MONOPOLES AND LANDAU-GINZBURG MODELS II: FLOER HOMOLOGY

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Abstract. This is the second paper of this series. We define the monopole Floer homology for 3-manifolds with torus boundary, extending the work of Kronheimer-Mrowka for closed 3-manifolds. The Euler characteristic of this Floer homology recovers the Milnor torsion invariant of the 3-manifold by a theorem of Meng-Taubes.

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Part 1. Introduction

1.1. **Motivations.** The Seiberg-Witten Floer homology of a closed oriented 3-manifold is defined by Kronheimer-Mrowka [KM07] and has greatly influenced the study of 3-manifold topology. The aim of this current paper is to generalize their construction for any compact oriented 3-manifold $(Y, \partial Y)$ with torus boundary, with the potential to recover the knot Floer homology (for a knot in S^3), both the hat-version $\widehat{\mathrm{HFK}}$ and the minus-version HFK^- as special cases. The Euler characteristic of this Floer homology group will recover the Milnor torsion invariant of $(Y, \partial Y)$ by a theorem of Meng-Taubes [MT96].

In the first paper of this series [Wan20], we discussed an infinite dimensional gauged Landau-Ginzburg model for any Riemannian 2-torus (Σ, g_Σ)

$$(M(\Sigma), W_\lambda, \mathcal{G}(\Sigma)) \tag{1.1}$$

whose gauged Witten equations on the complex plane $\mathbb{C}$ recover the Seiberg-Witten equations on the product manifold $\mathbb{C}\times\Sigma$. This allows us to borrow many ideas from symplectic topology and interpret our construction as Lagrangian Floer homology in the infinite dimensional setting. The author would like to refer readers to [Wan20, Section 2] for more details on this heuristic. For the present paper, we focus on the analytic details that implement these ideas. The use of Landau-Ginzburg models will be minimized.

One motivation of this work is to define invariants for knots and links inside S^3. Within the framework of Heegaard Floer Homology, this goal has been accomplished via the construction of knot Floer homology by Ozsváth-Szabó [OS04] and independently Rasmussen [Ras03]. See [Man16] for a nice survey on their constructions. A long term goal of our program is to interpret their works in the context of gauge theory and hopefully provide new insights for future research.

It has been believed [Man16] that the knot Floer homology of (S^3, K) encodes something about the Seiberg-Witten equations on $\mathbb{R}_t$ times the knot complement $S^3\backslash N(K)$. This heuristic can be approached using the invariants constructed in this paper, which apply to any knot and link complements. The **conjectural** relation is as follows

$$\begin{aligned} &HM_*(Y) \rightsquigarrow \mathrm{HFK}^-_*(S^3, K) && \text{if } Y = S^3\backslash N(K), \\ &HM_*(Y) \rightsquigarrow \widehat{\mathrm{HFK}}_*(S^3, K) \text{ or } KHM_*(S^3, K) && \text{if } Y = S^3\backslash N(K\cup m), \end{aligned}$$

where m is a meridian of $K \subset S^3$. The evidence is not strong, but they do have the same Euler characteristics.

Some constructions of knot Floer homology that uses gauge theory already exist in the literature. Motivated by the sutured manifolds technique developed by Juhász [Juh06, Juh08], Kronheimer-Mrowka defined the monopole knot Floer homology KHM_* in [KM10], as the analogue of $\widehat{\mathrm{HFK}}$ in Heegaard Floer homology. By further exploring this idea, Li [Li19] proposed a construction of HFK^- in the Seiberg-Witten theory using a direct system of sutures on the knot complement.

Our construction will follow a more direct approach. We will make $(Y, \partial Y)$ into a complete Riemannian manifold by attaching cylindrical ends, and define the monopole Floer homology as an infinite dimensional Morse theory, as we explain in the next section. In particular, it is reminiscent of the original construction of Kronheimer-Mrowka [KM07] for closed 3-manifolds.

1.2. **The Setup.** To state our results, let Y be a compact oriented 3-manifold whose boundary $\partial Y \cong \Sigma := \coprod_{1\leqslant i\leqslant n} \mathbb{T}^2_i$ is a disjoint union of 2-tori. Throughout this paper, we assume that Y is connected and its boundary ∂Y is non-empty. The Floer homology of $(Y, \partial Y)$ that we construct will rely on some auxiliary data on the boundary Σ including

- a choice of flat metric g_Σ of Σ;
- an imaginary-valued harmonic 1-form $0 \neq \lambda \in \Omega^1_h(\Sigma, i\mathbb{R})$;
- an imaginary-valued harmonic 2-form $\mu \in \Omega^2_h(\Sigma, i\mathbb{R})$ such that the triple (g_Σ, λ, μ) satisfies conditions (P4)(P5)(P7) in Section 3;

We denote such a quadruple $(Y, g_\Sigma, \lambda, \mu)$ by a thickened letter $\mathbb{Y}$. The boundary data (g_Σ, λ, μ) will play essential roles in the proof of compactness theorems. One may think of them as a way to close up the boundary of Y, so analytically it behaves like a closed 3-manifold. The monopole Floer homology $HM_*(\mathbb{Y})$ can be viewed as an invariant of Y relative to the gauged Landau-Ginzburg model (1.1).

We are only interested in the *spin*^{c} structure $\widehat{\mathfrak{s}}_{std}$ on Σ such that

$$c_1(S^+)[\mathbb{T}^2_i] = 0$$

on each connected component of Σ. A relative *spin*^{c} structure $\widehat{\mathfrak{s}}$ of Y is a *spin*^{c} structure $\mathfrak{s}$ together with an identification of $\mathfrak{s}$ with $\widehat{\mathfrak{s}}_{std}$ on the boundary Σ. For each relative *spin*^{c} manifold $(\mathbb{Y}, \widehat{\mathfrak{s}})$, we will associate a finitely generated module over a base ring $\mathcal{R}$:

$$HM_*(\mathbb{Y}, \widehat{\mathfrak{s}}). \tag{1.2}$$

called *the monopole Floer homology group* of $(\mathbb{Y}, \widehat{\mathfrak{s}})$. This group will be constructed as an infinite dimensional Morse theory of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega$ on the complete Riemannian manifold:

$$\widehat{Y} := Y \coprod_{\Sigma} [0, +\infty)_s \times \Sigma,$$

where the metric on the cylindrical end is $d^2 s + g_\Sigma$ and $\mathcal{L}_\omega$ is perturbed by a closed 2-form $\omega \in \Omega^2(\widehat{Y}, i\mathbb{R})$ (cf. Definition 3.8) such that

$$\omega = \mu + ds \wedge \lambda$$

on $[0,+\infty)_s \times \Sigma$. We will always work with irreducible configurations: there is no need to blow up the configuration space. Critical points of $\mathcal{L}_\omega$ are solutions to the perturbed 3-dimensional Seiberg-Witten equations on $\widehat{Y}$, while the Floer differential is defined by counting solutions on $\mathbb{R}_t \times \widehat{Y}$.

Remark 1.1. The Seiberg-Witten invariant $\underline{\mathrm{SW}}$ of the 3-manifold $(Y, \partial Y)$ is defined in [MT96] as the signed count of critical points of $\mathcal{L}_\omega$ on $\widehat{Y}$. We are using exactly the same setup here. $\diamondsuit$

The set of isomorphism classes of relative *spin*^{c} structures on Y:

$$\mathrm{Spin}^c_R(Y)$$

is a principal homogeneous space over $H^2(Y,\partial Y;\mathbb{Z})$. The desired invariant of $\mathbb{Y}$ is obtained by forming the direct sum,

$$HM_*(\mathbb{Y}) := \bigoplus_{\widehat{\mathfrak{s}}\in \mathrm{Spin}^c_R(Y)} HM_*(\mathbb{Y},\widehat{\mathfrak{s}}), \tag{1.3}$$

which admits an additional homology grading (cf. Section 18 for more details):

- the monopole Floer homology group $HM_*(\mathbb{Y})$ carries a canonical grading by homotopy classes of oriented relative 2-plane fields on Y (i.e. oriented 2-plane fields that take a standard form near Σ); If $\widehat{\mathfrak{s}}$ and $\widehat{\mathfrak{s}}'$ come down to the same underlying *spin*^{c} structure, then their grading sets are the same;
- a homology orientation of Y determines a canonical mod 2 grading of $HM_*(\mathbb{Y})$.

As for the base ring $\mathcal{R}$ in the definition of $HM_*(\mathbb{Y})$,

- we take $\mathcal{R} = \mathbb{Z}$ if $\mu = 0$ and the perturbation is monotone in the sense of Definition 17.3;
- we take $\mathcal{R}$ to be a Novikov ring over $\mathbb{Z}$ otherwise.

1.3. The Euler Characteristic.

By the work of Meng-Taubes [MT96], for any closed 3-manifold Y_0 with the first Betti number $b_1(Y_0) > 0$, the Euler characteristic of the reduced monopole Floer homology $HM^{red}_*(Y_0)$ defined by Kronheimer-Mrowka recovers the Milnor torsion invariant of Y_0. The same statement continues to hold in our case. Since we have followed the same setup of Meng-Taubes in [MT96], the Euler characteristic of $HM_*(\mathbb{Y})$ recovers the Seiberg-Witten invariant $\underline{\mathrm{SW}}(Y,\partial Y)$ defined in their paper. In particular, it is independent of the choice of (g_Σ, λ, μ).

Theorem 1.2 (Theorem 1.1 [MT96])**.** *For any compact oriented 3-manifold $(Y,\partial Y)$ with torus boundary, the Euler characteristic $\chi(HM_*(\mathbb{Y}))$ recovers the Milnor torsion invariant of $(Y,\partial Y)$; in particular, $\chi(HM_*(\mathbb{Y},\widehat{\mathfrak{s}}))$ is non-zero only for finitely many relative spin^c structures $\widehat{\mathfrak{s}} \in \mathrm{Spin}^c_R(Y)$ if $b_1(Y) > 1$.*

Remark 1.3. It is not clear to the author whether $HM_*(\mathbb{Y},\widehat{\mathfrak{s}}) \neq 0$ only for finitely many relative *spin*^{c} structures if $b_1(Y) > 1$. $\diamondsuit$

Remark 1.4. Turaev [Tur98] later refined their result by showing that $\chi(HM_*(\mathbb{Y}))$ as a map

$$\mathrm{Spin}^c_R(Y) \to \mathbb{Z}$$

agrees with the Milnor-Turaev invariant of $(Y, \partial Y)$ up to an overall sign ambiguity. The version proved in [MT96] is slightly weaker: relative *spin*^{c} structures differed by a torsion line bundle in $H^2(Y, \partial Y; \mathbb{Z})$ are not distinguished. Readers are referred to their original paper for the precise statements. ♢

1.4. **The TQFT Property and Invariance.** To state the $(3+1)$ TQFT property enjoyed by HM_*, we introduce a class of cobordisms between 3-manifolds with torus boundary:

$$\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2 \tag{1.4}$$

which are subject to certain constraints. On the level of manifolds, the cobordism

$$(X, W) : (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$$

is a manifold with corners carrying a cobordism $W : \partial Y_1 \to \partial Y_2$ of boundaries. We will require W to be the product cobordism $[-1, 1]_t \times \Sigma$ between ∂Y_1 and ∂Y_2, and as such must have the same number of components. They form the so-called *strict cobordism category* Cob_s. The precise definition is given in Section 3. In the theorem below, we will work instead with SCob_s: each object $(\mathbb{Y}, \widehat{\mathfrak{s}})$ of SCob_s is coupled with a relative *spin*^{c} structure, while morphism sets of SCob_s are the same as those of Cob_s.

Theorem 1.5. *Let $\mathcal{R}$ be the Novikov ring with integral coefficients, then the monopole Floer homology HM_* extends to a functor:*

$$HM_* : \text{SCob}_s \to \mathcal{R}\text{-Mod}$$

from the strict cobordism category SCob_s *to the category of $\mathcal{R}$-modules.*

Remark 1.6. The strict cobordism category Cob_s or SCob_s will be large enough to prove the invariance of $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$ under

- the change of tame perturbations of the Chern-Simons-Dirac functional $\mathcal{L}_\omega$,
- the change of interior metrics of Y and
- the isotopy of the identification map $\partial Y \cong \Sigma$,

as a corollary of Theorem 1.5. In the actual construction of HM_*, we will use a formal enlargement of SCob_s to deal with the orientation issue; see Section 19. ♢

Remark 1.7. Although it is believed that the monopole Floer homology $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$ is independent of the flat metric g_Σ of Σ, this is not proved in this paper. The author wishes to come back to the invariance of g_Σ as well as general cobordism maps in a future paper. If the restriction of $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ on the boundary $W : \partial Y_1 \to \partial Y_2$ or

$$\mathbb{W} : (\partial Y_1, g_{\Sigma_1}, \lambda_1, \mu_1) \to (\partial Y_2, g_{\Sigma_2}, \lambda_2, \mu_2)$$

is a general cobordism, then one would hope to construct a map:

$$HM_*(\mathbb{Y}_1) \otimes HM_*(\mathbb{W}) \to HM_*(\mathbb{Y}_2).$$

When $\mathbb{W}$ is the product cobordism $[-1, 1]_t \times (\Sigma, g_\Sigma)$, it recovers the functor in Theorem 1.5 by inserting the canonical generator $1 \in HM_*(\mathbb{W}) \cong \mathcal{R}$. ♢

1.5. **Some Speculations: Relations with Knot Floer Homology.** The simplest examples of $(Y, \partial Y)$ arise from the knot complements of knots inside S^3. In this case, there exist a unique *spin*^{c} structure $\mathfrak{s}$ on $(Y, \partial Y)$ and $\mathrm{Spin}^c_R(Y)$ is a torsor over

$$H^1(\partial Y; \mathbb{Z})/\operatorname{Im} H^1(Y; \mathbb{Z}) \cong \mathbb{Z}. \tag{1.5}$$

The technical conditions (P4)(P5)(P7) on the boundary data (g_Σ, λ, μ) now require that $\mu = 0$ and

- $[*_\Sigma \lambda] \in \operatorname{Im}(H^1(Y, \mathbb{R}) \to H^1(\Sigma, \mathbb{R}))$ and $[\lambda] \in H^1(\Sigma, \mathbb{R})$ is not any multiple of an integral class. In particular, $\lambda \neq 0$.

The second condition is essentially a constraint on the flat metric g_Σ. The choice of λ will pick up an isomorphism of (1.5). Despite these unpleasant limitations, the monopole Floer homology group $HM_*(\mathbb{Y})$ carries a bi-grading of $\mathbb{Z} \oplus \mathbb{Z}$. The first grading arises from relative *spin*^{c} structures, and

$$HM_*(\mathbb{Y}, \widehat{\mathfrak{s}} + n) = \{0\}$$

when $n \gg 1$ under (1.5). The second one arises from the homology grading by oriented relative 2-plane fields. $HM_*(\mathbb{Y})$ is analogous to $\mathrm{HFK}^-(S^3, K)$ in Heegaard Floer homology, but one important structure is missing: $\mathrm{HFK}^-(S^3, K)$ is an $\mathbb{F}_2[U]$-module with $\deg U = (-1, -1)$.

As noted in the first paper [Wan20, Section 2.3], we would hope to assign an A_∞-algebra $\mathcal{A}$ to the fundamental Landau-Ginzburg model (1.1) and enhance $HM_*(\mathbb{Y})$ into an A_∞-module over $\mathcal{A}$. By passing to the homology category, $HM_*(\mathbb{Y})$ becomes a module over the algebra $H_*(\mathcal{A})$. This is one way that U-action might arise in our picture. However, it would require some new ideas and analytic tools to fully implement this picture, since the proposals of Haydys [Hay15] and Gaiotto-Moore-Witten [GMW15] do not apply directly here.

On the other hand, we pick a meridian m of the knot $K \subset S^3$ and consider the link complement $Y_K := S^3 \backslash N(K \cup m)$. By gluing the two boundary components of Y_K (using any orientation reversing diffeomorphism), we obtain a closed 3-manifold $\tilde{Y}_K$. An internal gluing theorem may then relate $HM_*(\mathbb{Y}_K)$ with the monopole Floer homology of the closure $\tilde{Y}_K$, which is isomorphic to the monopole knot Floer homology $KHM_*(S^3, K)$ by [KM10]. Interested readers are referred to [Wan20, Section 2] for more heuristics on this gluing formula. It is left as an interesting future project and will not be explored in the present paper. At this point, the only computation that we can make is for the unknot $U \subset S^3$, so $Y_U = [-1, 1]_s \times \mathbb{T}^2$ is a finite cylinder and $\widehat{Y}_U = \mathbb{R}_s \times \mathbb{T}^2$.

Proposition 1.8. *For $Y_U = [-1, 1]_s \times \mathbb{T}^2$, the monopole Floer homology $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$ is isomorphic to $\mathcal{R}$ for the standard relative spin^c structure $\widehat{\mathfrak{s}} = \widehat{\mathfrak{s}}_{std}$ and is trivial when $\widehat{\mathfrak{s}} = \widehat{\mathfrak{s}}_{std} \otimes L$ and $[L] \neq 0 \in H^2(Y_U, \partial Y_U; \mathbb{Z})$, regardless of the choice of the boundary data $(g_{\mathbb{T}^2}, \lambda, \mu)$. However, we insist here that the restriction of the metric of Y_U on $\{1\} \times \mathbb{T}^2$ and $\{-1\} \times \mathbb{T}^2$ are the same.*

Proof. This can be checked directly by working with the product metric on $\widehat{Y}_U = \mathbb{R}_s \times \mathbb{T}^2$. □

1.6. **Organizations.** To define the monopole Floer homology $HM_*(\mathbb{Y})$ and implement the construction sketched in Subsection 1.2, we address five analytic problems in this paper, as summarized below. We will follow closely the plotline of the book [KM07].

Compactness. To obtain the right compactification of moduli spaces on $\mathbb{R}_t \times \widehat{Y}$, we have to address the planar end of $\mathbb{R}_t \times \widehat{Y}$:

$$\mathbb{H}^2_+ \times \Sigma := \mathbb{R}_t \times [0, \infty)_s \times \Sigma, \tag{1.6}$$

where the upper half plane $\mathbb{H}^2_+$ is furnished with the Euclidean metric. At this point, we make essential use of results from the first paper [Wan20]. Our constraints on the boundary data (g_Σ, λ, μ) are intended to make the following properties hold:

- finite energy solutions are trivial on $\mathbb{C} \times \Sigma$, namely, they have to be $\mathbb{C}$-translation invariant up to gauge [Wan20, Theorem 1.2 or 8.1].
- finite energy solutions on $\mathbb{R}_s \times \Sigma$ are trivial, namely, they have to be $\mathbb{R}_s$-translation invariant up to gauge. This result is due to Taubes; see [Tau01, Proposition 4.4 & 4.7] or [Wan20, Proposition 10.1 & 10.3] for a version that we exploit.

In Part 2, we will first set up the strict cobordism category Cob_s and derive an energy identity for the Seiberg-Witten equations. Combining results from the first paper [Wan20], this will lead us to the compactness theorem in Section 6. Part 2 is the counterpart of [KM07, Section 4, 5, 16] of the book.

Perturbations. To make moduli spaces regular, we have to apply a further perturbation to the Chern-Simons-Dirac functional $\mathcal{L}_\omega$. Any additional perturbations will happen within the compact region

$$Y = \{s \leqslant 0\}$$

of $\widehat{Y}$. In particular, the monopole equations are always unperturbed on the planar end $\mathbb{H}^2_+ \times \Sigma$. The cylinder functions that we use here are slightly different from those in [KM07, Section 11] since global gauge fixing conditions never give rise to compactly supported perturbations, in the sense of Definition 7.1. Inspired by holonomy perturbations from instanton Floer homology, we will look at embeddings of $S^1 \times D^2$ into Y instead. The construction is carried out in details in Part 3, as the counterpart of [KM07, Section 10, 11].

Linear Analysis. This part is more or less standard. The extended Hessian of $\mathcal{L}_\omega$ on $\widehat{Y}$ as a self adjoint operator has essential spectrum, since $\widehat{Y}$ is a non-compact manifold. This is a main difference of our case from that of closed 3-manifolds. Fortunately, the essential spectrum of $\mathcal{L}_\omega$ is still away from the origin, allowing us to speak of spectrum flow and construct Fredholm operators once we stick to compact perturbations. We will follow the setup of [RS95] and summarize relevant results in Part 4, as the counterpart of [KM07, Section 17] of the book.

Unique Continuation. As our perturbation space is not large enough, we need a better unique continuation property to attain transversality. The non-linear version is stated as follows: if two solutions γ_1, γ_2 to the perturbed monopole equations on $\mathbb{R}_t \times \widehat{Y}$

are gauge equivalent on the slice

$$\{0\} \times Y \text{ where } Y = \{s \leqslant 0\} \subset \widehat{Y},$$

then they are gauge equivalent on the whose space. The proof will rely on the Carleman estimates from [Kim95].

Part 5 is the counterpart of [KM07, Section 7, 12, 15] of the book. The proof of transversality will be accomplished in Section 16.

Orientations. To work with a Novikov ring $\mathcal{R}$ defined over $\mathbb{Z}$ (instead of $\mathbb{F}_2$), we have to orient moduli spaces in a consistent way. For closed 3-manifolds, this is done by first looking at reducible configurations in the blown-up configuration space. See [KM07, Section 20] for details. In our case, we have to adopt a different approach as configurations are never reducible and the action of the gauge group is free.

The situation we have here is similar to that of [KM97] in which case a Riemannian 4-manifold with a conic end is considered, so one may follow the argument of [KM97, Appendix] to orient moduli spaces consistently. The key ingredients are relative determinant line bundles or **relative orientations** that compare two Fredholm operators. We will adopt a more direct approach to this notion without referring to either K-theory or the proof of the index theorem [AS68]. This combinatoric construction is based on a simple proof of excision principle due to Mrowka and is carried out in Appendix B.

Part 6 is the counterpart of [KM07, Section 20, 22, 28] of the book. The canonical grading of $HM_*(\mathbb{Y})$ by homotopy classes of oriented relative 2-plane fields is introduced in Section 18. We will first define monopole Floer homology of $\widehat{Y}$ using $\mathbb{F}_2$-coefficient in Section 17, while orientations are addressed in Section 19.

Most results and proofs in the present paper are intended to generalize the ones in [KM07]. Readers are assumed to have a reasonable understanding of the monopole Floer homology of closed 3-manifolds, at least in the case when $c_1(\mathfrak{s})$ is non-torsion.

Remark 1.9. On the other hand, we point out what will **not** be proved in the present work:

- the exponential decay of solutions in the time-direction, cf. [KM07, Section 13];
- the gluing theorem, cf. [KM07, Section 18, 19].

Once we have set up the rest of the theory correctly, these results follow immediately from corresponding sections of [KM07]. ♢

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Part 2. Three-Manifolds with Torus Boundary

In this part, we define the strict cobordism category Cob_s of oriented 3-manifolds with torus boundary and study the Seiberg-Witten equations on their completions. Throughout this paper, we will use (Σ, g_Σ) to denote a disjoint union of 2-tori. Although most results in the first paper [Wan20] do not require the metric g_Σ to be flat, we will also always assume that g_Σ is flat in this paper so that we can exploit Theorem 2.6 in our construction.

For any compact oriented 3-manifold $(Y, \partial Y)$ with torus boundary $\partial Y \cong \Sigma$, we attach a cylindrical end to obtain a complete Riemannian 3-manifold

$$\widehat{Y} := Y \coprod_{\Sigma} [0,\infty)_s \times \Sigma.$$

For any strict cobordism between two such manifolds,

$$(X, [-1,1]\times\Sigma) : (Y_1, \partial Y_1) \to (Y_2, \partial Y_2),$$

we associate a complete Riemannian manifold $\mathcal{X}$ with a planar end:

$$\mathcal{X} = (-\infty,-1]_t \times \widehat{Y}_1 \cup \widehat{X} \cup [1,+\infty)_t \times \widehat{Y}_2 \text{ where}$$
$$\widehat{X} = X \cup [-1,1]_t \times [0,\infty)_s \times \Sigma.$$

The end point of this part is to prove the compactness theorem (Theorem 6.1) for the Seiberg-Witten moduli spaces on $\mathbb{R}_t \times \widehat{Y}$ and $\mathcal{X}$, which is the cornerstone in any Floer theory. The proof relies on three key ingredients:

- (K1) a uniform upper bound on the analytic energy;
- (K2) finite energy solutions are trivial on $\mathbb{C}\times\Sigma$; in other words, they are gauge equivalent to the unique $\mathbb{C}$-translation invariant solution on $\mathbb{C}\times\Sigma$; see Theorem 2.4 below.
- (K3) finite energy solutions on $\mathbb{R}_s \times \Sigma$ are trivial; in other words, they are gauge equivalent to the unique $\mathbb{R}_s$-translation invariant solution on $\mathbb{R}_s\times\Sigma$; see Theorem 2.6 below. This result is due to Taubes and requires g_Σ to be flat.

Part 2 is organized as follows. In Section 2, we give a brief review of the Seiberg-Witten equations and summarize results from the first paper [Wan20], which gives (K2) and (K3). In Section 3, we define the strict cobordism category and set up the configuration spaces on $\widehat{Y}$ and $\widehat{X}$ respectively. In Section 4, we prove that the quotient configuration space in our case is still Hausdorff and remains a Hilbert manifold after Sobolev completions.

Section 5 is devoted to the derivation of energy equations, which gives (K1). At this point, the existence of certain bounded harmonic forms on $\widehat{Y}$ or $\widehat{X}$ is crucial (see Lemma 3.2 and 3.5 below), and relevant results are summarized in Appendix A. Finally, the compactness theorems are stated and proved in Section 6.

2. Results from the First Paper

In this section, we summarize results from the first paper [Wan20], which are essential to the proof of the compactness theorem (Theorem 6.1) in Section 6. In particular, they ensure properties (K2) and (K3). Throughout this section, we will work primarily with the product manifold $X = \mathbb{C}\times\Sigma$ or $\mathbb{H}^2_+ \times \Sigma$.

2.1. **Review.** Recall that a *spin*^{c} structure $\mathfrak{s}$ on a smooth Riemannian 4-manifold X is a pair (S_X, ρ_4) where $S_X = S^+ \oplus S^-$ is the spin bundle, and the bundle map $\rho_4 : T^*X \to \mathrm{Hom}(S_X, S_X)$ defines the Clifford multiplication. A configuration $\gamma = (A, \Phi) \in \mathcal{C}(X, \mathfrak{s})$ consists of a smooth *spin*^{c} connection A and a smooth section Φ of S^+. Use A^t to denote the induced connection of A on $\bigwedge^2 S^+$. Let ω be a closed 2-form on X and ω^+ denote its self-dual part. The Seiberg-Witten equations perturbed by ω are defined on $\mathcal{C}(X, \mathfrak{s})$ by the formula:

$$\begin{cases} \frac{1}{2}\rho_4(F^+_{A^t}) - (\Phi\Phi^*)_0 - \rho_4(\omega^+) = 0, \\ D^+_A \Phi = 0, \end{cases} \tag{2.1}$$

where $D^+_A : \Gamma(S^+) \to \Gamma(S^-)$ is the Dirac operator and $(\Phi\Phi^*)_0 = \Phi\Phi^* - \frac{1}{2}|\Phi|^2 \otimes \mathrm{Id}_{S^+}$ denotes the traceless part of the endomorphism $\Phi\Phi^* : S^+ \to S^+$.

The gauge group $\mathcal{G}(X) = \mathrm{Map}(X, S^1)$ acts naturally on $\mathcal{C}(X, \mathfrak{s})$:

$$\mathcal{G}(x) \ni u : \mathcal{C}(X, \mathfrak{s}) \to \mathcal{C}(X, \mathfrak{s}),\ (A, \Phi) \mapsto (A - u^{-1}du, u\Phi).$$

The monopole equations (2.1) is invariant under gauge transformations.

Let $\Sigma = (\mathbb{T}^2, g_\Sigma)$ be a 2-torus with a **flat** metric. In the special case when $X = \mathbb{C} \times \Sigma$ is a Kähler manifold furnished with the product metric and the complex orientation, the equations (2.1) can be understood more explicitly, as we explain now.

Let $dvol_{\mathbb{C}}$ and $dvol_\Sigma$ denote volume forms on $\mathbb{C}$ and Σ respectively. The symplectic form on X is given by the sum $\omega_{sym} := dvol_{\mathbb{C}} + dvol_\Sigma$. The spin bundle S^+ splits as $L^+ \oplus L^-$: they are $\mp 2i$ eigenspaces of the bundle map $\rho_4(\omega_{sym}) : S^+ \to S^+$. The spin section Φ decomposes as (Φ_+, Φ_-) with $\Phi_\pm \in \Gamma(X, L^\pm)$. We are only interested in the *spin*^{c} structure on $\mathbb{C} \times \Sigma$ with

$$c_1(S^+)[\Sigma] = 0,$$

so both L^+ and L^- are topologically trivial.

Let $z = t + is$ be the coordinate function on $\mathbb{C}$. The Clifford multiplication $\rho_4 : T^*X \to \mathrm{Hom}(S, S)$ can be constructed by setting:

$$\rho_4(dt) = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix},\ \rho_4(ds) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} :\ S^+ \oplus S^- \to S^+ \oplus S^-,$$

where $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : S^+ = L^+ \oplus L^- \to L^+ \oplus L^-$ is the first Pauli matrix. If we identify $L^+ \cong \mathbb{C}$ and $L^- \cong \bigwedge^{0,1} \Sigma$, then

$$\rho_3(w) := \rho_4(dt)^{-1} \cdot \rho_4(w) = \begin{pmatrix} 0 & -\iota(\sqrt{2}w^{0,1}) \cdot \\ \sqrt{2}w^{0,1} \otimes \cdot & 0 \end{pmatrix} : S^+ \to S^+,$$

for any $x \in \Sigma$ and $w \in T_x\Sigma$.

Remark 2.1. We will frequently work with Clifford multiplications in dimension 3 and 4, denoted by ρ_3 and ρ_4 respectively. Identify $\mathbb{C}$ as $\mathbb{R}_t \times \mathbb{R}_s$, then they are related by

$$\rho_3(w) = \rho_4(dt)^{-1} \cdot \rho_4(w) : S^+ \to S^+,$$

for any $w \in T^*(\mathbb{R}_s \times \Sigma)$. In particular, $\rho_3(ds) = \sigma_1$. $\diamondsuit$

The symplectic form ω_{sym} is parallel, so is the decomposition $S^+ = L^+ \oplus L^-$. Any *spin*^{c} connection A must then split as

$$\nabla_A = \begin{pmatrix} \nabla_{A_+} & 0 \\ 0 & \nabla_{A_-} \end{pmatrix}.$$

We regard $L^+ = \mathbb{C}$ and $L^- = \bigwedge^{0,1}\Sigma$ as bundles over Σ, and they pull back to spin bundles over X via the projection map $X \to \Sigma$. Let $\check{B}_* = (d, \nabla^{LC})$ be the reference connection on $\mathbb{C} \oplus \bigwedge^{0,1}\Sigma \to \Sigma$. We obtain a reference connection A_* on $S^+ \to X$ by setting

$$\nabla_{A_*} = dt \otimes \frac{\partial}{\partial t} + ds \otimes \frac{\partial}{\partial s} + \nabla_{\check{B}_*}. \tag{2.2}$$

One can easily check that A_* is a *spin*^{c} connection. Any other *spin*^{c} connection A differs from A_* by an imaginary valued 1-form $a = A - A_* \in \Gamma(X, iT^*X)$. Their curvature tensors are related by

$$F_A = F_{A_*} + d_X a \otimes \mathrm{Id}_S, \text{ so } F_{A^t} = F_{A^t_*} + 2d_X a.$$

2.2. **Point-Like Solutions.** For this subsection, we will always work with the product 4-manifold $X = \mathbb{C} \times \Sigma$. For our primary applications, the closed 2-form ω in the Seiberg-Witten equations (2.1) will take the special form

$$\omega := \mu + ds \wedge \lambda$$

where

- $\lambda \in \Omega^1_h(\Sigma, i\mathbb{R})$ is an imaginary-valued harmonic 1-form on Σ, and
- $\mu \in \Omega^2_h(\Sigma, i\mathbb{R})$ is an imaginary-valued harmonic 2-form.

Since the metric g_Σ is flat, the 2-form ω is parallel on $X = \mathbb{C} \times \Sigma$.

Assumption 2.2. *The pair* $(\lambda, \mu) \in \Omega^1_h(\Sigma; i\mathbb{R}) \times \Omega^2_h(\Sigma; i\mathbb{R})$ *is said to be admissible if* $\lambda \neq 0$ *and one of the following two conditions holds:*

($V1$) $\mu \neq 0$*;*
($V2$) λ *is not a multiple of any integral class in* $H^1(\Sigma; i\mathbb{Z}) \subset \Omega^1_h(\Sigma, i\mathbb{R})$.

We always assume (λ, μ) *is admissible in this paper.*

For the rest of this section, we will recollect a few theorems from [Wan20] and explain why Assumption 2.2 is crucial. Before that, let us first introduce the notion of local energy functional associated to a configuration (A, Φ) on X.

Definition 2.3 ([Wan20] Definition 8.3)**.** For any region $\Omega \subset \mathbb{C}$ and any configuration $\gamma = (A, \Phi)$ on $\mathbb{C} \times \Sigma$, define *the local energy functional* of γ over Ω as

$$\mathcal{E}_{an}(A, \Phi; \Omega) := \int_\Omega \int_\Sigma \frac{1}{4}|F_{A^t}|^2 + |\nabla_A \Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2. \qquad \diamond$$

A solution γ to the Seiberg-Witten equations (2.1) is called **point-like** if its global energy $\mathcal{E}_{an}(\gamma; \mathbb{C})$ is finite. Let us first describe a constant solution $\gamma_* = (A_*, \Phi_*)$ to (2.1) with $\mathcal{E}_{an}(\gamma_*;\mathbb{C}) = 0$. The *spin*^{c} connection of γ_* is provided by the formula (2.2), while the spinor Φ_* can be written as

$$(r_+, \sqrt{2}\lambda^{0,1}r_-) \in \Gamma(X, \mathbb{C} \oplus \Lambda^{0,1}\Sigma),$$

where $r_\pm$ are real numbers subject to relations:

$$r_+r_- = 1 \text{ and } \frac{i}{2}(|r_+|^2 - |r_-|^2|\lambda|^2) = - *_\Sigma \mu.$$

In particular, Φ_* is a parallel section with respect to A_*.

One consequence of Assumption 2.2 is that γ_* will be the only point-like solution on X up to gauge. For practical reasons, we give a more general statement. Let $I_n = [n-2, n+2]_t \subset \mathbb{R}_t$. Choose a compact domain $\Omega_0 \subset I_0 \times [0, \infty)_s$ with a smooth boundary such that

$$I_0 \times [1,3]_s \subset \Omega_0 \subset I_0 \times [0,4]_s. \tag{2.3}$$

For any $n \in \mathbb{Z}$ and $S \in \mathbb{R}_s$, define $\Omega_{n,S} \subset \mathbb{C}$ to be the translated domain

$$\Omega_{n,S} := \{(t,s) : (t-n, s-S) \in \Omega_0\} \subset I_n \times [0,\infty)_s. \tag{2.4}$$

Theorem 2.4 ([Wan20] Proposition 8.3)**.** *If $\lambda \neq 0$, then there exists a constant $\epsilon_* > 0$ depending only on (g_Σ, λ, μ) with following significance. If a solution $\gamma = (A, \Phi)$ to* (2.1) *on $X = \mathbb{C} \times \Sigma$ satisfies the estimate*

$$\mathcal{E}_{an}(A, P; \Omega_{n,S}) < \epsilon_*$$

when $|n| + |S| \gg 1$, then γ is gauge equivalent to the constant configuration (A_*, Φ_*). In particular, a point-like solution on X is necessarily trivial.

On the other hand, we are also interested in solutions on $\mathbb{H}^2_+ \times \Sigma$ where the upper half plane $\mathbb{H}^2_+ = \mathbb{R}_t \times [0, +\infty)_s$ is furnished with the Euclidean metric. The next theorem says that if a solution γ on $\mathbb{H}^2_+ \times \Sigma$ is close to γ_* everywhere, then γ converges to γ_* exponentially in the spatial direction:

Theorem 2.5 ([Wan20] Theorem 9.1)**.** *There exists constants $\epsilon, \zeta > 0$ depending only the boundary data $(g_\Sigma, \lambda \neq 0, \mu)$ with the following significance. Suppose a configuration $\gamma = (A, \Phi)$ solves the Seiberg-Witten equations* (2.1) *on $\mathbb{H}^2_+ \times \Sigma$ and $\mathcal{E}_{an}(\gamma; \Omega_{n,S}) < \epsilon$ for any $n \in \mathbb{Z}$ and $S \geqslant 0$, then*

$$\mathcal{E}_{an}(\gamma; \Omega_{n,S}) < e^{-\zeta S}.$$

We will improve this theorem in terms of Sobolev norms of $\gamma - \gamma_*$ in Section 6; see Theorem 6.2.

2.3. **Solutions on** $\mathbb{R}_s \times \Sigma$**.** We also study the dimensional reduction of (2.1), the 3-dimensional Seiberg-Witten equations, defined on $\mathbb{R}_s \times \Sigma$:

$$\begin{cases} \frac{1}{2}\rho_3(F_{B^t}) - (\Psi\Psi^*)_0 - \rho_3(\omega) = 0, \\ D_B\Psi = 0. \end{cases} \tag{2.5}$$

where $\omega = \mu + ds \wedge \lambda$ and $\check{\gamma} = (B, \Psi)$ is a configuration on the 3-manifold. To go back to the 4-dimensional case, one may set

$$A = dt \otimes \frac{\partial}{\partial t} + B,\ \Phi(t) = \Psi \text{ on } \mathbb{R}_t \times \mathbb{R}_s \times \Sigma.$$

Then $\mathcal{E}_{an}(A, \Phi; [0,1]_t \times \mathbb{R}_s)$ comes down to the energy of (B, Ψ):

$$\mathcal{E}_{an}(B, \Psi; \mathbb{R}_s) = \int_{\mathbb{R}_s \times \Sigma} \frac{1}{4}|F_{B^t}|^2 + |\nabla_B \Psi|^2 + |(\Psi\Psi^*)_0 + \rho_3(\omega^+)|^2$$

The trivial solution $\check{\gamma}_* = (B_*, \Psi_*)$ of (2.5) can be written as

$$B_* = ds \otimes \frac{\partial}{\partial s} + \begin{pmatrix} d & 0 \\ 0 & \nabla^{LC} \end{pmatrix}, \Psi_* = (r_+, \sqrt{2}\lambda^{0,1} r_-), \tag{2.6}$$

in which case with $\mathcal{E}_{an}(\check{\gamma}_*; \mathbb{R}) = 0$. In fact, this is the only solution with finite energy if Assumption 2.2 holds.

Theorem 2.6 ([Tau01], Proposition 4.4 & 4.7)**.** *If g_Σ is flat and Assumption* 2.2 *holds, then any solution $\check{\gamma}$ of* (2.5) *with $\mathcal{E}_{an}(\check{\gamma}; \mathbb{R}_s) < \infty$ is gauge equivalent to the unqiue $\mathbb{R}_s$-translation solution $\check{\gamma}_*$.*

Remark 2.7. This result is due to Taubes. Readers can find a short discussion on its proof in [Wan20, Section 10]. Theorem 2.6 is the only reason why we insist that g_Σ is flat. In fact, Theorem 2.4 and 2.5 also hold for any non-flat metric g_Σ of Σ with a slightly different expression of $\mathcal{E}_{an}$; see [Wan20]. ♢

3. The Strict Cobordism Category

The cobordism category Cob_s is said to be strict, because objects and morphisms are subject to certain constraints. Roughly speaking, each object of Cob_s is a 3-manifold $(Y, \partial Y)$ with torus boundary together with a choice of cylindrical metric g_Y and boundary data (g_Σ, λ, μ). A morphism of Cob_s is a manifold with corners

$$(X, W) : (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$$

together with some coherence conditions on boundary data $(g_{\Sigma_i}, \lambda_i, \mu_i)$. The restriction of a strict cobordism between boundaries is required to be a product, so $W = [-1,1]_t \times \Sigma_1$ and $\Sigma_1 = \Sigma_2$. Some of these constraints might be circumvented in the future by looking at the Seiberg-Witten moduli spaces on 4-manifolds with more complicated geometry. For now, we restrict attention to this smaller category Cob_s for the sake of simplicity.

Subsection 3.1 and 3.2 are devoted to the definition of Cob_s. Once this is done, we will continue to set up the configuration spaces on $\widehat{Y}$ and $\widehat{X}$ respectively in Subsection 3.3.

3.1. **Objects.** Let $(\Sigma, g_\Sigma) = \coprod_{i=1}^n (\mathbb{T}^2_i, g_i)$ be a disjoint union of 2-tori with a prescribed **flat** metric. Each object of the strict cobordism category Cob_s is a quintuple $\mathbb{Y} = (Y, \psi, g_Y, \omega, \mathfrak{q})$ satisfying the following properties:

- (P1) Y is a compact oriented 3-manifold with boundary and $\psi : \partial Y \to \Sigma$ is an orientation preserving diffeomorphism. The identification map ψ might be dropped from our notations when it is clear from the context.

(P2) The metric g_Y of Y is cylindrical, i.e. g_Y is the product metric

$$ds^2 + \psi^* g_\Sigma$$

within a collar neighborhood $(-2,0]_s \times \partial Y$ of ∂Y. We form a complete Riemannian 3-manifold $\widehat{Y}$ by attaching cylindrical ends along Σ:

$$\widehat{Y} = Y \cup_\psi [-1,\infty)_s \times \Sigma,$$

whose metric is denoted also by g_Y.

(P3) $\omega \in \Omega^2(Y, i\mathbb{R})$ is an imaginary valued **closed** 2-form on Y such that within the collar neighborhood $[-1,0]_s \times \partial Y$, ω restricts to an s-independent form

$$\mu + ds \wedge \lambda,$$

so ω extends naturally to a closed 2-form on $\widehat{Y}$, denoted also by ω.

(P4) $\lambda \in \Omega^1_h(\Sigma, i\mathbb{R})$ is an imaginary-valued harmonic 1-form on Σ. Moreover, $*_\Sigma\lambda$ lies in the image

$$\mathrm{Im}(H^1(Y, i\mathbb{R}) \to H^1(\Sigma, i\mathbb{R})).$$

(P5) $\mu \in \Omega^2_h(\Sigma, i\mathbb{R})$ is an imaginary-valued harmonic 2-form on Σ. Moreover, μ lies in the image

$$\mathrm{Im}(H^2(Y, i\mathbb{R}) \to H^2(\Sigma, i\mathbb{R})).$$

(P6) The cohomology class $[\omega] \in H^2(Y, i\mathbb{R})$ is called the period class. Let $i : \Sigma \to Y$ be the inclusion map, then $i^*([\omega]) = [\mu] \in H^2(\Sigma, i\mathbb{R})$. The closed 2-form ω in (P3) can be reconstructed from $(\lambda, \mu, [\omega])$ as follows. Choose a cut-off function $\chi_1 : [0,\infty)_s \to \mathbb{R}$ such that

$$\chi_1(s) \equiv 1 \text{ if } s \geqslant -1;\ \chi_1(s) \equiv 0 \text{ if } s \leqslant -3/2.$$

By Corollary A.5, we can find a closed 2-form $\overline{\omega}$ on Y in the class $[\omega]$ such that $\overline{\omega} \equiv \mu$ on $[-1,0]_s \times \Sigma$. Set

$$\omega = \overline{\omega} + \chi_1(s) ds \wedge \lambda. \tag{3.1}$$

The period class $[\omega]$ is independent of λ.

