}{|z|} \partial_r \phi_R) = -\frac{1}{R^2} \psi'' \left(\frac{|z|}{R} \right) - \frac{1}{|z|R} \psi' \left(\frac{|z|}{R} \right). \end{aligned}$$

Suppose $u(z_0) > 0$ for some $z_0 \in \mathbb{H}_+^2$. Consider $u_R(z) := u(z)\phi_R(z - z_0)$. Then $u_R(z) \equiv 0$ when $|z - z_0| > 2R$ and

$$u_R(t, 0) \leq 0.$$

Hence, $\max u_R$ is attained at some z_1 in the interior of $\mathbb{H}_+^2$. Let $N = \|u\|_{\infty}$, so

$$(A.1) \quad 0 < u(z_0) = u_R(z_0) \leq u_R(z_1) \leq N\phi_R(z_1 - z_0).$$

At $z_1 \in \mathbb{H}_+^2$, we have

$$0 = (\nabla u_R)(z_1) = (\nabla u \cdot \phi_R(z_1 - z_0) + u \nabla \phi_R(z_1 - z_0)),$$

so $\nabla u(z_1) = -u(z_1) \nabla \phi_R(z_1 - z_0) / \phi_R(z_1 - z_0)$. Then the relation $\Delta_{\mathbb{C}} u \leq -\zeta^2 u$ gives:

$$\begin{aligned} 0 &\leq \Delta_{\mathbb{C}} u_R(z_1) = \phi_R(z_1 - z_0) (\Delta_{\mathbb{C}} u) + u (\Delta_{\mathbb{C}} \phi_R(z_1 - z_0)) - 2 \nabla \phi_R(z_1 - z_0) \cdot \nabla u, \\ &\leq u(-\zeta^2 \phi_R + \Delta_{\mathbb{C}} \phi_R + \frac{2|\nabla \phi_R|^2}{\phi_R})(z_1 - z_0). \end{aligned}$$

However, this inequality is violated when $R \gg 0$ which yields a contradiction. By (A.1), when $R \gg 1$,

$$|\Delta_{\mathbb{C}} \phi_R + \frac{2|\nabla \phi_R|^2}{\phi_R}|(z_1 - z_0) \leq \frac{L}{R^2} + \frac{2L^2}{R^2} \cdot \frac{N}{u(z_0)} < \frac{\zeta^2 u(z_0)}{N} \leq \zeta^2 \phi_R(z_1 - z_0).$$

Moreover, $u(z_1) > 0$. This completes the proof. $\square$

Corollary A.2. Take $\zeta > 0$. Suppose $u : \mathbb{H}_+^2 = \mathbb{R}_t \times [0, \infty)_s \rightarrow \mathbb{R}$ is a bounded $\mathcal{C}^2$ -function on the upper half plane such that

- (1) $(\Delta_{\mathbb{C}} + \zeta^2)u \leq 0$, and
- (2) $u(t, 0) \leq M_1$ for some $M_1 > 0$ and any $t \in \mathbb{R}_t$.

Then $u(t, s) \leq M_1 e^{-\zeta s}$ for any $(t, s) \in \mathbb{H}_+^2$.

Proof. Let $v(t, s) = M e^{-\zeta s}$. Then $(\Delta_{\mathbb{C}} + \zeta^2)v = 0$ and $v(t, 0) = M$ for any $t \in \mathbb{R}_t$. Apply Proposition A.1 to $u - v$ to conclude. $\square$

There are analogous statements for a strip of finite length. Their proofs are similar and omitted here.

Proposition A.3. Take $\zeta > 0$. Suppose $u : \mathbb{R}_t \times [0, 2R]_s \rightarrow \mathbb{R}$ is a bounded $\mathcal{C}^2$ -function such that

- (1) $(\Delta_{\mathbb{C}} + \zeta^2)u \leq 0$, and
- (2) $u(t, s) \leq 0$ for any $t \in \mathbb{R}_t$ and $s \in \{0, 2R\}$.

Then $u(t, s) \leq 0$ for any $(t, s) \in \mathbb{R}_t \times [0, 2R]_s$.

Corollary A.4. Take $\zeta > 0$. Suppose $u : \mathbb{R}_t \times [0, 2R]_s \rightarrow \mathbb{R}$ is a bounded $\mathcal{C}^2$ -function such that

- (1) $(\Delta_{\mathbb{C}} + \zeta^2)u \leq 0$, and
- (2) $u(t, s) \leq M_1$ for any $t \in \mathbb{R}_t$ and $s \in \{0, 2R\}$.

Then $u(t, s) \leq M_1 \cdot \frac{\cosh(\zeta(s-R))}{\cosh(\zeta R)}$ for any $(t, s) \in \mathbb{R}_t \times [0, 2R]_s$.

APPENDIX B. A BOCHNER-TYPE FORMULA

The purpose of this appendix is to summarize some useful formulae from Riemannian geometry for a gauged Landau-Ginzburg model (M, W, G, ρ) . In particular, we will prove a Bochner-type formula for a generalized vortex (A, P) on $\mathbb{H}_+^2$. Some formulae become more transparent when M is a complex linear space. Readers are recommended to skim these formulae quickly when first reading and come back to their proofs when it is necessary.