(P7) Let (λ_i, μ_i) be the restriction of (λ, μ) on each connected component $(\mathbb{T}^2_i, g_i)$ of Σ. Then Assumption 2.2 holds for (λ_i, μ_i) for any $1 \leqslant i \leqslant n$. In particular, $\lambda_i \neq 0$.

(P8) $\{\mathfrak{q}\}$ is a collection of admissible perturbations (in the sense of Definition 13.3) of the Chern-Simons-Dirac functional $\mathcal{L}_\omega$ for each relative *spin*^{c} structures $\widehat{\mathfrak{s}}$.

Remark 3.1. The closed 2-form ω is used to perturb the Chern-Simons-Dirac functional, see Definition 3.8 below. (P7) will allow us to apply Theorem 2.4−2.6 in Section 6, so the Seiberg-Witten moduli spaces will have the right compactness property. We will address the issue of perturbations in Part 3, so readers may ignore the last property (P8) at this point. ♢

Properties (P5) requires some further explanation: it is used to find certain bounded **harmonic forms** on $\widehat{Y}$, which play essential roles in the energy equations in Section 5, cf. Theorem 5.4. The next lemma is a consequence of (P5) and Corollary A.6.

Lemma 3.2. *For any object $\mathbb{Y} \in \mathrm{Cob}_s$, there exists a bounded harmonic 2-form ω_h on $\widehat{Y}$ such that ω_h converges exponentially to $ds \wedge \lambda$ as $s \to \infty$. In particular, $\omega_h - \chi_1(s) ds \wedge \lambda \in L^2(\widehat{Y})$. Such a harmonic 2-form ω_h is unique up to an L^2-harmonic form. By* [APS75, Proposition 4.9]*, the space of L^2-harmonic forms on $\widehat{Y}$ is isomorphic to*

$$\mathrm{Im}(H^*(Y, \Sigma; i\mathbb{R}) \to H^*(Y; i\mathbb{R})).$$

3.2. **Morphisms.** Having described objects in the strict cobordism category Cob_s, we now turn to describe the set of morphisms in this subsection. Since each object $\mathbb{Y}$ is coupled with a closed 2-form ω, morphisms must take these forms into account. Given two objects $\mathbb{Y}_i = (Y_i, \psi_i, g_i, \omega_i, \mathfrak{q}_i), i = 1, 2$ in Cob_s, a morphism

$$\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$$

is a quadruple $\mathbb{X} = (X, \psi_X, W, [\omega_X]_{cpt})$ with the following properties.

(Q1) X is a manifold with corners, i.e. X is a space stratified by manifolds

$$X \supset X_{-1} \supset X_{-2} \supset X_{-3} = \varnothing$$

such that the co-dimensional 1 stratum X_{-1} consists of three parts

$$X_{-1} = (-Y_1) \cup (Y_2) \cup W_X.$$

where W_X is an oriented 3-manifold with boundary $\partial W_X = \partial Y_1 \cap \partial Y_2$. Moreover, $\partial Y_i = Y_i \cap W_X$ and $X_{-2} = \partial Y_1 \cup \partial Y_2$. For more details on the definition, see Definition A.7.

(Q2) $W = [-1, 1]_t \times \Sigma$ is the product cobordism of Σ to itself.

(Q3) $\psi_X : W_X \to W$ is an orientation preserving diffeomorphism compatible with ψ_1 and ψ_2. To be more precise, we require that

$$\psi_X|_{\partial Y_1} = \psi_1 : \partial Y_1 \to \{-1\} \times \Sigma,$$
$$\psi_X|_{\partial Y_2} = \psi_2 : \partial Y_2 \to \{1\} \times \Sigma,$$

which also hold in a collar neighborhood of ∂W_X. When no chance of confusion is possible, ψ_X might be dropped from our notations. Such a pair (X, ψ_X) is called **a strict cobordism** from (Y_1, ψ_1) to (Y_2, ψ_2).

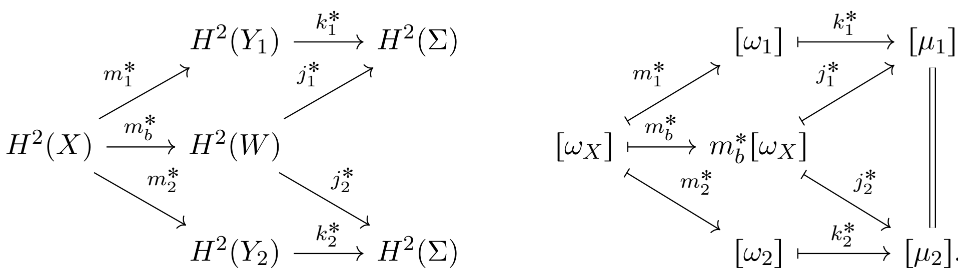
(Q4) The closed 2-form ω_i on Y_i contains a bit more information than the period class $[\omega_i] \in H^2(Y_i)$. We first require that

$$\mu_1 = \mu_2 = \mu \in \Omega_h^2(\Sigma, i\mathbb{R}),$$

then the triple $(\omega_1, \mu, \omega_2)$ determines a class $[\alpha]$ in $H^2((-Y_1) \cup W \cup Y_2, i\mathbb{R})$. $[\alpha]$ is required to lie in the image

$$\mathrm{Im}\left(m_0^* : H^2(X) \to H^2((-Y_1) \cup W \cup Y_2), i\mathbb{R})\right),$$

where $m_0 : (-Y_1) \cup W \cup Y_2 \hookrightarrow X$ is the inclusion map, and let $[\omega_X]$ be a lift of $[\alpha]$. As a result, $[\omega_X]$ generates all cohomology classes in the diagram below:



(Q5) $[*_2\lambda_1] = [*_2\lambda_2] \in H^1(\Sigma; i\mathbb{R})$.

(Q6) There exists a closed 2-form $\overline{\omega}_X \in \Omega^2(X)$ on X with the following properties:
- $\overline{\omega}_X$ realizes the class $[\omega_X] \in H^2(X, i\mathbb{R})$;
- $\overline{\omega}_X = \overline{\omega}_i$ (see (P6)) within a collar neighborhood of $Y_i \subset X_{-1}$ for $i = 1, 2$;
- within a collar neighborhood of $W \subset X_{-1}$, $\overline{\omega}_X = \mu$.

The existence of such a form $\overline{\omega}_X$ is equivalent to the cohomological condition defined in (Q4). Finally, set $\omega_\lambda = \chi_1(s)ds \wedge \lambda$ (with $\lambda = \lambda_i$) and

$$\omega_X := \overline{\omega}_X + \omega_\lambda = \overline{\omega}_X + \chi_1(s)ds \wedge \lambda \text{ on } X.$$

(Q7) For any two closed forms ω_X and ω'_X satisfying the condition in (Q6), they are said to be equivalent if $\omega'_X - \omega_X = da$ for a compactly supported smooth 1-form $a \in \Omega^1(X, i\mathbb{R})$. Denote by $[\omega_X]_{cpt}$ the equivalence classes of ω_X.

Example 3.3. The product cobordism $\mathbb{X} = [-1,1] \times \mathbb{Y} : \mathbb{Y} \to \mathbb{Y}$. In this case, $X = [-1,1]_t \times Y$ and $\psi_X = \mathrm{Id}_{[-1,1]_t} \times \psi$ is the product map. We obtain ω_X by pulling back the 2-form ω from Y. ♢

Example 3.4. Take $\mathbb{Y}_1, \mathbb{Y}_2 \in \mathrm{Cob}_s$ with $Y_1 = Y_2 = Y$ and ψ_1 isotopic to ψ_2. Suppose in addition that $\omega_2 - \omega_1 = d_Y b$ for a compactly supported 1-from $b \in \Omega^1(Y, i\mathbb{R})$, then one may construct a cobordism $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ as follows. Let $X = [-1,1]_t \times Y$ and ψ_X be an isotopy from ψ_1 to ψ_2. Set $\omega_X = d_X(\chi(t)b) + \omega_1$ where $\chi(t)$ is a cut-off function such that

$$\chi(t) \equiv 0 \text{ if } t \leqslant -1/2; \chi(t) \equiv 1 \text{ if } t \geqslant 1/2.$$ ♢

Similar to the definition of $\widehat{Y}$, for each strict cobordism $X : Y_1 \to Y_2$, we obtain a cobordism between $\widehat{Y}_1$ and $\widehat{Y}_2$ by attaching a cylindrical end to X:

$$\widehat{X} := X \cup_{\psi_X} [-1,1]_t \times [-1,\infty)_s \times \Sigma : \widehat{Y}_1 \to \widehat{Y}_2.$$

A planar metric g_X on X is a metric compatible with the corner structure (see Definition A.8). We insist that the metric g_W of $W = [-1,1]_t \times \Sigma$ is the product metric

$$T^2 dt^2 + g_\Sigma$$

for some constant $T > 0$. One might alternatively normalize T to be 1 by rescaling the interval $[-1,1]_t$. For the sake of simplicity, we set $T = 1$ in the sequel.

The planar metric g_X is required to be the product metric

$$dt^2 + ds^2 + g_\Sigma$$

in a neighborhood $(-\epsilon,0]_t\times(-1,0]_s\times X_{-2}$ of the co-dimension 2 stratum $X_{-2}=(-\Sigma)\cup\Sigma$. For a strict cobordism $X:Y_1\to Y_2$, g_X is also required to be cylindrical near the co-dimensional 1 stratum X_{-1}:

$$g_X|_{[-1,-1+\epsilon)\times Y_1}=d^2t+g_1, \qquad g_X|_{(1-\epsilon,1]\times Y_2}=d^2t+g_2,$$
$$g_X|_{[-1,1]_t\times(-1,0]_s\times\Sigma}=ds^2+g_Z=d^2t+d^2s+g_\Sigma.$$

Such a metric extends to a cylindrical metric on $\widehat{X}$ compatible with that of $(-\widehat{Y}_1)\cup\widehat{Y}_2$. When it is clear from the context, we also use g_X to denote this extended metric on $\widehat{X}$.

Although a planar metric g_X of X is **not** encoded in the definition of a morphism $\mathbb{X}:\mathbb{Y}_1\to\mathbb{Y}_2$, it is used to define the functor HM_* in Theorem 1.5. Nevertheless, the resulting maps on morphism sets are independent of the choice of g_X.

The property (Q5) is also used to find certain bounded harmonic 2-forms on $(\widehat{X},g_X)$, which is crucial for the energy equations in Section 5, cf. Theorem 5.1. The next lemma follows from Lemma 3.6 and Corollary A.15 in which we set

$$Y_e=[-1,1]_t\times\Sigma \text{ and } Y_b=(-Y_1)\cup Y_2.$$

Lemma 3.5. *For any morphism $\mathbb{X}\in\mathrm{Cob}_s$, there exists a bounded harmonic 2-form $\omega_{X,h}$ on $\widehat{X}$ such that $\omega_{X,h}$ converges exponentially to $ds\wedge\lambda$ as $s\to\infty$ and*

$$*_4\,\omega_{X,h}=0 \text{ on } (-\widehat{Y}_1)\cup\widehat{Y}_2, \tag{3.2}$$

so $\omega_{X,h}$ satisfies the Neumann boundary condition. In particular, $\omega_{X,h}-\omega_\lambda\in L^2(\widehat{X})$.

Lemma 3.6. *For any morphism $\mathbb{X}:\mathbb{Y}_1\to\mathbb{Y}_2$, the class*

$$[dt\wedge *_2\lambda]\in H^2(W,\partial W;i\mathbb{R})$$

lies in the image $\mathrm{Im}\left(H^2(X,Y_1\cup Y_2;i\mathbb{R})\to H^2(W,\partial W;i\mathbb{R})\right)$ *where $\lambda=\lambda_1=\lambda_2$.*

Proof of Lemma 3.6. By (P4), take z to be a lift of $[*_\Sigma\lambda]$ in $H^1(Y,i\mathbb{R})$. In the diagram below, all cohomology groups take value in $i\mathbb{R}$:

$$\begin{array}{ccccc}
H^1(Y_1) & \xrightarrow{z\mapsto(z,0)} & H^1(Y_1)\oplus H^1(Y_2) & \xrightarrow{\delta} & H^2(X,Y_1\cup Y_2)\\
\downarrow & & \downarrow & & \downarrow\\
H^1(\Sigma) & \xrightarrow{[*_2\lambda]\mapsto([*_2\lambda],0)} & H^1(\{-1\}\times\Sigma)\oplus H^1(\{1\}\times\Sigma) & \xrightarrow{\delta} & H^2(W,\partial W)
\end{array}$$

□

3.3. **Relative *spin^c* Structures and Configuration Spaces.** Let $\mathfrak{s}_{std}=(S_{std},\rho_{std,3})$ be the standard *spin^c* structure on $\mathbb{R}_s\times\Sigma$ as described in Section 2 with

$$S_{std}=\mathbb{C}\oplus\Lambda^{0,1}\Sigma.$$

For each object $\mathbb{Y}=(Y,\psi,g_Y,\omega,\mathfrak{q})\in\mathrm{Cob}_s$, a relative *spin^c* structure $\widehat{\mathfrak{s}}$ is a pair $(\mathfrak{s},\phi)$ where $\mathfrak{s}=(S,\rho_3)$ is a *spin^c* structure on Y and

$$\varphi:(S,\rho_3)|_{\partial Y}\to\psi^*\mathfrak{s}_{std}|_{\partial Y}$$

is an isomorphism of *spin*^{c} structures near the boundary that is compatible with ψ. The set of isomorphism classes of relative *spin*^{c} structures on Y

$$\mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(Y)$$

is a torsor over $H^2(Y, \partial Y; \mathbb{Z})$. There is a natural forgetful map from $\mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(Y)$ to the set of isomorphism classes of *spin*^{c} structures:

$$\mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(Y) \to \mathrm{Spin}^{\mathrm{c}}(Y),\ \widehat{\mathfrak{s}} = (\mathfrak{s}, \phi) \mapsto \mathfrak{s},$$

whose fiber is acted on freely and transitively by $H^1(\Sigma, \mathbb{Z})/\operatorname{Im}(H^1(Y, \mathbb{Z}))$ reflecting the change of boundary trivializations. Any $\widehat{\mathfrak{s}} \in \mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(Y)$ extends to a relative *spin*^{c} structure on $\widehat{Y}$, denoted also by $\widehat{\mathfrak{s}}$.

Let (B_*, Ψ_*) be the translation invariant configuration on $\mathbb{R}_s \times \Sigma$ such that the restriction

$$(B_*, \Psi_*)|_{\mathbb{R}_s \times \mathbb{T}^2_i} \tag{3.3}$$

on each connected component is defined by the formula (2.6) for any $1 \leqslant i \leqslant n$. Take (B_0, Ψ_0) to be a smooth configuration on $\widehat{Y}$ which agrees with (B_*, Φ_*) on the cylindrical end $[0, \infty)_s \times \Sigma$. Recall from (P3) that the closed 2-form $\omega \in \Omega^2(Y, i\mathbb{R})$ defined on Y extends to a closed 2-form on the completion $\widehat{Y}$ by setting

$$\omega|_{[-1,\infty)\times\Sigma} = \mu + ds \wedge \lambda,$$

and $[\omega] \in H^2(Y; i\mathbb{R})$ is the period class of ω.

Consider the configuration space for any $k > \frac{1}{2}$:

$$\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) = \{(B, \Psi) : (b, \psi) = (B, \Psi) - (B_0, \Psi_0) \in L^2_k(\widehat{Y}, iT^*\widehat{Y} \oplus S)\}.$$

Remark 3.7. Since $\widehat{Y}$ is non-compact, the condition that $(b, \psi) \in L^2_k$ includes a mild decay condition on the section (b, ψ) on the cylindrical end of $\widehat{Y}$. It turns out that this decay is always exponential for solutions to the Seiberg-Witten equations, cf. Theorem 6.2. ◊

Definition 3.8. The perturbed Chern-Simons-Dirac functional on $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ is defined as

$$\mathcal{L}_\omega(B, \Psi) = -\frac{1}{8}\int_{\widehat{Y}} (B^t - B^t_0) \wedge (F_{B^t} + F_{B^t_0}) + \frac{1}{2}\int_{\widehat{Y}} \langle D_B\Psi, \Psi\rangle + \frac{1}{2}\int_{\widehat{Y}} (B - B_0)^t \wedge \omega. \tag{3.4}$$ ◊

Remark 3.9. $\mathcal{L}_\omega$ is the analogue of the gauged action functional $\mathcal{A}_H$ in the context of gauged Witten equations, see [Wan20, Definition 4.1]. ◊

The configuration space $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ is acted on freely by the gauge group

$$\mathcal{G}_{k+1}(\widehat{Y}) = \{u : \widehat{Y} \to S^1 \subset \mathbb{C} : u - 1 \in L^2_{k+1}(\widehat{Y}, \mathbb{C})\},$$

via the formula:

$$u(B, \Psi) = (B - u^{-1}du, u\Psi).$$

The Lie algebra of $\mathcal{G}_{k+1}$ is $\mathrm{Lie}(\mathcal{G}_{k+1}) = L^2_{k+1}(\widehat{Y}, i\mathbb{R})$. The exponential map $f \mapsto e^f$ is surjective onto the identity component $\mathcal{G}^e_{k+1}$of $\mathcal{G}_{k+1}$; they fit to a short exact sequence:

$$0 \to \mathcal{G}^e_{k+1} \to \mathcal{G}_{k+1} \to \pi_0(\mathcal{G}_{k+1}) \cong H^1(Y, \Sigma; \mathbb{Z}) \to 0.$$

The Chern-Simons-Dirac functional $\mathcal{L}_\omega$ is not fully gauge-invariant in general:

Lemma 3.10. *For any $\gamma = (B, \Psi) \in \mathcal{C}_k(\widehat{Y}, \hat{\mathfrak{s}})$ and $u \in \mathcal{G}_{k+1}(\widehat{Y})$, we have*

$$\mathcal{L}_\omega(u \cdot \gamma) - \mathcal{L}_\omega(\gamma) = (2\pi^2[u] \cup c_1(S) - 2\pi i[u] \cup [\omega])[Y, \partial Y],$$

where $[u] = [\frac{u^{-1}du}{2\pi i}] \in H^1(Y, \partial Y; \mathbb{Z})$ is the relative cohomology class determined by u and $[\omega]$ is the period class of ω.

The tangent space at each $\gamma \in \mathcal{C}_k(\widehat{Y}, \hat{\mathfrak{s}})$ is naturally identified with $L^2_k(\widehat{Y}, iT^*\widehat{Y} \oplus S)$. We compute the gradient of $\mathcal{L}_\omega$ with respect to the L^2 inner product:

$$\operatorname{grad} \mathcal{L}_\omega(B, \Psi) = (\frac{1}{2} *_3 F_{B^t} + \rho_3^{-1}(\Psi\Psi^*)_0 - *_3\omega, D_B\Psi). \tag{3.5}$$

Hence, a configuration $\gamma \in \mathcal{C}_k(\widehat{Y}, \hat{\mathfrak{s}})$ is a critical point of $\mathcal{L}_\omega$ if and only if it solves the perturbed Seiberg-Witten equations on $\widehat{Y}$:

Definition 3.11. For any object $\mathbb{Y} = (Y, \psi, g_Y, \omega, \mathfrak{q}) \in \mathrm{Cob}_s$, the Seiberg-Witten map defined on $\mathcal{C}_k(\widehat{Y}, \hat{\mathfrak{s}})$ is given by (ignoring the perturbation $\mathfrak{q}$ for a moment)

$$\mathfrak{F}_\omega(B, \Psi) = (\frac{1}{2}\rho_3(F_{B^t} - 2\omega) - (\Psi\Psi^*)_0, D_B\Psi).$$

and the equation

$$\mathfrak{F}_\omega(B, \Psi) = 0 \tag{3.6}$$

is called the 3-dimensional Seiberg-Witten equations. ♢

Remark 3.12. The reference configuration (B_*, Ψ_*) defined in (3.3) is the unique $\mathbb{R}_s$-translation invariant solution of (3.6) on $\mathbb{R}_s \times \Sigma$ up to gauge. ♢

The downward gradient flowline equation of $\mathcal{L}_\omega$

$$\frac{d}{dt}(B(t), \Psi(t)) = -\operatorname{grad} \mathcal{L}_\omega(B(t), \Psi(t))$$

can be cast into the 4-dimensional Seiberg-Witten equations:

$$\begin{cases} \frac{1}{2}\rho_4(F^+_{A^t} - 2\omega^+_X) - (\Phi\Phi^*)_0 = 0, \\ D^+_A\Phi = 0, \end{cases} \tag{3.7}$$

on $\mathbb{R}_t \times \widehat{Y}$ with $A = \frac{d}{dt} + B(t), \Phi = \Psi(t)$ and $\omega_X = \pi^*\omega$ where $\pi : \mathbb{R}_t \times \widehat{Y} \to \widehat{Y}$ denotes the projection map. This corresponds to the product cobordism $[-1, 1] \times \widehat{Y}$ in Example 3.3.

In general, let (A_*, Φ_*) be the $\mathbb{C}$-translation-invariant solution on $\mathbb{C} \times \Sigma$ with

$$A_* = dt \otimes \frac{\partial}{\partial t} + B_*, \Phi_*(t) = \Psi_*. \tag{3.8}$$

Let $\mathbb{X} = (X, \psi_X, W, [\omega_X]_{cpt}) : \mathbb{Y}_1 \to \mathbb{Y}_2$ be a morphism in Cob_s and suppose $\widehat{X} : \widehat{Y}_1 \to \widehat{Y}_2$ extends to a *relative spin^c* cobordism:

$$(\widehat{X}, \hat{\mathfrak{s}}_X) : (\widehat{Y}_1, \hat{\mathfrak{s}}_1) \to (\widehat{Y}_2, \hat{\mathfrak{s}}_2). \tag{3.9}$$

Remark 3.13. For a *relative spin*^{c} cobordism, we insist that identification maps

$$(\widehat{X},\widehat{\mathfrak{s}}_X)|_{\widehat{Y}_i}\cong(\widehat{Y}_i,\widehat{\mathfrak{s}}_i), i=1,2$$

are implicitly baked in the definition. ♢

Let (A_0,Φ_0) be a reference configuration on $\widehat{X}$ whose restriction on $[-1,1]_t\times[0,\infty)_s\times\Sigma$ agrees with (A_*,Φ_*). For each $k\geqslant 1$, define

$$\mathcal{C}_k(\widehat{X},\widehat{\mathfrak{s}}_X)=\{(A,\Phi):(a,\phi)=(A,\Phi)-(A_0,\Phi_0)\in L^2_k(\widehat{X},iT^*\widehat{X}\oplus S^+)\}.$$

In this case, we take $\omega_X\in\Omega^2(\widehat{X},i\mathbb{R})$ to be the closed 2-form constructed in (Q6) and extended constantly over the cylindrical end $[-1,1]_t\times[0,\infty)_s\times\Sigma$; so for some $\epsilon>0$,

- $\omega_X=\omega_1$ on $\widehat{Y}_1\times[-1,-1+\epsilon)_t$;
- $\omega_X=\omega_2$ on $\widehat{Y}_2\times(1-\epsilon,1]_t$;
- $\omega_X=\mu+ds\wedge\lambda$ on $[-1,1]_t\times[0,\infty)_s\times\Sigma$.

Then the left hand side of (3.7) defines a smooth map:

$$\mathfrak{F}_X:\mathcal{C}_k(\widehat{X},\widehat{\mathfrak{s}}_X)\to L^2_{k-1}(\widehat{X},i\mathfrak{su}(S^+)\oplus S^-) \tag{3.10}$$

called the Seiberg-Witten map on $\widehat{X}$. For $0\leqslant j\leqslant k$, let $\mathcal{V}_j$ be the trivial vector bundle with fiber $L^2_j(i\mathfrak{su}(S^+)\oplus S^-)$ over $\mathcal{C}_k(\widehat{X},\widehat{\mathfrak{s}})$:

$$\mathcal{V}_j:=L^2_j(i\mathfrak{su}(S^+)\oplus S^-)\times\mathcal{C}_k(\widehat{X},\widehat{\mathfrak{s}}).$$

The Seiberg-Witten map $\mathfrak{F}_X$ defines a smooth section of $\mathcal{V}_{k-1}\to\mathcal{C}_k(\widehat{X},\widehat{\mathfrak{s}}_X)$.

3.4. **The Strict** ***spin***^{c} **Cobordism.** Now let us introduce the strict *spin*^{c} cobordism category SCob_s, which plays the central role in Theorem 1.5:

- each object of SCob_s is a pair $(\mathbb{Y},\widehat{\mathfrak{s}})$ where $\mathbb{Y}$ is an object of Cob_s and $\widehat{\mathfrak{s}}\in\mathrm{Spin}^c_R(Y)$ is a relative *spin*^{c} structure on Y;
- for any objects $(\mathbb{Y}_1,\widehat{\mathfrak{s}}_1)$ and $(\mathbb{Y}_2,\widehat{\mathfrak{s}}_2)$,

$$\mathrm{Hom}_{\mathrm{SCob}_s}((\mathbb{Y}_1,\widehat{\mathfrak{s}}_1),(\mathbb{Y}_2,\widehat{\mathfrak{s}}_2))=\mathrm{Hom}_{\mathrm{Cob}_s}(\mathbb{Y}_1,\mathbb{Y}_2).$$

3.5. **Homotopy Classes of Paths.** To define the monopole Floer homology $HM_*(\mathbb{Y},\widehat{\mathfrak{s}})$ for each object $(\mathbb{Y},\widehat{\mathfrak{s}})\in\mathrm{SCob}_s$, we will look at the moduli spaces of the Seiberg-Witten equations (3.7) on $\mathbb{R}_t\times(\widehat{Y},\widehat{\mathfrak{s}})$ and define a Floer chain complex:

$$\mathrm{CF}_*(\mathbb{Y},\widehat{\mathfrak{s}});$$

The underlying idea is an infinite dimensional Morse theory in the quotient configuration space:

$$\mathcal{B}_k(Y,\widehat{\mathfrak{s}}):=\mathcal{C}_k(Y,\widehat{\mathfrak{s}})/\mathcal{G}_{k+1}(Y).$$

For any $\mathfrak{a},\mathfrak{b}\in\mathcal{C}_k(Y,\widehat{\mathfrak{s}})$, the relative homotopy classes of paths $\pi_1(\mathcal{B}_k(Y,\widehat{\mathfrak{s}});[\mathfrak{a}],[\mathfrak{b}])$ is a torsor over

$$\pi_1(\mathcal{B}_k(Y,\widehat{\mathfrak{s}});[\mathfrak{b}])\cong\pi_0(\mathcal{G}_{k+1})\cong H^1(Y,\partial Y;\mathbb{Z}).$$

Moreover, for any $[\gamma] \in \pi_1(\mathcal{B}_k(Y,\hat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}])$, the relative loop space $\Omega_{[\gamma]}(\mathcal{B}_k(Y,\hat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}])$ in the class $[\gamma]$ is simply connected, since

$$\pi_2(\mathcal{B}_k(Y,\hat{\mathfrak{s}}); [\mathfrak{b}]) \cong \pi_1(\mathcal{G}_{k+1}) = \{0\}.$$

There are three additional ways to think of a path $\check{\gamma} : [-1,1] \to \mathcal{B}_k(Y,\hat{\mathfrak{s}})$ with $\check{\gamma}(-1) = \mathfrak{a}$ and $\check{\gamma}(1) = \mathfrak{b}$, and we shall use them interchangeably:

(1) a path $\check{\gamma}_1 : [-1,1] \to \mathcal{C}_k(Y,\hat{\mathfrak{s}})$ that connects $\mathfrak{a}$ and $u \cdot \mathfrak{b}$ for some $u \in \mathcal{G}_{k+1}(\hat{Y})$;

(2) a configuration γ on the 4-manifold $I \times (\hat{Y}, \hat{\mathfrak{s}})$ with $I = [-1,1]_t$ such that $\gamma|_{\{-1\}\times\hat{Y}} = \mathfrak{a}$ and $\gamma|_{\{1\}\times\hat{Y}} = u \cdot \mathfrak{b}$ for some $u \in \mathcal{G}_{k+1}(\hat{Y})$;

(3) a configuration γ' for a relative *spin*^{c} cobordism

$$(\hat{X} = I \times \hat{Y}, \hat{\mathfrak{s}}_X) : (\hat{Y}, \hat{\mathfrak{s}}) \to (\hat{Y}, \hat{\mathfrak{s}})$$

such that $\gamma|_{\{-1\}\times\hat{Y}} = \mathfrak{a}$ and $\gamma|_{\{1\}\times\hat{Y}} = \mathfrak{b}$. Indeed, all such relative *spin*^{c} structures on $I \times \hat{Y}$ form a torsor over

$$H^2(I \times Y, \partial(I \times Y)) \cong H^1(Y, \partial Y; \mathbb{Z}) \times H^1(I, \partial I; \mathbb{Z}) \cong H^1(Y, \partial Y; \mathbb{Z}).$$

The last standpoint makes it easier to think about a general morphism $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$. To make HM_* into a functor from SCob_s to $\mathcal{R}$-Mod as in Theorem 1.5, we attach cylindrical ends to $\hat{X}$ and obtain a complete Riemannian manifold $\mathcal{X}$:

$$\mathcal{X} := \Big((-\infty, -1]_t \times \hat{Y}_1\Big) \cup \hat{X} \cup \Big([1,\infty)_t \times \hat{Y}_2\Big).$$

The closed 2-form ω_X extend over $\mathcal{X}$ by setting

$$\omega_X = \omega_1 \text{ on } (-\infty,-1]_t \times Y_1;\ \omega_X = \omega_2 \text{ on } [1,\infty)_t \times Y_2. \tag{3.11}$$

The goal is to analyze the Seiberg-Witten equations (3.7) on $\mathcal{X}$ and construct a chain map:

$$\mathrm{CF}_*(\mathbb{X}) : \mathrm{CF}_*(\mathbb{Y}_1, \hat{\mathfrak{s}}_1) \to \mathrm{CF}_*(\mathbb{Y}_2, \hat{\mathfrak{s}}_2) \tag{3.12}$$

that is independent of the choice of

- the planar metric g_X compatible with $(g_{Y_1}, g_{Y_2}, g_\Sigma)$;
- the closed 2-form $\omega_X \in \omega^2(X, i\mathbb{R})$ in the class $[\omega_X]_{cpt}$;
- any auxiliary perturbation of (3.7) defined in Subsection 14.1;

up to chain homotopy. To do so, we have to take into account of all isomorphism classes of relative *spin*^{c} cobordisms:

$$\mathrm{Spin}^c_{\mathrm{R}}(X; \hat{\mathfrak{s}}_1, \hat{\mathfrak{s}}_2) := \{\text{all possible (3.9)} : (\mathbb{Y}_1, \hat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \hat{\mathfrak{s}}_2)\} \text{ modulo isomorphisms}$$

which is a torsor over $H^2(X, \partial X; \mathbb{Z})$. Indeed, any two relative *spin*^{c} cobordisms $\hat{\mathfrak{s}}_{X,1}, \hat{\mathfrak{s}}_{X,2}$ that cover the 4-manifold X with corners are related by a complex line bundle $L_{12} \to X$:

$$\hat{\mathfrak{s}}_{X,2} = \hat{\mathfrak{s}}_{X,1} \otimes L_{12},$$

and a trivialization $L_{12} \cong \mathbb{C}$ is specified along ∂X. Some of elements of $\mathrm{Spin}^c_{\mathrm{R}}(X; \hat{\mathfrak{s}}_1, \hat{\mathfrak{s}}_2)$ may arise from different underlying *spin*^{c} structures, but they all contribute to the chain map (3.12) and will not be separated from each other. For any $\mathfrak{a}_i \in \mathcal{C}_k(\hat{Y}_i, \hat{\mathfrak{s}}_i)$, $i = 1, 2$, an element of $\mathrm{Spin}^c_R(X;\hat{\mathfrak{s}}_1,\hat{\mathfrak{s}}_2)$ can be viewed a homotopy class of $\mathbb{X}$-paths that connect $\mathfrak{a}_1$ and $\mathfrak{a}_2$.

4. The Quotient Configuration Space and Slices

Configurations in $\mathcal{C}_k(\hat{Y},\hat{\mathfrak{s}})$ and $\mathcal{C}_k(\hat{X},\hat{\mathfrak{s}}_X)$ are required to converge to a fixed limit in the spatial direction, so by definition, they are never reducible, i.e. Ψ or $\Phi \not\equiv 0$. This prevents us from finding a global slice of the gauge action as in [KM07, Section 9.6] over the non-compact manifold $\hat{Y}$ or $\hat{X}$. Nevertheless, local slices always exists. In this section, we prove that:

Proposition 4.1. *For either $(M,\hat{\mathfrak{s}}_M) = (\hat{Y},\hat{\mathfrak{s}})$ or $(\hat{X},\hat{\mathfrak{s}}_X)$, the quotient space*

$$\mathcal{B}_k(M,\hat{\mathfrak{s}}_M) := \mathcal{C}_k(M,\hat{\mathfrak{s}}_M)/\mathcal{G}_{k+1}(M)$$

is a Hilbert manifold when $2(k+1) > \dim M$ and $k \in \mathbb{Z}$.

It is clear from the formula

$$(uv-1) = (u-1)(v-1) + (u-1) + (v-1),\ \forall u,v \in \mathcal{G}_{k+1}(M)$$

that $\mathcal{G}_{k+1}(M)$ is a Hilbert Lie group when $2(k+1) > \dim M$. Following the book [KM07, Section 9], we base the argument on a general principle:

Lemma 4.2 ([Pal68],[KM07] Lemma 9.3.2)**.** *Suppose a Hilbert Lie group G acts smoothly and freely on a Hilbert manifold C, and the quotient space C/G is Hausdorff. Suppose that at each $c \in C$, the differential*

$$d_c : T_eG \to T_cG$$

has closed range, then C/G is also a Hilbert manifold.

It remains to verify the condition of Lemma 4.2.

Lemma 4.3. *For either $(M,\hat{\mathfrak{s}}_M) = (\hat{Y},\hat{\mathfrak{s}})$ or $(\hat{X},\hat{\mathfrak{s}}_X)$, the quotient configuration space $\mathcal{B}_k(M,\hat{\mathfrak{s}}_M)$ is Hausdorff.*

Proof. Suppose we have a sequence of configurations $\gamma_n = (A_n,\Phi_n) \in \mathcal{C}_k(M,s)$ and a sequence of gauge transformations $u_n \in \mathcal{G}_{k+1}(M)$ such that

$$\gamma_n \to \gamma \text{ and } u_n \cdot \gamma_n \to \gamma'$$

for some $\gamma = (A,\Phi)$ and $\gamma' = (A',\Phi')$. We wish to show that $u\cdot\gamma = \gamma'$ for some $u \in \mathcal{G}_{k+1}(M)$. We prove that $v_n := 1-u_n$ has uniformly bounded L^2_{k+1} norm, so there is a weakly converging subsequence among $\{v_n\}$. Let v be the weak limit and define $u := 1-v$.

We begin with the L^2-norm of v_n. Since $\|v_n\|_\infty \leqslant 2$, $|v_n|_2^2$ contributes to a bounded integral over any compact region of M. It suffices to estimate $|v_n|_2^2$ over the cylindrical end of M. Note that

$$\|v_n\Phi\|_2 = \|(1-u_n)\Phi\|_2 \leqslant \|\Phi-\Phi'\|_2 + \|\Phi' - u_n\Phi_n\|_2 + \|u_n(\Phi_n-\Phi)\|_2,$$

which is uniformly bounded. As $s \to \infty$, Φ approximates the standard spinor and is non-vanishing everywhere. It follows that $\|v_n\|_2 \leqslant C$ for some uniform $C > 0$.

To deal with derivatives of v_n, let $w_n = u_n^{-1}du_n$. Then $\|w_n\|_{L^2_k} \leqslant \|u_n \cdot \gamma_n - \gamma_n\|_{L^2_k} \leqslant \|\gamma - \gamma'\|_{L^2_k} + 1$ when $n \gg 1$. The estimate for $\|\nabla^l v_n\|_{L^2}$ $(1 \leqslant l \leqslant k+1)$ now follows from the relation

$$\nabla v_n = \nabla u_n = w_n - v_n \cdot w_n$$

and an induction argument. If we already know $2k > \dim M$, then L^2_k is a Banach algebra itself; otherwise, the first a few steps in the induction requires special treatments. For instance, if $\dim M = 3$ and $k = 1$, then we have to bound

$$\|\nabla v_n\|_p \text{ for } 2 \leqslant p \leqslant 6 \text{ and } \|\nabla^2 v_n\|_2.$$

If $\dim M = 4$ and $k = 2$, then we have to bound

$$\|\nabla v_n\|_p \text{ for } 2 \leqslant p < \infty,\ \|\nabla^2 v_n\|_p \text{ for } 2 \leqslant p \leqslant 4 \text{ and } \|\nabla^3 v_n\|_2.$$

For the Sobolev embedding theorem on cylinders, see [KM07, Section 13.2]. □

Let $\mathcal{T}_k$ be the tangent space of $\mathcal{C}_k(M, \widehat{\mathfrak{s}}_M)$. For each configuration $\gamma = (A, \Phi) \in \mathcal{C}_k(M, \widehat{\mathfrak{s}}_M)$, let $\mathbf{d}_\gamma$ be the map obtained by linearizing the action of $\mathcal{G}_{k+1}(M)$, extended to lower Sobolev regularities $(0 \leqslant j \leqslant k)$:

$$\begin{aligned} \mathbf{d}_\gamma : L^2_{j+1}(M, i\mathbb{R}) &\to L^2_j(M, iT^*M \oplus S^+) = \mathcal{T}_{j,\gamma} \\ f &\mapsto (-df, f\Phi). \end{aligned}$$

Let $\mathcal{J}_{j,\gamma} \subset \mathcal{T}_{j,\gamma}$ be the image of $\mathbf{d}_\gamma$ and $\mathcal{K}_{j,\gamma}$ be the L^2-orthogonal complement of $\mathcal{J}_{j,\gamma}$:

$$\begin{aligned} \mathcal{K}_{j,\gamma} &:= \{v \in \mathcal{T}_{j,\gamma} : \langle v, \mathbf{d}_\gamma(f)\rangle_{L^2(M)} = 0, \forall f \in L^2_{j+1}(M, i\mathbb{R})\} \\ &= \{v = (\delta a, \delta\phi) \in L^2_j(M, iT^*M \oplus S^+) : \mathbf{d}^*_\gamma(v) = 0, \langle a, \vec{n}\rangle = 0 \text{ at } \partial M\} \end{aligned}$$

where $\vec{n}$ is the outward normal vector at ∂M and

$$\begin{aligned} \mathbf{d}^*_\gamma : L^2_j(M, iT^*M \oplus S^+) &\to L^2_{j-1}(M, i\mathbb{R}) \\ (\delta a, \delta\phi) &\mapsto -d^*\delta a + i \operatorname{Re}\langle i\Phi, \delta\phi\rangle. \end{aligned}$$

is the formal adjoint of $\mathbf{d}_\gamma$.

Lemma 4.4 (cf. [KM07] Proposition 9.3.4)**.** *As γ varies over $\mathcal{C}_k(M, \widehat{\mathfrak{s}}_M)$, $\mathcal{J}_{j,\gamma}$ and $\mathcal{K}_{j,\gamma}$ form complementary closed sub-bundles of $\mathcal{T}_j$, and we have a smooth decomposition*

$$\mathcal{T}_j|_{\mathcal{C}_k(M,\widehat{\mathfrak{s}})} = \mathcal{J}_j \oplus \mathcal{K}_j, 0 \leqslant j \leqslant k.$$

In particular, $T\mathcal{C}_k(M, \widehat{\mathfrak{s}}) = \mathcal{T}_k = \mathcal{J}_k \oplus \mathcal{K}_k$.

Proposition 4.1 now follows from Lemma 4.3 and 4.4.

Proof of Lemma 4.4. For any $v = (\delta a, \delta\phi) \in \mathcal{T}_{j,\gamma}$, we need to find the unique element $f \in L^2_{j+1}(M, i\mathbb{R})$ such that $v - \mathbf{d}_\gamma(f) \in \mathcal{K}_{j,\gamma}$. Such an element solves the Neumann boundary value problem:

$$\left\{ \begin{array}{rl} \Delta_M f + |\Phi|^2 f &= -\mathbf{d}^*_\gamma(v) \\ \langle df, \vec{n}\rangle &= \langle \delta a, \vec{n}\rangle \text{ at } \partial M. \end{array} \right. \tag{4.1}$$

The left hand side of (4.1) forms a Fredholm operator ($1 \leqslant j \leqslant k$):

$$(\Delta_M + |\Phi|^2, \frac{\partial}{\partial \vec{n}}\Big|_{\partial M}) : L^2_{j+1}(M, i\mathbb{R}) \to L^2_{j-1}(M, i\mathbb{R}) \times L^2_{j+1/2}(\partial M, i\mathbb{R}) \tag{4.2}$$

which is in fact invertible. If M is compact, this follows from [Tay11, Proposition 7.5]. In general, one may start with the special case when

$$(M, \Phi) = (\mathbb{R}_s \times \Sigma, \Psi_*) \text{ or } ([-1,1] \times \mathbb{R}_s \times \Sigma, \Phi_*)$$

using Fourier transformation on the real line $\mathbb{R}_s$ and the positivity of $|\Psi_*|^2$. To show (4.2) is Fredholm, apply the parametrix patching argument. To compute the index of 4.2, note that the restriction map

$$\frac{\partial}{\partial \vec{n}}\Big|_{\partial M} : L^2_{j+1}(M.i\mathbb{R}) \to L^2_{j+1/2}(\partial M, i\mathbb{R})$$

is surjective, and the operator

$$\Delta_M + |\Phi|^2 : \{f \in L^2_2(M, i\mathbb{R}) : \langle df, \vec{n}\rangle = 0\} \to L^2(M, i\mathbb{R})$$

is positive and self-adjoint. This proves that the operator (4.2) is invertible.

Alternatively, one may follow the proof of [Tay11, Proposition 7.5]. Details are left as exercises. □

We record the next proposition for convenience:

Proposition 4.5. *Over the configuration space $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$, the gradient* (3.5) *of the Chern-Simons-Dirac functional $\mathcal{L}_\omega$ defines a smooth section of $\mathcal{K}_{k-1} \to \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ when $k \geqslant 1$.*

5. Energy Equations

This section is devoted to the energy equations of the Seiberg-Witten equations (3.7) on $\widehat{X}$, which will play an important role in the proof of the Compactness Theorem 6.1 in Section 6. In particular, it gives property (K1). The main results of this section are Theorem 5.1 and Proposition 5.4. The existence of bounded harmonic forms on $\widehat{X}$ (cf. Lemma 3.5) is essential here.

5.1. **The 4-Dimensional Case.**

Following the book [KM07, Section 4], we prove an energy equation associated to the perturbed Seiberg-Witten equations (3.7):

Theorem 5.1 (cf. [KM07] P.593)**.** *For any morphism $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ in the strict cobordism category* Cob_s*, choose a planar metric g_X on X and consider a relative spin^c cobordism $(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_2, \widehat{\mathfrak{s}}_2)$. Then for any configuration $\gamma = (A, \Phi) \in \mathcal{C}(\widehat{X}, \widehat{\mathfrak{s}}_X)$, the L^2-norm of the Seiberg-Witten map $\mathfrak{F}_X(A, \Phi)$ can be expressed as*

$$\int_{\widehat{X}} |\mathfrak{F}_X(A, \Phi)|^2 = \mathcal{E}_{an}(A, \Phi) - \mathcal{E}_{top}(A, \Phi),$$

where

$$\mathcal{E}_{an}(A,\Phi) := \int_{\widehat{X}} \frac{1}{4}|F_{A^t}|^2 + |\nabla_A\Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega_X^+)|^2 + \frac{s}{4}|\Phi|^2 - \langle F_{A^t}, \overline{\omega}_X\rangle$$
$$- \int_{\widehat{X}} \langle F_{A^t}, \omega_\lambda - \omega_{X,h}\rangle - \int_{\widehat{X}} F_{A_0^t} \wedge *_4\omega_{X,h}, \tag{5.1}$$

$$\mathcal{E}_{top}(A,\Phi) := 2\mathcal{L}_{\omega_1}(B_1,\Psi_1) - 2\mathcal{L}_{\omega_2}(B_2,\Psi_2) + \frac{1}{4}\int_{\widehat{X}} F_{A_0^t}\wedge F_{A_0^t} - \int_{\widehat{X}} F_{A_0^t}\wedge \omega_X, \tag{5.2}$$

and $(B_i,\Psi_i) = (A,\Phi)|_{\widehat{Y}_i}$ *are restrictions of* γ *at* $\widehat{Y}_i$ *for* $i = 1,2$. *Here,* $\omega_X = \overline{\omega}_X + \omega_\lambda$ *is the closed 2-form constructed in* (Q6) *with* $\omega_\lambda = \chi_1(s)ds\wedge\lambda$. *The bounded harmonic 2-form* $\omega_{X,h}$ *is subject to the Neumann boundary condition and* $\omega_\lambda - \omega_{X,h} \in L^2(\widehat{X})$. *Its existence is guaranteed by Lemma* 3.5.

Remark 5.2. Let us explain why (5.1) is a useful expression. Errors terms in the second line of (5.1) are bounded below by

$$-\frac{1}{16}\|F_{A^t}\|^2_{L^2(\widehat{X})} - C(A_0,\omega_X,g_X)$$

for some constant $C(A_0,\omega_X,g_X) > 0$.

The first line of (5.1) is consistent with the *local energy functional* $\mathcal{E}_{an}(A,\Phi;\Omega)$ in Definition 2.3. Indeed, over the cylindrical end $I\times[0,\infty)_s\times\Sigma$, (5.1) becomes (with $I = [-1,1]_t$):

$$\int_{I\times[0,\infty)_s}\int_\Sigma \frac{1}{4}|F_{A^t}|^2 + |\nabla_A\Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2 - \langle F^\Sigma_{A^t},\mu\rangle \tag{5.3}$$

where $\omega = \mu + ds\wedge\lambda$. The last term in (5.3)

$$-\int_\Sigma \langle F^\Sigma_{A^t},\mu\rangle$$

is always zero. Indeed, if we write $a = A - A_0 \in L^2(\widehat{X}, iT^*\widehat{X})$, then $F^\Sigma_{A^t} = 2d_\Sigma a$ is an exact form on the surface Σ. Since μ is harmonic on Σ, their inner product is always zero. Hence, (5.3) has a definite sign. The integral in (5.1) over the compact region $X = \{s \leqslant 0\} \subset \widehat{X}$ can be treated in the usual way. We summarize this remark into a lemma. ◇

Lemma 5.3. *Under the assumption of Theorem* 5.1*, there exists a constant* $C_2(A_0,\omega_X,g_X)$ *independent of* (A,Φ) *such that*

$$\mathcal{E}_{an}(A,\Phi) + C_2 > \int_{\widehat{X}} \frac{1}{8}|F_{A^t}|^2 + |\nabla_A\Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega_X^+)|^2 + \frac{s}{4}|\Phi|^2.$$

Proof. Note that

$$\left|\int_{\widehat{X}}\langle F_{A^t},\overline{\omega}_X\rangle\right| = \left|\int_X \langle F_{A^t},\overline{\omega}_X\rangle\right| \leqslant \frac{1}{16}\|F_{A^t}\|^2_{L^2(\widehat{X})} + C_3(A_0,\omega_X,g_X).$$ □

Proof of Proposition 5.1. Let $\gamma_0 = (A_0,\Phi_0)$ be the reference configuration in $\mathcal{C}(\widehat{X},\hat{\mathfrak{s}}_X)$. For convenience, take its restrictions at the boundary

$$(B_{i0},\Phi_{i0}) = \gamma_0|_{\widehat{Y}_i} \in \mathcal{C}(\widehat{Y}_i)$$

as reference configurations in the definition of $\mathcal{L}_{\omega_i}$ for $i = 1, 2$. It suffices to prove the theorem when the section

$$(a, \phi) = (A, \Phi) - (A_0, \Phi_0) \in \mathcal{C}_c^\infty(\widehat{X}, iT^*\widehat{X} \oplus S),$$

is compactly support, and the rest will follow by continuity. Let $X_S = \{s \leqslant S\} \subset \widehat{X}$ be the truncated manifold and $Y_{i,S} = \widehat{Y}_i \cap X_S$. The boundary of X_S consist of three parts:

$$-Y_{1,S}, Y_{2,S} \text{ and } \{S\} \times W = [-1,1]_t \times \{S\} \times \Sigma.$$

Since (a, ϕ) is compactly supported, we may discard any boundary integrals over $\{S\} \times W \subset \partial X_S$ when $S \gg 1$. By the Lichnerowicz-Weizenböck formula [KM07, (4.15)], we have

$$\int_{X_S} |D_A^+\Phi|^2 = \int_{X_S} |\nabla_A\Phi|^2 + \frac{1}{2}\langle\rho_4(F_{A^t}^+)\Phi, \Phi\rangle + \frac{s}{4}|\Phi|^2 - \int_{Y_{1,S}} \langle D_{B_1}\Phi_1, \Phi_1\rangle + \int_{Y_{2,S}} \langle D_{B_2}\Phi_2, \Phi_2\rangle. \tag{5.4}$$

Now consider the first equation of (3.7):

$$\int_{X_S} |\frac{1}{2}\rho_4(F_{A^t}^+ - 2\omega_X^+) - (\Phi\Phi^*)_0|^2 = \int_{X_S} \frac{1}{4}|F_{A^t}|^2 - \frac{1}{2}\langle\rho_4(F_{A^t}^+)\Phi, \Phi\rangle + |(\Phi\Phi^*)_0 + \rho_4(\omega_X^+)|^2 - \frac{1}{4}\int_{X_S} F_{A^t} \wedge F_{A^t} - 2\int_{X_S} \langle F_{A^t}, \omega_X^+\rangle. \tag{5.5}$$

Only the second line requires some further work. Note that

$$-\frac{1}{4}\int_{X_S} F_{A^t} \wedge F_{A^t} = -\frac{1}{4}\int_{X_S} F_{A_0^t} \wedge F_{A_0^t} - \frac{1}{2}\int_{\partial X_S} a \wedge (F_{A^t} + F_{A_0^t}).$$

Finally, using the relation $\omega_X = \overline{\omega}_X + \omega_\lambda$, we compute

$$\begin{aligned} 2\int_{X_S} \langle F_{A^t}, \omega_X^+\rangle &= \int_{X_S} \langle F_{A^t}, \omega_X + *_4\omega_X\rangle \\ &= \int_{X_S} \langle F_{A^t}, \overline{\omega}_X\rangle + \langle F_{A^t}, \omega_\lambda\rangle + \langle F_{A_0^t}, *_4\omega_X\rangle + \langle 2da, *_4\omega_X\rangle \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

J_1 and J_3 already show up in (5.1) and (5.2). As for J_2 and J_4, note that

$$\begin{aligned} J_4 &= -2\int_{\partial X_S} a \wedge \omega_X = \int_{Y_{1,S}} (B_1^t - B_{10}^t) \wedge \omega_1 + \int_{Y_{2,S}} (B_2^t - B_{20}^t) \wedge \omega_2, \\ J_2 &= \int_{X_S} \langle F_{A^t}, \omega_\lambda\rangle = -\int_{X_S} F_{A^t} \wedge *_4\omega_{X,h} + \int_{X_S} \langle F_{A^t}, \omega_\lambda - \omega_{X,h}\rangle. \end{aligned}$$

Since $\omega_{X,h}$ is harmonic, the first term in J_2 is a pairing in cohomology:

$$[\frac{i}{2\pi}F_{A^t}] \cup [\frac{i}{2\pi} *_4 \omega_{X,h}] \in H^4(X, \partial X) \xleftarrow{\cup} H^2(X, Z) \otimes H^2(X, Y_1 \cup Y_2),$$

so one may replace A by A_0. Now the energy identity follows by adding (5.4) and (5.5) together. $\square$

5.2. **The 3-Manifold Case.** Let $I = [t_1, t_2]_t$. In the special case when $\mathbb{X} = I \times \mathbb{Y} : \mathbb{Y} \to \mathbb{Y}$ is the product morphism, Theorem 5.1 takes a simpler form.

The 4-manifold $\widehat{X} = I \times \widehat{Y}$ is furnished with the product metric. Let $\omega_X = \pi^*\omega$ be the pull-back of ω where $\pi : \widehat{X} \to \widehat{Y}$ is the projection map. Any *spin*^{c} connection A on $\widehat{X}$ can be written as

$$A = \frac{d}{dt} + B(t) + c(t)dt \otimes \mathrm{Id}_S \,. \tag{5.6}$$

where $B(t)$ is a path of *spin*^{c} connections on $(\widehat{Y}, \widehat{\mathfrak{s}})$ and $c(t) \in L^2_k(\widehat{Y}, i\mathbb{R})$. Any configuration $\gamma \in (A, \Phi) \in \mathcal{C}_k(\widehat{X}, \widehat{\mathfrak{s}}_X)$ gives rise to a path $\check{\gamma}(t) = (B(t), \Psi(t))$ in $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ by setting

$$\Psi(t) = \Phi|_{\{t\}\times\widehat{Y}}.$$

Moreover, γ solves the Seiberg-Witten equations (3.7) on $\widehat{X}$ if and only if the path $(\check{\gamma}(t), c(t))$ forms a downward gradient flowline of $\mathcal{L}_\omega$:

$$\frac{d}{dt}\gamma(t) = -\operatorname{grad}\mathcal{L}_\omega(\gamma(t)) - \mathbf{d}_{\gamma(t)}\, c(t).$$

Let $A_0 = \frac{d}{dt} + B_0$ be the reference connection on $(\widehat{X}, \widehat{\mathfrak{s}}_X) = I \times (\widehat{Y}, \widehat{\mathfrak{s}})$. The curvature form $F_{A_0^t}$ does not involve any dt-component, so $F_{A_0^t} \wedge F_{A_0^t} \equiv 0$.

Proposition 5.4. *For any configuration $\gamma = (A, \Phi)$ on $(\widehat{X}, \widehat{\mathfrak{s}}_X) = I \times (\widehat{Y}, \widehat{\mathfrak{s}})$, the L^2-norm of the Seiberg-Witten map $\mathfrak{F}_X(A, \Phi)$ can be expressed as*

$$\int_{\widehat{X}} |\mathfrak{F}_X(A, \Phi)|^2 = \mathcal{E}_{an}(A, \Phi) - \mathcal{E}_{top}(A, \Phi)$$

where $\mathcal{E}_{top}(A, \Phi) := 2\mathcal{L}_\omega(\check{\gamma}(t_1)) - 2\mathcal{L}_\omega(\check{\gamma}(t_2))$ and

$$\begin{aligned}\mathcal{E}_{an}(A, \Phi) &:= \int_I \|\frac{d}{dt}\check{\gamma}(t) + d_{\check{\gamma}(t)}c(t)\|^2_{L^2(\widehat{Y})} + \|\operatorname{grad}\mathcal{L}_\omega(\check{\gamma}(t))\|^2_{L^2(\widehat{Y})} \\ &= \int_{I\times\widehat{Y}} \frac{1}{4}|F_{A^t}|^2 + |\nabla_A\Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2 + \frac{s}{4}|\Phi|^2 - \langle F_{A^t}, \omega\rangle.\end{aligned} \tag{5.7}$$

The last term can be written as

$$\int_{I\times\widehat{Y}} \langle F_{A^t}, \omega\rangle = \int_{I\times\widehat{Y}} \langle F_{A^t}, \overline{\omega}\rangle + \int_{I\times\widehat{Y}} \langle F_{A^t}, \omega_\lambda - \omega_h\rangle - |I| \int_{\widehat{Y}} F_{B_0^t} \wedge *_3\omega_h,$$

where $\omega = \overline{\omega} + \omega_\lambda$ and $\omega_\lambda = \chi_1(s)ds \wedge \lambda$. The bounded harmonic 2-form ω_h is constructed by Lemma 3.2 such that $\omega_\lambda - \omega_h \in L^2(\widehat{Y})$. In particular, for any $(B, \Psi) \in \mathcal{C}_1(\widehat{Y}, \widehat{\mathfrak{s}})$,

$$\|\operatorname{grad}\mathcal{L}_\omega(B, \Psi)\|^2_{L^2(\widehat{Y})} = \int_{\widehat{Y}} \frac{1}{4}|F_{B^t}|^2 + |\nabla_B\Psi|^2 + |(\Psi\Psi^*)_0 + \rho_3(\omega)|^2 + \frac{s}{4}|\Psi|^2 - \langle F_{B^t}, \omega\rangle.$$

6. Compactness

6.1. **Statements.** With all machinery developed so far, we are ready to state and prove the compactness theorem for the (unperturbed) Seiberg-Witten equations on $\mathbb{R}_t \times \widehat{Y}$. The result easily generalizes to a complete Riemannian manifold $\mathcal{X}$ induced from a morphism $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ in Cob_s. Nevertheless, we will focus on the first case for the sake of simplicity. The analogous results for perturbed equations will be addressed in Section 9, after we set up tame perturbations in the next part. Now let

$$\gamma_0 := (A_0, \Phi_0) \text{ with } A = \frac{d}{dt} + B_0,\ \Phi(t) = \Psi_0,$$

be the reference configuration on $\mathbb{R}_t \times \widehat{Y}$, then it agrees with the standard configuration (A_*, Φ_*) over the planar end $\mathbb{R}_t \times [0, \infty)_s \times \Sigma$. For any $k \geqslant 2$, define

$$\mathcal{C}_{k,loc}(\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})) = \{(A, \Phi) : (A, \Phi)\big|_{I\times\widehat{Y}} \in \mathcal{C}_k(I \times (\widehat{Y}, \widehat{\mathfrak{s}})), \forall \text{ finite interval } I \subset \mathbb{R}_t\}$$

and $\mathcal{G}_{k+1,loc}(\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}}))$ in a similar manner. We will set up the Fredholm theory of moduli spaces in a different way in Section 13. For now, let us stick to these loosely defined spaces.

For any $\gamma \in \mathcal{C}_{k,loc}$ and $I \subset \mathbb{R}_t$, define the analytic energy $\mathcal{E}_{an}(\gamma; I)$ over the interval I to be the integral of (5.7) over $I \times \widehat{Y}$ and

$$\mathcal{E}_{an}(\gamma) := \mathcal{E}_{an}(\gamma, \mathbb{R}_t).$$

One standard assumption below is the finiteness of the total energy $\mathcal{E}_{an}$. Since $\mathcal{E}_{an}(\gamma; I)$ is alway non-negative, it implies that

$$\mathcal{E}_{an}(\gamma; I) < \mathcal{E}_{an}(\gamma; \mathbb{R}_t) < \infty \text{ for any } I \subset \mathbb{R}_t.$$

The primary result of this section is the compactness theorem.

Theorem 6.1. *Suppose $\{\gamma_n = (A_n, \Phi_n)\} \subset \mathcal{C}_{k,loc}$ is a sequence of solutions to the Seiberg-Witten equations* (3.7) *on $\mathbb{R}_t \times \widehat{Y}$ and their analytic energy*

$$\mathcal{E}_{an}(\gamma_n) := \mathcal{E}_{an}(\gamma_n, \mathbb{R}_t) < C$$

is uniformly bounded by a positive constant $C > 0$. Then we can find a sequence of gauge transformations $u_n \in \mathcal{G}_{k+1,loc}(\mathbb{R}_t \times \widehat{Y})$ with the following properties. For a subsequence $\{\gamma_n'\}$ of $\{u_n(\gamma_n)\}$ and any finite interval $I \subset \mathbb{R}_t$, the restriction of each γ_n' on $I \times \widehat{Y}$

$$\gamma_n'|_{I\times\widehat{Y}}$$

lies in $\mathcal{C}_l(I \times (\widehat{Y}, \widehat{\mathfrak{s}}))$. In addition, they converge in $L^2_l(I \times \widehat{Y})$-topology for any $l \geqslant 2$.

The main difficulty is to deal with the cylindrical end of $\widehat{Y}$ and the proof relies on the exponential decay of L^2_l-norms. To state the result, recall that $\Omega_{n,S}$ ($n \in \mathbb{Z},\ S \in \mathbb{R}_s$) defined in (2.4) is a bounded sub-domain of $\mathbb{C}$ with smooth boundary, which is centered at $(n, S) \in \mathbb{R}_t \times \mathbb{R}_s$.

Theorem 6.2. *For any $C > 0$ and $l \in \mathbb{Z}_{\geqslant 1}$, there exists constants $\zeta(\widehat{Y}, \widehat{\mathfrak{s}}), M_l(C, \widehat{Y}, \widehat{\mathfrak{s}}) > 0$ with the following significance. For any solution $\gamma = (A, \Phi) \in \mathcal{C}_{k,loc}(\mathbb{R}_t \times \widehat{Y})$ to the Seiberg-Witten equations (3.7) on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with analytic energy $\mathcal{E}_{an}(A, \Phi) < C$, we can find a gauge transformation $u \in \mathcal{G}_{k+1,loc}(\mathbb{R}_t \times \widehat{Y})$ such that*

$$\|u(\gamma) - \gamma_0\|_{L^2_{l,A_0}(\Omega_{n,S} \times \Sigma)} \leqslant M_l e^{-\zeta S}, \tag{6.1}$$

for any $l \geqslant 1$, $n \in \mathbb{Z}$ and $S \geqslant 0$. Here γ_0 is the reference configuration in $\mathcal{C}_{k,loc}(\mathbb{R}_t \times \widehat{Y})$.

Theorem 6.1 is an easy corollary of Theorem 6.2.

Proof of Theorem 6.1. It suffices to prove the case when $I = [-2, 2]$. The rest will follow by a patching argument (cf. [KM07, Section 13.6]). By Theorem 6.2, for any γ_n in that sequence, we may assume the exponential decay (6.1) holds for $\gamma_n - \gamma_0$. Take $S \gg 1$ and let $Y_S = \{s \leqslant S\}$ be the truncated 3-manifold.

With the energy equation in Proposition 5.4, the classical compactness theorem [KM07, Theorem 5.2.1] implies that a subsequence of $\{\gamma_n\}$ converges smoothly (up to gauge) in the interior of the compact manifold $I \times Y_S$. Suppose $\{u_n : I \times Y_S \to S^1\}$ is the sequence of gauge transformations, then the restriction

$$u_n : I \times [S-1, S]_s \times \Sigma \to S^1$$

must lie in the same homotopy class when $n \gg 1$ (by (6.1)). We may correct $\{u_n\}$ so their restrictions lie in the trivial homotopy class. By a patching argument, we extend u_n over the whole space $I \times \widehat{Y}$ by setting $u_n \equiv 1$ when $s \geqslant S + 1$. By Theorem 6.2, a subsequence of $\{u_n(\gamma_n)\}$ converges in fact in L^2_l-topology on $[-2 + \epsilon, 2 - \epsilon] \times \widehat{Y}$ for some small $\epsilon > 0$. This completes the proof of the theorem (some details are left to the readers). ☐

The proof of Theorem 6.2 will dominate the rest of the section.

6.2. **Decay of Local Energy Functional.** Recall from Definition 2.3 that the local energy functional of $\gamma = (A, \Phi)$ over $\Omega_{n,S} \subset \mathbb{H}^2_+$ is defined as

$$\mathcal{E}_{an}(A, \Phi; \Omega) := \int_\Omega \int_\Sigma \frac{1}{4}|F_{A^t}|^2 + |\nabla_A \Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2.$$

with $\omega = \mu + ds \wedge \lambda$. We wish to first get an estimate on $\mathcal{E}_{an}(A, \Phi; \Omega_{n,S})$ for a solution (A, Φ) to (3.7) on $\mathbb{R}_t \times \widehat{Y}$ when $S \gg 1$. The main results are as follows.

Theorem 6.3. *For any $C, \epsilon > 0$, there exists a constant $R_0(\epsilon, C, \widehat{Y}, \widehat{\mathfrak{s}}) > 0$ with the following significance. For any solution $(A, \Phi) \in \mathcal{C}_k(\mathbb{R}_t \times \widehat{Y})$ to the Seiberg-Witten equations (3.7) on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with analytic energy $\mathcal{E}_{an}(A, \Phi) < C$ and any $S > R_0$, we have*

$$\mathcal{E}_{an}(A, \Phi; \Omega_{n,S}) < \epsilon.$$

The uniform decay in Theorem 6.3 can be improved into exponential decay using Theorem 2.5:

Theorem 6.4. *For any $C > 0$, there exists constants $\zeta(\widehat{Y}, \widehat{\mathfrak{s}}), M_0(C, \widehat{Y}, \widehat{\mathfrak{s}}) > 0$ with the following significance. For any solution $(A, \Phi) \in \mathcal{C}_k(\mathbb{R}_t \times \widehat{Y})$ to the Seiberg-Witten equations* (3.7) *on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with analytic energy $\mathcal{E}_{an}(A, \Phi) < C$, any $n \in \mathbb{Z}$ and $S > 0$,*

$$\mathcal{E}_{an}(A, \Phi; \Omega_{n,S}) < M_0 e^{-\zeta S}.$$

The proof of Theorem 6.3 will dominate the rest of Subsection 6.2 and it relies on Theorem 2.4 and 2.6 in an essential way. Let us first state a lemma in which we set $\Omega_S := \Omega_{0,S}$.

Lemma 6.5. *Let $J = [-3, 3] \supset I = [-2, 2]$. For any $\epsilon > 0$, there exists constants $R_0(\widehat{Y}, \epsilon), \eta(\widehat{Y}, \epsilon) > 0$ with the following significance. For any solution (A, Φ) to the Seiberg-Witten equations* (3.7) *on $J \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with $\mathcal{E}_{an}(A, \Phi; J) < \eta$ and any $S > R_0$, we must have*

$$\mathcal{E}_{an}(A, \Phi; \Omega_S) < \epsilon.$$

Proof. Suppose on the contrary that there exists a sequence $\{(A_n, \Phi_n)\}_{n \geqslant 1}$ of solutions to the Seiberg-Witten equations (3.7) on $J \times (\widehat{Y}, \widehat{\mathfrak{s}})$, a sequence of numbers $\eta_n \to 0$ and $R_n \to \infty$ such that

$$\mathcal{E}_{an}(A, \Phi; J) < \eta_n \text{ and } \mathcal{E}_{an}(A_n, \Phi_n; \Omega_{R_n}) \geqslant \epsilon.$$

By Proposition 5.4 and Lemma 5.3,

$$\mathcal{E}_{an}(A_n, \Phi_n; J \times [0, \infty)_s) \leqslant C_2'$$

for some uniform constant $C_2' > 0$. Let $\beta_n = (A_n', \Phi_n')(t, s) = (A_n, \Phi_n)(t, s - R_n)$ be the translated configuration defined on $J \times [-R_n, R_n] \times \Sigma$. Since we have a uniform bound on

$$\mathcal{E}_{an}(\beta_n; J \times [-R_n, R_n]),$$

the classical compactness theorem [KM07, Theorem 5.2.1] ensures that there is a subsequence of $\{\beta_n\}$ that converges in $\mathcal{C}^\infty_{loc}$ topology to a solution $\beta_\infty = (A_\infty, \Phi_\infty)$ on $J \times \mathbb{R}_s \times \Sigma$. On the other hand, if we write β_∞ as

$$(\check{\gamma}(t), c(t)) = (B(t), \Psi(t), c(t)),$$

then Proposition 5.4 implies

$$\partial_t \check{\gamma}(t) + \mathbf{d}_{\check{\gamma}(t)}\, c(t) = -\operatorname{grad} \mathcal{L}_\omega(\check{\gamma}(t)) = 0,$$

since $\eta_n \to 0$ as $n \to \infty$. By making β_∞ into temporal gauge (i.e $c(t) \equiv 0$), we conclude that $\check{\gamma}(t)$ is independent of $t \in I$ and solves the 3-dimensional Seiberg-Witten equations (3.6) or (2.5).

This is the place where the property (P7) is used. By Theorem 2.6, up to gauge, $\gamma(t)$ has to be $\mathbb{R}_s$-translation invariant, so

$$\mathcal{E}_{an}(\beta_\infty; I \times [-3, 3]) = 0.$$

This contradicts the assumption that $\mathcal{E}_{an}(A_n, \Phi_n; \Omega_{R_n}) \geqslant \epsilon$ for each n. □

Proof of Theorem 6.3. Suppose on the contrary that there exists a sequence

$$\{\beta_m = (A_m, \Phi_m)\}_{m \geqslant 1} \subset \mathcal{C}_{k,loc}(\mathbb{R}_t \times \widehat{Y})$$

of solutions to the Seiberg-Witten equations (3.7) on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$, a sequence of integers $n_m \geqslant 0$ and numbers $R_m \to \infty$ such that

$$\mathcal{E}_{an}(\beta_m) < C \text{ and } \mathcal{E}_{an}(A_m, \Phi_m, \Omega_{n_m, R_m}) \geqslant \epsilon.$$

Let $J_n = [n-3, n+3]$ for each $n \in \mathbb{Z}$. For each m, define the significant set of β_m as

$$K_m = \{n \in \mathbb{Z} : \mathcal{E}_{an}(\beta_m, J_n) > \eta\},$$

where $\eta = \eta(\epsilon, \widehat{Y}, \widehat{\mathfrak{s}})$ is the constant obtained in Lemma 6.5. Then $n_m \in K_m$. Since there is a uniform upper bound on $\mathcal{E}_{an}(\beta_m, \mathbb{R}_t)$, we know that

$$|K_m| < C_1 := 6C/\eta.$$

By passing to a subsequence, we assume $|K_m|$ are the same for all m. Place elements of K_m in the increasing order:

$$a_1^m < a_2^m < \cdots < a_k^m, \ k = |K_m|.$$

By passing to a further subsequence, we require that $\lim_{m\to\infty} |a_{i+1}^m - a_i^m|$ exists (either finite or infinite) for each $1 \leqslant i \leqslant k$ and it is infinite precisely when i is one of

$$i_0 := -1 < i_1 < i_2 < \cdots < i_l < i_{l+1} := k.$$

Let $N = \max_{0 \leqslant j \leqslant l, m \geqslant 0} |a_{i_{j+1}}^m - a_{i_j+1}^m|$. Now consider the translated configuration

$$\beta_m' = (A_m', \Phi_m') \text{ with } (A_m', \Phi_m')(t,s) = (A_m, \Phi_m)(t - n_m, s - R_m)$$

defined on $\mathbb{R}_t \times [-R_m, R_m] \times \Sigma$. What we have shown so far implies that

- $\mathcal{E}_{an}(\beta_m', [-N, N]_t \times [-R_m, R_m]_s)$ is bounded above by a constant C_2 independent of β_m'. This follows from energy equations and the assumption that $\mathcal{E}_{an}(\beta_m) \leqslant C$.
- For any $j \in \mathbb{Z}$ with $|j| \geqslant N$ and any $S \in \mathbb{R}_s$, $\mathcal{E}_{an}(\beta_m', \Omega_{j,S}) < \epsilon$ when $m \gg 1$. Indeed, by the choice of N, when $m \gg 1$, $n_m + j \notin K_m$ and $R_m \gg R_0 - S$. Now apply lemma 6.5

By the classical compactness theorem [KM07, Theorem 5.2.1], up to gauge, a subsequence of $\{\beta_m'\}$ will converge in $\mathcal{C}_{loc}^\infty$-topology to a solution $\beta_\infty = (A_\infty, \Phi_\infty)$ defined on $\mathbb{R}_t \times \mathbb{R}_s \times \Sigma$. Moreover, we have the following estimates on its analytic energy:

- For some large constant $M > 0$, $\mathcal{E}_{an}(\beta_\infty, \Omega_{j,S}) < \epsilon$ whenever $|j| > N$ or $|S| > M$;
- $\mathcal{E}_{an}(\beta_\infty, [-N, N]_t \times [-M, M]_s) < \infty$;
- $\mathcal{E}_{an}(\beta_\infty, \Omega_{0,0}) \geqslant \epsilon$.

Now we draw a contradiction from Theorem 2.4 which rules out such solutions. $\square$

6.3. **Decay of L^2_k-norm.** Having addressed the exponential decay of the local energy functional

$$\mathcal{E}_{an}(A,\Phi;\Omega_{n,S})$$

in Theorem 6.4, let us estimate the L^2_k-norm of (A,Φ) over the sub-domain $\Omega_{n,S}$ in terms of $\mathcal{E}_{an}(A,\Phi;\Omega_{n,S})$. Aside from Remark 5.2, this is the second reason why the local energy functional is useful. For the sake of simplicity, let us state the results for the compact domain

$$\Omega_0 \subset [-2,2]_t \times \mathbb{R}_s \subset \mathbb{C}$$

defined in (2.3). Let $M = \Omega_0 \times \Sigma$. Recall that $\gamma_* = (A_*, \Phi_*)$ defined by (3.8) is the standard configuration on $\mathbb{C} \times \Sigma$. For any smooth $\gamma = (A,\Phi) \in \mathcal{C}(M)$, set $(a,\phi) = \gamma - \gamma_*$ and consider the gauge fixing condition

$$\left\{ \begin{array}{ll} \mathbf{d}^*_{\gamma_*}(a,\phi) & := -d^*a + i\,\mathrm{Re}\langle \phi, i\Phi_*\rangle = 0 \\ \langle a, \vec{n}\rangle & = 0 \text{ at } \partial M. \end{array} \right. \tag{6.2}$$

The proof of Theorem 6.2 requires three additional lemmas, summarized as follows:

- Lemma 6.6: put γ into the Coulomb-Neumann gauge slice of γ_*;
- Lemma 6.7: once γ is in the slice, estimate the L^2_{1,A_*}-norm of $(a,\phi) = \gamma - \gamma_*$ in terms of $\mathcal{E}_{an}(\gamma;\Omega_0)$;
- Lemma 6.8: once γ is in the slice, estimate the L^2_{l,A_*}-norm of $(a,\phi) = \gamma - \gamma_*$ in terms of $\mathcal{E}_{an}(\gamma;\Omega_0)$ for any $l \geqslant 1$.

Lemma 6.6. *There exist constants $\epsilon_0, C_0 > 0$ with the following significance. For any configuration $\gamma \in \mathcal{C}(\Omega_0 \times \Sigma)$ with*

$$\|\gamma - \gamma_*\|_{L^2_{2,A_*}(M)} < \epsilon_0 \tag{6.3}$$

then we can find a smooth function $f : M \to i\mathbb{R}$ such that $e^f \cdot \gamma$ satisfies the Coulomb-Neumann gauge fixing condition (6.2)*. Moreover,*

$$\|e^f \cdot \gamma - \gamma_*\|_{L^2_{2,A_*}(M)} \leqslant C_0 \|\gamma - \gamma_*\|_{L^2_{2,A_*}(M)}.$$

Proof. Let $\mathcal{K}_2$ be the subspace of $\mathcal{T}_{2,\gamma_*} := L^2_2(M, iT^*M \oplus S^+)$ subject to the gauge fixing condition (6.2). Consider the non-linear map:

$$\begin{aligned} U : L^2_3(M, i\mathbb{R}) \times \mathcal{K}_2 &\to \mathcal{T}_{2,\gamma_*} \\ (f,(a,\phi)) &= (a - df, (e^f - 1)\cdot \Phi_* + e^f \cdot \phi). \end{aligned}$$

The linearized operator $\mathcal{D}_0 U$ of U at $(0,(0,0))$ is invertible. Now our lemma follows from the implicit function theorem. □

Suppose now that γ already lies in the Coulomb-Neumann gauge slice of γ_*. The next step is to estimate $\|(a,\phi)\|_{L^2_{1,A_*}}$ in terms of the local energy functional $\mathcal{E}_{an}(A,\Phi;\Omega_0)$.

Lemma 6.7. *There exist constants $\epsilon_1, C_1 > 0$ with the following significance. For any γ subject to the gauge fixing condition* (6.2)*, if $\|(a,\phi)\|_{L^2_{1,A_*}} < \epsilon_1$, then*

$$\|(a,\phi)\|^2_{L^2_{1,A_*}} \leqslant C_1 \cdot \mathcal{E}_{an}(\gamma, \Omega_0).$$

Proof. Consider the non-linear operator:

$$\begin{aligned}\mathcal{F}(a,\phi) &= \mathcal{F}_1+\mathcal{F}_2 \text{ where}\\ \mathcal{F}_1(a,\phi) &= (da, \nabla_{A_*}\phi + a\otimes\Phi_*, (\Phi_*\phi^*+\phi\Phi_*^*)_0, \mathbf{d}^*_{\gamma_*}(a,\phi)),\\ \mathcal{F}_2(a,\phi) &= (0, a\otimes\phi, (\phi\phi^*)_0, 0),\end{aligned}$$

so $\mathcal{F}_1$ is the linear part of $\mathcal{F}$ and $\|\mathcal{F}(a,\phi)\|^2_{L^2(M)} = \mathcal{E}_{an}(\gamma,\Omega_0)$ by Definition 2.3. Using the identity

$$|(\Phi_*\phi^*+\phi\Phi_*^*)_0|^2 + |\operatorname{Im}\langle\phi,\Phi_*\rangle|^2 = |\Phi_*|^2|\phi|^2,$$

we calculate that

$$\begin{aligned}\|\mathcal{F}_1(a,\phi)\|^2_{L^2(M)} &= \|da\|_2^2+\|d^*a\|_2^2+\|\nabla_{A_*}\phi\|_2^2+\|a\otimes\Phi_*\|_2^2+\||\phi||\Phi_*|\|_2^2+K_3 \text{ where}\\ K_3 &= 2\operatorname{Re}\int_M \langle\nabla_{A_*}\phi, a\otimes\Phi_*\rangle - \langle\phi,(d^*a)\Phi_*\rangle\\ &= 2\operatorname{Re}\int_M d^*(\langle\phi,\Phi_*\rangle\cdot a) + \langle a\otimes\phi, \nabla_{A_*}\Phi_*\rangle = 0.\end{aligned}$$

In the last step, we used the facts that Φ_* is ∇_{A_*}-parallel and $\langle a,\vec{n}\rangle = 0$ at ∂M. Hence,

$$\|\mathcal{F}_1(a,\phi)\|_{L^2(M)} \geqslant c_1\|(a,\phi)\|_{L^2_{1,A_*}},$$

for some $c_1>0$. Finally,

$$\|\mathcal{F}\|_2 \geqslant \|\mathcal{F}_1\|_2 - \|\mathcal{F}_2\|_2 \geqslant c_1\|(a,\phi)\|_{L^2_{1,A_*}} - m_3\|(a,\phi)\|^2_{L^2_{1,A_*}} \geqslant \frac{c_1}{2}\|(a,\phi)\|_{L^2_{1,A_*}}$$

if $\|(a,\phi)\|_{L^2_{1,A_*}} \leqslant c_1/2m_3$, where m_3 is the constant that appears in the Sobolev embedding $L^2_1\times L^2_1 \to L^4$. □

Now we come to estimate the L^2_k-norm of (a,ϕ). Consider a closed subset $\Omega_0'\subset\Omega_0$ with a smooth boundary such that

$$[-1,1]_t\times[1,3]\subset(\Omega_0')^\circ\subset\Omega_0'\subset(\Omega_0)^\circ.$$

Lemma 6.8. *There exist constants $\epsilon_k, C_k>0$ for each $k\geqslant 1$ with the following significance. For any smooth solution $\gamma\in\mathcal{C}(M)$ to the Seiberg-Witten equations* (3.7)*, if γ is subject to the gauge fixing condition* (6.2) *and $\|(a,\phi)\|_{L^2_{1,A_*}(M)}<\epsilon_k$, then*

$$\|(a,\phi)\|^2_{L^2_{k,A_*}(\Omega_0'\times\Sigma)} \leqslant C_k\cdot\mathcal{E}_{an}(\gamma,\Omega_0).$$

Proof. The case when $k=1$ is settled in Lemma 6.7. For $k>1$, this follows from the standard bootstrapping argument [KM07, P.107]. To illustrate, consider the case when $1<k<2$. Take a cut-off function χ_4 such that

$$\chi_4\equiv 1 \text{ on } \Omega_0';\ \operatorname{supp}\chi_4\subset(\Omega_0)^\circ.$$

The section $v:=(a,\phi)\in C^\infty(M, iT^*M\oplus S)$ is subject to a non-linear elliptic equation:

$$Dv + v\# v = 0$$

where # stands for a certain bilinear form that involves only point-wise multiplication. By Gårding's inequality, for any $0 < \eta < 1$,

$$\begin{aligned}\|\chi_4 v\|_{L^2_{1+\eta}(M)} &\leqslant \|D(\chi_4 v)\|_{L^2_\eta(M)} + \|v\|_2 \leqslant m_4\|v\|_{L^2_1} + \|(\chi_4 v)\#v\|_{L^2_\eta}\\ &\leqslant m_4\|v\|_{L^2_1} + m_5\|\chi_4 v\|_{L^2_{1+\eta}}\|v\|_{L^2_1}\end{aligned}$$

If $\|v\|_{L^2_1} < 1/(2m_5)$, then we use the rearrangement argument to show that

$$\|v\|_{L^2_{1+\eta}(\Omega_0'\times\Sigma)} \leqslant \|\chi_4 v\|_{L^2_{1+\eta}(M)} \leqslant 2m_4\|v\|_{L^2_1} \leqslant 2m_4\sqrt{C_1}\cdot\sqrt{\mathcal{E}_{an}(\gamma,\Omega_0)},$$

so we set $\epsilon_{1+\eta} = \min\{\epsilon_1, 1/(2m_5)\}$. In the last step, we used Lemma 6.7 to estimate $\|v\|_{L^2_{1,A_*}}$ in terms of $\mathcal{E}_{an}(\gamma,\Omega_0)$. When $k \geqslant 2$, we need more cut-off functions to separate Ω_0' from Ω_0 and use inductions. In fact, we can take

$$\epsilon_k = \min\{\epsilon_1, 1/(2m_5)\}$$

for any $k > 1$. □

Proof of Theorem 6.2. We divide the proof into three steps. Lemma 6.6 and 6.8 will be used only in the last step. In *Step* 1 and *Step* 2, we arrange so that the assumptions of these lemmas can be satisfied.

Step 1. By the classical compactness theorem [KM07, Theorem 5.2.1], for any $\epsilon > 0$, we can find a constant $\eta(\epsilon) > 0$ with the following property. Under the assumption of Theorem 6.2, if $\mathcal{E}_{an}(\gamma,\Omega_0) < \eta(\epsilon)$, then there exists a gauge transformation $u' : \Omega_0 \to S^1$ such that

$$\|u'(\gamma) - \gamma_*\|_{L^2_2(\Omega_0'\times\Sigma)} < \epsilon.$$

At this point, we have no controls of the function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$.

Step 2. We wish to find a gauge transformation $u_1 \in \mathcal{G}_{k+1,loc}(\mathbb{R}_t \times \widehat{Y})$ such that

$$\|u_1(\gamma) - \gamma_0\|_{L^2_{2,A_*}(\Omega_{n,S}\times\Sigma)} < \min\{\epsilon_0, \frac{\epsilon_l}{C_0}\}. \tag{6.4}$$

for any $n \in \mathbb{Z}$ and $S \gg 1$, where ϵ_0 and ϵ_l are positive constants constructed in Lemma 6.6 and 6.8. (6.4) is provided by the uniform L^∞ decay of the local energy functional. Let $S = m \in \mathbb{Z}_{\geqslant 0}$ be an integer and apply *Step* 1 to the domain

$$\Omega_{n,m}, \forall n \in \mathbb{Z}, m > R_0(\eta(\epsilon), C),$$

where R_0 is the constant obtained in Theorem 6.3. We find gauge transformations $u_{n,m} \in \mathcal{G}^e(\Omega_{n,m}\times\Sigma)$ such that

$$\|u_{n,m}(\gamma) - \gamma_0\|_{L^2_2(\Omega_{n,m}'\times\Sigma)} < \epsilon.$$

Here $\Omega_{n,m}'$ is the translated domain of $\Omega_0' \subset \Omega_0$:

$$\Omega_{n,m}' = \{(t,s) : (t-n, s-m) \in \Omega_0'\} \subset \Omega_{n,m}.$$

The collection of domains $\{(\Omega_{n,m}')^\circ\}$ still forms an open cover of $\mathbb{R}_t \times [R_0+1)_s \times \Sigma$. By a patching argument (cf. [KM07, Section 13.6]), we can find a global gauge transformation u_1 such that

$$\|u_1(\gamma) - \gamma_0\|_{L^2_1(\Omega_{n,m}\times\Sigma)} < N_1\epsilon.$$

for a constant $N_1 > 0$. Then one may achieve (6.4) by starting with ϵ small enough.

Step 3. Now apply Lemma 6.6 to $u_1(\gamma)$ on each $\Omega_{n,m}$ with $m > R_0$. We find some smooth functions $f_{n,m} : \Omega_{n,m} \times \Sigma \to i\mathbb{R}$ such that

$$\begin{aligned}\|e^{f_{n,m}} \cdot u_1(\gamma) - \gamma_0\|_{L^2_{1,A_*}(\Omega_{n,m}\times\Sigma)} &\leqslant \|e^{f_{n,m}} \cdot u_1(\gamma) - \gamma_0\|_{L^2_{2,A_*}(\Omega_{n,m}\times\Sigma)} \\ &\leqslant C_0\|u_1(\gamma) - \gamma_0\|_{L^2_{2,A_*}(\Omega_{n,m}\times\Sigma)} \leqslant \epsilon_l.\end{aligned}$$

and $e^{f_{n,m}} \cdot u_1(\gamma)$ lies in the Coulomb gauge slice (6.2) of γ_*. Using Lemma 6.8 and Theorem 6.4, we estimate the L^2_{l,A_*}-norm of the resulting configuration:

$$\|e^{f_{n,m}} \cdot u_1(\gamma) - \gamma_0\|^2_{L^2_{l,A_*}(\Omega'_{n,m}\times\Sigma)} \leqslant C_l \cdot \mathcal{E}_{an}(\gamma, \Omega_{n,m}) \leqslant C_l M_0 e^{-\zeta m}.$$

Finally, using the patching argument once again, we find a global gauge transformation $u \in \mathcal{G}_{k+1,loc}(\mathbb{R}_t \times \widehat{Y})$ such that

$$\|u(\gamma) - \gamma_0\|^2_{L^2_{l,A_*}(\Omega_{n,m}\times\Sigma)} \leqslant N_2 C_l M_0 e^{-\zeta m}.$$

for a constant $N_2 > 0$. This completes the proof of Theorem 6.2. □

Part 3. Perturbations

In order to make the moduli spaces on $\mathbb{R}_t \times \widehat{Y}$ smooth and define the Floer homology of the 3-manifold $(Y, \partial Y = \Sigma)$, a suitable perturbation $\cancel{\mathcal{L}}_\omega = \mathcal{L}_\omega + f$ of the Chern-Simons-Dirac functional $\mathcal{L}_\omega$ is needed. We follow the construction of tame perturbations in [KM07, Section 10-11]. However, there is one distinct feature of our situation, which requires some technical tricks to deal with:

($\star$) We want the perturbation supported within **a compact region** of $\widehat{Y}$ so that the Seiberg-Witten equations (3.7) defined on $\mathbb{R}_t \times \widehat{Y}$ remains unperturbed on the planar end $\mathbb{H}^2_+ \times \Sigma$, and Theorem 2.5 is applicable.