B.1. Some Useful Formulae. Recall that (M, ω, J, g) is a Kähler manifold and G is a compact abelian Lie group acting on M holomorphically and isometrically. (G, ρ) admits a moment map $\mu : M \rightarrow \mathfrak{g}$ which is G -invariant. $W = L + iH$ is a $G_{\mathbb{C}}$ -invariant holomorphic function on M , called the superpotential.

For any $\xi \in \mathfrak{g}$, let $\tilde{\xi}$ be the vector field on M induced from the action (G, ρ) . The convention of the moment map used in our paper is that

$$\iota(\tilde{\xi})\omega = -d\langle \mu, \xi \rangle_{\mathfrak{g}}$$

Since $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, we will frequently use the following equivalent equation:

$$(B.1) \quad \tilde{\xi} = J\langle \nabla \mu, \xi \rangle_{\mathfrak{g}},$$

where $\nabla \mu \in \Gamma(M, TM \otimes \mathfrak{g})$ is viewed as a $\mathfrak{g}$ -valued vector field on M .

Lemma B.1. For a gauged Landau-Ginzburg model (M, W, G, ρ) as defined in Definition 3.1, we have the following identities:

- (1) $\nabla L + J\nabla H = 0$.
- (2) $\text{Hess } L + J \circ \text{Hess } H = 0$.
- (3) $J \circ \text{Hess } H + \text{Hess } H \circ J = 0$, i.e. the Hessian $\text{Hess } H$ of H anti-commutes with J .
- (4) $J \circ \text{Hess } \mu = \text{Hess } \mu \circ J$, i.e. the Hessian $\text{Hess } \mu$ commutes with J .
- (5) $\langle \nabla \mu, \nabla H \rangle = \langle J\nabla \mu, \nabla H \rangle = 0$.
- (6) $\langle \nabla \mu, \tilde{\xi} \rangle = 0$ for any $\xi \in \mathfrak{g}$.

Proof. The first identity (1) is the Cauchy-Riemann equation.

Since M is kähler, the almost complex structure J is parallel, i.e. $\nabla J = 0$, so (2) follows from (1) by taking the covariant derivative.

Both $\text{Hess } L$ and $\text{Hess } H$ are symmetric operators with respect to the metric g , so by (2), we have

$$J \circ \text{Hess } H = (J \circ \text{Hess } H)^T = (\text{Hess } H)^T \circ J^T = -\text{Hess } H \circ J.$$

Since the metric g is G -invariant, for any $\xi \in \mathfrak{g}$, the Lie derivative of g is zero:

$$\mathcal{L}_\xi g = 0.$$

This implies that for any vector fields V, U , $\langle \nabla_V \tilde{\xi}, U \rangle + \langle \nabla_U \tilde{\xi}, V \rangle = 0$. Using the defining equation (B.1), we conclude that $\langle J \text{Hess } \mu, \tilde{\xi} \rangle_{\mathfrak{g}}$ is an anti-symmetric operator, so $\text{Hess } \mu$ commutes with J . This proves (4).

Finally, since H is G -invariant, $\langle \tilde{\xi}, \nabla H \rangle = 0$. By (B.1), $\langle J \nabla \mu, \nabla H \rangle = 0$. The other identity in (5) follows from the G -invariance of L and the first identity (1).

(6) follows from the fact that $\mu : M \rightarrow \mathfrak{g}$ is G -invariant. $\square$

A smooth connection $A = d + a$ in the trivial principal bundle $G \times \mathbb{H}_+^2 \rightarrow \mathbb{H}_+^2$ allows us to take covariant derivatives of a map $P : \mathbb{H}_+^2 \rightarrow M$. It is also important to know covariant derivatives of a vector field v along P (for each $x \in \mathbb{H}_+^2$, $v(x)$ is a tangent vector at $P(x) \in M$).

Recall that for any tangent vector $(x, V) \in T\mathbb{H}_+^2$, $\nabla_V^A P$ is defined by the property:

$$(B.2) \quad \nabla_V^A P = \left. \frac{d}{dt} \rho(e^{t \cdot a(V)}) P(\gamma(t)) \right|_{t=0}$$

where $\gamma : [0, 1] \rightarrow \mathbb{H}_+^2$ is a path with $\gamma(0) = x$ and $\gamma'(0) = V$. The action of G extends to the tangent bundle TM of M :

$$\rho_{TM}(g)(p, v) \mapsto (\rho(g)p, \rho(g)_* v).$$

If v is a vector field along $P \circ \gamma(t)$, it is reasonable to define:

$$(B.3) \quad \nabla_V^A v := \nabla_{\nabla_V^A P} \rho(e^{t \cdot a(V)})_* v.$$

It is enlightening to find a concrete formula of $\nabla_V^A v$ without using the group action. By the defining property of the moment map (B.1), we have

$$(B.4) \quad \nabla_V^A P = V \cdot P + \tilde{a}(V) = P_*(V) + J \langle \nabla \mu, a(V) \rangle.$$

Lemma B.2. *The covariant derivative of a vector field v equals:*

$$\nabla_V^A v = \nabla_{P_* V} v + J \langle \text{Hess } \mu(v), a(V) \rangle.$$

where $\nabla_{P_* V} v$ denotes the covariant derivative with respect to the Levi-Civita connection.