Hence, the error term f must factorize through the restriction map to the truncated manifold $Y_n := \{s \leqslant n\} \subset \widehat{Y}$ for some $n \geqslant 0$:

$$\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{C}_{k-1/2}(Y_n, \widehat{\mathfrak{s}}).$$

As a result, the perturbation space is not large enough to separate all tangent vectors and points of $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ as in [KM07, Proposition 11.2.1]. Nevertheless, we can still achieve the transversality of moduli spaces on $\mathbb{R}_t \times \widehat{Y}$, even with this smaller perturbation space. In fact, one may even require that $n = 0$, so $Y_n = Y = \{s \leqslant 0\}$.

Part 3 is organized as follows. In Section 7, we introduce the so-called tame perturbations (Definition 7.3) and state the formal mapping properties that they enjoy.

In Section 8, we take up the task to construct tame perturbations. The separation properties are examined carefully in Subsection 8.2. The Banach space $\mathcal{P}$ of tame perturbations is constructed in Subsection 8.5.

Section 9 is devoted to the compactness theorems for perturbed Seiberg-Witten equations. Since tame perturbations are made compactly supported, the proofs in Section 6 apply verbatim to this case.

7. Abstract Perturbations

The perturbation that we deal with is a continuous section $(k > 1)$

$$\mathfrak{q} : \mathcal{C}_{k-\frac{1}{2}}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{T}_0$$

where $\mathcal{T}_0$ is the L^2-completion of the tangent bundle $T\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ introduced in Section 4. The perturbation $\mathfrak{q}$ is required to be the formal gradient of a $\mathcal{G}_{k+1/2}(\widehat{Y})$-invariant continuous function $f : \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathbb{R}$, and we write $\mathfrak{q} = \operatorname{grad} f$. This means that

$$f(\check{\gamma}(1)) - f(\check{\gamma}(0)) = \int_0^1 \langle \dot{\check{\gamma}}, \mathfrak{q}(\check{\gamma}(t)) \rangle_{L^2} dt$$

for any smooth path $\check{\gamma} : [0, 1] \to \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$. Take

$$\cancel{\mathcal{L}}_\omega = \mathcal{L}_\omega + f$$

to be the perturbed Chern-Simons-Dirac functional. Let $I = [t_1, t_2]$ and $\widehat{Z}$ be the product $spin^c$ manifold $I \times (\widehat{Y}, \widehat{\mathfrak{s}})$. The down-ward gradient flowline equation of $\mathcal{L}_\omega$ becomes

$$\begin{aligned} \frac{d}{dt}\check{\gamma}(t) &= -\operatorname{grad} \mathcal{L}_\omega(\check{\gamma}(t)) - \mathbf{d}_{\check{\gamma}(t)}\, c(t) \\ &= -\operatorname{grad} \mathcal{L}_\omega(\check{\gamma}(t)) - \mathbf{d}_{\check{\gamma}(t)}\, c(t) - \mathfrak{q}(\check{\gamma}(t)), \end{aligned} \tag{7.1}$$

where $\check{\gamma}(t) = (B(t), \Psi(t))$ is a underlying path in $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ and

$$A = \frac{d}{dt} + B(t) + c(t)dt \otimes \mathrm{Id}_S,\ \Phi|_{\{t\}\times Y} = \Psi(t) \tag{7.2}$$

is the corresponding 4-dimensional configuration $\gamma = (A, \Phi)$ in $\mathcal{C}(\widehat{Z})$. In this way, the continuous section $\mathfrak{q}$ extends to a section of the trivial bundle $\mathcal{V}_0$ over $\mathcal{C}(\widehat{Z})$:

$$\widehat{\mathfrak{q}} : \mathcal{C}(\widehat{Z}) \to \mathcal{V}_0 = L^2(\widehat{Z}, i\mathfrak{su}(S^+) \oplus S^-) \times \mathcal{C}(\widehat{Z}) \tag{7.3}$$

by sending $\gamma = (A, \Phi)$ to $\mathfrak{q}(\check{\gamma}(t))$ at each time slice $t \in I$. Here we use the 3-dimensional Clifford multiplication ρ_3 to identify the bundle $iT^*\widehat{Y}$ with $i\mathfrak{su}(S^+)$ over $\widehat{Z}$. We wish that this section $\widehat{\mathfrak{q}}$ extends to a smooth section of $\mathcal{V}_k \to \mathcal{C}_k(\widehat{Z})$ for any $k \geqslant 2$, so (7.1) is cast into the perturbed Seiberg-Witten equation $\mathfrak{F}_{\widehat{Z},\mathfrak{q}} = 0$ where

$$\mathfrak{F}_{\widehat{Z},\mathfrak{q}} := \mathfrak{F}_{\widehat{Z}} + \widehat{\mathfrak{q}} : \mathcal{C}_k(\widehat{Z}) \to \mathcal{V}_{k-1},$$

and $\mathfrak{F}_{\widehat{Z}}$ is defined as in (3.7).

We do not have a canonical L^2_j norm on the space $\Gamma(\widehat{Z}, i\mathfrak{su}(S^+) \oplus S^-)$. For each $\gamma = (A, \Phi) \in \mathcal{C}_k(\widehat{Z})$, we define a norm at the fiber $\mathcal{V}_j|_\gamma$ using A as the covariant derivative, i.e.

$$\|v\|^2_{L^2_{j,A}} := \sum_{n=0}^{j} \|\nabla^n_A v\|^2$$

for any $v \in \mathcal{V}_j|_\gamma$. This family of norms on $\mathcal{V}_j$ is equivariant under the gauge action of $\mathcal{G}_{k+1}(\widehat{Z})$. Similarly, we define the $L^2_{j,A}$ norm on $\mathcal{T}_j \to \mathcal{C}_k(\widehat{Z})$. Then the l-th derivative of $\widehat{\mathfrak{q}}$ at γ is a bounded multi-linear map:

$$\begin{aligned} \mathcal{D}^l_\gamma \widehat{\mathfrak{q}} &\in \mathrm{Mult}^l \left(\bigtimes_l L^2_{k,A}(\widehat{Z}, iT^*\widehat{Z} \otimes S^+), L^2_{k,A}(i\mathfrak{su}(S^+) \oplus S^-) \right) \\ &= \mathrm{Mult}^l(\bigtimes_l \mathcal{T}_k, \mathcal{V}_k). \end{aligned}$$

The bundle map $\mathcal{D}^l_\gamma \mathfrak{q}$ might not be a local operator: it does not necessarily send compactly supported sections on $\widehat{Y}$ to another section with the same or smaller support. However, this is a property enjoyed by derivatives $\mathcal{D}^l_\gamma \mathfrak{F}_{\widehat{Z}}$ of the unperturbed Seiberg-Witten map $\mathfrak{F}_{\widehat{Z}}$, which motivates the next definition:

Definition 7.1. For any closed subset $\Omega \subset \widehat{Y}$, a perturbation $\mathfrak{q}$ is said to be supported on Ω if supp $\mathfrak{q}(\check{\gamma}) \subset \Omega$ for any $\check{\gamma} \in \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ and

$$\mathfrak{q}(\check{\gamma}_1) = \mathfrak{q}(\check{\gamma}_2)$$

for any configurations $\check{\gamma}_1, \check{\gamma}_2 \in \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ such that $\check{\gamma}_1 = \check{\gamma}_2$ on Ω. $\diamondsuit$

We are primarily interested in the case when $\Omega = Y_n = \{s \leqslant n\}$ for some $n \geqslant 0$. It turns out that the choice of the integer n is inconsequential for the Floer homology, so we may safely set $n = 0$ and focus on the case when $\Omega = Y$.

Remark 7.2. One may even take $\Omega = [0,1]_s \times \Sigma \subset \widehat{Y}$ and the construction in Section 8 would be simplified if one uses the gauge fixing condition along each fiber $\{s\} \times \Sigma$. ◇

For technical reasons, we also need completions of bundles and the configuration space with respect to other Sobolev norms L^p_k with $p \neq 2$. Let

$$\mathcal{C}_k^{(p)},\ \mathcal{T}_k^{(p)},\ \mathcal{V}_k^{(p)}$$

be the resulting space and bundles when $k \geqslant 1$ and $1 \leqslant p \leqslant \infty$. Note that $\mathcal{C}_k^{(2)}(\widehat{Z}) = \mathcal{C}_k(\widehat{Z})$ and so on.

Let us state the constraints on the perturbation $\mathfrak{q} = \operatorname{grad} f$.

Definition 7.3. Let Y' be a smooth co-dimension 0 submanifold of $\widehat{Y}$ with possibly non-empty boundary. We usually take Y' to be either $Y = \{s \leqslant 0\}$ or $\widehat{Y}$. For each integer $k \geqslant 2$, a perturbation $\mathfrak{q}$ given as a section

$$\mathfrak{q} : \mathcal{C}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{T}_0.$$

is called k**-tame in** Y' if it is the formal gradient of a continuous $\mathcal{G}(\widehat{Y})$-invariant function f on $\mathcal{C}(\widehat{Y})$ such that

(A1) the corresponding 4-dimensional perturbation $\widehat{\mathfrak{q}}$ defines an element:

$$\widehat{\mathfrak{q}} \in C^\infty(\mathcal{C}_j(\widehat{Z}), \mathcal{V}_j)$$

for any integer $j \in [2, k]$;

(A2) When $p > 3$, $\widehat{\mathfrak{q}}$ also defines an element in

$$C^\infty(\mathcal{C}_j^{(p)}(\widehat{Z}), \mathcal{V}_j^{(p)})$$

for any integer $j \in [1, k]$;

(A3) $\widehat{\mathfrak{q}}$ extends to a continuous map:

$$\mathcal{C}_1(\widehat{Z}) \to \mathcal{V}_0^{(m)}$$

for any $2 \leqslant m < 4$.

(A4) for each integer $j \in [-k, k]$, the first derivative

$$\mathcal{D}\widehat{\mathfrak{q}} \in C^\infty(\mathcal{C}_k(\widehat{Z}), \operatorname{Hom}(T\mathcal{C}_k(\widehat{Z}), \mathcal{V}_k))$$

extends to a smooth map

$$\mathcal{D}\widehat{\mathfrak{q}} \in C^\infty(\mathcal{C}_k(\widehat{Z}), \operatorname{Hom}(\mathcal{T}_j, \mathcal{V}_j));$$

(A5) for any $(B, \Psi) \in \mathcal{C}_k(\widehat{Y})$, the L^2_k-section $\mathfrak{q}(B, \Psi)$ is supported on Y':

$$\operatorname{supp} \mathfrak{q}(B, \Psi) \subset Y'.$$

Moreover, there exists a constant $m_2 > 0$ such that

$$\|\mathfrak{q}(B, \Psi)\|_{L^2(Y')} \leqslant m_2(\|\Psi\|_{L^2(Y')} + 1),$$

for any $(B, \Psi) \in \mathcal{C}_k(\widehat{Y})$.

(A6) For any $0 \leqslant \epsilon < \frac{1}{2}$, $\widehat{\mathfrak{q}}$ extends to a continuous map

$$\mathcal{C}_{1-\epsilon}(\widehat{Z}) \to \mathcal{V}_0.$$

(A7) the 3-dimensional perturbation $\mathfrak{q}$ defines a C^1-section

$$\mathfrak{q} : \mathcal{C}_1(\widehat{Y}) \to \mathcal{T}_0.$$

We simply say that $\mathfrak{q}$ is tame in Y' if $\mathfrak{q}$ is k-tame in Y' for any $k \geqslant 2$. We may not mention the support Y' when $Y' = \widehat{Y}$. ♢

Remark 7.4. When $Y' = \widehat{Y}$, Definition 7.3 agrees with [KM07, Definition 10.5.1], with some minor changes in properties (A2)(A3)(A5)(A6). Our construction of tame perturbations in Section 8 ends up with weaker mapping properties, in exchange for having them compactly supported. ♢

Remark 7.5. Let us briefly explain where these properties will be used:

- (A1)(A2)(A3)(A6) will be used in the compactness theorem for the perturbed Seiberg-Witten equations, i.e. Theorem 9.5. They give intermediate steps in the bootstrapping arguments;
- (A5) is used in the energy equation for the perturbed Seiberg-Witten equations, i.e. Proposition 9.1;
- (A4) is relevant with the linear theory in Part 4;
- (A7) will be used in the proof of the exponential decay result in time direction, which we will not actually work out in this paper, cf. [KM07, Section 13.4], in particular [KM07, Lemma 13.4.3]. ♢

8. Constructing Tame Perturbations

8.1. **Cylinder Functions.** The construction of cylinder functions in the book [KM07, Section 11] involves a global gauge slice, which prevents perturbations being local. Instead, we adopt a variation that is reminiscent of the holonomy perturbations in instanton Floer homology to achieve our goal.

First, we fix a smooth embedding of $S^1 \times D^2$ into $\widehat{Y}$, where $D^2 = B(0,1) \subset \mathbb{R}^2$ is the unit disk:

$$\iota : S^1 \times D^2 \to \widehat{Y}.$$

To find such an ι, one may first embed the core $S^1 \times \{0\}$ into $\widehat{Y}$ and extend this map to a tubular neighborhood of the image. We pull back the metric and the spin bundle $S \to \widehat{Y}$ via ι. The induced Riemannian metric $g_1 := \iota^* g_Y$ might not agree with the product metric

$$g_{std} := \iota^* g_{\widehat{Y}}|_{S^1 \times \{0\}} + g_{D^2},$$

on $S^1 \times D^2$, where g_{D^2} is the standard Euclidean metric of D^2. They are related by a smooth symmetric bundle map $K : T^*(S^1 \times D^2) \to T^*(S^1 \times D^2)$ (with respect to g_{std}) such that

$$\langle b_1, b_2 \rangle_1 = \langle K(b_1), b_2 \rangle_{std}.$$

for any co-vectors b_1 and b_2. The volume forms of g_1 and g_{std} differ by a smooth positive function $\eta > 0$:

$$dvol_1 = \eta \cdot dvol_{std}.$$

It is only important to know that K and η are smooth; the Clifford multiplication ρ_3 is never needed for the purpose of perturbations.

Let (B_0, Ψ_0) be the reference configuration in $\mathcal{C}_k(Y)$. For any $(B, \Psi) \in \mathcal{C}_k(\widehat{Y})$, take the difference

$$(b, \psi) := (B, \Psi) - (B_0, \Psi_0) \in L^2_k(\widehat{Y}, iT^*\widehat{Y} \oplus S).$$

There are three classes of perturbations to be considered. The first two concern the imaginary valued 1-form b. The last one deals with the spin section Ψ.

(B1) For any compactly supported 1-form $c \in \Omega^1_c(S^1 \times D^2, i\mathbb{R})$, define

$$\begin{aligned} r_c : \mathcal{C}_k(\widehat{Y}) &\to \mathbb{R} \\ (b, \psi) &\mapsto \int_{S^1 \times D^2} b \wedge d\bar{c} \\ &= \int_{S^1 \times D^2} \langle b, *_1 dc\rangle_{g_1} dvol_1 = \int_{S^1 \times D^2} \langle b, *_{std} dc\rangle_{g_{std}} dvol_{std}, \end{aligned}$$

where $*_1$ and $*_{std}$ stand for the Hodge star operators of g_1 and g_{std} respectively. The formal gradient of r_c is

$$\operatorname{grad} r_c = *_1 dc,$$

while using g_{std} we obtain

$$\operatorname{grad}_{std} r_c := *_{std} dc = \eta K(\operatorname{grad} r_c).$$

(B2) Fix a compactly supported 2-form $\nu \in \Omega^1_c(D^2, i\mathbb{R})$ on the disk D^2 with

$$\int_{D^2} \nu = i,$$

and define

$$\begin{aligned} r_\nu : \mathcal{C}_k(\widehat{Y}) &\to \mathbb{R} \\ (b, \psi) &\mapsto \int_{S^1 \times D^2} b \wedge \pi^* \overline{\nu}, \end{aligned}$$

where $\pi : S^1 \times D^2 \to D^2$ is the projection map. Unlike r_c, r_ν is not fully gauge-invariant. For any $u \in \mathcal{G}_{k+1}(\widehat{Y})$,

$$r_\nu(u(b, \psi)) - r_\nu(b, \psi) = -2\pi \deg(u \circ \iota : S^1 \times \{0\} \to S^1) \in 2\pi\mathbb{Z}.$$

Hence, r_ν descends to a circle valued function

$$[r_\nu] : \mathcal{C}_k(\widehat{Y}) \to \mathbb{R}/(2\pi\alpha\mathbb{Z})$$

where $\alpha \in \mathbb{Z}_{\geqslant 0}$ is the multiplicity of $\iota_*([S^1 \times \{0\}])$ in $H_1(Y, \Sigma; \mathbb{Z})$, i.e $\iota_*([S^1 \times \{0\}])$ is α times a primitive class in $H_1(Y, \Sigma; \mathbb{Z})$. Using the Euclidean metric of D^2, one may conveniently set

$$\nu = i\chi_2(z) dvol_{D^2}$$

where χ_2 is a cut-off function on D^2 with $\chi_2(z) \equiv 1$ when $|z| \leqslant \frac{1}{2}$.

(B3) Fix a gauge transformation $u_1 : \widehat{Y} \to S^1$ with the following properties:
- u_1 is smooth on $\widehat{Y}$;
- The composition $u_1 \circ \iota : S^1 \times \{0\} \to S^1$ is harmonic and has degree α.
- $u_1 \circ \iota : S^1 \times D^2 \to S^1$ is constant in D^2.

Let the transformation u_1 act on the bundle $\mathbb{R}_x \times S \to \mathbb{R}_x \times (S^1 \times D^2)$ by the formula:

$$u_1^n(x, \Phi) \mapsto (x - 2\pi n\alpha, u_1^n \Phi).$$

Passing to the quotient space, we obtain a bundle $\mathbb{S}$ over $(\mathbb{R}/2\pi\alpha\mathbb{Z}) \times (S^1 \times D^2)$. If Υ is a compactly supported smooth section of $\mathbb{S}$, let $\tilde{\Upsilon}$ denote its lift as a section of $\mathbb{R}_x \times S \to \mathbb{R}_x \times (S^1 \times D^2)$. Then $\tilde{\Upsilon}$ is an equivariant section, as

$$\tilde{\Upsilon}(x - 2\pi n\alpha, \theta, z) = u_1^n \tilde{\Upsilon}(x, \theta, z)$$

for any $(\theta, z) \in S^1 \times D^2$ and $x \in \mathbb{R}_x$. Let $b_z = b|_{S^1 \times \{z\}}$ be the restriction of the 1-form b over the S^1-fiber at $z \in D^2$. Using the product metric g_{std}, we write

$$b_z = b_z^1 + b_z^h$$

in terms of the Hodge decomposition along each fiber $S^1 \times \{z\}$ with

$$b_z^1 \text{ exact and } b_z^h \text{ harmonic (the coexact part } b_z^2 = 0).$$

Let $d^*_{S^1}$ be the adjoint of the exterior differential d_{S^1} over $S^1 \times \{0\}$ and

$$G : C^\infty(S^1, i\mathbb{R}) \to C^\infty(S^1, i\mathbb{R})$$

be the Green operator. Then the exact part b_z^1 can be explicitly written as

$$b_z^1 = d_{S^1} G d^*_{S^1} b_z,$$

and b_z^h stands for the harmonic part of b_z. It is tempting to form the map:

$$\begin{aligned} \Upsilon^\dagger : \mathcal{C}(\widehat{Y}) &\to C^\infty(S^1 \times D^2, S) \\ (b, \psi) &\mapsto e^{-Gd^*_{S^1} b_z} \tilde{\Upsilon}(r_\nu(b), \theta, z) \text{ on } S^1 \times \{z\}, \end{aligned}$$

which is **equivariant** under the action of u_1^n. However, $\Upsilon^\dagger$ is **not** equivariant under the action of the full gauge group $\mathcal{G}(\widehat{Y})$ (compare [KM07, P.173]). In fact, $\Upsilon^\dagger$ is invariant under $\mathrm{Map}(D^2, S^1)$, the space of gauge transformations that are constant along each fiber $S^1 \times \{z\}$.

To circumvent this problem, let Ψ_z and $\Upsilon_z^\dagger$ be the restriction of Ψ and $\Upsilon^\dagger$ along the fiber $S^1 \times \{z\}$ for any $z \in D^2$. Fix an S^1-invariant function $h : \mathbb{C}_w \to \mathbb{R}$. For instance, set

$$h(w) = \chi_3(|w|^2), \forall w \in \mathbb{C},$$

for some cut-off function $\chi_3 : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ such that

$$\chi_3(t) \equiv 1 \text{ if } t \leqslant 1;\ \chi_3(t) \equiv 0 \text{ if } t \geqslant 2.$$

Then the composition $h(\sigma(z)) : \mathcal{C}(\widehat{Y}) \to \mathbb{R}$ is fully gauge invariant, where

$$\sigma(z) := \int_{S^1\times\{z\}} \langle \Psi_z, \Upsilon_z^\dagger \rangle.$$

Finally, define

$$q_\Upsilon(b,\psi) = \int_{D^2} h(\sigma(z))\chi_2(z) dvol_{D^2},$$

where χ_2 is the cut-off function on D^2 defined in (B2).

By choosing a finite collection of 1-forms $c_1, \cdots, c_n$ and smooth sections $\Upsilon_1, \cdots, \Upsilon_m$ of $\mathbb{S}$, we obtain a map

$$\Xi = (r_{c_1}, \cdots, r_{c_n}, [r_\nu], q_{\Upsilon_1}, \cdots, q_{\Upsilon_m}) : \mathcal{C}(\widehat{Y}) \to \mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m.$$

Definition 8.1. A function f defined on $\mathcal{C}(\widehat{Y})$ is called *a cylinder function* if it arises as the composition $g \circ \Xi$ where

- the map $\Xi : \mathcal{C}(\widehat{Y}) \to \mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m$ is defined as above, using any compactly supported forms c_i $(1 \leqslant i \leqslant n)$ defined on $S^1 \times D^2$ and compactly supported sections Υ_j $(1 \leqslant j \leqslant m)$ on $(\mathbb{R}/2\pi\alpha\mathbb{Z}) \times (S^1 \times D^2)$, for any $n, m \geqslant 0$;
- the function
$$g : \mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m \to \mathbb{R}$$
is any smooth function with compact support.

A cylindrical function is fully gauge invariant. ♢

Theorem 8.2. *For any cylinder function $f : \mathcal{C}(\widehat{Y}) \to \mathbb{R}$, its formal gradient*

$$\operatorname{grad} f : \mathcal{C}(\widehat{Y}) \to \mathcal{T}_0$$

is a perturbation tame in $Y' = \operatorname{Im} \iota$, in the sense of Definition 7.3, where $\iota : S^1 \times D^2 \hookrightarrow \widehat{Y}$ is the embedding used to define f.

We will prove Theorem 8.2 in Subsection 8.4.

8.2. **Cylinder Functions and Embeddings.** In this subsection, we examine the separating property of cylinder functions. The main results are Proposition 8.4 and 8.6.

Fix an embedding $\iota : S^1 \times D^2 \hookrightarrow \widehat{Y}$, and define

$$\operatorname{Cylin}(\iota) := \{f : f \text{ is a cylinder function defined via } \iota\}.$$

It is reasonable to ask: to what extend elements of $\operatorname{Cylin}(\iota)$ separate points and tangent vectors of $\mathcal{C}(\widehat{Y})$. Apparently, if (B_1, Ψ_1) is identical to (B_2, Ψ_2) over the image of ι up to gauge, then they can not be separated by any element of $\operatorname{Cylin}(\iota)$, because only local information is employed when defining cylinder functions. In addition, they can not be separated if $B_1 = B_2$ and

$$e^{i\theta(z)}\Psi_1 = \Psi_2$$

for some smooth function $\theta : D^2 \to \mathbb{R}$ as the function $h(\sigma(z))$ defined in (B3) is fully gauge invariant. In fact, this is the worst case that can happen:

Proposition 8.3. *Take $\gamma_i = (B_i, \Psi_i) \in \mathcal{C}(\widehat{Y})$ $(i = 1, 2)$. Suppose for any cylinder function $f \in \mathrm{Cylin}(\iota)$, we always have*

$$f(\gamma_1) = f(\gamma_2),$$

then there exists a gauge transformation $v \in \mathcal{G}(\widehat{Y})$ and some function $\theta : B(0, 1/3) \to \mathbb{R}$ such that

$$v(B_1) = B_2,\ e^{i\theta(z)} v \cdot \Psi_1 = \Psi_2$$

over the smaller solid torus $\iota(S^1 \times B(0, 1/3))$. The function θ might not be continuous because of the zero locus of Ψ_1.

Proof. Take $(b_i, \psi_i) = (B_i, \Psi_i) - (B_0, \Psi_0)$ and set

$$\delta b = b_2 - b_1.$$

By our assumptions, γ_1 and γ_2 can not be separated by any functions of classes (B1)(B2) and (B3). First, we claim that δb is closed on $S^1 \times D^2$, since

$$0 = r_c(b_2) - r_c(b_1) = r_c(\delta b) = \int_{S^1 \times D^2} \delta b \wedge d\bar{c} = \int_{S^1 \times D^2} d(\delta b) \wedge \bar{c}$$

for any compactly supported 1-form c. Moreover,

$$r_\nu(\delta b) = r_\nu(b_2) - r_\nu(b_1) = 2\pi n\alpha \in \mathbb{R}$$

for some $n \in \mathbb{Z}$, since $[r_\nu](b_1) = [r_\nu](b_2)$. Using the gauge transformation u_1 from (B3), we may place γ_1 by

$$u_1^{-n}(\gamma_1)$$

to make $r_\nu(b_2) - r_\nu(b_1)$ zero. From now on, let us assume $r_\nu(\delta b) = 0$.

This allows us to conclude that δb is exact on $S^1 \times D^2$, so $\delta b = d\xi$ for some function $\xi : S^1 \times D^2 \to i\mathbb{R}$. By cutting off ξ outside $B(0, 2/3)$, we extend ξ to the whole manifold $\widehat{Y}$ (by zero outside of $\mathrm{Im}\,\iota$). Finally, replace γ_1 by $e^{-\xi} \cdot \gamma_1$.

It remains to show that $\Psi_1 = \Psi_2$ along the core $S^1 \times \{0\}$ up to an overall phase $e^{i\theta} \in S^1$ when $\delta b = 0$ on $S^1 \times B(0, 1/2)$. Let

$$\Psi_{1,0}, \Psi_{2,0}$$

be their restriction along the core $S^1 \times \{0\}$. If they do not generate the same complex plane in $\Gamma(S^1 \times \{0\}, S)$, then we can always find a section $\Upsilon_0 \in \Gamma(S^1 \times \{0\}, S)$ such that

$$\Psi_{1,0} \perp \Upsilon_0 \text{ and } \Psi_{2,0} \not\perp \Upsilon_0$$

or the other way around. Extending Υ_0 to a section Υ of

$$\mathbb{S} \to (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times S^1 \times D^2,$$

supported near $\{r_\nu(b_1)\} \times S^1 \times \{0\}$ will result in a function q_Υ of class (B3) that separates γ_1 and γ_2.

When $\Psi_{1,0}$ and $\Psi_{2,0}$ do generate the same complex plane, but $\|\Psi_{1,0}\|_{L^2(S^1)} \neq \|\Psi_{2,0}\|_{L^2(S^1)}$, one can construct Υ in a similar way.

We obtain the function $\theta : B(0, 1/3) \to \mathbb{R}$, by applying the same argument to the fiber $S^1 \times \{z\}$ for any $z \in B(0, 1/3)$. □

Hence, it is necessary to take into account all possible embeddings of $S^1 \times D^2$ into $\widehat{Y}$ in order to obtain the desired separating property:

Proposition 8.4. *Recall that* $Y = \{s \leqslant 0\} \subset \widehat{Y}$. *Let*

$$\mathrm{Cylin}(Y) := \bigcup_{\mathrm{Im}\,\iota \subset Y} \mathrm{Cylin}(\iota)$$

be the union of all possible cylinder functions with $\mathrm{Im}\,\iota \subset Y$. *If* γ_1 *and* $\gamma_2 \in \mathcal{C}(\widehat{Y})$ *can not be separated by any element in* $\mathrm{Cylin}(Y)$, *then there is a gauge transformation* $v \in \mathcal{G}(\widehat{Y})$ *that identifies* γ_1 *with* γ_2 *over* Y, *i.e.*

$$v(\gamma_1) = \gamma_2 \text{ on } Y.$$

Proof. Again, take $(b_i, \psi_i) = (B_i, \Psi_i) - (B_0, \Psi_0)$ and set

$$\delta b = b_2 - b_1.$$

By the proof of Proposition 8.3, we deduce that δb is closed over Y, and there is a gauge transformation $v \in \mathcal{G}(\widehat{Y})$ such that $v(B_1) = B_2$. The remaining step is to verify

$$v \cdot \Phi_1 = \Phi_2$$

up to a global constant $e^{i\theta} \in S^1$. By Proposition 8.3, the equality $|\Phi_1| = |\Phi_2|$ holds point-wise on Y, and

$$e^{i\theta(y)} v \cdot \Phi_1 = \Phi_2$$

for some function $\theta : Y^\circ \to \mathbb{R}$ defined in the interior of Y. Suppose for some $y_1, y_2 \in Y^\circ$, $\Phi_1(y_1), \Phi_1(y_2) \neq 0$. Choose an embedding $S^1 \times \{0\} \hookrightarrow Y$ that passes y_1, y_2 and extend it into an embedding of the solid torus:

$$\iota : S^1 \times D^2 \to Y \subset \widehat{Y}.$$

By Proposition 8.3, the function $e^{i\theta}$ has to be constant along the core $S^1 \times \{0\}$, so $e^{i\theta(y_1)} = e^{i\theta(y_2)}$. This allows us to modify θ to be a constant function $\theta \equiv \theta_0$, so

$$e^{i\theta_0} v \cdot \Phi_1 = \Phi_2. \qquad \square$$

Now we state the infinitesimal version of Proposition 8.3 and 8.4 concerning the separating property of tangent vectors. They are essential for the proof of transversality in Section 16. Proposition 8.6 is a direct consequence of Proposition 8.5, so we focus on the proof of the latter.

Proposition 8.5. *Take* $\gamma = (B, \Psi) \in \mathcal{C}(\widehat{Y})$ *and* $V = (\delta b, \delta\psi) \in T_\gamma \mathcal{C}(\widehat{Y})$. *For a fixed embedding* $\iota : S^1 \times D^2 \hookrightarrow \widehat{Y}$ *and any* $f \in \mathrm{Cylin}(\iota)$, *suppose we always have*

$$df(V) = 0,$$

then either

- *there exists some* $\xi \in \mathrm{Lie}(\mathcal{G}(\widehat{Y}))$ *and some function* $\theta : B(0, 1/3) \to \mathbb{R}$ *such that*

$$(\delta b, \delta\psi) = (-d\xi, (\xi + i\theta(z))\Psi)$$

  over the smaller solid torus $\iota(S^1 \times B(0, 1/3))$; *or*
- $\Psi \equiv 0$ *on* $S^1 \times \{z\}$ *for some* $z \in B(0, 1/3)$.

Proposition 8.6. *Suppose for some* $\gamma = (B, \Psi) \in \mathcal{C}(\widehat{Y})$ *and some tangent vector* $V \in T_\gamma \mathcal{C}(\widehat{Y})$*, we always have*

$$df(V) = 0$$

for any $f \in \mathrm{Cylin}(Y)$*. Then either*

- $\Psi \equiv 0$ *on* Y*, or*
- *for some* $\xi \in \mathrm{Lie}(\mathcal{G}(\widehat{Y}))$*,* V *is generated by the infinitesimal action of* ξ *over* Y*, i.e.*
$$V = (-d\xi, \xi\Psi) \text{ on } Y.$$

Proof of Proposition 8.5. Since $V = (\delta b, \delta\psi)$ can not be separated by any functions in classes (B1)(B2), δb has to be an exact 1-form on $S^1 \times D^2$, so $\delta b = -d\xi$ for some $\xi : S^1 \times D^2 \to i\mathbb{R}$. Since this problem is linear and the vector $(-d\xi, \xi\Psi)$ can not be separated, it remains to deal with the case when $\delta b = 0$ and show

$$\delta\psi = i\theta(z)\Psi$$

on $S^1 \times B(0, 1/3)$ for some function $\theta : B(0, 1/3) \to \mathbb{R}$. For a fixed section Υ of $\mathbb{S}$, consider functions $\sigma, \sigma_1 : D^2 \to \mathbb{C}$:

$$\sigma(z) := \int_{S^1 \times \{z\}} \langle \Psi_z, \Upsilon_z^\dagger \rangle, \ \sigma_1(z) := \int_{S^1 \times \{z\}} \langle \delta\psi_z, \Upsilon_z^\dagger \rangle.$$

Then the differential of q_Υ along $V = (0, \delta\psi)$ can be computed directly as

$$dq_\Upsilon(0, \delta\psi) = \int_{D^2} 2\chi_2(z)\chi_3'(|\sigma_1|^2)\, \mathrm{Re}(\sigma(z)\overline{\sigma_1}(z)) dvol_{D^2},$$

where χ_3 is the cut-off function used to define the S^1-invariant function h in (B3). For any $z \in B(0, 1/3)$, if Ψ_z and $\delta\psi_z$ do not lie in the same complex direction in $\Gamma(S^1 \times \{z\}, S)$, then for some section $\Upsilon_z^\dagger \in \Gamma(\{r_\nu(b)\} \times S^1 \times \{z\}, S)$, $\mathrm{Re}(\sigma(z)\overline{\sigma_1}(z))$ is non-zero (it suffices to verify this statement for two vectors in $\mathbb{C}^2$). By properly extending $\Upsilon_z^\dagger$ to a section Υ of $\mathbb{S}$, we can make $dq_\Upsilon(0, \delta\psi) \neq 0$.

Finally, if $\Psi_z \not\equiv 0$ and $\delta\psi_z = w\Psi_z$ for some $w \in \mathbb{C}$, then w has to be imaginary for the same reason. This proves the existence of $\theta(z) \in \mathbb{R}$ when $\Psi_z \not\equiv 0$. □

8.3. **Estimates of Perturbations on Cylinders.** In this subsection, we take up the proof of Theorem 8.2. Unlike the case of closed 3-manifolds (cf. [KM07, Section 11.3]), gradients and Hessians of f can not be estimated in a straightforward way; the use of anisotropic Sobolev spaces is already necessary. We will only state the estimates for the 3-manifold $\widehat{Y}$, whose proof will follow from their analogue on the 4-manifold $[t_1, t_2] \times \widehat{Y}$:

Proposition 8.7 (cf. Proposition 11.3.3 in [KM07])**.** *For any* $k \geqslant 2$ *and any cylinder function* f *defined using an embedding* $\iota : S^1 \times D^2 \to \widehat{Y}$*,* $\mathfrak{q} = \mathrm{grad}\, f$ *determines a smooth vector field on* $\mathcal{C}_k(\widehat{Y})$*, and for each* $l \geqslant 0$*, there is a constant* C *with*

$$\|\mathcal{D}^l_{(B,\Psi)}\mathfrak{q}\| \leqslant C(1 + \|b\|_{L^2_{k-1}(Y')})^{2k(l+1)}(1 + \|\Psi\|_{L^2_{k,B}(Y')})^{l+1},$$

where $\mathcal{D}^l_{(B,\Psi)}\mathfrak{q}$ *is viewed as an element of* $\mathrm{Mult}_l(\times_l \mathcal{T}_k, \mathcal{T}_k)$ *and* $Y' = \mathrm{Im}\,\iota$*.*

In addition, for any $j \in [-k,k]$, the first derivative $\mathcal{D}\mathfrak{q}$ extends to a smooth map

$$\mathcal{D}\mathfrak{q} : \mathcal{C}_k(\widehat{Y}) \to \mathrm{Hom}(\mathcal{T}_j, \mathcal{T}_j)$$

whose $(l-1)$-th derivative viewed as an element of $\mathrm{Mult}_l(\bigtimes_{l-1}\mathcal{T}_k \times \mathcal{T}_j, \mathcal{T}_j)$ satisfies the same bound.

Remark 8.8. The author was unable to prove this proposition when $k = 1$. We will come back to this point in Subsection 8.4. ♢

Let $I = [t_1, t_2] \subset \mathbb{R}_t$ and $\widehat{Z} = I \times \widehat{Y}$. As described in the beginning of Section 7, each smooth perturbation $\mathfrak{q}$ gives arise to a section

$$\widehat{\mathfrak{q}} : \mathcal{C}_k(\widehat{Z}) \to \mathcal{V}_0$$

of the trivial bundle

$$\mathcal{V}_0 = L^2(\widehat{Z}, i\mathfrak{su}(S^+) \oplus S^-) \times \mathcal{C}_k(\widehat{Z}) \to \mathcal{C}_k(\widehat{Z}),$$

where the bundle $iT^*Y \oplus S^+$ is identified with $(i\mathfrak{su}(S^+) \oplus S^-)$ using the bundle map

$$(\rho_3, \rho_4(dt)),$$

over the 4-manifold $\widehat{Z}$. For any $\gamma = (A, \Phi) \in \mathcal{C}_k(\widehat{Z})$, write

$$(a, \phi) = (A, \Phi) - (A_0, \Phi_0) \in L^2_k(\widehat{Z}, iT^*\widehat{Z} \oplus S^+),$$

where $\gamma_0 = (A_0, \Phi_0)$ is the reference configuration of $\mathcal{C}_k(\widehat{Z})$.

Proposition 8.9 (cf. [KM07] Proposition 11.4.1)**.** *For any $k \geqslant 2$ and any cylinder function f defined via the embedding $\iota : S^1 \times D^2 \to \widehat{Y}$, consider its induced perturbation on the 4-manifold $\widehat{Z}$:*

$$\widehat{\mathfrak{q}} = \mathrm{grad}\, f : \mathcal{C}_k(\widehat{Z}) \to \mathcal{V}_0.$$

(C1) The map $\widehat{\mathfrak{q}}$ extends to a smooth map

$$\mathcal{C}_k(\widehat{Z}) \to \mathcal{V}_k,$$

whose l-th derivative regarded as a multi-linear map

$$\mathcal{D}^l_{(A,\Phi)}\widehat{\mathfrak{q}} \in \mathrm{Mult}^l(\bigtimes_l \mathcal{T}_k(\widehat{Z}), \mathcal{V}_k),$$

satisfies the estimate:

$$\|\mathcal{D}^l_{(A,\Phi)}\widehat{\mathfrak{q}}\| \leqslant C(1 + \|a\|_{L^2_k(\Omega)})^{2k(l+1)}(1 + \|\Phi\|_{L^2_{k,A}(\Omega)})^{l+1},$$

where $\Omega = I \times \mathrm{Im}\,\iota \subset \widehat{Z}$.

(C2) For any $j \in [-k,k]$, the first derivative $\mathcal{D}\widehat{\mathfrak{q}}$ extends to a smooth map

$$\mathcal{D}\widehat{\mathfrak{q}} : \mathcal{C}_k(\widehat{Z}) \to \mathrm{Hom}(\mathcal{T}_j(\widehat{Z}), \mathcal{V}_j)$$

whose $(l-1)$-the derivative regarded as a multi-linear map

$$\mathcal{D}^l_{(A,\Phi)}\widehat{\mathfrak{q}} \in \mathrm{Mult}^l(\bigtimes_{l-1} \mathcal{T}_k(\widehat{Z}) \times \mathcal{T}_j(\widehat{Z}), \mathcal{V}_j),$$

satisfies the same bound as in (C1)*.*

(C3) When $p > 3$ and $k \geqslant 1$, the map $\widehat{\mathfrak{q}}$ extends to a smooth map

$$\mathcal{C}_k^{(p)}(\widehat{Z}) \to \mathcal{V}_k^{(p)},$$

whose l-th derivative regarded as a multi-linear map

$$\mathcal{D}^l_{(A,\Phi)}\widehat{\mathfrak{q}} \in \mathrm{Mult}^l(\bigtimes_l \mathcal{T}_k^{(p)}(\widehat{Z}), \mathcal{V}_k^{(p)}),$$

satisfies the estimate:

$$\|\mathcal{D}^l_{(A,\Phi)}\widehat{\mathfrak{q}}\| \leqslant C(1+\|a\|_{L^p_k(\Omega)})^{2k(l+1)}(1+\|\Phi\|_{L^p_{k,A}(\Omega)})^{l+1}.$$

(C4) For any $2 \leqslant p < 4$, the map $\widehat{\mathfrak{q}}$ satisfies the estimate

$$\|\widehat{\mathfrak{q}}\|_{L^{\boldsymbol{n}(p)}} \leqslant C(1+\|(a,\phi)\|_{L^p_{1,A}(\Omega)}) \text{ with } \boldsymbol{n}(p) = 4p/(4-p).$$

(C5) When $2 \leqslant p < 4$, the map $\widehat{\mathfrak{q}}$ extends to a continuous map from

$$\mathcal{C}_1^{(p)}(\widehat{Z}) \to \mathcal{V}_0^{(m)} \text{ for any } m < \boldsymbol{n}(p).$$

(C6) For any $0 \leqslant \epsilon < \frac{1}{2}$, the map $\widehat{\mathfrak{q}}$ extends to a continuous map from

$$\mathcal{C}_{1-\epsilon}(\widehat{Z}) \to \mathcal{V}_0.$$

Remark 8.10. Properties (C1)(C3)(C5)(C6) are essential in the proof of compactness of perturbed Seiberg-Witten equations in Section 9. Starting with $p = 2$, we have $\boldsymbol{n}(p) = 4 > 3$. ♢

Before we proceed to the proof, let us add a few remarks to simplify the situation. For a fixed cylinder function f, one can either compute its gradient using the pull-back metric g_1 on $S^1 \times D^2$, or using the standard product metric g_{std}:

$$\mathrm{grad}_{std} f \text{ or } \mathfrak{q} := \mathrm{grad} f.$$

If we write $\mathrm{grad} f = (\mathrm{grad}^0 f, \mathrm{grad}^1 f)$ as entries of $L^2_k(\widehat{Y}, iT^*\widehat{Y} \oplus S)$, then

$$\mathrm{grad}_{std} f = (\eta K(\mathrm{grad}^0 f), \eta\, \mathrm{grad}^1 f),$$

where the function η and the bundle map K were introduced in Section 8.1. Since they are related by a smooth bundle map of $iT^*\widehat{Y} \oplus S|_{\mathrm{Im}\,\iota}$, it suffices to prove estimates for $\mathrm{grad}_{std} f$. The change of metrics of $S^1 \times D^2$ will also affect the $L^2_{j,A}$-norms on $\mathcal{T}_j$ and $\mathcal{V}_j$, which is again inconsequential for our estimates.

From now on, we assume $g_1 = g_{std}$, and the length of the core $S^1 \times \{0\}$ is 2π.

The second remark concerns the anisotropic Sobolev spaces, which involves different orders of differentiability in different directions. In what follows, let

$$\begin{aligned} Y' &= S^1 \times D^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times D^2 \subset \widehat{Y}, \\ \Omega &= I \times S^1 \times D^2 = [t_1, t_2] \times Y' \subset \widehat{Z}, \\ M &= I \times D^2. \end{aligned}$$

Within the product manifold Ω only the direction along S^1-fibers is special. Let θ be the coordinate function of the circle $\mathbb{R}/2\pi\mathbb{Z}$, and define the $L^2_{m,l}$ norm ($l \leqslant m$) of functions on Ω to be

$$\|\xi\|^p_{L^2_{m,l}(\Omega)} = \sum_{\substack{i+j\leqslant m,\\ i\leqslant l}} \int_\Omega |(\frac{\partial}{\partial\theta})^j \nabla^i_M \xi|^p$$

and let $L^p_{m,l}(\Omega)$ be the completion of smooth functions (or sections) with respect to this norm. We are mostly interested in the case when $p = 2$. There are two useful lemmas:

Lemma 8.11. *Consider the Banach space $L^p_{k+1,k}$ with $k \geqslant 2$ if $p = 2$ and $k \geqslant 1$ if $p > 3$. Then $L^p_{k+1,k}$ is an algebra under the point-wise multiplication and $L^p_{k+1,k} \subset C^0$; Moreover, for any $|r| \leqslant k+1$ and $|q| \leqslant k$, $L^p_{r,q}(\Omega)$ is a module of $L^p_{k+1,k}$.*

Proof. Note that $L^2_{k+1,k}(\Omega) \hookrightarrow L^2_1(S^1, L^2_k(M)) \hookrightarrow C^0(S^1, C^0(M))$ when $k \geqslant 2$, and

$$L^p_{k+1,k}(\Omega) \hookrightarrow L^p_1(S^1, L^p_k(M)) \hookrightarrow C^0(S^1, C^0(M))$$

when $k \geqslant 1$ and $p > 3$. □

Lemma 8.12. *For any (m, l) and $p \in [1, \infty)$, the slicewise operator $d_{S^1}G$ and $Gd^*_{S^1}$ are bounded linear operators from $L^p_{m,l}(\Omega) \to L^p_{m+1,l}(\Omega)$, where*

$$G : C^\infty(S^1) \to C^\infty(S^1)$$

is the Green operator associated to the Hodge Laplacian operator.

Proof. It follows from the fact that G extends to a bounded linear operator

$$G : L^p_m(S^1, \mathbb{R}) \to L^p_{m+2}(S^1, \mathbb{R})$$

for any $p \in [1, \infty)$ and $m \geqslant 0$. □

Proof of Proposition 8.9. Suppose the cylinder function f arises as the composition $g \circ \Xi$:

$$\mathcal{C}_k(\widehat{Y}) \xrightarrow{\Xi} \mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m \xrightarrow{g} \mathbb{R}$$

where $\Xi = (r_{c_1}, \cdots, r_{c_n}, [r_\nu], q_{\Upsilon_1}, \cdots, q_{\Upsilon_m})$ is induced from a collection of 1-forms $c_1, c_2, \cdots c_n$ and sections $\Upsilon_1, \cdots, \Upsilon_m$. Let x_i ($1 \leqslant i \leqslant n$), x and y_j be the coordinate functions on $\mathbb{R}^n$, $\mathbb{R}/2\pi\alpha\mathbb{Z}$ and $\mathbb{R}^m$ respectively. Then set

$$\begin{aligned} X_i &:= \mathrm{grad}(x_i \circ \Xi) = (*_3 dc_i, 0), \\ X_\nu &:= \mathrm{grad}(x \circ \Xi) = (*_3 \pi^*\nu, 0) \text{ and} \\ Y_j &:= \mathrm{grad}(y_j \circ \Xi). \end{aligned}$$

The expression of Y_j requires some further work. First, we compute the differential:

$$d(y_j \circ \Xi)(\delta b, \delta\psi) = 2\,\mathrm{Re} \int_{D^2} \chi_2(z) \frac{\partial h}{\partial w}(\sigma(z)) dvol_{D^2} \cdot d(\sigma(z))(\delta b, \delta\psi)$$

and

$$d(\sigma(z))(\delta b, \delta\psi) = \int_{S^1 \times \{z\}} \langle \delta\psi, \Upsilon^\dagger_{j,z}\rangle + \langle \Psi_z, (\partial_x \Upsilon_j)^\dagger_z\rangle \langle \delta b, X_\nu\rangle_{Y'} + \langle \Psi_z, (-Gd^*_{S^1}\delta b_z)\Upsilon^\dagger_{j,z}\rangle.$$

where $Y' = S^1 \times D^2 \subset \widehat{Y}$. This allows us to write $Y_j = (Y_j^0, Y_j^1) = 2(\operatorname{Im} W_j^0, W_j^1)$ with

$$W_j = \chi_2(z)\frac{\partial h}{\partial w}(\sigma(z))((-d_{S^1}G)\langle \Psi, \Upsilon_j^\dagger\rangle + \langle \Psi, (\partial_x \Upsilon_j)^\dagger\rangle_{Y'} X_\nu, \Upsilon_j^\dagger). \tag{8.1}$$

As sections of $\mathbb{S} \to (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times (S^1 \times D^2)$, $\partial_x \Upsilon_j$ denotes the derivative of Υ_j along the first factor. Finally, we obtain that

$$\mathfrak{q} = \operatorname{grad} f = \sum_{i=1}^n (\frac{\partial g}{\partial x_i} \circ \Xi) X_i + (\frac{\partial g}{\partial x} \circ \Xi) X_\nu + \sum_{j=1}^m (\frac{\partial g}{\partial y_j} \circ \Xi) Y_j. \tag{8.2}$$

To study the mapping properties of $\mathfrak{q}$, we first examine the map:

$$\Upsilon^\dagger : \mathcal{C}_k(\hat{Y}) \to L^2(S^1 \times D^2, S)$$

and its extension in dimension 4:

$$\Upsilon^\ddagger : \mathcal{C}_k(\widehat{Z}) \to L^2(\Omega, S^-) \text{ where } \Omega = I \times S^1 \times D^2,$$
$$(A, \Phi) \mapsto \Upsilon^\dagger(\check{A}(t), \check{\Phi}(t)), \ \forall t \in I = [t_1, t_2].$$

for any compactly supported section Υ of $\mathbb{S} \to (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times S^1 \times D^2$.

Lemma 8.13 (cf. Lemma 11.4.4 in [KM07])**.** *For any $k \geqslant 2$ and any $j \in [-k, k]$, $\Upsilon^\ddagger$ extends to a smooth map*

$$\mathcal{C}_k(\widehat{Z}) \to L^2_{j+1,j,A}(\widehat{Z}, S^-)$$

with the following properties.

(D1) For each $l \geqslant 0$, there is a constant $C > 0$ such that the differential

$$\mathcal{D}^l_{(A,\Phi)}\Upsilon^\ddagger \in \operatorname{Mult}^l(\bigtimes_l \mathcal{T}_k(\widehat{Z}), L^2_{j+1,j,A}(\widehat{Z}, S^-))$$

satisfies the bound

$$\|\mathcal{D}^l_{(A,\Phi)}\Upsilon^\ddagger\| \leqslant C(1 + \|a\|_{L^2_j})^j (1 + \|a\|_{L^2_k})^k, \ \forall (A, \Phi) \in \mathcal{C}_k(\widehat{Z}).$$

(D2) The l-th derivative extends to an element of

$$\operatorname{Mult}^l(\bigtimes_{l-1} \mathcal{T}_k(\widehat{Z}) \times \mathcal{T}_j(\widehat{Z}), L^2_{j+1,j,A}(\widehat{Z}, S^-))$$

whose norm satisfies the bound

$$\|\mathcal{D}^l_{(A,\Phi)}\Upsilon^\ddagger\| \leqslant C(1 + \|a\|_{L^2_k})^{2k}, \ \forall (A, \Phi) \in \mathcal{C}_k(\widehat{Z}).$$

(D3) For any $k \geqslant 1$ and $p > 3$, $\Upsilon^\ddagger$ extends to a smooth map

$$\mathcal{C}_1^{(p)}(\widehat{Z}) \to L^{(p)}_{j+1,j,A}(\widehat{Z}, S^-).$$

whose l-th derivative extends to an element of

$$\operatorname{Mult}^l(\bigtimes_{l-1} \mathcal{T}_k^{(p)}(\widehat{Z}) \times \mathcal{T}_j^{(p)}(\widehat{Z}), L^p_{j+1,j,A}(\widehat{Z}, S^-))$$

with norm bounded by

$$\|\mathcal{D}^l_{(A,\Phi)}\Upsilon^\ddagger\| \leqslant C(1 + \|a\|_{L^p_k})^{2k}, \ \forall (A, \Phi) \in \mathcal{C}_k^{(p)}(\widehat{Z}).$$

(D4) For $i=0,1$ and any $p\in[2,\infty]$, we have the bound

$$\|\Upsilon^\ddagger\|_{L^p_{A,i}}\leqslant C(1+\|a\|_{L^p_i})^i,\ \forall(A,\Phi)\in\mathcal{C}_k(\widehat{Z}).$$

(D5) For any $1\leqslant m<p$, $\Upsilon^\ddagger$ extends to a continuous map from

$$\mathcal{C}_1^{(p)}(\widehat{Z})\to L^m_1(\widehat{Z},S^-).$$

(D6) For any $1\leqslant p',p<\infty$, $\Upsilon^\ddagger$ extends to a continuous map from

$$\mathcal{C}^{(p)}(\widehat{Z})\to L^{p'}(\widehat{Z},S^-).$$

Proof. The proof of (D1)(D2)(D3) carries though with little changes as in [KM07, Lemma 11.4.4], using Lemma 8.3 in place of [KM07, Lemma 11.4.3]. In what follows, we will focus on (D4)(D5)(D6).

As this point, it is convenient to have a lemma that is slightly stronger than [KM07, Lemma 11.4.5]:

Lemma 8.14. *Let $\mathcal{H}_1,\mathcal{H}_2$ be any separable Banach spaces and $\dim\mathcal{H}_1<\infty$. Suppose $\chi:\mathcal{H}_1\to\mathcal{H}_2$ be a smooth map with bounded C^1-norm. Then the composition map $\chi^*:\xi\mapsto\chi\circ\xi$ is continuous from*

$$L^1(\Omega_*,\mathcal{H}_1)\to L^p(\Omega_*,\mathcal{H}_2)$$

for any finite measure space Ω_* and any $1\leqslant p<\infty$. Moreover, $\|\chi\circ\xi\|_\infty\leqslant\|\chi\|_\infty$.

Proof of Lemma. It is clear that $\chi\circ\xi$ lies in $L^\infty(\Omega_*,\mathbb{C})$ with $\|\chi\circ\xi\|_\infty\leqslant\|\chi\|_\infty$. Since Ω_* has a finite measure, $\chi\circ\xi\in L^p$. We prove that χ^* is Hölder continuous. For any $\xi_1,\xi_2\in L^1(\Omega_*,\mathcal{H}_1)$,

$$\begin{aligned}\|\chi\circ\xi_1-\chi\circ\xi_2\|_p^p&=\int_{\Omega_*}\|\chi\circ\xi_1-\chi\circ\xi_2\|^p_{\mathcal{H}_2}\leqslant\|2\chi\|_\infty^{p-1}\int_{\Omega_*}|\chi\circ\xi_1-\chi\circ\xi_2|_{\mathcal{H}_2}\\&=\|2\chi\|_\infty^{p-1}\|\nabla\chi\|_\infty\int_{\Omega_*}|\xi_1-\xi_2|_{\mathcal{H}_1}=\|2\chi\|_\infty^{p-1}\|\nabla\chi\|_\infty\|\xi_1-\xi_2\|_{L^1(\Omega_*,\mathcal{H}_1)}.\end{aligned}$$ ☐

Back to the proof of Lemma 8.13. Let $(a,\phi)=(A,\Phi)-(A_0,\Phi_0)\in L^p_1(\widehat{Z},iT^*\widehat{Z}\oplus S^+)$, then $\Upsilon^\ddagger(A,\Phi)$ is defined as

$$e^{-Gd^*_{S^1}a}\tilde{\Upsilon}(r_\nu(a)) \tag{8.3}$$

as a section supported on

$$\Omega=I\times S^1\times D^2$$

with $r_\nu(a)=r_\nu(a|_{\{t\}\times\hat{Y}})\in L^p(I,\mathbb{R})$.

Step 1. Proof of (D6). It follows from Lemma 8.14 directly: the exponential map

$$\xi\mapsto e^\xi$$

is continuous from $L^p(\Omega,i\mathbb{R})\to L^{2p'}(\Omega,\mathbb{C})$ for any $1\leqslant p,p'<\infty$, so the map

$$\varphi:a\mapsto\exp(-Gd^*_{S^1}a)$$

is continuous from $L^p \to L^{2p'}$. On the other hand, we view the map $a \mapsto \tilde{\Upsilon}(r_\nu(a))$ as the composition

$$L^p(\widehat{Z}) \to L^p(I,\mathbb{R}) \to L^{2p'}(I, L^{2p'}(\widehat{Y})) = L^{2p'}(\widehat{Z}),$$
$$a \mapsto r_\nu(a) \mapsto \tilde{\Upsilon}(r_\nu(a)),$$

so Lemma 8.14 applies. Finally, $L^{2p'} \times L^{2p'} \to L^{p'}$ is continuous.

Step 2. Proof of (D5). Now we deal with the first derivative of $\Upsilon^\ddagger$. Write $\nabla_A \Upsilon^\ddagger = K_1 + K_2 + K_3 + K_4$ with

$$K_1 = (-d_{S^1} G d^*_{S^1} a)\Upsilon^\ddagger, \qquad K_3 = (e^{-Gd^*_{S^1} a})\nabla_{A_0} \tilde{\Upsilon}(r_\nu(a)),$$
$$K_2 = (-G d^*_{S^1} d_M a)\Upsilon^\ddagger, \qquad K_4 = a \otimes \Upsilon^\ddagger, \tag{8.4}$$

where $M = I \times D^2$. To prove (D5), we verify that each K_i is continuous from $L^p_1 \to L^m$ for any $m < p$. It is clear that each of the following terms:

$$-d_{S^1} G d^*_{S^1} a,\ -G d^*_{S^1} d_M a,\ a$$

is continuous from L^p_1 to L^p. To analyze K_3, we expand $\nabla_{A_0} \tilde{\Upsilon}(r_\nu(a))$ as

$$(\nabla_{B_0} \tilde{\Upsilon})(r_\nu(a)) + (\widetilde{\partial_x \Upsilon})(r_\nu(a)) \langle \frac{d}{dt} a, X_\nu \rangle_{Y'},$$

which is continuous from $L^p_1 \to L^{p'}$ for any $1 \leqslant p' < p$. Now we use *Step* 1 to complete the proof of (D5).

Step 3. Proof of (D4). It follows directly from the expression of $\Upsilon^\ddagger$ and $\nabla_A \Upsilon^\ddagger$, (8.3) and (8.4), using the fact that $\|\varphi(a)\|_\infty = 1$. □

Back to the proof of Proposition 8.9. The proof of (C1)∼(C3) follows from (D1)∼(D3) in the same line as [KM07, Proposition 11.4.1], using Lemma 8.12.

In what follows, we will explain how (C4)(C5)(C6) follow from (D4) and (D6). In fact, (D6) provides better bounds than (D5). To estimate $\widehat{\mathfrak{q}}$, we investigate the section

$$W_j = \chi_2(z) \frac{\partial h}{\partial w}(\sigma(z))((-d_{S^1} G)\langle \Phi, \Upsilon^\ddagger_j \rangle + \langle \Phi, (\partial_x \Upsilon_j)^\ddagger \rangle_{Y'} X_\nu, \Upsilon^\ddagger_j),$$

in place of Y_j, so

$$\widehat{\mathfrak{q}} = \sum_{i=1}^n (\frac{\partial g}{\partial x_i} \circ \Xi) X_i + (\frac{\partial g}{\partial x} \circ \Xi) X_\nu + 2 \sum_{j=1}^m (\frac{\partial g}{\partial y_j} \circ \Xi)(\operatorname{Im} W^0_j, W^1_j).$$

We break W into four simpler pieces: $W = \varpi(V_1 + V_2 + V_3)$ where

$$\varpi = \chi_2(z) \frac{\partial h}{\partial w}(\sigma(z)), \qquad V_1 = (-d_{S^1} G)\langle \Phi, \Upsilon^\ddagger \rangle,$$
$$V_2 = \langle \Phi, (\partial_x \Upsilon)^\ddagger \rangle_{Y'} X_\nu, \qquad V_3 = \Upsilon^\ddagger.$$

Step 1. Proof of (C5). The map $V_1 : \mathcal{C}_1^{(p)}(\widehat{Z}) \to L^m(\widehat{Z}, iT^*\widehat{Z})$ is continuous for any $m < \boldsymbol{n}(p)$ when $2 \leqslant p < 4$. Indeed, V_1 can be viewed as the composition

$$(\Phi, \Upsilon^\ddagger) \in L_1^p \times L^{p'} \to L^{\boldsymbol{n}(p)} \times L^{p'} \xrightarrow{\times} L^m \xrightarrow{-d_{S^1}G} L^m,$$

when p' is sufficiently large.

For any $p' \gg 1$, the map $V_2 : \mathcal{C}_1^{(p)} \to L^{p'}(\widehat{Z}, iT^*\widehat{Z})$ is also continuous since the map $(\Phi, (\partial_x\Upsilon)^\ddagger) \to \langle \Phi, (\partial_x \Upsilon)^\ddagger\rangle_{Y'}$ can be viewed as the composition:

$$\begin{aligned} L_1^p \times L^{p'} \to L_1^p(I, L^p(\hat{Y})) \times L^{p'} \to C^0(I, L^p(\hat{Y})) \times L^{p'}(I, L^{p'}(\hat{Y})) \\ \xrightarrow{\times} L^{p'}(I, L^1(\hat{Y})) \xrightarrow{\int} L^{p'}(I). \end{aligned}$$

By Lemma 8.13 (D6), V_3 is a continuous map into $L^{\boldsymbol{n}(p)}$. It remains to deal with ϖ, which is viewed as the composition of $\frac{\partial h}{\partial w}$ with the map

$$\begin{aligned} \sigma : \mathcal{C}_1^{(p)} &\to L^1(M, \mathbb{C}),\ M = I \times D^2, \\ (A, \Phi) &\mapsto \left((t,z) \mapsto \int_{\{t\}\times S^1 \times \{z\}} \langle \Phi, \Upsilon^\ddagger\rangle \right). \end{aligned}$$

The map σ is continuous, since it is the composition:

$$(\Phi, \Upsilon^\ddagger) \in L_1^p \times L^4 \xrightarrow{\times} L^1 = L^1(M, L^1(S^1)) \xrightarrow{\int_{S^1}} L^1(M, \mathbb{C}).$$

Since $\frac{\partial h}{\partial w} : \mathbb{C}_w \to \mathbb{C}$ is a smooth function with compact support, it follows from Lemma 8.14 that $\varpi : C_1^{(p)} \to L^{p'}$ is continuous for any $1 \leqslant p' < \infty$.

The same argument shows that

$$\frac{\partial g}{\partial x_i} \circ \Xi, \frac{\partial g}{\partial x} \circ \Xi, \frac{\partial g}{\partial y_j} \circ \Xi$$

are continuous functions into $L^{p'}(I, \mathbb{R})$ for any $1 \leqslant p' < \infty$. This completes the proof of (C5).

Step 2. Proof of (C4). It follows by replacing $L^{p'}$ by L^∞ through out *Step* 1, using (D4) from Lemma 8.13.

Step 3. Proof of (C6). It follows by replacing L_1^p by $L_{1-\epsilon}^2$ through out *Step* 1 with $0 \leqslant \epsilon < \frac{1}{2}$.

The proof of Proposition 8.9 is now completed. □

8.4. **Proof of Theorem 8.2.** In this subsection, we verify that a cylinder function f satisfies conditions in Definition 7.3 and prove Theorem 8.2.

- (A1) and (A2) follows from (C1) and (C3).
- (A3) is satisfied on account of (C5), as $\boldsymbol{n}(2) = 4$.
- (A4) is a consequence of (C2), while (A6) follows from (C6).
- As for (A5), the statement on the support of $\mathfrak{q} = \operatorname{grad} f$ is clear from the construction. The estimate on $\|\mathfrak{q}\|_2$ is a consequence of the explicit formulae (8.1) and (8.2).

Only (A7) requires some further explanation, as Proposition 8.7 does not extend to the case when $k = 1$. The proof of [KM07, Proposition 11.4.1] fails here, as $L^2_{2,1}(S^1 \times D^2)$ fails to be an algebra:

$$L^2_{2,1}(S^1 \times D^2) \hookrightarrow L^2_1(S^1, L^2_1(D^2)) \not\hookrightarrow C^0,$$

Nevertheless, it is at the borderline. As we are merely interested in $\mathcal{T}_0$, losing a tiny amount of regularity is affordable. In fact, one can still prove that

$$\mathfrak{q} : \mathcal{C}_1(\widehat{Y}) \to \mathcal{T}_0$$

is smooth. This completes the proof of Theorem 8.2.

8.5. **Banach Spaces of Tame Perturbations.** In this subsection, we construct a Banach space of tame perturbations as described in Section 7. Since only minor changes are needed, we will only state the theorem and refer to [KM07, Section 11.6] for the actual proof.

First, we introduce a broader class of functions defined on $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$, called generalized cylinder functions. In the definition of cylinder functions (cf. Definition 8.1), one may allow entries of Ξ to come from different embeddings of $S^1 \times D^2$ into $\widehat{Y}$. This motivates the next definition.

Definition 8.15. A function f' defined on $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ is called *a generalized cylinder function* if it arises as the composition $g' \circ \Xi'$ where

- the map Ξ' is defined using a collection of cylinder functions $f_1, \cdots, f_l$:
  $$\Xi' = (f_1, \cdots, f_l) : \mathcal{C}_{k-\frac{1}{2}}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathbb{R}^l.$$
  Their underlying embeddings $\iota_j : S^1 \times D^2 \to \widehat{Y}, 1 \leqslant j \leqslant l$ might be different.
- the function
  $$g' : \mathbb{R}^l \to \mathbb{R}$$
  is any smooth function with compact support. ♢

Theorem 8.16. *Let Y' is a smooth co-dimension 0 submanifold of $\widehat{Y}$. Suppose a generalized cylinder function f' is defined using a collection of embeddings $\{\iota_k\}_{1 \leqslant k \leqslant l}$ with* $\operatorname{Im} \iota_k \subset Y'$ *for all ι_k, then* $\operatorname{grad} f'$ *is a perturbation tame in Y' in the sense of Definition* 7.3.

The proof of Theorem 8.16 is not essentially different from that of Theorem 8.2.

Theorem 8.17. *Fix an open submanifold $Y' \subset \widehat{Y}$. Let $\mathfrak{q}^i$ $(i \in \mathbb{N})$ be any countable collection of tame perturbations arising as gradients of generalized cylinder functions on $\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ with support in Y'. Then there exists a separable Banach space $\mathcal{P}$ and a linear map:*

$$\begin{aligned} \mathfrak{O} : \mathcal{P} &\to C^0(\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}}), \mathcal{T}_0) \\ \lambda &\mapsto \mathfrak{q}^\lambda \end{aligned}$$

with the following properties:

(F1) *For each $\lambda \in \mathcal{P}$, the element $\mathfrak{q}^\lambda$ is a tame perturbation in Y' in the sense of Definition 7.3.*

(F2) *The image of $\mathfrak{O}$ contains all the perturbations $\mathfrak{q}^i$ from the given countable collections.*

(F3) *If $\widehat{Z} = [t_1, t_2] \times \widehat{Y}$ is a cylinder, then for all $k \geqslant 2$, the map*

$$\begin{aligned} \mathcal{P} \times \mathcal{C}_k(\widehat{Z}) &\to \mathcal{V}_k \\ (\lambda, \gamma) &\mapsto \widehat{\mathfrak{q}}^\lambda(\gamma) \end{aligned}$$

is a smooth map of Banach manifolds.

(F4) *For all $k \geqslant 1$ and $p = 7/2$, the map*

$$\begin{aligned} \mathcal{P} \times \mathcal{C}_k^{(p)}(\widehat{Z}) &\to \mathcal{V}_k^{(p)} \\ (\lambda, \gamma) &\mapsto \widehat{\mathfrak{q}}^\lambda(\gamma) \end{aligned}$$

is a smooth map of Banach manifolds.

(F5) *For $\epsilon = 1/4$, the map*

$$\begin{aligned} \mathcal{P} \times \mathcal{C}_{1-\epsilon}(Y) &\to \mathcal{T}_0(Y) \\ (\lambda, \beta) &\mapsto \mathfrak{q}^\lambda(\beta). \end{aligned}$$

is continuous and satisfies the bound:

$$\|\mathfrak{q}^\lambda(B, \Psi)\|_2 \leqslant \|\lambda\|_{\mathcal{P}} \cdot m_2(\|\Psi\|_{L^2(Y')} + 1).$$

Proof. See [KM07, Theorem 11.6.1]. □

We do not distinguish $\lambda \in \mathcal{P}$ with its image $\mathfrak{q}^\lambda$ in $C^0(\mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}}), \mathcal{T}_0)$.

Remark 8.18. In property (F4), any index $3 < p < 4$ will make the Compactness Theorem 9.5 work. In property (F5), one may take any $0 < \epsilon < 1/2$. ♢

Corollary 8.19. *Suppose $\{\mathfrak{q}_n\} \subset \mathcal{P}$ and $\|\mathfrak{q}_n\|_{\mathcal{P}} \to 0$ as $n \to \infty$. Then for any bounded region $\mathcal{O} \subset \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$, the C^l-norm of $\mathfrak{q}_n$ converges to zero, i.e.*

$$\|\mathfrak{q}_n\|_{C^l(\mathcal{O}, \mathcal{C}_k \to \mathcal{T}_k)} \to 0 \text{ as } n \to \infty.$$

Our primary interest is in the case when $Y' = Y = \{s \leqslant 0\}$, and let us specify the countable collection of tame perturbations associated to Y' in Theorem 8.17. We make the following choices in order:

- a positive integer l;
- a compact subset K' of $\mathbb{R}^l$;
- a smooth function g' on $\mathbb{R}^l$ with support in K'

and for each $j \in \{1, \cdots, l\}$,

- an embedding $\iota : S^1 \times D^2 \hookrightarrow (Y')^\circ$ into the interior of Y';
- a pair of positive integers n and m;
- compactly supported 1-forms $c_1, \cdots, c_n$ and compactly supported sections $\Upsilon_1, \cdots, \Upsilon_m$ of $\mathbb{S}$;
- a compact subset K of $\mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m$;

- a smooth function g on $\mathbb{R}^n \times (\mathbb{R}/2\pi\alpha\mathbb{Z}) \times \mathbb{R}^m$ with support in K.

We require the resulting collection $\{\mathfrak{q}^i\}_{i\in\mathbb{N}}$ to be dense in the space of all possible choices, in C^∞-topology; see [KM07, P. 192] for a complete description. For the rest of the paper, we presume that such a collection $\{\mathfrak{q}^i\}_{i\in\mathbb{N}}$ is chosen, once and for all, for $Y' = Y$. Let $\mathcal{P}$ be the resulting Banach spaces constructed by Theorem 8.17.

Each configuration and gauge transformation on $\widehat{Y}$ can be restricted to Y, giving rise to maps:

$$R_c : \mathcal{C}_{k-\frac{1}{2}}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{C}_{k-\frac{1}{2}}(Y, \widehat{\mathfrak{s}})$$
$$R_g : \mathcal{G}_{k+\frac{1}{2}}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{G}_{k+\frac{1}{2}}(Y, \widehat{\mathfrak{s}}).$$

Let $\mathcal{C}^*(Y, \widehat{\mathfrak{s}})$ be the irreducible part of $\mathcal{C}(Y, \widehat{\mathfrak{s}})$ and form the quotient configuration space:

$$\mathcal{B}^*(Y, \widehat{\mathfrak{s}}) = \mathcal{C}^*(Y, \widehat{\mathfrak{s}}) / \operatorname{Im}(R_g : \mathcal{G}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{G}(Y, \widehat{\mathfrak{s}})).$$

Let us now state the separating property enjoyed by $\mathcal{P}$: it is a direct consequence of Proposition 8.4 and 8.6 and the proof is omitted here.

Theorem 8.20. *Given a compact subset K of a finite dimensional C^1-submanifold $M \subset \mathcal{B}^*(\widehat{Y}, \widehat{\mathfrak{s}})$, suppose the restriction map to the truncated manifold Y*

$$[R_c] : \mathcal{B}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathcal{B}(Y, \widehat{\mathfrak{s}})$$

gives an embedding of K into $\mathcal{B}^*(Y, \widehat{\mathfrak{s}})$. Then we can find a open neighborhood U of K in M, a collections of embeddings

$$\iota_j : S^1 \times D^2 \hookrightarrow Y, 1 \leqslant j \leqslant l$$

and cylinder functions f_k defined using ι_k such that the product map

$$\Xi' = (f_1, \cdots, f_l) : \mathcal{B}^*(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathbb{R}^l$$

gives an embedding of U into $\mathbb{R}^l$. If in addition, a tangent vector $V \in T_\beta \mathcal{B}^*(\widehat{Y}, \widehat{\mathfrak{s}})$ at some $\beta \in K$ is given (V is not necessarily tangential to M) and $[r_c]_*(V) \neq 0$, then we can arrange so that

$$\Xi'_*(V) \neq 0 \in T\mathbb{R}^l.$$

9. Compactness for Perturbed Seiberg-Witten Equations

With the Banach space $\mathcal{P}$ of tame perturbations defined as in Subsection 8.5, we start to analyze the moduli space of perturbed Seiberg-Witten equations. The primary goal of this section is to prove the compactness theorem for solutions on $\mathbb{R}_t \times \widehat{Y}$. Before that, we have to generalize results from Section 5 and 6 for the perturbed equations.

9.1. **Energy Equations.** Choose a tame perturbation $\mathfrak{q} = \operatorname{grad} f \in \mathcal{P}$ with

$$\|\mathfrak{q}\|_{\mathcal{P}} < 1. \tag{9.1}$$

For all estimates and theorems below, (9.1) will be a standard assumption. Following the notations in Section 7, let $I = [t_1, t_2]_t$ and $\widehat{Z} = I \times (\widehat{Y}, \widehat{\mathfrak{s}})$. Consider a solution $\gamma \in \mathcal{C}_k(\widehat{Z})$ to the perturbed Seiberg-Witten equations

$$0 = \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(\gamma) := \mathfrak{F}_{\widehat{Z}}(\gamma) + \widehat{\mathfrak{q}}(\gamma). \tag{9.2}$$

Write γ as $(c(t), B(t), \Psi(t))$ where $\check{\gamma}(t) = (B(t), \Psi(t))$ is the underlying path in $\mathcal{C}_{k-1/2}(\widehat{Y})$. Then the equation (9.2) can be cast into the form

$$\frac{d}{dt}\check{\gamma}(t) = -\operatorname{grad} \mathcal{L}_\omega(\check{\gamma}(t)) - \mathbf{d}_{\check{\gamma}(t)}\, c(t) - \mathfrak{q}(\check{\gamma}(t)). \tag{9.3}$$

Proposition 9.1. *For any perturbation $\mathfrak{q} = \operatorname{grad} f \in \mathcal{P}$ with $\|\mathfrak{q}\|_{\mathcal{P}} < 1$ and any configuration $\gamma = (A, \Phi)$ on $\widehat{Z} = I \times (\widehat{Y}, \widehat{\mathfrak{s}})$, the L^2-norm of the perturbed Seiberg-Witten map $\mathfrak{F}_{\widehat{Z},\mathfrak{q}}(A, \Phi)$ can be expressed as*

$$\int_Z |\mathfrak{F}_{\widehat{Z},\mathfrak{q}}(A, \Phi)|^2 = \mathcal{E}^{\mathfrak{q}}_{an}(A, \Phi) - \mathcal{E}^{\mathfrak{q}}_{top}(A, \Phi)$$

where

$$\begin{aligned}
\mathcal{E}^{\mathfrak{q}}_{top}(A, \Phi) &:= 2\rlap{\,/}\mathcal{L}_\omega(\check{\gamma}(t_1)) - 2\rlap{\,/}\mathcal{L}_\omega(\check{\gamma}(t_2)),\\
\mathcal{E}^{\mathfrak{q}}_{an}(A, \Phi) &:= \int_I \|\frac{d}{dt}\check{\gamma}(t) + d_{\check{\gamma}(t)}c(t)\|^2_{L^2(\widehat{Y})} + \|\operatorname{grad} \rlap{\,/}\mathcal{L}_\omega(\check{\gamma}(t))\|^2_{L^2(\widehat{Y})},
\end{aligned}$$

and $\rlap{\,/}\mathcal{L}_\omega = \mathcal{L}_\omega + f$ is the perturbed Chern-Simons functional. Moreover, there exist constants $C_1, C_2 > 0$ such that

$$\mathcal{E}_{an}(A, \Phi) < C_1 \cdot \mathcal{E}^{\mathfrak{q}}_{an}(A, \Phi) + C_2,$$

where $\mathcal{E}_{an}$ is the analytic energy defined in Proposition 5.4.

Proof. Only the last clause requires some work. By the Cauchy-Schwartz inequality, we have

$$2\mathcal{E}^{\mathfrak{q}}_{an}(\gamma) \geqslant \mathcal{E}_{an}(\gamma) - 2\int_I \|\mathfrak{q}(\check{\gamma}(t))\|^2_{L^2(\widehat{Y})}. \tag{9.4}$$

since $\operatorname{grad} \rlap{\,/}\mathcal{L}_\omega = \operatorname{grad} \mathcal{L}_\omega + \mathfrak{q}$. By the property (F5) from Theorem 8.17,

$$\int_I \|\mathfrak{q}(\check{\gamma}(t))\|^2_{L^2(\widehat{Y})} \leqslant 2m_2^2(1 + \|\Phi\|^2_{L^2(I\times Y)}). \tag{9.5}$$

Hence, it remains to estimate $\|\Phi\|^2_{L^2(I\times Y)}$ in terms of $\mathcal{E}^{\mathfrak{q}}_{an}(\gamma)$. Recall from Lemma 5.3 that

$$\begin{aligned}
\mathcal{E}_{an}(A, \Phi) + C_2' &\geqslant \int_{I\times\widehat{Y}} \frac{1}{8}|F_{A^t}|^2 + |\nabla_A\Phi|^2 + |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2 + \frac{s}{4}|\Phi|^2,\\
&\geqslant \int_{I\times Y} |(\Phi\Phi^*)_0 + \rho_4(\omega^+)|^2 + \frac{s}{4}|\Phi|^2.
\end{aligned} \tag{9.6}$$

for some $C_2' > 0$. Combining (9.4)(9.5)(9.6) together, we obtain that

$$2\mathcal{E}_{an}^{\mathfrak{q}}(\gamma) + C_2'' \geqslant \int_{I\times Y} \frac{1}{4}|\Phi|^4 - C_3|\Phi|^2 \geqslant \int_{I\times Y} |\Phi|^2 - C_4. \tag{9.7}$$

for some $C_2'', C_3, C_4 > 0$. This completes the proof. □

Now the proof of Lemma 6.5 and Theorem 6.3 can proceed with no difficulty. Let us record the results for perturbed equations:

Theorem 9.2. *For any $C, \epsilon > 0$, there exists a constant $R_0(\epsilon, C, \widehat{Y}, \widehat{\mathfrak{s}}) > 0$ with the following significance. For any tame perturbation $\mathfrak{q} \in \mathcal{P}$ with $\|\mathfrak{q}\|_{\mathcal{P}} < 1$, let $\gamma = (A, \Phi)$ be a solution to the perturbed Seiberg-Witten equations (9.3) on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with analytic energy $\mathcal{E}_{an}^{\mathfrak{q}}(A, \Phi) < C$. Then for any $n \in \mathbb{Z}$ and $S > R_0$, we have*

$$\mathcal{E}_{an}(A, \Phi; \Omega_{n,S}) < \epsilon.$$

Here $\Omega_{n,S} \subset \mathbb{C}_z$ is the translated region of Ω_0 defined in (2.4).

Theorem 9.3. *For any $C > 0$, there exist constants $M_0(C, \widehat{Y}, \widehat{\mathfrak{s}}), \zeta(C, \widehat{Y}, \widehat{\mathfrak{s}}) > 0$ with the following significance. For any perturbation $\mathfrak{q} \in \mathcal{P}$ with $\|\mathfrak{q}\|_{\mathcal{P}} < 1$, suppose (A, Φ) is a solution to the perturbed Seiberg-Witten equations (9.3) on $\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ with analytic energy $\mathcal{E}_{an}^{\mathfrak{q}}(A, \Phi) < C$, then for any $n \in \mathbb{Z}$ and $S > 0$*

$$\mathcal{E}_{an}(A, \Phi, \Omega_{n,S}) < M_0 e^{-\zeta S}.$$

Remark 9.4. The analogous result for the exponential decay in the time direction follows from the standard argument as in [KM07, Section 13], assuming the non-degeneracy of critical points (cf. Definition 12.2). Indeed, once we obtain the exponential decay of $\mathcal{L}_\omega$, one starts to estimate the L^2_1-norm and L^2_k-norm of (A, Φ) as in Subsection 6.3. The proof is omitted here. ♢

9.2. **Compactness.** The next theorem is the analogue of Theorem 6.1 when $\mathfrak{q} \neq 0$.

Theorem 9.5. *For any perturbation $\mathfrak{q} \in \mathcal{P}$ with $\|\mathfrak{q}\|_{\mathcal{P}} < 1$, suppose $\{\gamma_n = (A_n, \Phi_n)\} \subset \mathcal{C}_{k,loc}(\mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}}))$ is a sequence of solutions to the perturbed Seiberg-Witten equations (9.3) on $\mathbb{R}_t \times \widehat{Y}$ and their analytic energy*

$$\mathcal{E}_{an}^{\mathfrak{q}}(\gamma_n) := \mathcal{E}_{an}^{\mathfrak{q}}(\gamma_n, \mathbb{R}_t) < C$$

is uniformly bounded. Then we can find a sequence of gauge transformations $u_n \in \mathcal{G}_{k+1,loc}(\mathbb{R}_t \times \widehat{Y})$ with the following properties. For a subsequence $\{\gamma_n'\}$ of $\{u_n(\gamma_n)\}$ and any finite interval $I \subset \mathbb{R}_t$, the restriction of each γ_n' on $I \times \widehat{Y}$

$$\gamma_n'|_{I\times\widehat{Y}}$$

lies in $\mathcal{C}_l(I \times (\widehat{Y}, \widehat{\mathfrak{s}}))$. Additionally, they converge in $L^2_l(I \times \widehat{Y})$-topology for any $l > 1$.

Proof. It suffices to deal with the compact region $I \times Y_1$ where $Y_1 = \{s \leqslant 1\}$ is the truncated 3-manifold. Fix a reducible configuration γ_0' on $I \times Y_1$ as reference. The bootstrapping argument works as follows: by passing to a subsequence and applying appropriate gauge transformations, we obtain that

$$\begin{aligned}
&\gamma_n - \gamma_0' \text{ bounded in } L^2_1 \Rightarrow \gamma_n \to \gamma_\infty \text{ weakly in } L^2_1 \text{ for some } \gamma_\infty\\
\Rightarrow &\gamma_n \to \gamma_\infty \text{ in } L^2_{3/4} \Rightarrow \widehat{\mathfrak{q}}(\gamma_n) \to \widehat{\mathfrak{q}}(\gamma_\infty) \text{ in } L^2 \text{ by (A6) with } \epsilon = 1/4\\
\Rightarrow &\gamma_n \to \gamma_\infty \text{ in } L^2_1 \text{ on interior domains} \Rightarrow \widehat{\mathfrak{q}}(\gamma_n) \to \widehat{\mathfrak{q}}(\gamma_\infty) \text{ in } L^{7/2} \text{ by (A3)}\\
\Rightarrow &\gamma_n \to \gamma_\infty \text{ in } L^{7/2}_1 \text{ on interior domains} \Rightarrow \widehat{\mathfrak{q}}(\gamma_\infty) \to \widehat{\mathfrak{q}}(\gamma_n) \text{ in } L^{7/2}_1 \text{ by (A2)}\\
\Rightarrow &\gamma_n \to \gamma_\infty \text{ in } L^{7/2}_2 \text{ on interior domains} \Rightarrow \widehat{\mathfrak{q}}(\gamma_n) \to \widehat{\mathfrak{q}}(\gamma_\infty) \text{ in } L^{7/2}_2 \hookrightarrow L^2_2 \text{ by (A1)}\\
\Rightarrow &\gamma_n \to \gamma_\infty \text{ in } L^2_3 \text{ on interior domains} \cdots
\end{aligned}$$

Once we arrive at L^2_3, one may proceed as in [KM07, Theorem 10.7.1]. To conclude convergence of γ_n on interior domains from the convergence of $\widehat{\mathfrak{q}}(\gamma_n)$, we use the properness of the Seiberg-Witten map, cf. Theorem [KM07, Theorem 5.2.1]. □

Remark 9.6. It is not clear to the author whether the L^2_1-norm of $\widehat{\mathfrak{q}}(\gamma)$ can be estimated in terms of the L^2_1-norm of $\gamma - \gamma_0$, so we adopt a different approach to arrive at the L^2-convergence of $\widehat{\mathfrak{q}}(\gamma_n)$, cf. [KM07, Theorem 10.7.1]. ♢

Proposition 9.7. *Suppose* $\{\mathfrak{q}_i\} \subset \mathcal{P}$ *is a convergent sequence in* $\mathcal{P}$ *with* $\|\mathfrak{q}_i\|_{\mathcal{P}} < 1$ *and let* $\beta_i \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ *be solutions of the equation*

$$(\operatorname{grad} \mathcal{L}_\omega + \mathfrak{q}_i)(\beta_i) = 0.$$

Then there is a sequence of gauge transformations $u_i \in \mathcal{G}_{k+1}(\widehat{Y})$ *such that the transformed solutions* $u_i(\beta_i)$ *have a convergent subsequence in* $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$.

Proof. The proof follows the same line of argument of Theorem 9.5. To conclude the convergence of

$$\mathfrak{q}_i(\beta_i) \to \mathfrak{q}_\infty(\beta_\infty),$$

use (F3)(F4)(F5) from Theorem 8.17. □

Part 4. Linear Analysis

Over the non-compact manifold $\widehat{Y}$, the inclusion map

$$L^2_{k+1}(\widehat{Y}) \hookrightarrow L^2_k(\widehat{Y})$$

is no longer compact. As a result, the spectrum of the extended Hessian of the Chern-Simons-Dirac functional $\mathcal{L}_\omega$, as a unbounded self-adjoint operator, is not discrete.

The goal of this part to understand the essential spectrum of extended Hessians and show that it is disjoint from the origin, in which case one can still speak of the spectrum flow. Moreover, we will show the linearization of the Seiberg-Witten equations together with the linearized gauge fixing equation form a Fredholm operator on the complement Riemannian 4-manifolds $\mathbb{R}_t \times \widehat{Y}$ and $\mathcal{X}$; so we have a well-posed moduli problem.

Part 4 is organized as follows. In Section 10, we review an abstract formalism of spectral flow following the work of Robbin-Salamon [RS95]. In Section 11 we collect some criterion from functional analysis that computes the essential spectrum following the textbook [HS96] by Hislop and Sigal. These results will be applied to the extended Hessian $\widehat{\text{Hess}}$ of $\mathcal{L}_\omega$ in Section 12. The key observation here is that $\widehat{\text{Hess}}$ can be cast into the form (up to a compact perturbation):

$$\sigma(\partial_s + D_\Sigma) : \Gamma(\mathbb{R}_s \times \Sigma, E) \to \Gamma(\mathbb{R}_s \times \Sigma, E)$$

such that $\sigma^2 = -\operatorname{Id}_E$ and $D_\Sigma : \Gamma(\Sigma, E) \to \Gamma(\Sigma, E)$ is a first order self-adjoint operator that anti-commutes with σ, i.e.

$$\sigma D_\Sigma + D_\Sigma \sigma = 0.$$

This observation was due to Yoshida [Yos91]. A short discussion in the context of the gauged Witten equations can be found in [Wan20, Subsection 4.2].

Section 13 and 14 are devoted to the linearization of the Seiberg-Witten map on $\mathbb{R}_t \times \widehat{Y}$ and $\mathcal{X}$ respectively. We will study the Fredholm property and the Atiyah-Patodi-Singer boundary value problem following the book [KM07, Section 17].

10. Spectral Flow and Fredholm Index

In the section, we summarize the axioms that characterize the spectral flow. Let us first introduce a few notations before we state the main result: Theorem 10.1.