Remark B.3. The correction term $J \langle \text{Hess } \mu(v), a(V) \rangle$ reflects the dependence on the connection 1-form a . It is linear in a , v and V as expected. $\diamond$

Proof. The formal proof is to linearize the equation (B.4) along the tangent vector $v(x) \in T_{P(x)}M$. Let us make this intuition precise. Consider a variation of $P \circ \gamma$ along the vector field v :

$$Q(r, t, s) = \rho(e^{r \cdot a(V)}) \exp_{P \circ \gamma(t)}(sv(t)).$$

When $r \equiv 0$, $Q(0, t, s)$ is a variation of the path $P \circ \gamma(t)$. Indeed, $Q(0, t, 0) \equiv P \circ \gamma(t)$. When $s \equiv 0$, the covariant derivative of P is defined as (comparing (B.2)):

$$\begin{aligned} \left. \frac{d}{dt} Q(t, t, 0) \right|_{t=0} &= \nabla_V^A P = \frac{d}{dt} Q(0, t, 0) + \frac{d}{dt} Q(t, 0, 0) \Big|_{t=0} \\ &= P_*(V) + J\langle \nabla \mu, a(V) \rangle. \end{aligned}$$

Let $U_1 = \frac{d}{dt} Q(t, t, s)$ and $U_2 = \frac{d}{ds} Q(t, t, s)$. Then $U_1 = U_3 + U_4$ with

$$U_3 = (\partial_2 Q)(t, t, s), U_4 = (\partial_1 Q)(t, t, s).$$

$U_4 = J\langle \nabla \mu, a(V) \rangle$. When $t = s = 0$, $U_3 = P_* V$. Moreover, $[U_2, U_4] = 0$. By (B.4), we have

$$\begin{aligned} \nabla_V^A v &= \nabla_{U_1} U_2 \Big|_{t=s=0} = \nabla_{U_3} U_2 + \nabla_{U_4} U_2 \Big|_{t=s=0} = \nabla_{P_* V} v + \nabla_{U_4} U_2 \Big|_{t=s=0} \\ &= \nabla_{P_* V} v + J\langle \text{Hess } \mu(v), a(V) \rangle. \quad \square \end{aligned}$$

The next lemma concerns the curvature tensor of ∇^A . Since we are merely interested in the manifold $\mathbb{H}_+^2$, it suffices to work with vector fields ∂_t and ∂_s .

Lemma B.4. *Write $T = \nabla_{\partial_t}^A P$ and $S = \nabla_{\partial_s}^A P$ for short. The following properties hold for any configuration (A, P) and any vector field v along P :*

- (1) *The connection ∇^A is equivariant under the gauge transformation $u(A, P) = (A - u^{-1}du, u \cdot P)$, i.e.*

$$u_*(\nabla_V^A P) = \nabla_V^{u(A)} u(P), \quad u_*(\nabla_V^A v) = \nabla_V^{u(A)} u_* v.$$

where $u_ v$ is the vector field along $u(P)$.*

- (2) *If v is induced from a G -invariant vector field on M , then $\nabla_{\partial_s}^A v = \nabla_S v$.*
(3) *$(\nabla_{\partial_t}^A \nabla_{\partial_s}^A - \nabla_{\partial_s}^A \nabla_{\partial_t}^A)P = J\langle \nabla \mu, F_A(\partial_t, \partial_s) \rangle = -\tilde{F}$ where $F = -*_2 F_A$.*
(4) *For any vector fields v, w on $\text{Im } P \subset M$,*

$$\partial_s \langle v, w \rangle = \langle \nabla_{\partial_s}^A v, w \rangle + \langle v, \nabla_{\partial_s}^A w \rangle,$$

i.e. the connection ∇^A is unitary.

- (5) *The curvature tensor of ∇^A is given by*

$$(\nabla_{\partial_t}^A \nabla_{\partial_s}^A - \nabla_{\partial_s}^A \nabla_{\partial_t}^A)v = R_M(T, S)v + J\langle \text{Hess } \mu(v), F_A(\partial_t, \partial_s) \rangle.$$

where $R_M(\cdot, \cdot)$ denotes the Riemannian curvature tensor on M .

Proof. The property (1) follows from the defining property (B.2) and (B.3) of ∇_A .