Let H_0 be a real separable Hilbert space and $\mathbb{A}_0 : H_0 \to H_0$ be a self-adjoint operator with domain $W_0 := D(\mathbb{A}_0)$ dense in H_0. We assume that 0 does not lie in the essential spectrum of $\mathbb{A}_0$:

$$0 \notin \sigma_{ess}(\mathbb{A}_0). \tag{10.1}$$

W_0 becomes a Hilbert space with respect to the graph norm

$$\|x\|^2_{W_0} := \|\mathbb{A}_0 x\|^2_{H_0} + \|x\|^2_{H_0}, \ \forall x \in W_0.$$

The inclusion map $W_0 \hookrightarrow H_0$ is **not** assumed to be compact, so $\sigma_{ess}(\mathbb{A}_0)$ might be non-empty. A pair (W, H) of Hilbert spaces is called admissible if one can find a finite dimensional space $V = \mathbb{R}^n$ such that

$$W = W_0 \oplus V, \ H = H_0 \oplus V.$$

A symmetric operator $\mathbb{A} : W \to H$ is called admissible if one can find a symmetric **compact** operator $K : W \to H$ such that

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_0 & 0 \\ 0 & 0 \end{pmatrix} + K.$$

By the Kato-Rellich theorem, $\mathbb{A}$ is self-adjoint with domain $D(\mathbb{A}) = W$. Let $\mathscr{L}_{sym}(W, H)$ be the affine space of all admissible operators between (W, H). It is topologized using the operator norm on the compact perturbation K. Let $\mathscr{B}(\mathbb{R}, W, H)$ be the space of continuous maps $\mathbb{A} : \mathbb{R} \to \mathscr{L}_{sym}$ such that the limits

$$\mathbb{A}^{\pm} = \lim_{t \to \pm\infty} \mathbb{A}(t) : W \to H$$

exist. The $\mathcal{C}^k$-distance between two paths $\mathbb{A}_1$ and $\mathbb{A}_2$ is defined as

$$d_k(\mathbb{A}_1, \mathbb{A}_2) := \sup_{t \in \mathbb{R}} \sum_{0 \leqslant j \leqslant k} \| \frac{d^j}{dt^j} (\mathbb{A}_1(t) - \mathbb{A}_2(t)) \|_{W \to H}.$$

Denote by $\mathscr{B}^k(\mathbb{R}, W, H) \subset \mathscr{B}(\mathbb{R}, W, H)$ be the subspace consisting of paths having finite $\mathcal{C}^k$-distance with a constant path, endowed with $\mathcal{C}^k$-topology. Note that $\mathscr{B}^0(\mathbb{R}, W, H) = \mathscr{B}(\mathbb{R}, W, H)$. Finally, define an open subset

$$\mathscr{A} = \mathscr{A}(\mathbb{R}, W, H) := \{\mathbb{A} \in \mathscr{B}(\mathbb{R}, W, H) : \mathbb{A}^{\pm} \text{ invertible}\}$$

and set $\mathscr{A}^k = \mathscr{A} \cap \mathscr{B}^k$. Given paths $\mathbb{A}, \mathbb{A}_l, \mathbb{A}_r \in \mathscr{A}(\mathbb{R}, W, H)$ such that $\mathbb{A}_l(t) = \mathbb{A}(0) = \mathbb{A}_r(-t),\ t \geqslant 0$, $\mathbb{A}$ is said to be the catenation of $\mathbb{A}_l$ and $\mathbb{A}_r$ and write

$$\mathbb{A} = \mathbb{A}_l \# \mathbb{A}_r$$

if

$$\mathbb{A}(t) = \begin{cases} \mathbb{A}_l(t) & \text{if } t \leqslant 0 \\ \mathbb{A}_r(t) & \text{if } t \geqslant 0 \end{cases}$$

Given any two reference operators $(\mathbb{A}_{01}, W_{01}, H_{01})$ and $(\mathbb{A}_{02}, W_{02}, H_{02})$ satisfying the condition (10.1) and any two paths $\mathbb{A}_i \in \mathscr{A}(\mathbb{R}, W_i, H_i),\ i = 1, 2$, one can form the direct sum

$$\mathbb{A}_1 \oplus \mathbb{A}_2 \in \mathscr{A}(\mathbb{R}, W_1 \oplus W_2, H_1 \oplus H_2).$$

Let us now state the axioms that characterize the spectrum flow along a path $\mathbb{A} \in \mathscr{A}(\mathbb{R}, W, H)$.

Theorem 10.1 (cf. [RS95] Theorem 4.23)**.** *For any reference operator $(\mathbb{A}_0, W_0, H_0)$ satisfying the condition* (10.1) *and any finite dimensional auxiliary space V, there exists a unique map*

$$\mu : \mathscr{A}(\mathbb{R}, W, H) \to \mathbb{Z}$$

satisfying the following axioms

- (*Homotopy*) *μ is constant on the connected components of $\mathscr{A}(\mathbb{R}, W, H)$;*
- (*Constant*) *If $\mathbb{A}$ is a constant path, then $\mu(\mathbb{A}) = 0$;*
- (*Direct Sum*) *$\mu(\mathbb{A}_1 \oplus \mathbb{A}_2) = \mu(\mathbb{A}_1) + \mu(\mathbb{A}_2)$;*
- (*Catenation*) *If $\mathbb{A} = \mathbb{A}_r \# \mathbb{A}_r$, then $\mu(\mathbb{A}) = \mu(\mathbb{A}_l) + \mu(\mathbb{A}_r)$;*
- (*Normalization*) *For $W = H = \mathbb{R}$ and $\mathbb{A}(t) = \arctan(t)$, $\mu(\mathbb{A}) = 1$.*

The integer $\mu(\mathbb{A})$ *is called the spectral flow of* $\mathbb{A} \in \mathscr{A}(\mathbb{R}, W, H)$.

Proof. The proof follows the same line of argument as [RS95, Theorem 4.23]. The idea for existence works as follows. Define

$$\mathscr{L}_k = \{\mathbb{A} \in \mathscr{L}_{sym}(W, H) : \dim\ker \mathbb{A} = k\},$$

then $\mathscr{L}_k$ is a smooth Banach submanifold of $\mathscr{L}_{sym}$ of real co-dimension $k(k+1)/2$. For any path $\mathbb{A} \in \mathscr{A}$, find a $\mathcal{C}^1$-path $\mathbb{A}' \in \mathscr{A}^1$ that is homotopic to $\mathbb{A}$ and intersects each $\mathscr{L}_k,\ k \geqslant 1$ transversely. Then $\mu(\mathbb{A})$ is defined as the algebraic intersection of $\mathbb{A}'$ with $\mathscr{L}_1$. For details, see [RS95]. □

There is another way to think of the spectral flow. For any path $\mathbb{A} \in \mathscr{A}^k$, define the differential operator:

$$D_{\mathbb{A}} : \mathcal{W}_k := L^2_k(\mathbb{R}, W) \cap L^2_{k+1}(\mathbb{R}, H) \to L^2_k(\mathbb{R}, H)$$
$$\xi(t) \mapsto \frac{d}{dt}\xi(t) + \mathbb{A}(t)\xi(t),$$

where the $\mathcal{W}_k$-norm is defined as

$$\|\xi\|^2_{\mathcal{W}_k} = \int_{\mathbb{R}} \left(\sum_{0\leqslant j\leqslant k+1} \|\frac{d^j}{dt^j}\xi\|^2_H + \sum_{0\leqslant j\leqslant k} \|\frac{d^j}{dt^j}\xi\|^2_W \right) dt \text{ for all } \xi \in C^\infty_0(\mathbb{R}, W).$$

Theorem 10.2 (cf. [RS95] Theorem 3.12)**.** *For any* $k \geqslant 0$ *and any* $\mathbb{A} \in \mathscr{A}^k$ *such that*

$$\mathbb{A}(t) \to \mathbb{A}^\pm \text{ in } \mathcal{C}^k_{loc}\text{-topology as } t \to \pm\infty,$$

then $D_{\mathbb{A}} : \mathcal{W}_k \to L^2_k(\mathbb{R}, H)$ *is a Fredholm operator of the index* $\mu(\mathbb{A})$.

Proof. As our situation is slightly simpler than [RS95, Theorem 3.12], we present a direct proof using parametrix patching argument. The theorem holds when $\mathbb{A}(t) \equiv \mathbb{A}^+$ is a constant path and $\mathbb{A}^+$ is invertible. Indeed,

$$\|(\frac{d}{dt} + \mathbb{A}^+)\xi\|^2_{L^2_k(\mathbb{R},H)} = \sum_{0\leqslant j\leqslant k} \int_{\mathbb{R}} \|\frac{d^j}{dt^j}(\frac{d}{dt} + \mathbb{A}^+)\xi\|^2_H = \sum_{0\leqslant j\leqslant k} \int_{\mathbb{R}} \|\frac{d^{j+1}}{dt^{j+1}}\xi\|^2_H + \|\mathbb{A}^+(\frac{d^j}{dt^j}\xi)\|^2_H$$
$$= \sum_{0\leqslant j\leqslant k} \int_{\mathbb{R}} \|\frac{d^{j+1}}{dt^{j+1}}\xi\|^2_H + \|\frac{d^j}{dt^j}\xi\|^2_W \gtrsim \|\xi\|^2_{\mathcal{W}_k}.$$

In general, let $\mathbb{A}^\pm = \lim_{t\to\pm\infty} \mathbb{A}(t)$ be the limiting operators of $\mathbb{A}$ and $Q^\pm : L^2_k(\mathbb{R}, H) \to \mathcal{W}_k$ be the inverse of $D_{\mathbb{A}^\pm}$. Choose cut-off functions $\beta_\pm$ on $\mathbb{R}_t$ such that

- $\beta_- + \beta_+ = 1$;
- $\beta_+(t) \equiv 1$ when $t > 1$; $\beta_+(t) \equiv 0$ when $t < -1$.

Take $Q_L = Q^-\beta_- + Q^+\beta_+$ and $K^\pm = D_{\mathbb{A}} - D_{\mathbb{A}^\pm} = \mathbb{A} - \mathbb{A}^\pm$. We compute:

$$\begin{aligned} Q_L D_A &= Q^- D_A \beta_- + Q^-[\beta_-, D_A] + Q^+ D_A \beta_+ + Q^+[\beta_+, D_A] \\ &= \mathrm{Id}_{\mathcal{W}_k} + Q^-(K^-\beta_-) + Q^+(K^+\beta_+) + (Q^+ - Q^-)\partial_t\beta_- \\ &= \mathrm{Id}_{\mathcal{W}_k} + Q^-(K^-\beta_-) + Q^+(K^+\beta_+) - Q^+((\mathbb{A}^+ - \mathbb{A}^-)\partial_t\beta_-)Q^-. \end{aligned}$$

(For the right parametrix, take $Q_R = \beta_- Q^- + \beta_+ Q^+$).

To show that each error term gives arise to a compact operator, apply the next lemma to operators:

$$K^-\beta_-, K^+\beta_+ \text{ and } (\mathbb{A}^+ - \mathbb{A}^-)\partial_t\beta_-.$$

Lemma 10.3 ([RS95] Lemma 3.18)**.** *For any $k \geqslant 0$, suppose $K(t): W \to H$ is a $\mathcal{C}^k$-family of compact operators that converges to zero in $\mathcal{C}^k_{loc}$-topology as $t \to \pm\infty$, i.e.*

$$\lim_{t\to\pm\infty} \|K(t+\cdot)\|_{\mathcal{C}^k([-1,1])} = 0.$$

Then the multiplication operator $K_*: \xi(t) \mapsto K(t)\xi(t)$ is compact from $\mathcal{W}_k$ to $L^2_k(\mathbb{R}, H)$.

Proof of the Lemma. We follow the argument of [RS95, Lemma 3.18]. It suffices to show the operator $\xi(t) \mapsto \frac{d^j}{dt^j}(K(t)\xi(t))$ is compact from $\mathcal{W}_k$ to $L^2(\mathbb{R}, H)$ for any $0 \leqslant j \leqslant k$. This reduces the problem to the case when $k = 0$.

Let $\mathrm{Comp}(W, H)$ be the space of compact operators from W to H. The function $K: \mathbb{R} \to \mathrm{Comp}(W, H)$ can be approximated in $\mathcal{C}^0$-topology by linear combinations of characteristic functions. Each approximation K_n is a finite sum

$$\sum_{j=0}^{n} \chi_{I_j} K_n^{(j)}$$

where χ_{I_j} is the characteristic function of a finite interval $I_j \subset \mathbb{R}$ and $K_n^{(j)} \in \mathrm{Comp}(W, H)$ is a compact operator. As $(K_n)_* \to K_*$ in the norm topology, it suffices to prove each $(K_n)_*$ is compact. We reduce to the case when $K = \chi_{I_1} K^{(1)}$ consists of a single term.

The final step is to approximate $K^{(1)}$ by a sequence of finite rank operators. When $K^{(1)}$ is a finite rank operator, K_* is the composition of three operators:

$$\mathcal{W}_0 \xrightarrow{K_*} L^2_1(I_1, U) \to L^2(I_1, U) \to L^2(\mathbb{R}, H),$$

where $U = \operatorname{Im} K^{(1)}$ is a finite dimensional real vector subspace of H, so the middle map is compact. This completes the proof of the lemma. □

Back to the proof of Theorem 10.2. To prove $\mathrm{Ind}(D_{\mathbb{A}}) = \mu(\mathbb{A})$, it remains to verify the assignment $\mathbb{A} \mapsto \mathrm{Ind}(D_{\mathbb{A}})$ satisfies all axioms of spectral flow in Theorem 10.1 when $k = 0$. Only the catenation axiom is not obvious. However, by [RS95, Proposition 4.26], the catenation axiom follows from the homotopy, direct sum and constant axioms. This completes the proof of Theorem 10.2 □

11. Essential Spectrum

To apply the general theory from the previous section, it is important to verify the condition (10.1) for operators of interest. In this section, we discuss a class of model operators following the setup of [Yos91]. The main result is Proposition 11.2. This general formalism will be applied to the extended Hessians $\widehat{\mathrm{Hess}}$ of $\mathcal{L}_\omega$ in the next section.

Recall that $\widehat{Y} = Y \cup [-1, \infty)_s \times \Sigma$ is a 3-manifold with cylindrical ends. Suppose $E \to \widehat{Y}$ is a real vector bundle over $\widehat{Y}$ such that

$$E|_{[-1,\infty)_s\times\Sigma} = \pi^* E_0$$

and $E_0 \to \Sigma$ is a vector bundle over Σ. Here $\pi : [-1, \infty)_s \times \Sigma \to \Sigma$ is the projection map. Bundles E and E_0 are endowed with Riemannian metrics. We investigate a special class of first order differential operators

$$D_Y : C_0^\infty(\widehat{Y}, E) \to C_0^\infty(\widehat{Y}, E);$$

satisfying the following constraints on D_Y:

- D_Y is elliptic and symmetric with respect to the L^2-inner product;
- $D_Y = \sigma(\frac{d}{ds} + D_\Sigma)$ on the cylindrical end $[-1, \infty)_s \times \Sigma$, where
- $\sigma : E_0 \to E_0$ is skew-symmetric bundle map of $E_0 \to \Sigma$, i.e. $\sigma + \sigma^* = 0$; moreover, $\sigma^2 = -\operatorname{Id}_{E_0}$;
- $D_\Sigma : C^\infty(\Sigma, E_0) \to C^\infty(\Sigma, E_0)$ is a first order self adjoint elliptic differential operator; moreover, D_Σ anti-commutes with σ, i.e. $\sigma D_\Sigma + D_\Sigma \sigma = 0$.

Example 11.1. The simplest example of D_Y is the Dirac operator. Let $E = S$ be the spin bundle and $D_Y = \sum_{1 \leqslant i \leqslant 3} \rho_3(e_i)\nabla^B_{e_i}$ for some *spin*^{c} connection B. On the cylindrical end $[-1, \infty)_s \times \Sigma$, we require B to take the form

$$B = \frac{d}{ds} + \check{B}$$

for some *spin*^{c} connection $\check{B}$ on Σ. Set $\sigma = \rho_3(ds)$ on $[-1, \infty)_s \times \Sigma$. ♢

Proposition 11.2. *Under above assumptions, D_Y is a unbounded self-adjoint operator on $L^2(\widehat{Y}, E)$ with domain $L^2_1(\widehat{Y}, E)$. Moreover, the essential spectrum σ_{ess} of D_Y is*

$$(-\infty, -\lambda_1] \cup [\lambda_1, \infty)$$

where λ_1 is the first non-negative eigenvalue of D_Σ. In particular, if D_Σ is invertible, then $0 \notin \sigma_{ess}(D_Y)$.

Remark 11.3. Since D_Σ anti-commutes with σ, $-\lambda_1$ is also the first non-positive eigenvalue of D_Σ. The spectrum of D_Σ is symmetric with respect to the origin. ♢

The proof of Proposition 11.2 will dominate the rest of this section. To compute the essential spectrum of D_Y, we need two additional results from functional analysis: Weyl's criterion and Zhislin's criterion.

Definition 11.4. Suppose $\mathbb{A} : H \to H$ is a self-adjoint operator with domain $W := D(\mathbb{A}) \subset H$. For any $\lambda \in \mathbb{C}$, a sequence $\{u_n\}$ is called a Weyl sequence for $(\mathbb{A}, \lambda)$ if $\{u_n\} \subset W$, $\|u_n\|_H = 1$, $u_n \xrightarrow{w} 0$ weakly in H and $(\mathbb{A} - \lambda)u_n \xrightarrow{s} 0$ strongly in H. ♢

Theorem 11.5 (Weyl's Criterion, [HS96] Theorem 7.2)**.** *Under the assumption of Definition* 11.4*, $\lambda \in \sigma_{ess}(\mathbb{A})$ if and only if there exists a Weyl sequence for $(\mathbb{A}, \lambda)$.*

When $H = L^2(\widehat{Y}, E)$, Weyl's criterion can be refined into Zhislin's criterion for locally compact operators.

Definition 11.6. Suppose $H = L^2(\widehat{Y}, E)$ and χ_B is the characteristic function for a subset $B \subset \widehat{Y}$. A self-adjoint operator $\mathbb{A}$ on H is called **locally compact** if the operator $\chi_B(\mathbb{A} - i)^{-1} : H \to H$ is compact for any compact subset $B \subset \widehat{Y}$. ♢

Definition 11.7. Let $Y_n = \{s \leqslant n\}, n \in \mathbb{Z}_{\geqslant 0}$ be the truncated 3-manifold. For any $\lambda \in \mathbb{C}$, a sequence $\{u_n\} \subset W$ is called **a Zhislin sequence** for $(\mathbb{A}, \lambda)$ if $\|u_n\|_H = 1$, $\mathrm{supp}(u_n) \subset Y_n^c$ and $(\mathbb{A} - \lambda)u_n \xrightarrow{s} 0$ in H. ♢

As u_n is supported on the complement of Y_n, $u_n \xrightarrow{w} 0$. As a result, a Zhislin sequence is always a Weyl sequence.

Theorem 11.8 (Zhislin's Criterion, [HS96] Theorem 10.6)**.** *Suppose $H = L^2(\widehat{Y}, E)$ and $\mathbb{A} : H \to H$ is self-adjoint and locally compact. If $\mathbb{A}$ satisfies the commutator estimate:*

$$\|[\mathbb{A}, \varphi_n](A - i)^{-1}\|_{H \to H} \to 0 \text{ as } n \to \infty, \tag{11.1}$$

where $\varphi_n = \varphi(s(\cdot)/n)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is some cut-off function such that $\varphi(r) \equiv 1$ when $r \leqslant 1$ and $\varphi(r) \equiv 0$ when $r \geqslant 2$, then $\lambda \in \sigma_{ess}(\mathbb{A})$ if and only if there exists a Zhislin sequence for $(\mathbb{A}, \lambda)$.

Idea of the Proof. The "if" part follows from Weyl's Criterion. Suppose $\lambda \in \sigma_{ess}(\mathbb{A})$ and $\{u_m\}$ is a Weyl sequence for $(\mathbb{A}, \lambda)$. We wish to construct a Zhislin sequence for $(\mathbb{A}, \lambda)$ out of $\{u_m\}$. For any $n \in \mathbb{Z}_{\geqslant 0}$, choose a large number $m(n)$ and define

$$v_n = (1 - \varphi_n)u_{m(n)}.$$

First of all, $(\mathbb{A} - i)u_m = (\mathbb{A} - \lambda)u_m + (\lambda - i)u_m \xrightarrow{w} 0$ as $m \to \infty$. Because $\varphi_n(\mathbb{A} - i)^{-1}$ is compact, $\varphi_n u_m = \phi_n(A - i)^{-1} \circ (A - i)u_m \xrightarrow{s} 0$ as $m \to \infty$ for any fixed n. By taking $m(n) \gg n$, we ensure that $\|v_n\|_H \geqslant \frac{1}{2}$.

The second step is to use the commutator (11.1) estimate to prove $(\mathbb{A} - \lambda)v_n \xrightarrow{s} 0$ as $n \to \infty$. Finally, $\{v_n/\|v_n\|_H\}$ is the desired Zhislin sequence. For details, see [HS96, Theorem 10.6] □

Remark 11.9. Zhislin's Criterion shows that the essential spectrum of $\mathbb{A}$ is determined completely by its behavior along the cylindrical end $[0, \infty)_s \times \Sigma$. ♢

Proof of Proposition 11.2. D_Y is a locally compact operator as $\chi_B(D_Y - i)^{-1} : L^2(\widehat{Y}) \to L^2(\widehat{Y})$ factorizes through $L^2_1(B)$ when $B = Y_n$. The commutator estimate is also satisfied as

$$[D_Y, \varphi_n] = \frac{1}{n} \cdot \frac{d\varphi}{dr}(\frac{s}{n})\rho(ds)$$

and its L^∞-norm decays to zero. Applying Zhislin's criterion, we reduce to the case when $\widehat{Y} = \mathbb{R}_s \times \Sigma$ is a cylinder and

$$D_Y = \sigma(\frac{d}{ds} + D_\Sigma).$$

To study the spectrum of D_Y in this case, apply Fourier transformation in $\mathbb{R}_s$-direction. Our goal is to find eigenvalues of

$$\widehat{D_Y}(\xi) = \sigma(i\xi + D_\Sigma) : \Gamma(\Sigma, E_0) \to \Gamma(\Sigma, E_0)$$

for any fixed $\xi \in \mathbb{R}_\xi$. Let ϕ_λ be an eigenvector of D_Σ with eigenvalue $\lambda > 0$. As D_Σ anti-commutes with σ, $-\lambda$ is also an eigenvalue; indeed,

$$D_\Sigma(\sigma(\phi_\lambda)) = -\lambda\sigma(\phi_\lambda).$$

As a result, $\mathrm{span}_{\mathbb{C}}\{\phi_\lambda, \sigma(\phi_\lambda)\}$ is an invariant subspace of $\widehat{D_Y}(\xi)$:

$$\widehat{D_Y}(\xi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda + i\xi & 0 \\ 0 & -\lambda + i\xi \end{pmatrix} = \begin{pmatrix} 0 & \lambda - i\xi \\ \lambda + i\xi & 0 \end{pmatrix}$$

whose eigenvalues are $\pm\sqrt{\xi^2 + \lambda^2}$. Let $\phi_\lambda^\pm(\xi)$ be their associated eigenvectors respectively and set

$$\phi_n(s) := (\varphi(s - 2n) - \varphi(s - n))\phi_\lambda^\pm(\xi)\exp(i\xi s).$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the cut-off function defined in Theorem 11.8. Then $\{\phi_n / \|\phi_n\|_2\}$ is a Zhislin sequence for $(D_Y, \pm\sqrt{\xi^2 + \lambda^2})$, and $\pm\sqrt{\xi^2 + \lambda^2} \in \sigma_{ess}(D_Y)$ by Theorem 11.8.

When $\lambda' \in (-\lambda_1, \lambda_1)$, $(\widehat{D_Y}(\xi) - \lambda')$ is invertible for each $\xi \in \mathbb{R}_\xi$; their inverses are uniformly bounded. As a result, the operator

$$D_Y - \lambda'$$

is invertible, so $\lambda' \notin \sigma_{ess}(D_Y)$. This completes the proof of Proposition 11.2. □

12. Extended Hessians

In this section, we apply the abstract formalisms in Section 11 to the extended Hessians of $\mathcal{L}_\omega$ and compute its essential spectrum. The main result is Proposition 12.1. The proof relies on the key observation from the first paper [Wan20, Proposition 7.4]: the Seiberg-Witten equations on $\mathbb{C} \times \Sigma$ is secretly the gauged Witten equations on $\mathbb{C}$. The structural results from [Wan20, Subsection 4.2] then becomes essential here. The formalism from Section 11 in fact applies to any gauged Witten equations.

Recall from Section 4 that the quotient configuration space

$$\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}) = \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})/\mathcal{G}_{k+1}(\widehat{Y})$$

is a Hilbert manifold when $k > \frac{1}{2}$. For any $\gamma \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$, denote by $[\gamma]$ its gauge equivalent class in $\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}})$. By Lemma 4.4 the tangent space of $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ at γ admits a decomposition:

$$\mathcal{T}_{k,\gamma} := T_\gamma \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) = \mathcal{J}_{k,\gamma} \oplus \mathcal{K}_{k,\gamma}$$

where

$$\mathcal{J}_{k,\gamma} = \mathrm{Im}(\mathbf{d}_\gamma : L^2_{k+1}(\widehat{Y}, i\mathbb{R}) \to \mathcal{T}_{k,\gamma}) \text{ and}$$
$$\mathcal{K}_{k,\gamma} = \ker(\mathbf{d}^*_\gamma : \mathcal{T}_{k,\gamma} \to L^2_{k-1}(\widehat{Y}, i\mathbb{R}))$$

form L^2-complementary sub-bundles of $\mathcal{T}_k \to \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$. Moreover,

$$T_{[\gamma]}\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}) = \mathcal{K}_{k,\gamma}.$$

Take a tame perturbation $\mathfrak{q} = \mathrm{grad} f \in \mathcal{P}$. As the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega = \mathcal{L}_\omega + f$ is invariant under the identity component of $\mathcal{G}_{k+1}(\widehat{Y})$, its gradient

$$\mathrm{grad}\, \mathcal{L}_\omega = \mathrm{grad}\, \mathcal{L}_\omega + \mathfrak{q}$$

defines a smooth section of $\mathcal{K}_{k-1} \to \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ and its Hessian is a symmetric bundle map:

$$\mathcal{D}\,\mathrm{grad}\, \mathcal{L}_\omega : \mathcal{T}_k \to \mathcal{T}_{k-1}$$

which is equivariant under the action of $\mathcal{G}_{k+1}(\widehat{Y})$. As $\mathcal{C}_k(\widehat{Y},\widehat{\mathfrak{s}})$ is an affine space, the tangent bundle $\mathcal{T}_k \to \mathcal{C}_k(\widehat{Y},\widehat{\mathfrak{s}})$ is endowed with the trivial flat connection, but the decomposition $\mathcal{T}_k = \mathcal{J}_k \oplus \mathcal{K}_k$ is not parallel. Consider the composition of maps:

$$\mathrm{Hess}_{\mathfrak{q}} := \Pi_{\mathcal{K}_{k-1}} \circ \mathcal{D}\,\mathrm{grad}\,\mathcal{L}_\omega : \mathcal{K}_k \to \mathcal{K}_{k-1},$$

and write $\mathcal{D}\,\mathrm{grad}\,\mathcal{L}_\omega$ into a block form:

$$\mathcal{D}\,\mathrm{grad}\,\mathcal{L}_\omega = \begin{pmatrix} y & x \\ x^* & \mathrm{Hess}_{\mathfrak{q}} \end{pmatrix} : \mathcal{J}_k \oplus \mathcal{K}_k \to \mathcal{J}_{k-1} \oplus \mathcal{K}_{k-1}, \tag{12.1}$$

where $x = \Pi_{\mathcal{J}_{k-1}} \circ \mathcal{D}\,\mathrm{grad}\,\mathcal{L}_\omega|_{\mathcal{K}_k}$ and $y = \Pi_{\mathcal{J}_{k-1}} \circ \mathcal{D}\,\mathrm{grad}\,\mathcal{L}_\omega|_{\mathcal{J}_k}$. Note that

$$x = 0, y = 0$$

when $\gamma \in \mathrm{Crit}(\mathcal{L}_\omega)$ is a critical point. Here is the another way to think of $\mathrm{Hess}_{\mathfrak{q}}$. $\mathcal{L}_\omega$ descends to a circle valued functional $\overline{\mathcal{L}_\omega}$ on the quotient configuration space $\mathcal{B}_k(\widehat{Y},\widehat{\mathfrak{s}})$. The Hessian of $\overline{\mathcal{L}_\omega}$ at $[\gamma] \in \mathcal{B}_k(\widehat{Y},\widehat{\mathfrak{s}})$ regarded as a map

$$\mathcal{K}_{k,\gamma} = T_{[\gamma]}\mathcal{B}_k(\widehat{Y},\widehat{\mathfrak{s}}) \to \mathcal{K}_{k-1,\gamma}$$

is precisely given by $\mathrm{Hess}_{\mathfrak{q}}$. However, $\mathrm{Hess}_{\mathfrak{q}}$ is not the convenient notion to work with from the gauge theoretic point of view. One looks instead at **the extended Hessian** $\widehat{\mathrm{Hess}}_{\mathfrak{q}}$ of $\mathcal{L}_\omega$ whose expression at $\gamma \in \mathcal{C}_k(\widehat{Y},\widehat{\mathfrak{s}})$ is defined by

$$\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma} := \begin{pmatrix} 0 & \mathbf{d}_\gamma^* \\ \mathbf{d}_\gamma & \mathcal{D}_\gamma\,\mathrm{grad}\,\mathcal{L}_\omega \end{pmatrix} : L^2_k(\widehat{Y}, i\mathbb{R}) \oplus \mathcal{T}_{k,\gamma} \to L^2_{k-1}(\widehat{Y}, i\mathbb{R}) \oplus \mathcal{T}_{k-1,\gamma}.$$

Proposition 12.1 (cf. [KM07] Proposition 12.3.1)**.** *The operator* $\mathrm{Hess}_{\mathfrak{q},\gamma} : \mathcal{K}_k \to \mathcal{K}_{k-1}$ *is symmetric. If* γ *is a critical point of* $\mathcal{L}_\omega$*, then it is invertible if and only if the extended Hessian* $\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}$ *at* γ *is invertible. Moreover, the spectrum of* $\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}$ *is real and*

$$\sigma_{ess}(\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}) = (-\infty, -\lambda_1] \cup [\lambda_1, \infty)$$

where $\lambda_1 > 0$ *is a positive number depending only on the boundary data* (g_Σ, λ, μ) *of* $\mathbb{Y} \in \mathrm{Cob}_s$*. In particular,* $\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}$ *is a Fredholm operator of index* 0 *for any* $k \geqslant 1$*.*

Definition 12.2. A critical point $\gamma \in \mathcal{C}_k(\widehat{Y},\widehat{\mathfrak{s}})$ of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega = \mathcal{L}_\omega + f$ is called **non-degenerate** if the extended Hessian $\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}$ at γ is invertible. ♢

The proof of Proposition 12.1 will dominate the rest of this section.

Proof of Proposition 12.1. We focus on the essential spectrum of $\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma}$; the rest of statements follows from the same line of argument of [KM07, Proposition 12.3.1].

Let $\gamma_0 = (B_0, \Psi_0)$ be the reference configuration of $\mathcal{C}_k(\widehat{Y},\widehat{\mathfrak{s}})$. Then $\gamma - \gamma_0 = (b, \psi) \in L^2_k(\widehat{Y}, iT^*\widehat{Y} \oplus S)$ and

$$\widehat{\mathrm{Hess}}_{\mathfrak{q},\gamma} = \widehat{\mathrm{Hess}}_{0,\gamma_0} + h(b,\psi) + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}_\gamma \mathfrak{q} \end{pmatrix}.$$

where $h(b,\psi)$ is an operator that involves only point-wise multiplication of (b,ψ). When $g \in L^2_k(\widehat{Y})$ is fixed, the Sobolev multiplication

$$\begin{aligned} L^2_k(\widehat{Y}) \times L^2_k(\widehat{Y}) &\to L^2_{k-1} \\ (f,g) &\mapsto fg \end{aligned}$$

is a compact operator in the first argument when $k \geqslant 1$ (see [KM07, Theorem 13.2.2]), so the error $h(b,\psi)$ is compact. As $\mathfrak{q}$ is tame, by property (A4), $\mathcal{D}_\gamma\mathfrak{q} : L^2_k \to L^2_k$ is bounded linear. In addition, since its image is supported on $Y \subset \widehat{Y}$, the operator $\mathcal{D}_\gamma\mathfrak{q} : L^2_k \to L^2_{k-1}$ is also compact.

By the Kato-Rellich theorem, the essential spectrum is invariant under compact perturbations. It suffices to compute the essential spectrum of $\widehat{\text{Hess}}_{0,\gamma_0}$. The general theory from Section 11 applies here, so we may concentrate on the special case when $\widehat{Y} = \mathbb{R}_s \times \Sigma$ is a cylinder and $\gamma_0 = (B_*, \Psi_*)$ is the $\mathbb{R}_s$-translation invariant solution defined by (2.6).

At this point, we have to recall some results [Wan20, Subsection 4.2]. The extended Hessian $\widehat{\text{Hess}}_{\gamma_0}$ can be cast into the form $\sigma(\partial_s + \hat{D}_\kappa)$ as an operator

$$L^2_1(\widehat{Y}, i\mathbb{R} \oplus (i\mathbb{R} \otimes ds) \oplus iT^*\Sigma \oplus S) \to L^2(\widehat{Y}, i\mathbb{R} \oplus (i\mathbb{R} \otimes ds) \oplus iT^*\Sigma \oplus S)$$

with

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & *_\Sigma & 0 \\ 0 & 0 & 0 & \rho_3(ds) \end{pmatrix}$$

and $\hat{D}_\kappa$ defined as in [Wan20, P.36]. It is shown in [Wan20, Proposition 7.9] that $\hat{D}_\kappa$ is an invertible operator. Now we use Proposition 11.2 to conclude. $\square$

13. Linearized Operators on Cylinders

In this section, we study the Seiberg-Witten moduli space on the cylinder $\mathbb{R}_t \times \widehat{Y}$ and prove the Fredholm property of the linearized operator using the formalism of Section 10. In Subsection 13.2, we will prove a separating property of the cokernel of the linearized operator, which will be crucial in the proof of transversality in Theorem 16.1.

We have to justify that the proof of gluing theorem in [KM07, Section 18, 19] continue to work in our case, in the presence of essential spectra. This is done in Subsection 13.3 and 13.4, where the relevant Atiyah-Patodi-Singer boundary value theory is also developed.

13.1. **Linearized Operators.** Here is the second reason why the extended Hessian is a natural object: it is more consistent with the 4-manifold theory. Suppose

$$\mathfrak{a}, \mathfrak{b} \in \text{Crit}(\mathcal{L}_\omega) \subset \mathcal{C}(\widehat{Y}, \widehat{\mathfrak{s}})$$

are non-degenerate critical points of the perturbed Chern-Simons functional $\mathcal{L}_\omega$ in the sense of Definition of 12.2. To describe the moduli space of flowlines from $\mathfrak{a}$ to $\mathfrak{b}$, we fix a smooth configuration γ on $\widehat{Z} := \mathbb{R}_t \times \widehat{Y}$ such that γ is in the temporal gauge and

$$\check{\gamma}(t) = \mathfrak{a} \text{ if } t < -1,$$
$$\check{\gamma}(t) = \mathfrak{b} \text{ if } t > 1.$$

Consider the configuration space

$$\mathcal{C}_k(\mathfrak{a},\mathfrak{b}) = \{(A,\Phi) = \gamma_0 + (a,\phi) : (a,\phi) \in L^2_k(\widehat{Z}, iT^*\widehat{Z} \oplus S^+)\}.$$

and the gauge group

$$\mathcal{G}_{k+1}(\widehat{Z}) = \{u : \widehat{Z} \to S^1 : u - 1 \in L^2_{k+1}(\widehat{Z}, \mathbb{C})\}.$$

We are interested in solutions of the perturbed Seiberg-Witten equations on $\widehat{Z}$:

$$0 = \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(\gamma) := \mathfrak{F}_{\widehat{Z}}(\gamma) + \widehat{\mathfrak{q}}(\gamma), \tag{13.1}$$

where $\mathfrak{F}_{\widehat{Z}}$ is defined by (3.7) and $\widehat{\mathfrak{q}}$ is defined as in (7.3). We form the moduli space

$$\mathcal{M}_k(\mathfrak{a},\mathfrak{b}) := \{\gamma \in \mathcal{C}_k(\mathfrak{a},\mathfrak{b}) : \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(\gamma) = 0\}/\mathcal{G}_{k+1}(\widehat{Z}).$$

We focus on the linearized theory of the moduli space in this section. Take a configuration $\gamma = (A,\Phi) \in \mathcal{C}_k(\mathfrak{a},\mathfrak{b})$, then a tangent vector V at γ is a section

$$(\delta c(t), \delta b(t), \delta\psi(t)) \in L^2_k(\mathbb{R}_t \times \widehat{Y}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S).$$

It lies in the kernel of the linearized operator $\mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}$ (i.e. the tangent map) of $\mathfrak{F}_{\widehat{Z},\mathfrak{q}}$ if and only if it solves the equation

$$\frac{d}{dt}\begin{pmatrix}\delta b(t)\\ \delta\psi(t)\end{pmatrix} + \mathcal{D}_{\check{\gamma}(t)} \operatorname{grad} \mathcal{L}_\omega \begin{pmatrix}\delta b(t)\\ \delta\psi(t)\end{pmatrix} + \mathbf{d}_{\check{\gamma}(t)}\, \delta c(t) = 0, \ \forall t \in \mathbb{R}. \tag{13.2}$$

(13.2) is obtained by formally linearizing the equation (7.1). The convention of (7.2) is also adopted here: $\check{\gamma}(t)$ stands for the underlying path in $\mathcal{C}(\widehat{Y}, \widehat{\mathfrak{s}})$.

On the other hand, the linearized action of $\mathcal{G}(\widehat{Z})$ at γ is given by:

$$\mathbf{d}_\gamma : \operatorname{Lie}(\mathcal{G}_{k+1}(\widehat{Z})) = L^2_{k+1}(\widehat{Z}, i\mathbb{R}) \to T_\gamma\mathcal{C}(\mathfrak{a},\mathfrak{b})$$
$$f(t) \mapsto (-\frac{d}{dt} f(t), \mathbf{d}_{\check{\gamma}(t)}\, f(t)),$$

whose L^2-formal adjoint is

$$\mathbf{d}^*_\gamma : T_\gamma\mathcal{C}(\mathfrak{a},\mathfrak{b}) \to L^2_{k-1}(\widehat{Z}, i\mathbb{R})$$
$$V(t) = (\delta c(t), \delta b(t), \delta\psi(t)) \mapsto \frac{d}{dt}\delta c(t) + \mathbf{d}^*_{\check{\gamma}(t)} \begin{pmatrix}\delta b(t)\\ \delta\psi(t)\end{pmatrix}.$$

It follows that $\mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}$ together with the linearized gauge fixing operator $\mathbf{d}^*_\gamma$ can be cast into the form:

$$V(t) \mapsto \frac{d}{dt}V(t) + \widehat{\operatorname{Hess}}_{\mathfrak{q},\check{\gamma}(t)} V(t), \tag{13.3}$$

for $V(t) = (\delta c(t), \delta b(t), \delta\psi(t))$. By Theorem 10.2, we have

Proposition 13.1. *For any $\gamma \in \mathcal{C}_k(\mathfrak{a}, \mathfrak{b})$, the operator*

$$(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}) : L^2_k(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S) \to L^2_{k-1}(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S)$$

is Fredholm. Its index is independent of γ and equals the spectrum flow from $\widehat{\mathrm{Hess}}_{\mathfrak{q},\mathfrak{a}}$ to $\widehat{\mathrm{Hess}}_{\mathfrak{q},\mathfrak{b}}$.

Proof. The operator $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}})$ differs from the $(\mathbf{d}^*_{\gamma_0}, \mathcal{D}_{\gamma_0} \mathfrak{F}_{\widehat{Z},\mathfrak{q}})$ by a compact term. When $\gamma = \gamma_0$ is the reference configuration, apply Theorem 10.2. □

Definition 13.2. The moduli space $\mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$ is called regular if the linearized operator $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}})$ at γ is surjective for any $[\gamma] \in \mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$. ♢

Definition 13.3. A tame perturbation $\mathfrak{q} = \operatorname{grad} f \in \mathcal{P}$ is called admissible if

(E1) all critical points of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega = \mathcal{L}_\omega + f$ are non-degenerate in the sense of Definition 12.2;

(E2) for any pair of critical points $\mathfrak{a}, \mathfrak{b} \in \mathrm{Crit}(\mathcal{L}_\omega)$, the moduli space $\mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$ is regular in the sense of Definition 13.2. ♢

One may think of $\mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$ as the moduli space of down-ward gradient flowlines in the quotient space $\mathcal{B}_{k-1/2}(\widehat{Y}, \hat{\mathfrak{s}})$. The reference configuration γ_0 determines a homotopy class of paths connecting $[\mathfrak{a}]$ and $[\mathfrak{b}]$, so it is more appropriate to write

$$\mathcal{M}_{[\gamma]}([\mathfrak{a}], [\mathfrak{b}]) := \mathcal{M}_k(\mathfrak{a}, \mathfrak{b}),\ \ [\gamma] \in \pi_1(\mathcal{B}_{k-1/2}(\widehat{Y}, \hat{\mathfrak{s}}), [\mathfrak{a}], [\mathfrak{b}]). \tag{13.4}$$

By Theorem 9.5, this space is independent of the Sobolev completion that we choose, so the subscript k is dropped in our notation.

Remark 13.4. To identify a finite energy solution γ in Theorem 9.5 with an element of $\mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$, we have to know the exponential decay of γ in the time direction using the non-degeneracy of critical points, which is omitted in this paper; cf. Remark 9.4. ♢

Since the Seiberg-Witten equations on $\widehat{Z} = \mathbb{R}_t \times \widehat{Y}$ has an apparent $\mathbb{R}_t$-translation symmetry, $\mathcal{M}_{[\gamma]}([\mathfrak{a}], [\mathfrak{b}])$ is acted on freely by $\mathbb{R}_t$ if the topological energy $\mathcal{E}_{top}$ along the path

$$[\gamma] \in \pi_1(\mathcal{B}_{k-1/2}(\widehat{Y}, \hat{\mathfrak{s}}), [\mathfrak{a}], [\mathfrak{b}])$$

is positive. We form the unparameterized moduli space by taking the quotient space

$$\breve{\mathcal{M}}_{[\gamma]}([\mathfrak{a}], [\mathfrak{b}]) := \mathcal{M}_{[\gamma]}([\mathfrak{a}], [\mathfrak{b}])/\mathbb{R}_t. \tag{13.5}$$

When $\mathfrak{q}$ is admissible, $\breve{\mathcal{M}}_{[\gamma]}([\mathfrak{a}], [\mathfrak{b}])$ is a smooth manifold of dimension $\mathrm{Ind}(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}) - 1$.

13.2. **Sections in the Cokernel.** Our ultimate goal is to show that admissible perturbations, in the sense of Definition 13.3, are generic, cf. Theorem 16.1. To do this, we have to understand sections in the cokernel of $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathcal{F}_{\widehat{Z},\mathfrak{q}})$, when it is not surjective.