If v is induced from a G -variant vector field on M , then for any $g \in G$, $\rho(g)_* v = v$. By (B.3),

$$\nabla_V^A v := \nabla_{\nabla_V^A P} \rho(e^{t \cdot a(V)})_* v = \nabla_{\nabla_V^A P} v.$$

This proves (2). For (3), if $F_A \equiv 0$ near a point $x \in \mathbb{H}_+^2$, then we apply a gauge transformation u so that the connection 1-form $a \equiv 0$ near x . Thus,

$$u_*(\nabla_{\partial_t}^A \nabla_{\partial_s}^A - \nabla_{\partial_s}^A \nabla_{\partial_t}^A)P = \nabla_{\partial_t} \partial_s u(P) - \nabla_{\partial_s} \partial_t u(P) = 0.$$

This shows the commutator is at least proportional to $*F_A$. To work out the general case, we apply (B.4) and Lemma B.2:

$$\begin{aligned}\nabla_{\partial_t}^A \nabla_{\partial_s}^A P &= \nabla_{\partial_t P} S + J \langle \text{Hess } \mu(S), a(\partial_t) \rangle \\ &= \nabla_{\partial_t P} \partial_s P + \nabla_{\partial_t P} J \langle \nabla \mu, a(\partial_s) \rangle + J \langle \text{Hess } \mu(S), a(\partial_t) \rangle \\ &= \nabla_{\partial_t P} \partial_s P + J \langle \nabla \mu, \partial_t a(\partial_s) \rangle - \langle \text{Hess } \mu(\langle \nabla \mu, a(\partial_s) \rangle), a(\partial_t) \rangle \\ &\quad + J \langle \text{Hess } \mu(\partial_t P), a(\partial_s) \rangle + J \langle \text{Hess } \mu(\partial_s P), a(\partial_t) \rangle.\end{aligned}$$

At this point, we need the following fact. For any $\xi, \eta \in \mathfrak{g}$,

$$\langle \text{Hess } \mu(\langle \nabla \mu, \xi \rangle), \eta \rangle = \langle \text{Hess } \mu(\langle \nabla \mu, \eta \rangle), \xi \rangle.$$

This follows from the fact that $\nabla_{\tilde{\xi}} \tilde{\eta} - \nabla_{\tilde{\eta}} \tilde{\xi} = [\tilde{\xi}, \tilde{\eta}] = \widetilde{[\xi, \eta]} = 0$. This proves (3).

As for (4), we apply the gauge invariance of ∇^A . Alternatively, one uses Lemma B.2 and the fact that

$$(B.5) \quad \langle J \text{Hess } \mu(v), w \rangle + \langle J \text{Hess } \mu(w), v \rangle = 0.$$

since the metric g is G -invariant.

The expression of the curvature tensor (5) requires some work. Again, if $F_A \equiv 0$, we use the gauge invariance of ∇_A , and (5) follows from the definition of R_M . The actually proof is not very tidy. We follow the strategy of (3):

$$\begin{aligned}\nabla_{\partial_t}^A \nabla_{\partial_s}^A v &= \nabla_{\partial_t P} \nabla_{\partial_s}^A v + J \langle \text{Hess } \mu(\nabla_{\partial_s}^A v), a(\partial_t) \rangle \\ &= \nabla_{\partial_t P} \nabla_{\partial_s P} v + J \nabla_{\partial_t P} \langle \text{Hess } \mu(v), a(\partial_s) \rangle + J \langle \text{Hess } \mu(\nabla_{\partial_s}^A v), a(\partial_t) \rangle \\ &= \nabla_{\partial_t P} \nabla_{\partial_s P} v + J \langle \text{Hess } \mu(v), \partial_t a(\partial_s) \rangle + J \langle (\nabla_{\partial_t P} \text{Hess } \mu)(v), a(\partial_s) \rangle \\ &\quad + J \langle \text{Hess } \mu(\nabla_{\partial_t P} v), a(\partial_s) \rangle + J \langle \text{Hess } \mu(\nabla_{\partial_s P} v), a(\partial_t) \rangle \\ &\quad + J \langle \text{Hess } \mu(\langle J \text{Hess } \mu(v), a(\partial_s) \rangle), a(\partial_t) \rangle\end{aligned}$$

There are six terms in the expression. The fourth and fifth ones will also occur in that of $\nabla_{\partial_s}^A \nabla_{\partial_t}^A v$, so canceled out. The second term contributes to

$$J \langle \text{Hess } \mu(v), F_A(\partial_t, \partial_s) \rangle.$$

The first one contributes to $R_M(\partial_t P, \partial_s P)$. To better organize the proof, we point out two lemmas from which the identity (5) will follow. Note that

$$R_M(T, S) - R_M(\partial_t P, \partial_s P) = R_M(\partial_t P, \tilde{a}(\partial_s)) + R_M(\tilde{a}(\partial_t), \partial_s P) + R_M(\tilde{a}(\partial_t), \tilde{a}(\partial_s)).$$

We have to identify these terms in the expression of $\nabla_{\partial_t}^A \nabla_{\partial_s}^A v - \nabla_{\partial_s}^A \nabla_{\partial_t}^A v$. $\square$

Lemma B.5. *For any $\xi \in \mathfrak{g}$ and vector fields u, w on M , we have*

$$R_M(w, \tilde{\xi})u = \langle \nabla_w(J \text{Hess } \mu)(u), \xi \rangle.$$

Proof of Lemma. Differentiating (B.5) yields that

$$\langle \nabla_u(J \text{Hess } \mu)(v), w \rangle + \langle \nabla_u(J \text{Hess } \mu)(w), v \rangle = 0.$$

The key observation is that for any vectors u, v, w , we have

$$\langle R_M(u, v)w, J \nabla \mu \rangle = -\langle \nabla_u(J \text{Hess } \mu)(v), w \rangle - \langle \nabla_v(J \text{Hess } \mu)(w), u \rangle.$$

Indeed, we use the symmetry of curvature tensor to compute:

$$\begin{aligned}\langle R_M(u, v)w, J\nabla\mu \rangle &= -\langle R_M(u, v)J\nabla\mu, w \rangle \\ &= -\langle \nabla_u(J \operatorname{Hess} \mu)(v), w \rangle + \langle \nabla_v(J \operatorname{Hess} \mu)(u), w \rangle \\ &= -\langle \nabla_u(J \operatorname{Hess} \mu)(v), w \rangle - \langle \nabla_v(J \operatorname{Hess} \mu)(w), u \rangle.\end{aligned}$$

This expression is unchanged if we permute (u, v, w) . Using the symmetry

$$R_M(u, v)w + R_M(v, w)u + R_M(w, u)v = 0$$

from Riemannian geometry, we conclude that

$$\langle \nabla_u(J \operatorname{Hess} \mu)(v), w \rangle + \langle \nabla_v(J \operatorname{Hess} \mu)(w), u \rangle + \langle \nabla_w(J \operatorname{Hess} \mu)(u), v \rangle = 0.$$

In particular, this implies $\langle R_M(u, v)w, J\nabla\mu \rangle = \langle \nabla_w(J \operatorname{Hess} \mu)(u), v \rangle$. Finally, note that $\tilde{\xi} = \langle \nabla\mu, \xi \rangle$ and

$$\langle R_M(w, \tilde{\xi})u, v \rangle = \langle R_M(u, v)w, \tilde{\xi} \rangle = \langle \nabla_w(J \operatorname{Hess} \mu)(u), v \otimes \xi \rangle. \quad \square$$

Lemma B.6. *For any $\xi, \eta \in \mathfrak{g}$ and any vector field v on M ,*

$$R_M(\tilde{\xi}, \tilde{\eta})v = J\langle \operatorname{Hess} \mu(\langle J \operatorname{Hess} \mu(v), \eta \rangle_{\mathfrak{g}}), \xi \rangle_{\mathfrak{g}} - J\langle \operatorname{Hess} \mu(\langle J \operatorname{Hess} \mu(v), \xi \rangle_{\mathfrak{g}}), \eta \rangle_{\mathfrak{g}}.$$

Proof of Lemma. This identity is equivalent to that

$$(B.6) \quad \langle R_M(\tilde{\xi}, \tilde{\eta})v, w \rangle = -\langle \nabla_v \tilde{\eta}, \nabla_w \tilde{\xi} \rangle + \langle \nabla_v \tilde{\xi}, \nabla_w \tilde{\eta} \rangle.$$

Recall that $\mathcal{L}_{\tilde{\xi}}g = 0$ implies $\langle \nabla_v \tilde{\xi}, w \rangle = -\langle \nabla_w \tilde{\xi}, v \rangle$. Hence, the right hand side of (B.6) equals

$$\begin{aligned}I &:= -\langle \nabla_v \tilde{\eta}, \nabla_w \tilde{\xi} \rangle + \langle \nabla_v \tilde{\xi}, \nabla_w \tilde{\eta} \rangle \\ &= -v \cdot \langle \tilde{\eta}, \nabla_w \tilde{\xi} \rangle + \langle \tilde{\eta}, \nabla_v \nabla_w \tilde{\xi} \rangle + w \cdot \langle \tilde{\eta}, \nabla_v \tilde{\xi} \rangle - \langle \tilde{\eta}, \nabla_w \nabla_v \tilde{\xi} \rangle\end{aligned}$$

Since $\nabla_{\tilde{\xi}}\tilde{\eta} = \nabla_{\tilde{\eta}}\tilde{\xi}$, we have

$$\langle \nabla_v \tilde{\xi}, \tilde{\eta} \rangle = -\langle \nabla_{\tilde{\eta}} \tilde{\xi}, v \rangle = -\langle \nabla_{\tilde{\xi}} \tilde{\eta}, v \rangle = \langle \nabla_v \tilde{\eta}, \tilde{\xi} \rangle.$$

Therefore, $v \cdot \langle \tilde{\eta}, \tilde{\xi} \rangle = 2\langle \nabla_v \tilde{\xi}, \tilde{\eta} \rangle$. Moreover,

$$\begin{aligned}-v \cdot \langle \tilde{\eta}, \nabla_w \tilde{\xi} \rangle + w \cdot \langle \tilde{\eta}, \nabla_v \tilde{\xi} \rangle &= -\frac{1}{2}v \cdot w \langle \tilde{\eta}, \tilde{\xi} \rangle + \frac{1}{2}w \cdot v \langle \tilde{\eta}, \tilde{\xi} \rangle. \\ &= -\frac{1}{2}[v, w] \langle \tilde{\eta}, \tilde{\xi} \rangle = -\langle \tilde{\eta}, \nabla_{[v, w]} \tilde{\xi} \rangle.\end{aligned}$$