Suppose $U \in L^2(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S)$ is L^2-orthogonal to the image of $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}})$ at a solution $[\gamma] \in \mathcal{M}_k(\mathfrak{a}, \mathfrak{b})$, then U solves the equation

$$-\frac{d}{dt}U(t) + \widehat{\mathrm{Hess}}_{\mathfrak{q},\check{\gamma}(t)} U(t) = 0 \text{ by (13.3).} \tag{13.6}$$

By elliptic regularity, U is smooth and $U \in L^2_1$. We write U as

$$U(t) = (\delta c'(t), \delta b'(t), \delta\psi'(t)).$$

The proof of Theorem 16.1 in Section 16 relies on a separating property of the section U:

Lemma 13.5. *Under above assumptions, $\delta c'(t) \equiv 0$. Moreover, if $U \neq 0$ and $\check{\gamma}(t)$ is never reducible on $\{t\} \times Y$, then there exists a time slice $t_0 \in \mathbb{R}$ such that the tangent vector $(\delta b'(t_0), \delta\psi'(t_0))$ at $\check{\gamma}(t_0)$ can be separated by a cylinder function f tame in Y. Here, $Y = \{s \leqslant 0\} \subset \widehat{Y}$ is the truncated 3-manifold.*

Remark 13.6. By the unique continuation property, cf. Theorem 15.3 below, if $\check{\gamma}(t)$ is reducible at some slice $\{t\} \times Y$, then a solution $\gamma \in \mathcal{C}_k(\mathfrak{a}, \mathfrak{b})$ has to reducible globally, which is absurd. So the condition of Lemma 13.5 is fulfilled. ♢

Proof of Lemma 13.5. Consider a smooth function $\xi \in L^2_{k+1}(\widehat{Z}, i\mathbb{R})$ and the section

$$V_\xi = (0, \mathbf{d}_\gamma \xi) \in L^2_k(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S).$$

Since $e^{r\xi} \cdot \gamma$ also solves the equation $\mathfrak{F}_{\widehat{Z},\mathfrak{q}} = 0$ for any $r \in \mathbb{R}$, taking the derivatives yields

$$\mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(\mathbf{d}_\gamma \xi) = 0,$$

so the vector

$$(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}})V_\xi = (\mathbf{d}^*_\gamma \mathbf{d}_\gamma \xi, 0, 0) \in L^2(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S)$$

is L^2-orthogonal to U. Since the composition $\mathbf{d}^*_\gamma \mathbf{d}_\gamma : L^2_2(\widehat{Z}, i\mathbb{R}) \to L^2(\widehat{Z}, i\mathbb{R})$ is an invertible operator and L^2_{k+1} is dense in L^2_2, $\delta c'(t) = 0$. Now (13.6) is reduced to a pair of equations:

$$0 = \mathbf{d}^*_{\check{\gamma}(t)}(\delta b'(t), \delta\psi'(t)), \tag{13.7}$$

$$\frac{d}{dt}(\delta b'(t), \delta\psi'(t)) = \mathcal{D}_{\check{\gamma}(t)} \operatorname{grad} \mathcal{L}_\omega(\delta b'(t), \delta\psi'(t)). \tag{13.8}$$

For the second clause of Lemma 13.5, suppose on the contrary that $U(t)$ can not be separated for any $t \in \mathbb{R}_t$. By Proposition 8.6, we can find a function $\xi(t) \in L^2_1(\widehat{Y}, i\mathbb{R})$ such that

$$(\delta b'(t), \delta\psi'(t)) = \mathbf{d}_{\check{\gamma}(t)} \xi(t) = (-d_{\widehat{Y}}\xi(t), \xi(t)\Psi(t)) \text{ on } \{t\} \times Y$$

for each $t \in \mathbb{R}_t$. If we write $\operatorname{grad} \mathcal{L}_\omega$ as

$$(\operatorname{grad} \mathcal{L}^0_\omega, \operatorname{grad}^1 \mathcal{L}_\omega) \in L^2(\widehat{Y}, iT^*\widehat{Y} \oplus S),$$

then

$$\operatorname{grad} \mathcal{L}_\omega(u \cdot \check{\gamma}) = (\operatorname{grad}^0 \mathcal{L}_\omega(\check{\gamma}), u \cdot \operatorname{grad}^1 \mathcal{L}_\omega(\check{\gamma})).$$

In particular,

$$\mathcal{D}_{\check{\gamma}(t)} \operatorname{grad} \mathcal{L}_\omega(\mathbf{d}_{\check{\gamma}(t)} \xi(t)) = (0, \xi(t) \cdot \operatorname{grad}^1 \mathcal{L}_\omega(\check{\gamma}(t)).$$

Even though $\mathbf{d}_{\check{\gamma}(t)} \xi(t)$ and $(\delta b'(t), \delta\psi'(t))$ only agree over $\{t\} \times Y$, we still have

$$\mathcal{D}_{\check{\gamma}(t)} \operatorname{grad} \mathcal{L}_\omega(\delta b'(t), \delta\psi'(t)) = (0, \xi(t) \cdot \operatorname{grad}^1 \mathcal{L}_\omega(\check{\gamma}(t)) \text{ on } \{t\} \times Y,$$

since the perturbation $\mathfrak{q}$ is supported on Y in the sense of Definition 7.1. The equation (13.8) then implies

$$\frac{d}{dt}\delta b' \equiv 0 \text{ on } \mathbb{R}_t \times Y.$$

As $U \in L^2$, $-d_{\widehat{Y}}\xi(t) = \delta b'(t) \equiv 0$. Now the equation (13.7) yields

$$0 = \Delta_{\widehat{Y}}\xi(t) + |\Psi(t)|^2\xi(t) = |\Psi(t)|^2\xi(t) \text{ on } \{t\} \times Y,$$

As a result, $U \equiv 0$ on $\mathbb{R}_t \times Y$. An elliptic operator of the form (13.6) satisfies the unique continuation property, so $U \equiv 0$ on $\mathbb{R}_t \times \widehat{Y}$. □

13.3. **Spectral Projections.** Having discussed the linearized operator on an infinite cylinder $\mathbb{R}_t \times \widehat{Y}$, we start to look at a finite interval $I = [t_1, t_2] \subset \mathbb{R}_t$ and consider the Atiyah-Patodi-Singer boundary-value problems. As noted in the beginning of Section 13, we have to justify that the proof of gluing theorem in [KM07, Section 18,19] remains valid in our case, in the presence of essential spectra. This subsection is devoted to an abstract formalism, while the application in gauge theory will be explained in Subsection 13.4. However, the results in these subsections will **not** be used elsewhere in this paper.

As we will work in a slightly abstract setting, define

$$E_0 := i\mathbb{R} \oplus iT^*\widehat{Y} \oplus S \to \widehat{Y}$$

Take a reference operator $\mathbb{A}_0$ that acts on sections of E_0, extending to bounded linear operators

$$\mathbb{A}_0 : L^2_j(\widehat{Y}, E_0) \to L^2_{j-1}(\widehat{Y}, E_0).$$

for any $j \geqslant 1$. Moreover, assume that $\mathbb{A}_0$ is a unbounded self-adjoint operators on L^2 and its spectrum is disjoint from the interval $(-\lambda_1/2, \lambda_1/2)$:

$$\sigma(\mathbb{A}_0) \subset (-\infty, -\lambda_1/2] \cap [\lambda_1/2, \infty) \text{ with} \tag{13.9}$$
$$\sigma_{ess}(\mathbb{A}_0) = (-\infty, -\lambda_1] \cap [\lambda_1, \infty),$$

for some $\lambda_1 > 0$ as in Proposition 12.1. One may think of $\mathbb{A}_0$ as a first-order self-adjoint elliptic differential operator plus a compact perturbation. For convenience, suppose the L^2_j-norm on $C^\infty_c(\widehat{Y}, E_0)$ is defined using $\mathbb{A}_0$:

$$\|s\|_{L^2_j(E_0)} = \|(1 + |\mathbb{A}_0|)^j s\|_{L^2(E_0)}, \forall s \in C^\infty(\widehat{Y}, E_0).$$

Let $K : C^\infty_c(\widehat{Y}, E_0) \to C^\infty(\widehat{Y}, E_0)$ be an operator acting on sections of E_0 extending to a compact operator:

$$K : L^2_j(\widehat{Y}, E_0) \to L^2_j(\widehat{Y}, E_0)$$

for any $j \geqslant 0$. Assume that K is self-adjoint on $L^2(\widehat{Y}, E_0)$. When the sum $\mathbb{A} := \mathbb{A}_0 + K$ is invertible, $L^2(\widehat{Y}, E_0)$ is the direct sum of the positive and negative spectral spaces of $\mathbb{A}$:

$$L^2(\widehat{Y}, E_0) = H^+_{\mathbb{A}} \oplus H^-_{\mathbb{A}},$$

and for any $j \geqslant 0$,

$$L^2_j(\widehat{Y}, E_0) = (H^+_{\mathbb{A}} \cap L^2_j) \oplus (H^-_{\mathbb{A}} \cap L^2_j). \tag{13.10}$$

Let $E \to \widehat{Z} := (-\infty, 0] \times \widehat{Y}$ be the pull-back bundle of E_0 over the half cylinder and consider the operator:

$$D_{\mathbb{A}} = \frac{d}{dt} + \mathbb{A} : C^\infty(\widehat{Z}, E) \to C^\infty(\widehat{Z}, E).$$

The next result is a direct consequence of Functional Calculus, cf. [KM07, Theorem 17.1.4].

Proposition 13.7. *Let $\widehat{Z} = (-\infty, 0] \times \widehat{Y}$ be the half cylinder. Suppose the operator*

$$\mathbb{A} = \mathbb{A}_0 + K : L^2_1(\widehat{Y}, E_0) \to L^2(\widehat{Y}, E_0)$$

is invertible, then the operator

$$D_{\mathbb{A}} \oplus \Pi^-_{\mathbb{A}} \circ r : L^2_k(\widehat{Z}, E) \to L^2_{k-1}(\widehat{Z}, E) \oplus (H^-_{\mathbb{A}} \cap L^2_{k-1/2}(\widehat{Y}, E_0))$$

is also invertible for any $k \geqslant 1$, where $r : L^2_k(\widehat{Z}, E) \to L^2_{k-1/2}(\widehat{Y}, E)$ is the restriction map at the boundary $\{0\} \times \widehat{Y}$ and

$$\Pi^-_{\mathbb{A}} : L^2_{k-1/2} \to H^-_{\mathbb{A}} \cap L^2_{k-1/2}(\widehat{Y}, E_0)$$

is the spectral projection. The subspace $H^-_{\mathbb{A}} \cap L^2_{k-1/2}$ is precisely the image of $\ker D_{\mathbb{A}}$ *under r.*

As $\mathbb{A}$ differs from $\mathbb{A}_0$ only by a compact operator, it is expected that $\Pi^-_{\mathbb{A}}$ forms a "compact" family as $\mathbb{A}$ varies. We make this precise in the next proposition.

Proposition 13.8. *Given an invertible operator $\mathbb{A} = \mathbb{A}_0 + K$, the difference of their spectral projections*

$$\Pi^-_{\mathbb{A}} - \Pi^-_{\mathbb{A}_0} : L^2_{k-1/2}(\widehat{Y}, E_0) \to L^2_{k-1/2}(\widehat{Y}, E_0)$$

is compact for any $k \geqslant 1$, i.e. $\Pi_{\mathbb{A}_0}$ and $\Pi_{\mathbb{A}}$ are k-commensurate in the sense of [KM07, Definition 17.2.1].

Proof. We follow the trick from [KM07, Proposition 17.2.4]. It suffices to show for any bounded sequence $\{w_i\} \subset L^2_{k-1/2}$, it image under $\Pi^-_{\mathbb{A}} - \Pi^-_{\mathbb{A}_0}$ contains a converging subsequence. In terms of the decomposition (13.10), we can deal with entries of $\{w_i\}$ separately. By the symmetry of $H^\pm_{\mathbb{A}}$, we focus on the case when $\{w_i\} \subset H^-_{\mathbb{A}} \cap L^2_{k-1/2}$. By Proposition 13.7, there exists sections $\{v_i\} \subset L^2_k(\widehat{Z}, E)$ such that

$$D_{\mathbb{A}} v_i = 0 \text{ and } r(v_i) = w_i.$$

Apply Proposition 13.7 again for $\mathbb{A}_0$ to find solutions $\{u_i\} \subset L^2_k(\widehat{Z}, E)$ with

$$D_{\mathbb{A}_0} u_i = -K(v_i) \text{ and } \Pi^-_{\mathbb{A}_0} \circ r(u_i) = 0.$$

Since $D_{\mathbb{A}_0}(u_i - v_i) = 0$, $r(u_i - v_i) \in H^-_{\mathbb{A}_0}$. So

$$(\Pi^-_{\mathbb{A}} - \Pi^-_{\mathbb{A}_0})(w_i) = (1 - \Pi^-_{\mathbb{A}_0})(w_i) = \Pi^+_{\mathbb{A}_0} \circ r(v_i) = \Pi^+_{\mathbb{A}_0} \circ r(u_i).$$

One may write the last term explicitly in terms of v_i using formulae on [KM07, P.299]:

$$v_i \mapsto \Pi^+_{\mathbb{A}_0} \circ r(u_i) = y_i := \int_{-\infty}^0 e^{t\mathbb{A}_0}(-K(v_i(t)))^+ dt. \tag{13.11}$$

where $(\cdot)^+$ denotes the positive part in $H^+_{\mathbb{A}_0}$. As this point, approximate K by finite rank operators. The operator $v \mapsto y$ defined by the expression (13.11) is also approximated by finite rank operators in the norm topology, so (13.11) is also compact.

Here is the main difference of this proof from that of [KM07, Proposition 17.2.4]: the operator

$$v \mapsto \int_{-\infty}^0 e^{t\mathbb{A}_0}(v(t))^+ dt,\ L^2_k(\widehat{Z}, E) \to L^2_{k-1/2}(\widehat{Y}, E_0),$$

is not compact as $\mathbb{A}_0$ has essential spectrum, so the compactness of $\Pi^-_{\mathbb{A}} - \Pi^-_{\mathbb{A}_0}$ really arises from K. □

With Proposition 13.8 in mind, we are ready to study the boundary value problem on a finite interval.

Proposition 13.9. *Let $I = [t_1, t_2]_t$ be a finite interval and $\widehat{Z} = I \times \widehat{Y}$. Given invertible operators $\mathbb{A}_i = \mathbb{A}_0 + K_i, i = 1, 2$ as compact perturbations of $\mathbb{A}_0$, consider the operator*

$$D = \frac{d}{dt} + \mathbb{A}_0 + K(t) : L^2_k(\widehat{Z}, E) \to L^2_{k-1}(\widehat{Z}, E)$$

on $\widehat{Z}$ and spectral projections

$$\Pi^+_{\mathbb{A}_1} \circ r_1 : L^2_k(\widehat{Z}, E) \to H^+_{\mathbb{A}_1} \cap L^2_{k-1/2}(\{t_1\} \times \widehat{Y}, E_0),$$
$$\Pi^-_{\mathbb{A}_2} \circ r_2 : L^2_k(\widehat{Z}, E) \to H^-_{\mathbb{A}_2} \cap L^2_{k-1/2}(\{t_2\} \times \widehat{Y}, E_0).$$

where $K : I \to \mathrm{Hom}(L^2_j, L^2_j), j \geqslant 0$ is a smooth family of self-adjoint compact operators. Then the operator

$$P := D \oplus (\Pi^+_{\mathbb{A}_1}, \Pi^-_{\mathbb{A}_2}) \circ (r_1, r_2)$$

is Fredholm, whose index is equal to the spectrum flow from $\mathbb{A}_1$ to $\mathbb{A}_2$. In particular, the restriction map on the kernel of D:

$$(\Pi^+_{\mathbb{A}_1}, \Pi^-_{\mathbb{A}_2}) \circ (r_1, r_2) : \ker D \to H^+_{\mathbb{A}_1} \cap L^2_{k-1/2}(\widehat{Y}, E_0) \oplus H^-_{\mathbb{A}_2} \cap L^2_{k-1/2}(\widehat{Y}, E_0)$$

is Fredholm of the same index.

In the sequel, we will abbreviate $H^+_{\mathbb{A}} \cap L^2_{k-1/2}(\widehat{Y}, E_0)$ into $H^+_{\mathbb{A}}$ when the regularity of sections is clear from the context.

Proof. We start with the model case when $K_1 = K_2 = K(t) \equiv 0$. The operator

$$P_0 = D_{\mathbb{A}_0} \oplus (\Pi^+_{\mathbb{A}_0} \circ r_1 \oplus \Pi^-_{\mathbb{A}_0} \circ r_2)$$

is then invertible by direct computation using Functional Calculus. For the general case, note that $D - D_{\mathbb{A}_0}$ is a compact operator. As for the boundary projections, Proposition 13.8 implies that

$$\Pi^+_{\mathbb{A}_0} : H^+_{\mathbb{A}_1} \to H^+_{\mathbb{A}_0},$$
$$\Pi^+_{\mathbb{A}_1} : H^+_{\mathbb{A}_0} \to H^+_{\mathbb{A}_1},$$

are mutual inverses modulo compact operators, which also holds for the negative projections $\{\Pi^-_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_1}\}$. To compute the index, we use the concatenation trick and compare P with the operator on the infinite cylinder:

$$\frac{d}{dt} + \mathbb{A}_0 + K'(t) : L^2_k(\mathbb{R}_t \times \widehat{Y}, E) \to L^2_{k-1}(\mathbb{R}_t \times \widehat{Y}, E).$$

where K' is a smooth path of compact operators connecting K_1 and K_2:

$$K(t) \equiv K_1 \text{ if } t \leqslant t_1;\ K(t) \equiv K_2 \text{ if } t \geqslant t_2.$$

Now apply Proposition 13.1 or Theorem 10.2. If we write $L^2_k(\widehat{Z}, E)$ as a direct sum

$$C \oplus \ker D$$

where C is the L^2_k-orthogonal complement of $\ker D$, then P is cast into a lower triangular metric

$$\begin{pmatrix} D & 0 \\ * & (\Pi^+_{\mathbb{A}_1}, \Pi^-_{\mathbb{A}_2}) \circ (r_1, r_2) \end{pmatrix}. \tag{13.12}$$

As $D|_C$ is a bijection by [KM07, Proposition 17.1.5] and the unique continuation property, the other diagonal entry has to be Fredholm of the same index as that of P. □

Remark 13.10. Here is a major difference of our case from [KM07, Proposition 17.2.5]: the projection map onto the complementary spectral subspaces:

$$(\Pi^-_{\mathbb{A}_1}, \Pi^+_{\mathbb{A}_2}) \circ (r_1, r_2) : \ker D \to H^-_{\mathbb{A}_1} \oplus H^+_{\mathbb{A}_2}$$

is not compact. To see this, consider the model case when $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}_0$ and $K(t) \equiv 0$, so $\ker D$ is parametrized by the image of $(\Pi^+_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_0}) \circ (r_1, r_2)$. Sticking to the positive part, the composition map

$$\begin{aligned} H^+_{\mathbb{A}_0} \cap L^2_{k-1/2}(\{t_1\} \times \widehat{Y}, E_0) &\to H^+_{\mathbb{A}_0} \cap L^2_{k-1/2}(\{t_2\} \times \widehat{Y}, E_0) \\ w &\mapsto v := P^{-1}(0, w, 0) \in \ker D \\ &\mapsto \Pi^+_{\mathbb{A}_0} \circ r_2(v). \end{aligned}$$

is simply $e^{-\mathbb{A}_0(t_2-t_1)}$ acting on $H^+_{\mathbb{A}_1}$ which has essential spectrum $[0, e^{-\lambda_1(t_2-t_1)}]$. As a result, it is never compact. ♢

To circumvent this problem, we have to refine the estimates when the 3-manifold $\widehat{Y}$ is not compact. Recall that a Fredholm operator P is invertible modulo compact operators. A right (left) parametrix Q is a right (left) inverse of P modulo compact operators, i.e.

$$PQ = \mathrm{Id} + \text{ a compact term.}$$

Such a Q is unique up to a compact term and is also a (two-sided) parametrix.

The difference up to a compact term is always insignificant. This motivates the next definition and lemma:

Lemma 13.11. *Let $H_i,\ i=1,2$ be Hilbert spaces. For any operator $Q: H_2 \to H_1$, define its essential norm as*

$$\|Q\|_{ess} := \inf_{K\ compact} \|Q+K\|_{H_2\to H_1}.$$

For any Fredholm operator $P: H_1 \to H_2$ with a parametrix Q, the perturbed operator $P+F$ is Fredholm if $\|FQ\|_{ess} < 1$.

Proof. As $(P+F)Q$ and Q are Fredholm, $P+F$ is Fredholm as well. □

Now let us recast Proposition 13.9 into a more convenient form for applications. Recall that the essential spectrum of $\mathbb{A}_0$ is away from the origin:

$$\sigma_{ess}(\mathbb{A}_0) = (-\infty, \lambda_1] \cup [\lambda_1, \infty),$$

for some $\lambda_1 > 0$.

Proposition 13.12. *Under the assumption of Proposition 13.9, the operator P is Fredholm. The essential norm of its parametrix $\widetilde{Q}$ is bounded by a constant C_1 that depends only on λ_1. The same conclusion applies to the projection map*

$$(\Pi^+_{\mathbb{A}_1}, \Pi^-_{\mathbb{A}_2})\circ(r_1,r_2): \ker D \to H^+_{\mathbb{A}_1} \cap L^2_{k-1/2}(\{t_1\}\times \widehat{Y}, E_0) \oplus H^-_{\mathbb{A}_2} \cap L^2_{k-1/2}(\{t_2\}\times\widehat{Y}\ E_0).$$

and its parametrix Q. Moreover, the essential norm of the complementary projection precomposed with Q:

$$(\Pi^-_{\mathbb{A}_0}, \Pi^+_{\mathbb{A}_0})\circ(r_1,r_2)\circ Q: H^+_{\mathbb{A}_1}\oplus H^-_{\mathbb{A}_2} \xrightarrow{Q} \ker D \to H^-_{\mathbb{A}_0}\oplus H^+_{\mathbb{A}_0}$$

is bounded above by $e^{-\lambda_1|I|}$, where $|I| = |t_2 - t_1|$ is the length of I.

Proof. We divide the proof into four steps:

Step 1. Estimate $\widetilde{Q}$. When $K_1 = K_2 = K(t) \equiv 0$, we obtain the model operator

$$P_0 = D_{\mathbb{A}_0} \oplus (\Pi^+_{\mathbb{A}_0}\circ r_1 \oplus \Pi^-_{\mathbb{A}_0}\circ r_2): L^2_k(\widehat{Z}, E) \to L^2_{k-1}(\widehat{Z},E)\oplus(H^+_{\mathbb{A}_0}\oplus H^-_{\mathbb{A}_0}).$$

Let $\widetilde{Q}_0 = (R, Q_0)$ be the inverse of P_0 with

$$Q_0: H^+_{\mathbb{A}_0}\oplus H^-_{\mathbb{A}_0} \to L^2_k(\widehat{Z}, E),$$
$$R: L^2_{k-1}(\widehat{Z}, E) \to L^2_k(\widehat{Z}, E).$$

The norm $\|\widetilde{Q}_0\|$ is bounded by a constant C_1 independent of the length $|I|$. In the general case, set $\widetilde{Q} := (R, Q_0 \circ (\Pi^+_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_0}))$ with

$$(\Pi^+_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_0}): H^+_{\mathbb{A}_1}\oplus H^-_{\mathbb{A}_2} \to H^+_{\mathbb{A}_0}\oplus H^-_{\mathbb{A}_0}.$$

Then $\|\widetilde{Q}\| \leqslant \|\widetilde{Q}_0\|$, since we have used $\mathbb{A}_0$ to define the L^2_j-norm on $C^\infty_c(\widehat{Y}, E_0)$. By Proposition 13.8, projection maps:

$$\Pi^\pm_{\mathbb{A}_i}: H^\pm_{\mathbb{A}_0} \to H^\pm_{\mathbb{A}_i},\ \Pi^\pm_{\mathbb{A}_0}: H^\pm_{\mathbb{A}_i} \to H^\pm_{\mathbb{A}_0}, i = 1,2$$

are mutual inverses modulo compact operators; so $\widetilde{Q}$ is a parametrix of P.

Step 2. Estimate Q. Using the block form (13.12), we write $\widetilde{Q}$ as a 2 by 2 matrix:

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Take $Q := Q_{22}$ to be the bottom right entry, then

$$Q : H^-_{\mathbb{A}_1} \oplus H^+_{\mathbb{A}_2} \to \ker D$$

is a left parametrix of $(\Pi^+_{\mathbb{A}_1}, \Pi^-_{\mathbb{A}_2}) \circ (r_1, r_2)$ and

$$\|Q\| \leqslant \|\widetilde{Q}\|,$$

since C is L^2_k-orthogonal to $\ker D$ in (13.12).

Step 3. Estimate the complementary projection. It suffices to estimate the norm of

$$M := (\Pi^-_{\mathbb{A}_0}, \Pi^+_{\mathbb{A}_0}) \circ (r_1, r_2) \circ Q.$$

First of all, the estimate holds for the model case when $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}_0$ and $K(t) \equiv 0$, by Remark 13.10. Define

$$M_0 := (\Pi^-_{\mathbb{A}_0}, \Pi^+_{\mathbb{A}_0}) \circ (r_1, r_2) \circ Q_0.$$

Now we allow $K(t) \neq 0$, but $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}_0$. Write Q' for the parametrix constructed in *Step* 2. We have to compare

$$M' := (\Pi^-_{\mathbb{A}_0}, \Pi^+_{\mathbb{A}_0}) \circ (r_1, r_2) \circ Q'.$$

with the model operator M_0, and show the difference $M - M_0$ is compact.

For any $(w_1, w_2) \in H^+_{\mathbb{A}_0} \oplus H^-_{\mathbb{A}_0}$, sections $u := Q'(w)$ and $v := Q_0(w)$ obey the following equations respectively:

$$\begin{cases} D_{\mathbb{A}_0}(u) &= -K(t)u \\ \Pi^+_{\mathbb{A}_0} \circ r_2(u) &= w_1 - k_1(w), \\ \Pi^-_{\mathbb{A}_0} \circ r_2(u) &= w_2 - k_2(w), \end{cases} \qquad \begin{cases} D_{\mathbb{A}_0}(v) &= 0, \\ \Pi^+_{\mathbb{A}_0} \circ r_2(v) &= w_1, \\ \Pi^-_{\mathbb{A}_0} \circ r_2(v) &= w_2, \end{cases}$$

where (k_1, k_2) is a compact operator acting on $H^+_{\mathbb{A}_0} \oplus H^-_{\mathbb{A}_0}$. It follows that

$$w \mapsto (Q' - Q_0)(w) = u - v = P_0^{-1}(-K(t)Q'(w), -k_1(w), -k_2(w))$$

is a compact operator.

Step 4. In the most general case, we allow $K(t) \neq 0$ and $\mathbb{A}_1, \mathbb{A}_2 \neq \mathbb{A}_0$. Recall that $Q = Q' \circ (\Pi^+_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_0})$, so $M = M' \circ (\Pi^+_{\mathbb{A}_0}, \Pi^-_{\mathbb{A}_0})$ and

$$\|M\|_{ess} \leqslant \|M'\|_{ess} = \|M_0\|_{ess} \leqslant e^{-\lambda_1 |I|}. \qquad \square$$

Spectral projections are not the most relevant boundary conditions for the main applications in gauge theory, although they serve important intermediate steps.

Proposition 13.13. *Under the assumption of Proposition* 13.7 *with* $\widehat{Z} = (-\infty, 0] \times \widehat{Y}$, *suppose* Π_1 *is any linear projection on* $L^2_{k-1/2}(\widehat{Y}, E_0)$ *whose kernel is a complement of* $H^-_{\mathbb{A}}$*:*

$$\ker(\Pi_1) \oplus (H^-_{\mathbb{A}} \cap L^2_{k-1/2}(\widehat{Y}, E_0)) = L^2_{k-1/2}(\widehat{Y}, E_0). \tag{13.13}$$

and let H_1^- be the image of Π_1. Then the operator

$$D_{\mathbb{A}} \oplus \Pi_1 \circ r : L^2_k(\widehat{Z}, E) \to L^2_{k-1}(\widehat{Z}, E) \oplus H_1^-$$

is an isomorphism.

Proof. See [KM07, Proposition 17.2.6] or [KM07, P.340-341]. □

Proposition 13.14 (cf. [KM07] Proposition 17.2.6)**.** *Under the assumption of Proposition 13.12 with $\widehat{Z} = I \times \widehat{Y}$ and $I = [t_1, t_2]$, suppose Π_1^+ and Π_2^- are any linear projections on $L^2_{k-1}(\widehat{Y}, E_0)$ whose kernels are complements of $H^+_{\mathbb{A}_1}$ and $H^-_{\mathbb{A}_2}$ respectively, i.e.* (13.13) *holds for $(\Pi_1^+, H^+_{\mathbb{A}_1})$ and $(\Pi_2^-, H^-_{\mathbb{A}_2})$. Let H_1^- and H_2^+ be images of Π_1^+ and Π_2^- respectively. Then there exists a constant $T_0(\Pi_1^+, \Pi_2^-) > 0$ such that the operator*

$$D \oplus (\Pi_1^+, \Pi_2^-) \circ (r_1, r_2) : L^2_k(\widehat{Z}, E) \to L^2_{k-1}(\widehat{Z}, E) \oplus H_1^+ \oplus H_2^-,$$

is Fredholm when $|I| > T_0$.

Proof. There are two ways to proceed. In the first approach, one may use Proposition 13.13 to construct a parametrix of $D \oplus (\Pi_1^+, \Pi_2^-)$; see Proposition 14.1 below. In the second approach, we use the estimate on essential operator norms from Proposition 13.12. It suffices to show the restriction map

$$(\Pi_1^+, \Pi_2^-) \circ (r_1, r_2) : \ker D \to H_1^+ \oplus H_2^-$$

is Fredholm. We focus on H_2^+ and pretend the other boundary does not exist. Write

$$\Pi_2^- = \Pi_2^- \circ \Pi^-_{\mathbb{A}_2} + \Pi_2^- \circ (\Pi^+_{\mathbb{A}_2} - \Pi^+_{\mathbb{A}_0}) + \Pi_2^- \circ \Pi^+_{\mathbb{A}_0}.$$

The middle term is compact. Since $\Pi_2^- : H^-_{\mathbb{A}_2} \to H_2^-$ is an isomorphism of Hilbert spaces, by Proposition 13.12,

$$\Pi_2^- \circ \Pi^-_{\mathbb{A}_2} \circ r_2 : \ker D \to H^-_{\mathbb{A}_2} \xrightarrow{\Pi_2^-} H_2^-$$

is Fredholm with parametrix $Q \circ (\Pi_2^-)^{-1}$. To apply Lemma 13.11, we have to estimate the essential norm of

$$(\Pi_2^- \circ \Pi^+_{\mathbb{A}_0}) \circ (Q \circ (\Pi_2^-)^{-1}) = \Pi_2^- \circ (\Pi^+_{\mathbb{A}_0} \circ Q) \circ (\Pi_2^-)^{-1}$$

which is bounded above by $C(\Pi_2^-) \cdot e^{-\lambda_1 |I|} < 1$ if $|I| \gg 1$. The constant $C(\Pi_2^-)$ depends only on the operator norms of

$$\Pi_2^- : H^-_{\mathbb{A}_2} \to H_2^- \text{ and } (\Pi_2^-)^{-1} : H_2^- \to H^-_{\mathbb{A}_2}.$$ □

13.4. **Applications in Gauge Theory.** Having developed the abstract theory in Subsection 13.3, let us explain now how various operators are defined in gauge theory. For each tame perturbation $\mathfrak{q} \in \mathcal{P}$ and a configuration $\mathfrak{a} \in \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$, consider the extended Hessian

$$\widetilde{\mathbb{A}} := \widehat{\mathrm{Hess}}_{\mathfrak{q},\mathfrak{a}},$$

The reference operator $\mathbb{A}_0$ is taken to be a compact perturbation of $\tilde{\mathbb{A}}$ such that the conditon (13.9) holds.

Recall that the space $L^2_{k-1/2}(\widehat{Y}, E_0)$ admits a decomposition for each $\mathfrak{a} \in \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$:

$$\begin{aligned} L^2_{k-1/2}(\widehat{Y}, E_0) &= L^2_{k-1/2}(\widehat{Y}, i\mathbb{R}) \oplus \mathcal{T}_{k-1/2,\mathfrak{a}}, \\ &= L^2_{k-1/2}(\widehat{Y}, i\mathbb{R}) \oplus \mathcal{J}_{k-1/2,\mathfrak{a}} \oplus \mathcal{K}_{k-1/2,\mathfrak{a}}, \end{aligned}$$

on which $\widehat{\mathrm{Hess}}_{\mathfrak{q},\mathfrak{a}}$ takes a block form:

$$\begin{pmatrix} 0 & \mathbf{d}^*_{\mathfrak{a}} & 0 \\ \mathbf{d}_{\mathfrak{a}} & 0 & 0 \\ 0 & 0 & \mathrm{Hess}_{\mathfrak{q},\mathfrak{a}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & x \\ 0 & x^* & 0 \end{pmatrix}$$

The operators x, y are defined as in (12.1) and they are compact. Denote the first matrix by $\mathbb{A}$ and consider its spectral decomposition:

$$\Pi^{\pm}_{\mathbb{A}} : L^2_{k-1/2}(\widehat{Y}, E_0) \to H^{\pm}_{\mathbb{A}}.$$

As $\mathrm{Hess}_{\mathfrak{q},\mathfrak{a}}$ acts on $\mathcal{K}_{k-1/2,\mathfrak{a}}$, we also have the spectral decomposition of $\mathrm{Hess}_{\mathfrak{q},\mathfrak{a}}$:

$$\mathcal{K}_{k-1/2,\mathfrak{a}} = \mathcal{K}^+_{\mathfrak{a}} \oplus \mathcal{K}^-_{\mathfrak{a}}.$$

Define subspaces:

$$H^{\pm}_{\mathfrak{a}} := L^2_{k-1/2}(\widehat{Y}, i\mathbb{R}) \oplus \{0\} \oplus \mathcal{K}^{\pm}_{\mathfrak{a}} \subset L^2_{k-1/2}(\widehat{Y}, E_0),$$

and the projection maps

$$\Pi^{\pm}_{\mathfrak{a}} : L^2_{k-1/2}(\widehat{Y}, E_0) \to H^{\pm}_{\mathfrak{a}},$$

whose kernels are

$$\{0\} \oplus \mathcal{J}_{k-1/2,\mathfrak{a}} \oplus \mathcal{K}^{\mp}_{\mathfrak{a}}.$$

The pairs $(\Pi^{\pm}_{\mathfrak{a}}, \Pi^{\pm}_{\mathbb{A}})$ that satisfy the condition (13.13), cf. [KM07, P.316]. By Proposition 13.14, the first statement of [KM07, Theorem 17.3.2] continues to hold in our case, and the proof the gluing theorem from [KM07, Section 17-19] remains valid. Proposition 13.12 is the replacement of [KM07, Proposition 17.2.5] in the presence of essential spectra.

Remark 13.15. In practice, we will take $\mathfrak{q}$ to be an admissible perturbation and $\mathfrak{a}$ to be a non-degenerate critical point of $\mathcal{L}_\omega$, in which case $\widetilde{\mathbb{A}} = \mathbb{A}$. Moreover, $\mathcal{L}_\omega$ has only finitely many critical points by the compactness theorem. Since only finitely many configurations are involved in the gluing theorem, we have a uniform upper bound on the constant T_0 in Proposition 13.14, so it does not cause a problem. ♢

Finally, let us compute the spectrum flow from $\mathrm{Hess}_{\mathfrak{q},\mathfrak{a}}$ to $\mathrm{Hess}_{\mathfrak{q},u\cdot\mathfrak{a}}$ as an application of Proposition 13.9.

Lemma 13.16 (cf. [KM07] Lemma 14.4.6)**.** *Consider the cylinder $\widehat{Z} = \mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}})$ and the operator $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}})$ defined in Proposition* 13.1 *with $\mathfrak{b} = u \cdot \mathfrak{a}$ and $u \in \mathcal{G}_{k+1}(\widehat{Y})$, then*

$$\mathrm{Ind}(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}) = ([u] \cup c_1(\widehat{\mathfrak{s}}))[Y, \partial Y] \in 2\mathbb{Z},\ \ \forall \gamma \in \mathcal{C}_k(\mathfrak{a}, u \cdot \mathfrak{a}).$$

Proof. We may use Proposition 13.9 and [KM07, Proposition 14.2.2] to identify this index to the index of an operator on $S^1 \times \widehat{Y}$. The spin bundle $S^+ \to S^1 \times \widehat{Y}$ is constructed as

$$[0,1] \times S/(0,v) \sim (1, u \cdot v).$$

Using the Atiyah-Patodi-Singer index theorem [APS75, Theorem 3.10] instead, the proof of [KM07, Lemma 14.4.6] can now proceed with no difficulty. Indeed, over the cylindrical end of $S^1 \times \widehat{Y}$, the operator is cast into the form (up to a compact term)

$$\partial_t + \sigma(\partial_s + D_\Sigma) = \sigma(\partial_s - \sigma \cdot \partial_t + D_\Sigma) \text{ on } S^1 \times [0, +\infty)_s \times \Sigma.$$

Following the proof of Proposition 11.2, the spectrum of $(-\sigma \cdot \partial_t + D_\Sigma)$ on $S^1 \times \Sigma$ is discrete and symmetric with respect to the origin, so its η-invariant is zero. Moreover, $(-\sigma \cdot \partial_t + D_\Sigma)$ is invertible, so its kernel is trivial. □

14. Linearized Operators on Cobordisms

Having addressed the linearized operators on the product manifold $\mathbb{R}_t \times \widehat{Y}$, in this section, we explore the case for a morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ in the strict cobordism category SCob_s. In this case, we have a relative *spin*^{c} cobordism

$$(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_1, \widehat{\mathfrak{s}}_2).$$

By attaching cylindrical ends, we obtain a complete Riemannian manifold

$$\mathcal{X} := \left((-\infty, -1]_t \times \widehat{Y}_1\right) \cup \widehat{X} \cup \left([1, \infty)_t \times \widehat{Y}_2\right)$$

together with a closed 2-form ω_X on $\mathcal{X}$ defined as in (3.11). There are two main tasks for this section:

- define the perturbation space of the Seiberg-Witten equaions on $\mathcal{X}$. This is crucial for the transversality result in Section 16, cf. Theorem 16.5;
- prove that the linearized operator on $\mathcal{X}$ is Fredholm.

They are addressed in Subsection 14.1 and 14.2 respectively.

14.1. Perturbations.

Given a morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ in the strict *spin*^{c} cobordism category SCob_s, the perturbation $\mathfrak{q}_i \in \mathcal{P}(Y_i)$ encoded in the definition of $(\mathbb{Y}_i, \widehat{\mathfrak{s}}_i)$ is admissible by (P8). Take a critial point

$$\mathfrak{a}_i \in \mathrm{Crit}(\mathcal{L}_{\omega_i, \widehat{Y}_i}) \subset \mathcal{C}_k(\widehat{Y}_i, \widehat{\mathfrak{s}}_i),$$

for each $i = 1, 2$. Pick a smooth configuration γ on $\mathcal{X}$ such that

$$\left\{ \begin{array}{ll} \check{\gamma}(t) & \equiv \mathfrak{a}_1 \text{ if } t < -1/2; \\ \check{\gamma}(t) & \equiv \mathfrak{a}_2 \text{ if } t > 1/2 \\ \gamma(t) & \text{is in the temporal gauge when } |t| > 1/2, \\ \gamma\big|_{\widehat{X}} & \in \mathcal{C}_k(\widehat{X}, \widehat{\mathfrak{s}}). \end{array} \right. \tag{14.1}$$

Now consider the configuration space on $\mathcal{X}$:

$$\mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2) := \{(A, \Phi) = \gamma_0 + (a, \phi) : (a, \phi) \in L^2_k(\mathcal{X}, iT^*\mathcal{X} \oplus S^+)\}.$$

and the gauge group

$$\mathcal{G}_{k+1}(\mathcal{X}) = \{u : \mathcal{X} \to S^1 : u - 1 \in L^2_{k+1}(\mathcal{X}, \mathbb{C})\}.$$

The linearized action of $\mathcal{G}_{k+1}(\mathcal{X})$ at $\gamma = (A, \Phi) \in \mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ is given by:

$$\begin{aligned}\mathbf{d}_\gamma : L^2_{k+1}(\mathcal{X}, i\mathbb{R}) &\to T_\gamma \mathcal{C}(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)\\ f(t) &\mapsto (-df, f\Phi)\end{aligned}$$

whose L^2-formal adjoint is

$$\begin{aligned}\mathbf{d}^*_\gamma : T_\gamma \mathcal{C}(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2) &\to L^2_{k-1}(\mathcal{X}, i\mathbb{R})\\ (\delta a, \delta\phi) &\mapsto -d^* a + i \operatorname{Re}\langle \delta\phi, i\Phi\rangle.\end{aligned}$$

Let us now specify the class of perturbations involved in the Seiberg-Witten equations. Choose a cut-off function $\beta : \mathbb{R}_t \to \mathbb{R}$ with $\beta(t) \equiv 1$ if $|t| > 3$ and $\beta(t) \equiv 0$ if $|t| < 2$. Pick another cut-off function $\beta_0 : \mathbb{R}_t \to \mathbb{R}$ supported on $[1, 2]_t \subset \mathbb{R}_t$, equal to 1 when $t \in [5/4, 7/4]$. Now consider the perturbed Seiberg-Witten equation:

$$\begin{aligned}&\mathfrak{F}_{\mathcal{X},\mathfrak{p}}(\gamma) = 0,\ \gamma \in \mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2),\\ &\mathfrak{F}_{\mathcal{X},\mathfrak{p}}(\gamma) := \mathfrak{F}_{\mathcal{X}}(\gamma) + \beta(t)[\widehat{\mathfrak{q}}_1(\gamma) + \widehat{\mathfrak{q}}_2(\gamma)] + \beta_0(t)(\widehat{\mathfrak{q}}_3(\gamma)) + (\rho_4(\omega_3^+), 0),\end{aligned} \tag{14.2}$$

where $\mathfrak{F}_{\mathcal{X}}$ is the unperturbed Seiberg-Witten map defined by the formula (3.7). Here $\mathfrak{p}$ denotes the quadruple

$$\mathfrak{p} := (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \omega_3) \in \mathcal{P}(Y_1) \times \mathcal{P}(Y_2) \times \mathcal{P}(Y_2) \times \Omega^2_c([1, 2] \times Y_2, i\mathbb{R}).$$

where $\mathfrak{q}_3 \in \mathcal{P}(Y_2)$ is a tame perturbation supported on Y_2 and ω_3 is an imaginary-valued exact 2-form compactly supported on $[1, 2] \times Y_2$. The effect of ω_3 is to deform ω_X into $\omega_X - \omega_3$, so the first equation of (3.7) is changed into

$$\frac{1}{2}\rho_4(F^+_{A^t} - 2\omega^+_X) - (\Phi\Phi^*)_0 = -\rho_4(\omega^+_3),$$

modulo perturbations from $\mathfrak{q}_i$'s. In practice, it suffices to consider ω_3 in the special form:

$$\omega_3 = d_{\mathcal{X}}(\beta_0(t) f_3 dt) = -\beta_0(t) dt \wedge d_{Y_2} f_3. \tag{14.3}$$

for a compactly supported smooth function $f_3 : [1, 2]_t \times Y_2 \to i\mathbb{R}$.