Finally, we conclude that $I = \langle R_M(v, w)\tilde{\xi}, \tilde{\eta} \rangle = \langle R_M(\tilde{\xi}, \tilde{\eta})v, w \rangle$. $\square$

B.2. A Bochner-Type Formula. It will be convenient to introduce the operator D_p for any $p \in M$. For any tangent vector $v \in T_p M$, its image is defined as:

$$v \in T_p M \mapsto D_p(v) := (\text{Hess}_p H(v), \langle \nabla \mu, v \rangle, \langle \nabla \mu, Jv \rangle) \in T_p M \oplus \mathfrak{g} \oplus \mathfrak{g}.$$

The family of operators D_p forms a bundle map $TM \rightarrow TM \oplus \mathfrak{g} \oplus \mathfrak{g}$, which is induced from the Hamiltonian H and the moment map μ .

The Bochner's formula [Pet16, P. 334] was originally stated for a harmonic function $u : M \rightarrow \mathbb{R}$ on a Riemannian manifold M . It computes the Laplacian of $|\nabla u|^2$:

$$0 = \frac{1}{2} \Delta_M |\nabla u|^2 + |\text{Hess } u|^2 + \text{Ric}(\nabla u, \nabla u).$$

We provide a formula in the same spirit for a generalized vortex on $\mathbb{H}_+^2$, with D_p playing the role of $\text{Ric}(\cdot, \cdot)$. The Laplacian operator Δ_M or $\Delta_{\mathbb{H}_+^2}$ is always assumed to have positive spectrum. In particular,

$$\Delta_{\mathbb{H}_+^2} = -(\partial_t^2 + \partial_s^2).$$

Theorem B.7. Write $T = \nabla_{\partial_t}^A P$ and $S = \nabla_{\partial_s}^A P$ for short. For a solution (A, P) to the gauged Witten equation (4.1) on $\mathbb{H}_+^2$, we have identities:

(1) The Laplacian $\frac{1}{2}(-\Delta_{\mathbb{H}_+^2})|T|^2 = \frac{1}{2}(\partial_s^2 + \partial_t^2)|T|^2$ of $|T|^2$ is equal to

$$\begin{aligned} & |\nabla_{\partial_s}^A T|^2 + |\nabla_{\partial_t}^A T|^2 + |D_P(T)|^2 + \langle R_M(S, T)S, T \rangle \\ & + \langle (\nabla_T \text{Hess } H)(\nabla H), T \rangle + \langle \text{Hess } \mu(2JS - T), T \otimes F \rangle. \end{aligned}$$

(2) Similarly, $\frac{1}{2}(-\Delta_{\mathbb{H}_+^2})|S|^2 = \frac{1}{2}(\partial_s^2 + \partial_t^2)|S|^2$ is equal to

$$\begin{aligned} & |\nabla_{\partial_t}^A S|^2 + |\nabla_{\partial_s}^A S|^2 + |D_P(S)|^2 + \langle R_M(T, S)T, S \rangle \\ & + \langle (\nabla_S \text{Hess } H)(\nabla H), S \rangle + \langle \text{Hess } \mu(-2JT - S), S \otimes F \rangle. \end{aligned}$$

(3) The Laplacian $\frac{1}{2}(-\Delta_{\mathbb{H}_+^2})|F|_{\mathfrak{g}}^2$ of $|F|_{\mathfrak{g}}^2$ is equal to

$$|\partial_s F|_{\mathfrak{g}}^2 + |\partial_t F|_{\mathfrak{g}}^2 + |\langle \nabla \mu, F \rangle_{\mathfrak{g}}|^2 + 2\langle \text{Hess } \mu(JS), T \otimes F \rangle$$

Define $\nabla^A P = dt \otimes T + ds \otimes S$ and $\text{Hess}_A P$ as $\omega_i \otimes \omega_j \otimes \nabla_{e_i}^A \nabla_{e_j}^A P$ for $(e_1, e_2) = (\partial_t, \partial_s)$ and $(\omega_1, \omega_2) = (dt, ds)$.

Corollary B.8. We have the following identity for $\Delta_{\mathbb{H}_+^2}(|\nabla^A P|^2 + |F|_{\mathfrak{g}}^2)$:

$$\begin{aligned} 0 = & \frac{1}{2} \Delta_{\mathbb{H}_+^2} (|\nabla^A P|^2 + |F|_{\mathfrak{g}}^2) + |\text{Hess}_A P|^2 + |\nabla F|_{\mathfrak{g}}^2 + |D_P(\nabla^A P)|^2 + |\langle \nabla \mu, F \rangle_{\mathfrak{g}}|^2 \\ & + 2\langle R(S, T)S, T \rangle + \langle (\nabla_T \text{Hess } H)(\nabla H), T \rangle + \langle (\nabla_S \text{Hess } H)(\nabla H), S \rangle \\ & + 6\langle \text{Hess } \mu(JS), T \otimes F \rangle - \langle \text{Hess } \mu(T), T \otimes F \rangle - \langle \text{Hess } \mu(S), S \otimes F \rangle. \end{aligned}$$

One thinks of $-(\nabla_{\partial_t}^A T + \nabla_{\partial_s}^A S)$ as the connection Laplacian of P . Let us define

$$-\Delta_A P := \nabla_{\partial_t}^A \nabla_{\partial_t}^A P + \nabla_{\partial_s}^A \nabla_{\partial_s}^A P = \nabla_{\partial_t}^A T + \nabla_{\partial_s}^A S.$$

We start with a useful formula of $\Delta_A P$.

Lemma B.9. $-\Delta_A P = J\tilde{F} + \text{Hess } H(\nabla H)$.

Proof. Apply the operator $-\nabla_{\partial_t}^A + J\nabla_{\partial_s}^A$ to the equation (4.7b). Using the fact that $\text{Hess } H$ anti-commutes with J (Lemma B.1 (3)) and Lemma B.4 (2)(3), we have

$$\begin{aligned} 0 &= (-\nabla_{\partial_t}^A + J\nabla_{\partial_s}^A)(T + JS + \nabla H) \\ &= -(\nabla_{\partial_t}^A T + \nabla_{\partial_s}^A S) - J[\nabla_{\partial_t}^A S - \nabla_{\partial_s}^A T] - \text{Hess } H(T + JS) \\ &= \Delta_A P + J(\tilde{F}) + \text{Hess } H(\nabla H). \end{aligned} \quad \square$$

Remark B.10. It is enlightening to work out Lemma B.9 and Theorem B.7 concretely in some special cases. In Example 3.2, we have $M = \mathbb{C}$, $W \equiv 0$ and $\mu = \frac{i}{2}|z|^2$. The metric on $\mathbb{C}$ is flat, and we identify $T_z M$ with $\mathbb{C}$ for each $z \in M$. Hence,

$$\nabla \mu(z) = z \otimes i \text{ and } \text{Hess } \mu = 1_{\mathbb{C}} \otimes i.$$

Take $\delta = \frac{i}{2} \in i\mathbb{R} = \mathfrak{g}$. In this case, the equation (4.1) recovers the vortex equation on $\mathbb{H}_+^2 \subset \mathbb{C}$:

$$\begin{cases} \bar{\partial}_A \Phi &= 0 \\ - * F_A + \mu &= \frac{i}{2} \end{cases}$$

where A is a unitary connection and $\Phi : \mathbb{H}_+^2 \rightarrow \mathbb{C}$ is a smooth function. In this case, Lemma B.9 says

$$\Delta_A \Phi = \langle F, i \rangle_{\mathfrak{g}} \Phi$$

which follows from the Weitzenböck formula. Since $\nabla H \equiv 0$, $T = -JS$. The first two identities in Theorem B.7 yield:

$$0 = \frac{1}{2} \Delta_{\mathbb{H}_+^2} |\nabla_A \Phi|^2 + |\text{Hess}_A \Phi|^2 + |D(\nabla_A \Phi)|^2 - 3 \langle F, i \rangle_{\mathfrak{g}} |\nabla_A \Phi|^2.$$

In this case, $D(v) = (\langle \nabla \mu, v \rangle, \langle \nabla \mu, Jv \rangle)$ and $|D(v)|_{\mathfrak{g}}^2 = |\Phi|^2 |v|^2$. In the meanwhile, we have

$$0 = \frac{1}{2} \Delta_{\mathbb{H}_+^2} |F|^2 + |\nabla F|_{\mathfrak{g}}^2 + |\Phi|^2 |F|_{\mathfrak{g}}^2 - |\nabla_A \Phi|^2 \langle F, i \rangle_{\mathfrak{g}}.$$

These formulae were first proved in [JT80, Proposition 6.1]. ◇

Proof of Theorem B.7. Let us start with $\partial_s^2 |T|^2$. By Lemma B.4 (4),

$$\frac{1}{2} \partial_s^2 |T|^2 = \partial_s \langle \nabla_{\partial_s}^A T, T \rangle = |\nabla_{\partial_s}^A T|^2 + \langle T, \nabla_{\partial_s}^A \nabla_{\partial_s}^A T \rangle.$$

By Lemma B.4 (3)(5), we have

$$\begin{aligned} \nabla_{\partial_s}^A \nabla_{\partial_s}^A T &= \nabla_{\partial_s}^A (\nabla_{\partial_t}^A S + \tilde{F}) \\ &= \nabla_{\partial_t}^A \nabla_{\partial_s}^A S + R_M(S, T)S + \langle J \text{Hess } \mu(S), F \rangle + \nabla_{\partial_s}^A \tilde{F}, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{2} (\partial_t^2 + \partial_s^2) |T|^2 &= |\nabla_{\partial_s}^A T|^2 + |\nabla_{\partial_t}^A T|^2 + \langle \nabla_{\partial_t}^A (-\Delta_A P), T \rangle + \langle R_M(S, T)S, T \rangle \\ &\quad + \langle J \text{Hess } \mu(S), F \otimes T \rangle + \langle \nabla_{\partial_s}^A \tilde{F}, T \rangle. \end{aligned}$$

By using the equation (4.7a), we attempt to extract some positivity from the last term:

$$\begin{aligned}\langle \nabla_{\partial_s}^A \tilde{F}, T \rangle &= \langle \nabla_{\partial_s}^A \langle J \nabla \mu, F \rangle, T \rangle. \\ &= \langle J \text{Hess } \mu(S), F \otimes T \rangle + \langle J \nabla \mu, \partial_s F \rangle, T \rangle. \\ &= \langle J \text{Hess } \mu(S), F \otimes T \rangle + \langle \langle \nabla \mu, JT \rangle, \langle \nabla \mu, S \rangle \rangle_{\mathfrak{g}}\end{aligned}$$

By Lemma B.1 (5), $\langle \nabla \mu, J \nabla H \rangle = 0$. Using (4.7b), we have

$$\langle \nabla \mu, JT \rangle = \langle \nabla \mu, S - J \nabla H \rangle = \langle \nabla \mu, S \rangle.$$

Hence, $\langle \nabla_{\partial_s}^A \tilde{F}, T \rangle = \langle J \text{Hess } \mu(S), F \otimes T \rangle + |\langle \nabla \mu, JT \rangle|_{\mathfrak{g}}$.

Now we deal with the term involving $\Delta_A P$, using Lemma B.9. We exploit the fact that $\text{Hess } H(\nabla H)$ is a G -invariant vector field on M and Lemma B.4 (2):

$$\begin{aligned}\nabla_{\partial_t}^A (-\Delta_A P) &= \nabla_{\partial_t}^A (J \tilde{F} + \text{Hess } H(\nabla H)) \\ &= -\nabla_{\partial_t}^A \langle \nabla \mu, F \rangle + (\nabla_T \text{Hess } H)(\nabla H) + \text{Hess } H^2(T). \\ &= -\langle \text{Hess } \mu(T), F \rangle - \langle \nabla \mu, \partial_t F \rangle + (\nabla_T \text{Hess } H)(\nabla H) + \text{Hess } H^2(T).\end{aligned}$$

Note that $-\langle \langle \nabla \mu, \partial_t F \rangle, T \rangle = \langle \langle \nabla \mu, \langle \nabla \mu, T \rangle \rangle, T \rangle = |\langle \nabla \mu, T \rangle|_{\mathfrak{g}}^2$.

Combining all these together, we obtain

$$\begin{aligned}\frac{1}{2}(\partial_t^2 + \partial_s^2)|T|^2 &= |\nabla_{\partial_t}^A T|^2 + |\nabla_{\partial_s}^A T|^2 + \langle R_M(S, T)S, T \rangle \\ &\quad + |\langle \nabla \mu, T \rangle|_{\mathfrak{g}}^2 + |\langle \nabla \mu, JT \rangle|_{\mathfrak{g}}^2 + |\text{Hess } H(T)|^2 \\ &\quad + \langle (\nabla_T \text{Hess } H)(\nabla H), T \rangle + \langle \text{Hess } \mu(2JS - T), F \otimes T \rangle.\end{aligned}$$

The formula of $\frac{1}{2}(\partial_t^2 + \partial_s^2)|S|^2$ is proved in a similar way.

Finally, let us deal with the Laplacian of $|F|_{\mathfrak{g}}^2$. By (4.7a), we have

$$\begin{aligned}\frac{1}{2}\partial_s^2|F|_{\mathfrak{g}}^2 &= |\partial_s F|_{\mathfrak{g}}^2 + \langle F, \partial_s^2(-\mu) \rangle_{\mathfrak{g}}. \\ &= |\partial_s F|_{\mathfrak{g}}^2 - \langle \text{Hess } \mu(S), S \otimes F \rangle - \langle \nabla \mu, \nabla_{\partial_s}^A S \otimes F \rangle.\end{aligned}$$

By Lemma B.9, we have

$$\begin{aligned}\langle -\nabla \mu, (\nabla_{\partial_t}^A T + \nabla_{\partial_s}^A S) \otimes F \rangle &= \langle -\nabla \mu, J \tilde{F} \otimes F \rangle + \langle -\nabla \mu, \text{Hess } H(\nabla H) \otimes F \rangle \\ &= |\langle \mu, F \rangle_{\mathfrak{g}}|^2 + \langle \text{Hess } \mu(\nabla H), \nabla H \otimes F \rangle.\end{aligned}$$

At the last step, we applied the identity:

$$(B.7) \quad \langle \text{Hess } \mu(X), \nabla H \rangle + \langle \nabla \mu, \text{Hess } H(X) \rangle = 0$$

for $X = \nabla H$. By Lemma B.1 (5), $\langle \nabla \mu, \nabla H \rangle \equiv 0$. Expanding the expression $X \cdot \langle \nabla \mu, \nabla H \rangle \equiv 0$ yields (B.7). $\square$

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