Within the space of all compactly supported smooth functions on $[1, 2]_t \times Y_2$, we choose a countable subset that is dense in C^∞-topology and form a Banach space as in Theorem 8.17:

$$\mathcal{P}_{\text{Form}}.$$

The space $\mathcal{P}_{\text{Form}}$ is dense in $C^\infty_c([1, 2]_t \times Y_2, i\mathbb{R})$, and we define ω_3 by the formula (14.3) with $f_3 \in \mathcal{P}_{\text{Form}}$. In all, the quadruple $\mathfrak{p}$ takes value in a Banach space

$$\mathfrak{p} = (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \omega_3) \in \mathcal{P}(Y_1) \times \mathcal{P}(Y_2) \times \mathcal{P}(Y_2) \times \mathcal{P}_{\text{Form}}.$$

Here $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are encoded in the cylindrical ends of $\mathcal{X}$; only the last two terms

$$(\mathfrak{q}_3, \omega_3)$$

give rise to the actual perturbation in (14.2), allowing us to achieve transversality in Section 16. Note that

$$\beta_0(t)\widehat{\mathfrak{q}}_3(\gamma) \text{ and } (\rho_4(\omega_3^+), 0)$$

are both supported in the compact region $[1,2]_t \times Y_2$. Finally, we form the moduli space $\mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ by taking the quotient space:

$$\mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2) := \{\mathfrak{F}_{\mathcal{X},\mathfrak{p}}(\gamma) = 0 : \gamma \in \mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)\}/\mathcal{G}_{k+1}(\mathcal{X}), \tag{14.4}$$

which is in fact independent of the subscript k, due to the exponential decay of the local energy functional, cf. Theorem 9.3.

14.2. **Linearized Operators.** Similar to the case for $\widehat{Z} = \mathbb{R}_t \times \widehat{Y}$, the linearization of $\mathfrak{F}_{\mathcal{X},\mathfrak{p}}$ together with $\mathbf{d}_\gamma^*$ forms a Fredholm operator. In particular, the cokernel is finite dimensional.

Proposition 14.1. *For any $i = 1, 2$, let $\mathfrak{a}_i$ be a smooth non-generate critical point of $\mathcal{L}_{\omega_i}$ in $\mathcal{C}_k(\widehat{Y}_i, \widehat{\mathfrak{s}}_i)$. Then for any $\gamma \in \mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$, the operator*

$$(\mathbf{d}_\gamma^*, \mathcal{D}_\gamma\mathfrak{F}_{\mathcal{X},\mathfrak{p}}) : L^2_k(\mathcal{X}, iT^*\mathcal{X} \oplus S^+) \to L^2_{k-1}(\mathcal{X}, i\mathbb{R} \oplus i\Lambda^+\mathcal{X} \oplus S^-)$$

is Fredholm.

Definition 14.2. The moduli space $\mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ is called regular, if the operator $(\mathbf{d}_\gamma^*, \mathcal{D}_\gamma\mathfrak{F}_{\mathcal{X},\mathfrak{p}})$ is surjective at any solution $[\gamma] \in \mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$, ◊

Proof of Proposition 14.1. It suffices to deal with the case for the reference configuration $\gamma = \gamma_0$ and when $(\mathfrak{q}_3, \omega_3) = 0$. As $\mathfrak{a}_i$ is non-degenerate, the operator on the infinite cylinder

$$D_i := \frac{d}{dt} + \widehat{\mathrm{Hess}}_{\mathfrak{q}_i,\mathfrak{a}_i} : L^2_k(\mathbb{R}_t \times \widehat{Y}_i, i\mathbb{R} \oplus iT^*\widehat{Y}_i \oplus S) \to L^2_{k-1}(\mathbb{R}_t \times \widehat{Y}_i, i\mathbb{R} \oplus iT^*\widehat{Y}_i \oplus S)$$

is invertible for $i = 1, 2$. Denote the inverse by Q_i. Unlike Theorem 10.2, the cut-off functions involved in the parametrix patching argument are more sophisticated, as we explain now. There are three of them:

$$\beta_1,\ \beta_2 \text{ and } \beta_X \text{ with } \beta_1 + \beta_2 + \beta_X \equiv 1 \text{ and } \beta_X \text{ compactly supported}$$

Over the region $\{s \leqslant 2\} \subset \mathcal{X}$, choose a partition of unity $\{\beta_1', \beta_2', \beta_X\}$ subordinate to the open cover $U_1 \cup U_2 \cup U_X$:

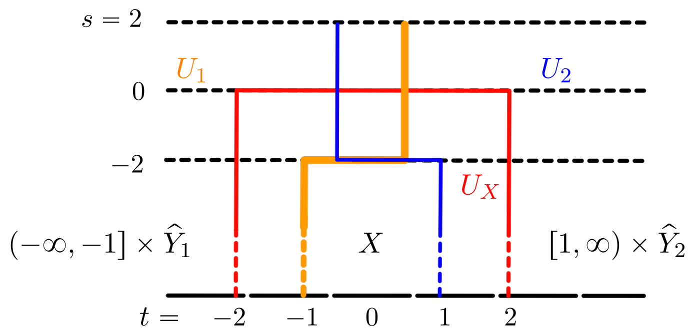


FIGURE 1. An open cover of $\{s \leqslant 2\}$

Over the region $\{s \geqslant 2\}$, $\beta_X \equiv 0$ and $\beta_i(s,t) = \beta_i^T(t), i = 1,2$ where $\{\beta_1^T, \beta_2^T\}$ is a partition of unity on the real line $\mathbb{R}_t$ subordinate to the cover

$$\mathbb{R}_t = (-\infty, T] \cup [-T, \infty),$$

such that $|d\beta_i^T| \leqslant 4/T$. The value of β_i in the transition area $\{1 \leqslant s \leqslant 2\}$ is filled in by interpolation. To be more precise, pick a partition of unity $\{\alpha^L, \alpha^U\}$ on $\mathbb{R}_s$ such that $\alpha^U(s) \equiv 1$ when $s \geqslant 2$ and $\alpha^U(s) \equiv 0$ when $s \leqslant 1$. Set

$$\beta_i = \alpha^L(s)\beta_i' + \alpha^U(s)\beta_i^T(t),\ i = 1,2.$$

Finally, we take

$$Q = \tilde{\beta}_1 Q_1 \beta_1 + \tilde{\beta}_2 Q_2 \beta_2 + \tilde{\beta}_X Q_X \beta_X,$$

with $\tilde{\beta}_i$ constructed in a similar manner. Here we require that $\tilde{\beta}_i \equiv 1$ on supp β_i so that $\tilde{\beta}_i \beta_i = \beta_i$. The same holds for $(\tilde{\beta}_X, \beta_X)$; and also supp $\tilde{\beta}_X$ is compact.

The parametrix Q_X is given by a local patching argument as usual. By taking $T \gg 0$, one verifies that Q is indeed a parametrix for the operator $(\mathbf{d}_\gamma^*, \mathcal{D}_\gamma \mathfrak{F}_{\mathcal{X},\mathfrak{p}})$. □

Part 5. Transversality

The primary goal of this part is to prove the key transversality result: Theorem 16.1, which states that admissible perturbations on $(\widehat{Y}, \widehat{\mathfrak{s}})$, in the sense of Definition 13.3, exist and are in fact generic. Because the perturbation space $\mathcal{P}(Y)$ that we consider are supported on the truncated 3-manifold $Y = \{s \leqslant 0\}$, only a weak separating property is satisfied, cf. Theorem 8.20. As a result, a stronger unique continuation property is required in order to achieve transversality.

Section 15 is devoted to the proof of unique continuation properties, which uses the Carleman estimates from [Kim95]. In Section 16, we prove Theorem 16.1 as well as its analogue for a general morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ in the SCob_s, cf. Theorem 16.5.

15. Unique Continuation

15.1. **Statements.** In this section, we prove the unique continuation properties of the perturbed Seiberg-Witten equations (13.1), which are crucial for the proof of Theorem 16.1. The main results are listed as follows:

- the non-linear version: Theorem 15.1;
- the linearized version: Theorem 15.2; and
- the irreducibility of spinors: Theorem 15.3.

These theorems are summarized in the first subsection, while the rest of section is devoted to their proofs. Let us start with the non-linear version of unique continuation:

Theorem 15.1. *Let $I = (t_1, t_2)_t$ be an open finite interval. Consider a tame perturbation $\mathfrak{q} \in \mathcal{P}$ supported on the truncated 3-manifold $Y = \{s \leqslant 0\} \subset \widehat{Y}$ and the perturbed Seiberg-Witten equations on $\widehat{Z} := I \times \widehat{Y}$:*

$$0 = \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(\gamma) := \mathfrak{F}_{\widehat{Z}}(\gamma) + \widehat{\mathfrak{q}}(\gamma). \tag{15.1}$$

If two solutions γ_1, γ_2 are gauge equivalent on the slice $\{t_0\} \times Y$ at some $t_0 \in I$, i.e there exists a gauge transformation $u \in \mathcal{G}(\widehat{Y})$ such that

$$u(\gamma_1|_{\{t_0\}\times\widehat{Y}}) = \gamma_2|_{\{t_0\}\times\widehat{Y}} \text{ on } Y,$$

then γ_1 and γ_2 are gauge equivalent over the whole manifold $\widehat{Z}$.

The analogous result for closed 3-manifolds is [KM07, Proposition 7.2.1]. The main difference here is that γ_1 and γ_2 are **not** assumed to be gauge equivalent on the whole time slice $\{t_0\} \times \widehat{Y}$; thus, the proof of [KM07, Proposition 7.2.1] does not apply directly here.

Theorem 15.1 will follow from the strong unique continuation of the Seiberg-Witten equations if $\mathfrak{q} = 0$. The problem arises from the tame perturbation $\mathfrak{q}$, which gives rise to non-local operators. We will provide a toy model in the next subsection to clarify this point, cf. Remark 15.5. It is essential here that the region $\{t_0\} \times Y$ over which γ_1 and γ_2 agree contains the support of $\mathfrak{q}$.

Before we proceed any further, let us state the linearized version of Theorem 15.1 and the version that concerns the irreducibility of spinors.

Theorem 15.2 (The Linearized Version)**.** *Let $I = (t_1, t_2)_t \subset \mathbb{R}_t$ be an open interval. Consider a tame perturbation $\mathfrak{q} \in \mathcal{P}$ supported on the truncated 3-manifold $Y = \{s \leqslant 0\} \subset \widehat{Y}$ and a smooth solution γ to the perturbed Seiberg-Witten equation* (15.1) *on the 4-manifold $\widehat{Z} = I \times (\widehat{Y}, \widehat{\mathfrak{s}})$. Suppose a smooth tangent vector at γ*

$$V(t) = (\delta c(t), \delta b(t), \delta\psi(t)) \in L^2_k(\widehat{Z}, iT^*\widehat{Z} \oplus S)$$

lies in the kernel of the linearized Seiberg-Witten map:

$$0 = \mathcal{D}_\gamma \mathfrak{F}_{\widehat{Z},\mathfrak{q}}(V), \tag{15.2}$$

or equivalently, it solves the equation (13.2)*. If V is generated by the linearized gauge action on $\{t_0\} \times Y$ at some $t_0 \in I$, i.e. there exists a smooth function $\xi \in L^2_{k+1/2}(\widehat{Y}, i\mathbb{R})$ such that*

$$(\delta b(0), \delta\psi(0)) = \mathbf{d}_{\check{\gamma}(t)}\, \xi \text{ on } \{t_0\} \times Y.$$

then V is generated by the linearized gauge action on the whole manifold $\widehat{Z}$, i.e. there exists a smooth function $\xi' \in L^2_{k+1}(\widehat{Z}, i\mathbb{R})$ such that

$$V = \mathbf{d}_\gamma\, \xi' \text{ on } \widehat{Z}.$$

Theorem 15.3 (Irreducibiliy of Spinors)**.** *Let $I = (t_1, t_2)_t \subset \mathbb{R}_t$ be an open interval. For any tame perturbation $\mathfrak{q} \in \mathcal{P}$ and a solution $\gamma = (A, \Phi)$ to the perturbed Seiberg-Witten equations* (15.1) *on the 4-manifold $\widehat{Z} = I \times \widehat{Y}$, if the spinor*

$$\Phi \equiv 0 \text{ on } \{t_0\} \times Y,$$

for some $t_0 \in I$, then $\Phi \equiv 0$ on $\widehat{Z}$.

The proofs of Theorem 15.1-15.3 will not be used elsewhere in this paper. They will dominate the rest of the section.

15.2. **A Motivating Problem.** To better explain the ideas and point out the difference from the standard theory [KM07, Section 7], let us first discuss a motivating problem that concerns the $\bar{\partial}$-operator on the complex plane. Let

$$f : \mathbb{C}_z \to \mathbb{C}$$

be a holomorphic function and $z = t + is$ be the complex coordinate of the domain. It is well-known that if f vanishes along the interval $\{0\} \times [0,1]_s$, then $f \equiv 0$ over $\mathbb{C}_z$.

We investigate a class of perturbations of the $\bar{\partial}-$operator. The equation $\bar{\partial} f = 0$ can be formally cast into an evolution equation:

$$\partial_t f = -D(f)$$

where $D(f) = i\partial_s f$ is a self-adjoint operator on $L^2(\mathbb{R}_s, \mathbb{C})$ (although we do not assume $f(t) \in L^2(\mathbb{R}_s, \mathbb{C})$ for any time slice $\{t\} \times \mathbb{R}_s$). Consider a smooth function $K_1 : \mathbb{R}_s \times \mathbb{R}_s \to \mathbb{C}$ with

$$\text{supp } K_1 \subset [0,1]_s \times [0,1]_s$$

and form the convolution operator

$$K : C^\infty(\mathbb{R}_s, \mathbb{C}) \to C^\infty(\mathbb{R}_s, \mathbb{C})$$
$$f \mapsto K(f)(s) = \int_{\mathbb{R}} K_1(s, s')f(s')ds'.$$

Then $D_K := D+K$ is a compact perturbation of D, not necessarily self-adjoint anymore. More generally, let $V : \mathbb{C}_z \to \mathbb{C}$ be any smooth function and consider the equation

$$\partial_t f = -D_K(f) - V \cdot f \text{ on } \mathbb{C} = \mathbb{R}_t \times \mathbb{R}_s. \tag{15.3}$$

The potential V can be viewed as a time-dependent perturbation of D_K.

Proposition 15.4. *Suppose $f \in C^\infty(\mathbb{C}_z, \mathbb{C})$ is a solution to the perturbed $\bar{\partial}$-equation (15.3) and $f(z) = 0$ for any $z \in \{0\} \times [0, 1]_s$, then $f \equiv 0$ on $\mathbb{C}_z$.*

Remark 15.5. If we only assume $f \equiv 0$ on $\{0\} \times [\epsilon, 1]_s$ for some small $\epsilon > 0$, then for some kernel K_1 and potential V, the conclusion fails. Indeed, set $f(t, s) \equiv g(s)$ and $V \equiv 0$. Let g be a cut-off function such that

$$g(s) \equiv 0, \forall s \geqslant \epsilon \text{ and } g(s) \equiv 1, \forall s < \epsilon/2.$$

Then one can find K_1 with $K_1 * g = -D(g) = -i\partial_s g$, so $g \in \ker D_K$. ♢

The problem here is that the convolution operator K is not local: even if a function $g : [0, 1]_s \to \mathbb{C}$ is supported on a small interval $[0, \epsilon] \subset [0, 1]_s$, $K(g) = K_1 * g$ might be non-vanishing on a much larger region. This is the analogue of the tame perturbation $\mathfrak{q}$ in the Seiberg-Witten equations.

The proofs of Theorem 15.1-15.3 are modeled on that of Proposition 15.4, which involves Carleman estimates, as we discuss in the next subsection.

15.3. **Carleman Estimates.** There are two classical ways to prove a strong unique continuation property like Proposition 15.4. The first follows Agmon and Nirenberg [AN67] and relies on a differential inequality. This is the approach adopted in the book [KM07, Section 7]. In this paper, we follow the second strategy and base our works on Carleman estimates [Car39]. The primary result that we consult is [Kim95, Theorem 1].

Let us first state a result in an abstract Hilbert space.

Proposition 15.6. *Let H be a Hilbert space and $L_i : H \to H$, $i = 1, 2$ be (unbounded) self-adjoint operators on H satisfying the relation*

$$(L_1 + rL_2 + \alpha)^2 - rL_2 \geqslant 0 \tag{15.4}$$

for any $r > 0$ and $\alpha > \alpha_0(H, L_1, L_2)$; or equivalently,

$$\|(L_1 + rL_2 + \alpha)v\|_H^2 - \mathrm{Re}\langle v, (rL_2)v\rangle \geqslant 0 \ \forall v \in D(L_1) \cap D(L_2). \tag{15.5}$$

Here, $\alpha_0 > 0$ is a fixed large number depending only on H, L_1 and L_2.

Suppose $w : [0, r_0]_r \to D(L_1) \cap D(L_2)$ is a smooth function such that

- *for a constant $C_0 > 0$, the following estimate holds for any $r \in (0, r_0]$:*

$$\|(\partial_r + \frac{1}{r} \cdot L_1 + L_2)w(r)\|_H \leqslant C_0\|w(r)\|_H; \tag{15.6}$$

- *$w(r)$ vanishes at the origin to the infinite order, i.e $(\partial_r^n w)(0) = 0$ for any $n \geqslant 0$. In practice, we will only need the property that*

$$\|w(r)\|_H, \|\partial_r w(r)\|_H = \mathcal{O}(r^n) \text{ as } r \to 0, \tag{15.7}$$

for any $n \geqslant 1$.

Then $w \equiv 0$.

With loss of generality, we assume $r_0 = 1$ and let $x := \ln r \in (-\infty, 0]$. Then the inequality (15.6) becomes

$$\begin{aligned} g(x) &:= (\partial_x + L_1 + e^x L_2) w(x), \\ \|g(x)\|_H &\leqslant C_0 e^x \|w(x)\|_H, \ \forall x \in (-\infty, 0]. \end{aligned} \tag{15.8}$$

The key ingredient is the Carleman estimate. We follow the idea from [AB80]. For any $\epsilon \in (0,1)$, consider the weight function $\varphi : (-\infty, 0] \to \mathbb{R}_+$ implicitly determined by the relation $-\varphi(x) + \exp(-\epsilon\varphi(x)) = x$, so $\varphi(x) \sim -x$ and

$$\partial_x \varphi(x) = -\frac{1}{1 + \epsilon e^{-\epsilon\varphi(x)}} \in (-1, -\frac{1}{2}), \tag{15.9}$$

$$\partial_x^2 \varphi(x) = \frac{\epsilon^2 e^{-\epsilon\varphi(x)}}{(1 + \epsilon e^{-\epsilon\varphi(x)})^3} \geqslant C_1 \epsilon^2 \cdot e^{2\epsilon x}, \tag{15.10}$$

for a constant $C_1 > 0$. In what follows, we will always treat $\epsilon \in (0,1)$ as a fixed constant.

Proposition 15.7 (Carleman Estimates, [Kim95] Theorem 1)**.** *Under the assumptions of Proposition* 15.6*, for any $\epsilon \in (0,1)$, there is a constant $C(\epsilon) > 0$ such that for any $\tau > 2\alpha_0$ and $u \in C_c^\infty((-\infty, 0), D(L_1) \cap D(L_2))$, we have*

$$\tau \int_{(-\infty,0)} \|e^{\tau\varphi(x) + \epsilon x} u(x)\|_H^2 dx \leqslant C(\epsilon) \int_{(-\infty,0)} \|e^{\tau\varphi(x)} (\partial_x + L_1 + e^x L_2) u(x)\|_H^2 dx.$$

This estimate is uniform in τ.

Proof of Proposition 15.6. Fix some $x_0 < 0$. To apply Carleman estimates, choose a cut-off function $\chi : (-\infty, 0]_x \to [0,1]$ such that $\chi(x) \equiv 1$ when $x < x_0$ and $\chi(0) = 0$. Set $u(x) = \chi(x) w(x)$. The function $u(x)$ is not compactly supported on $(-\infty, 0)$, but its decay is faster than any exponential function as $x \to -\infty$, by (15.7). In this case, Proposition 15.7 still applies, cf. Remark 15.8; so

$$\begin{aligned} \frac{\tau}{2C(\epsilon)} \int_{(-\infty,x_0]} \|e^{\tau\varphi(x)+\epsilon x} w(x)\|_H^2 dx &\leqslant \frac{\tau}{2C(\epsilon)} \int_{(-\infty,0]} \|e^{\tau\varphi(x)+\epsilon x} u(x)\|_H^2 dx, \\ &\leqslant \frac{1}{2} \int_{(-\infty,0]} \|e^{\tau\varphi(x)} (\partial_x + L_1 + e^x L_2) u(x)\|_H^2 dx, \\ &\leqslant \int_{(-\infty,0]} \|e^{\tau\varphi(x)} g(x)\|_H^2 dx + \int_{[x_0,0]} \|e^{\tau\varphi(x)} [\partial_x, \chi(x)] w(x)\|_H^2 dx, \\ \text{(by (15.8))} &\leqslant C_0 \int_{(-\infty,0]} \|e^{\tau\varphi(x)+x} w(x)\|_H^2 dx + C_2 e^{\tau\varphi(x_0)} \int_{[x_0,0]} \|w\|_H^2 dx, \end{aligned}$$

where $C_2 = \|\partial_x \chi\|_\infty^2$. The upshot is that this inequality holds for any $\tau \gg 1$, so when $\tau > 4C_0 C(\epsilon)$, we use the rearrangement argument to derive that

$$\frac{\tau}{2}\int_{(-\infty,x_0]} \|e^{\tau\varphi(x)+\epsilon x} w(x)\|_H^2 dx \leqslant 2C(\epsilon)C_2 e^{\tau\varphi(x_0)} \int_{[x_0,0]} \|w\|_H^2 dx.$$

Let $\tau \to \infty$. We conclude that $w(x) \equiv 0$ when $x < x_0$. Since $x_0 < 0$ is arbitrary, $w \equiv 0$ on $(-\infty, 0]$. □

To complete the proof of Proposition 15.6, it remains to prove Carleman estimates.

Proof of Proposition 15.7. It is essentially the same argument as [Kim95, Theorem 1]. We record the proof here because a slight modification will be made in our actual applications. Set $v(x) := e^{\tau\varphi(x)} u(x)$, then

$$e^{\tau\varphi(x)}(\partial_x + L_1 + e^x L_2)e^{-\tau\varphi(x)} v(x) = (\partial_x + L_1 + e^x L_2 + \tau(-\partial_x\varphi(x)))v(x).$$

Define $L(x) := L_1 + e^x L_2 + \tau(-\partial_x\varphi(x))$ and compute

$$\int_{(-\infty,0]} \|(\partial_x + L(x))v(x)\|_H^2 dx = \int_{(-\infty,0]} \|\partial_x v(x)\|_H^2 + \|L(x)v(x)\|_H^2 + \int_{(-\infty,0]} 2\,\mathrm{Re}\langle \partial_x v(x), L(x)v(x)\rangle_H dx.$$

Using the fact that $L(x) : H \to H$ is a self-adjoint operator, we integrate by parts:

(15.11)
$$\begin{aligned}\int_{(-\infty,0]} 2\,\mathrm{Re}\langle \partial_x v(x), L(x)v(x)\rangle_H &= -\int_{(-\infty,0]} \mathrm{Re}\langle v(x), (\partial_x L(x))v(x)\rangle \\ &= \int_{(-\infty,0]} \mathrm{Re}\langle v(x), e^x(-L_2)v(x)\rangle + \tau\int_{(-\infty,0]} \langle v(x), (\partial_x^2\varphi(x))v(x)\rangle \\ \text{(by (15.10))} &\geqslant \int_{(-\infty,0]} \mathrm{Re}\langle v(x), e^x(-L_2)v(x)\rangle + C_1\epsilon^2\tau \int_{(-\infty,0]} \|e^{\epsilon x}v(x)\|_H^2.\end{aligned}$$

Set $\alpha = \tau(-\partial_x\varphi)$. If $\tau > 2\alpha_0$, then by (15.9), $\alpha = \tau(-\partial_x\varphi) > \tau/2 > \alpha_0$. Now we use the relation (15.4) to conclude that

$$\int_{(-\infty,0]} \|(\partial_x + L(x))v(x)\|_H^2 dx \geqslant C_1\epsilon^2\tau \int_{(-\infty,0]} \|e^{\epsilon x}v(x)\|_H^2$$

for any $\tau > 2\alpha_0$ and $\epsilon \in (0,1)$. □

Remark 15.8. When $u(x) : (-\infty, 0) \to D(L_1) \cap D(L_2)$ is not compactly supported and yet $u(0) = 0$, we have to verify that the boundary term in (15.11) vanishes:

$$\lim_{x\to-\infty} \mathrm{Re}\langle v(x), L(x)v(x)\rangle.$$

Then one may assume that $\|u(x)\|_H$, $\|\partial_x u(x)\|_H$ and $\|(L_1 + e^x L_2)u(x)\|_H$ decay faster than any exponential functions as $x \to -\infty$. In Proposition 15.6, this is guaranteed by (15.8) and (15.7). ♢

15.4. **Applications.** In this subsection, we give a few examples of (H, L_1, L_2) for which the assumption (15.5) is fulfilled and derive Proposition 15.4 from the abstract Proposition 15.6. We will work out the Seiberg-Witten equations in the next subsection.

Lemma 15.9. *If self-adjoint operators $L_1, L_2 : H \to H$ anti-commute, i.e.*

$$\{L_1, L_2\} := L_1L_2 + L_2L_1 = 0,$$

then the condition (15.5) *holds.*

Proof. We rewrite the left hand side of (15.5) as

$$\|(L_1 + (1 - \frac{1}{2\alpha})rL_2 + \alpha)v\|_H^2 + (1 - (1 - \frac{1}{2\alpha})^2)\|(rL_2)v\|_H^2 + \frac{\mathrm{Re}\langle L_1v, (rL_2)v\rangle}{\alpha} \geqslant 0 \text{ if } \alpha > \frac{1}{4}.$$

The last term vanishes because $\{L_1, L_2\} = 0$. □

Example 15.10. The first example is the Dirac operator on $\mathbb{C}_z \times \Sigma$ where $\Sigma = \partial Y$ is a union of 2-tori endowed with a flat metric. We choose a *spin*^{c} connection A on $\mathbb{C}_z$ such that

$$A = \frac{d}{dt} + \frac{d}{ds} + \check{B}$$

for a fixed *spin*^{c} connection $\check{B}$ on the surface Σ.

Using the polar coordinate (r, θ) on the complex plane, the Dirac operator D_A^+ can be written as

$$D_A^+ = \rho_4(dr)(\partial_r + \rho_3(rd\theta) \cdot (\frac{1}{r}\partial_\theta + D^\Sigma_{\check{B}}))$$

where $D^\Sigma_{\check{B}}$ is the Dirac operator associated to $\check{B}$ on the surface. Unlike $\rho_4(rd\theta)$,

$$\rho_3(rd\theta) = \rho_4(dr)^{-1} \cdot \rho_4(rd\theta) = -\rho_4(dr \wedge rd\theta)$$

is a constant bundle map. Proposition 15.6 applies to the operator $\rho_4^{-1}(dr) \cdot D_A^+$ with

$$L_1^{\mathbb{D}} = \rho_3(rd\theta) \cdot \partial_\theta,\ L_2^{\mathbb{D}} = \rho_3(rd\theta)D^\Sigma_{\check{B}}.$$

and $H = L^2(S^1 \times \Sigma, S^+)$. Indeed, by Lemma 15.9, $\{L_1^{\mathbb{D}}, L_2^{\mathbb{D}}\} = 0$. ♢

Example 15.11. The second example concerns the self-dual operator

$$\Omega^1(X, i\mathbb{R}) \to \Omega^+(X, i\mathbb{R}),$$
$$b \mapsto d^+b,$$

on the 4-manifold $X = \mathbb{C}_z \times \Sigma$. Using the polar coordinate at the origin $0 \in \mathbb{C}_z$, we regard b as an 1-form on

$$X' = [0, r_0)_r \times S^1 \times \Sigma.$$

Suppose b does not contain the dr-component and write

$$b(r) = b_1(r)(rd\theta) + b_2(r)$$

with $b_1(r) \in H_1 := L^2(S^1 \times \Sigma, \mathbb{R})$ and $b_2(r) \in H_2 := L^2(S^1 \times \Sigma, T^*\Sigma)$. As the metric on X' is given by

$$dr^2 + (rd\theta)^2 + g_\Sigma,$$

the equation $d^+b = 0$ is equivalent to that

$$\partial_r \begin{pmatrix} b_1(r) \\ b_2(r) \end{pmatrix} + \left[\frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & *_\Sigma \partial_\theta \end{pmatrix} + \begin{pmatrix} 0 & *_\Sigma d_\Sigma \\ - *_\Sigma d_\Sigma & 0 \end{pmatrix} \right] \begin{pmatrix} b_1(r) \\ b_2(r) \end{pmatrix} = 0.$$

To apply Proposition 15.6, set $H = H_1 \oplus H_2$ and

$$L_1^{\mathbb{S}} = \begin{pmatrix} 1 & 0 \\ 0 & L_3 \end{pmatrix},\ L_2^{\mathbb{S}} = \begin{pmatrix} 0 & L_4 \\ L_4^* & 0 \end{pmatrix},$$

with $L_3 := *_\Sigma \partial_\theta : H_2 \to H_2$ and $L_4 := *_\Sigma d_\Sigma : H_2 \to H_1$. To verify the condition (15.5), we calculate for each $v = (b_1, b_2) \in H$ that

$$\begin{aligned}\|(L_1^{\mathbb{S}} + rL_2^{\mathbb{S}} + \alpha)v\|_H^2 - \langle v, rL_2^{\mathbb{S}} v\rangle = \|\alpha b_1 + rL_4 b_2\|_{H_1}^2 + \|rL_4^* b_1 + (L_3+\alpha) b_2\|_{H_2}^2 \\ + (2\alpha+1)\|b_1\|_{H_1}^2 \geqslant 0 \text{ if } \alpha > -\frac{1}{2}.\end{aligned}$$

In this case, Lemma 15.9 is not applicable because the anti-commutator $\{L_1^{\mathbb{S}}, L_2^{\mathbb{S}}\} \neq 0$. ♢

In the proof of Proposition 15.4 below, we will work with operator L_1, L_2 that are not self-adjoint on H. Nevertheless, the abstract Proposition 15.6 still applies, since we can verify the first step of (15.11) directly: this is the only place the self-adjointness was used.

Proof of Proposition 15.4. Let $I = [0,1]_s$. For any $r \geqslant 0$, consider the contour $\Gamma_r = \Gamma_r^{(1)} + \Gamma_r^{(2)} + \Gamma_r^{(3)} + \Gamma_r^{(4)}$ with

$$\Gamma_r^{(1)} = \{r\} \times I, \qquad \Gamma_r^{(2)} = \{i + re^{i\theta} : 0 \leqslant \theta \leqslant \pi\},$$
$$\Gamma_r^{(3)} = \{-r\} \times I, \qquad \Gamma_r^{(4)} = \{re^{i\theta} : \pi \leqslant \theta \leqslant 2\pi\},$$

and define

$$v_1(r) = f|_{\Gamma_r^{(1)} \coprod \Gamma_r^{(3)}} \in H_1 := L^2(I \coprod (-I), \mathbb{C}),$$
$$v_2(r) = f|_{\Gamma_r^{(2)} \coprod \Gamma_r^{(4)}} \in H_2 := L^2([0,\pi]_\theta \coprod [\pi, 2\pi]_\theta, \mathbb{C}).$$

where $(-I)$ stands for the orientation reversal of I. Finally, set

$$w(r) = (w_1(r), w_2(r)) := (v_1(r), \sqrt{r} v_2(r)) \in H := H_1 \oplus H_2.$$

Our assumptions imply that the function $w : [0,1) \to H$ vanishes to the infinite order at the origin. To apply Proposition 15.6, we look for the differential equation that governs $w(r)$. As the function f solves the perturbed $\bar{\partial}$-equation, we have

$$\partial_r w(r) + (\frac{1}{r} L_1 + L_2) w(r) = h(r) \tag{15.12}$$

with

$$L_1 = (0, i\partial_\theta - \frac{1}{2}) \text{ and } L_2 = (i\partial_s, 0) \text{ on } H = H_1 \oplus H_2.$$

The error term $h(r)$ in (15.12) is determined by the convolution operator K and the smooth potential V, so the assumption (15.6) is satisfied in our case.

Neither L_1 nor L_2 is a self-adjoint operator on H, but we still have

$$\mathrm{Re}\langle (\frac{1}{r} L_1 + L_2) w(r), \partial_r w(r)\rangle = \mathrm{Re}\langle w(r), (\frac{1}{r} L_1 + L_2)\partial_r w(r)\rangle \tag{15.13}$$

which justifies the equality (15.11) in the proof of Proposition 15.7. Indeed,

$$\langle(\frac{1}{r}L_1 + L_2)w(r), \partial_r w(r)\rangle - \langle w(r), (\frac{1}{r}L_1 + L_2)\partial_r w(r)\rangle = \frac{i}{2r}\int_{[0,\pi]\coprod[\pi,2\pi]} \partial_\theta |v_2(r,\theta)|^2 d\theta$$

is purely imaginary. As the relation

$$2\operatorname{Re}\langle L_1 v, L_2 v\rangle = 0, \forall v \in D(L_1) \cap D(L_2)$$

still holds in our case, the proof of Lemma 15.9 remains valid. Now we use Lemma 15.9 and Proposition 15.6 to complete the proof. □

15.5. **The Seiberg-Witten Equations.** Having discussed some toy problems, we are now ready to prove the strong unique continuation property for the perturbed Seiberg-Witten equations, by combining Example 15.10 and 15.11.

Proof of Theorem 15.1. With loss of generality, assume $I = [-1,1]$ and $t_0 = 0$. It suffices to show that γ_1 and γ_2 are gauge equivalent in an open neighborhood of $\{0\} \times Y$, then one may use induction to extend this neighborhood to the whole space $\widehat{Z} = I \times \widehat{Y}$.

To imitate the proof of Proposition 15.4, consider the closed 3-manifold $\mathcal{Y}_r = \mathcal{Y}_r^{(1)} \cup \mathcal{Y}_r^{(2)} \cup \mathcal{Y}_r^{(3)}$ where

$$\mathcal{Y}_r^{(1)} := \{r\} \times Y, \qquad \mathcal{Y}_r^{(2)} = \{re^{i\theta} : 0 \leqslant \theta \leqslant \pi\} \times \Sigma,$$
$$\mathcal{Y}_r^{(3)} := (-\{-r\} \times Y), \qquad \forall r \in [0,1].$$

Here $\mathcal{Y}_r^{(3)}$ is the orientation reversal of $\{-r\}\times Y$. Let B_0 be the reference $spin^c$ connection on $\widehat{Y}$, so B_0 agrees with the $\mathbb{R}_s$-invariant connection

$$\frac{d}{ds} + \check{B}$$

on the cylindrical end $[-1,\infty)_s \times \Sigma$. Set $\gamma_0 = (B_0, 0)$.

Extend the gauge transformation u constantly in the time direction and replace γ_1 by $u(\gamma_1)$. Construct gauge transformations u_i, $i = 1,2$ such that $u_i \equiv \mathrm{Id}$ on $\{0\} \times \widehat{Y}$ and $\gamma_i' := u_i(\gamma_i)$ is in the temporal gauge (the dt-component vanishes). Consider the difference

$$\delta_i(t) := \gamma_i'|_{\{t\}\times\widehat{Y}} - \gamma_0 \in C^\infty(\widehat{Y}, iT^*\widehat{Y} \oplus S)$$

Formally, δ_i is subject to an evolution equation:

$$\partial_t \delta_i(t) + L_2^Y \delta_i(t) + \delta_i(t)\#\delta_i(t) + \mathfrak{q}(\delta_i(t) + \gamma_0) = c.$$

where $\#$ is a symmetric bilinear form that involves only point-wise multiplications. Here c is a constant error term determined by γ_0 and

$$L_2^Y = \begin{pmatrix} *_3 d_{\widehat{Y}} & 0 \\ 0 & D_{B_0} \end{pmatrix}.$$

Now take the difference $\delta(t) := \delta_2(t) - \delta_1(t)$. Over the space $[-1,1]_t \times Y$, we have

$$\partial_t \delta(t) + L_2^Y(\delta(t)) = h_1(t) \in C^\infty(Y, iT^*Y \oplus S) \tag{15.14}$$

and $\|h_1(t)\|_{L^2(Y)} \leqslant C\|\delta(t)\|_{L^2(Y)}$ for a uniform constant $C > 0$. Moreover,

$$\partial_t^n \delta(0) \equiv 0 \text{ on } Y \text{ for any } n \geqslant 0.$$

When $n = 0$, this follows from the assumption that $\gamma_1 = \gamma_2$ on $\{0\} \times Y$. When $n \geqslant 1$, this is a consequence of the equation (15.14) and its higher time derivatives. As a result, all derivatives of δ vanish on $\{0\} \times Y$.

Set $H_1 = L^2(Y, iT^*Y \oplus S)$ and define

$$v_1(r) = (\delta(r)|_Y, \delta(-r)|_Y) \in H_1 \oplus H_1.$$

Then $\partial_r^n v_1(0) = 0$ for any $n \geqslant 0$.

To deal with the middle part $\mathcal{Y}_r^{(2)}$, consider the polar coordinate at $0 \in \mathbb{C}_z$ and restrict δ to a section of

$$iT^*X' \oplus S \to X' := [0,1]_r \times [0,\pi]_\theta \times \Sigma \subset \mathbb{R}_t \times \{s \geqslant 0\} \times \Sigma.$$

The section δ is not necessarily in the radial temporal gauge: the dr-component of δ only vanishes when $\theta = 0, \pi$. One has to construct gauge transformations $u_i' : X' \to S^1$ on X' such that $u_i'|_{\{0\}\times[0,\pi]_\theta\times\Sigma} \equiv \mathrm{Id}$ and $u_i'(\gamma_i')$ is the radial temporal gauge. Then we define

$$v_2(r) = u_2'(\gamma_2')(r) - u_1'(\gamma_1')(r) \in H_2 := L^2([0,\pi]_\theta \times \Sigma, i\mathbb{R} \oplus iT^*\Sigma \oplus S).$$

Then the path $v_2(r)$ is subject to the equation

$$\partial_r v_2(r) + \left(\frac{1}{r}\begin{pmatrix} L_1^{\mathbb{S}} & 0 \\ 0 & L_1^{\mathbb{D}} \end{pmatrix} + \begin{pmatrix} L_2^{\mathbb{S}} & 0 \\ 0 & L_2^{\mathbb{D}} \end{pmatrix}\right) v_2(r) = h_2(r) \in H_2.$$

and $\|h_2(r)\|_{H_2} \leqslant C\|v_2(r)\|_{H_2}$ for a constant $C > 0$. The Seiberg-Witten equations are not perturbed on X', so the error term $h_2(r)$ involves only point-wise multiplications with $v_2(r)$. Operators $L_i^{\mathbb{S}}$ and $L_i^{\mathbb{D}}, i = 1, 2$ are defined as in Example 15.10 and 15.11.

As all derivatives of δ vanish on $(0,0) \times \Sigma$, $\partial_r^n v_2(0) = 0$ for any $n \geqslant 0$.

Finally, let $H = (H_1 \oplus H_1) \oplus H_2$ and define

$$w(r) = (w_1(r), w_2(r)) := (v_1(r), \sqrt{r} v_2(r)) \in H.$$

Now the path $w : [0,1)_r \to H$ is subject to the equation

$$\partial_r w(r) + (\frac{1}{r} L_1 + L_2) w(r) = (h_1(r), -h_1(-r), \sqrt{r} h_2(r)). \tag{15.15}$$

with

$$L_1 = (0, 0, \begin{pmatrix} L_1^{\mathbb{S}} & 0 \\ 0 & L_1^{\mathbb{D}} \end{pmatrix} - \frac{1}{2}), L_2 = (L_2^Y, -L_2^Y, \begin{pmatrix} L_2^{\mathbb{S}} & 0 \\ 0 & L_2^{\mathbb{D}} \end{pmatrix}).$$

To apply Proposition 15.6, we have to verify:

- the positivity condition (15.5);
- the symmetry condition (15.13); note that neither L_1 nor L_2 is self-adjoint.

At this point, we have reduced the problem to some formal properties of L_1, L_2 and $w(r)$. We will treat the form component and the spinor component of (15.15) separately. The verification of (15.5) and (15.13) will dominate the rest of the proof.

Step 1. The Form Component and the Self-Dual operators. In this case, the positivity condition (15.5) follows from the same argument as in Example 15.11 and Proposition

15.4. It can be checked separately on each of $\mathcal{Y}_r^{(i)},\ 1 \leqslant i \leqslant 3$. As for (15.13), we focus on the common boundary of $\mathcal{Y}_r^{(1)}$ and $\mathcal{Y}_r^{(2)}$. Suppose the form components of $v_1(r)$ and $v_2(r)$ are given respectively by

$$\begin{aligned} v_1|_{\{r\}\times(-1,0]_s\times\Sigma} &\rightsquigarrow a_1 ds + a_2, & a_1(r) &\in C^\infty((-1,0]_s \times \Sigma, i\mathbb{R}),\\ && a_2(r) &\in C^\infty((-1,0]_s \times \Sigma, iT^*\Sigma),\\ v_2 &\rightsquigarrow b_1(rd\theta) + b_2, & b_1(r) &\in C^\infty([0,\pi]_\theta \times \Sigma, i\mathbb{R}),\\ && b_2(r) &\in C^\infty([0,\pi]_\theta \times \Sigma, iT^*\Sigma). \end{aligned}$$

Near the boundary of $\mathcal{Y}_r^{(1)}$, we have

$$(*_3 d_Y)\begin{pmatrix} a_1(r) \\ a_2(r)\end{pmatrix} = \begin{pmatrix} 0 & *_\Sigma d_\Sigma \\ -*_\Sigma d_\Sigma & *_\Sigma \partial_s \end{pmatrix}\begin{pmatrix} a_1(r) \\ a_2(r)\end{pmatrix}.$$

Then we calculate (the operator $L_2^{\mathbb{S}}$ is ignored here as it is always self-adjoint):

$$\langle (*_3 d_Y) v_1, (\partial_r v_1)\rangle_{\{r\}\times Y} - \langle v_1, (*_3 d_Y)(\partial_r v_1)\rangle_{\{r\}\times Y} = \langle *_\Sigma a_2(r,0), (\partial_r a_2)(r,0)\rangle_{(r,0)\times\Sigma}.$$

$$\begin{aligned} \langle \frac{1}{r} L_1^{\mathbb{S}} w_2, (\partial_r w_2)\rangle_{H_2} - \langle w_2, \frac{1}{r} L_1^{\mathbb{S}}(\partial_r w_2)\rangle_{H_2} &= -\langle *_\Sigma b_2(r,0), (\partial_r b_2)(r,0)\rangle_{(r,0)\times\Sigma} \\ &\quad + \frac{1}{r}\underbrace{\int_{[0,\pi]_\theta} \partial_\theta \langle *_\Sigma b_2, b_2\rangle_\Sigma}_{=0} + \cdots . \end{aligned}$$

It remains to verify that $a_2(r,0) = b_2(r,0)$ on $\mathcal{Y}_r^{(1)} \cap \mathcal{Y}_r^{(2)}$. Suppose the restriction of the form component of δ on $X' = [0,1]_r \times [0,\pi]_\theta \times \Sigma$ is $f dr + c_1(rd\theta) + c_2$ with

$$f(r), c_1(r) \in C^\infty([0,\pi]_\theta \times \Sigma, i\mathbb{R}),\ c_2(r) \in C^\infty([0,\pi]_\theta \times \Sigma, iT^*\Sigma).$$

It is clear that a_2 and c_2 agree along the common boundary of $\mathcal{Y}_r^{(1)}$ and $\mathcal{Y}_r^{(2)}$. Moreover,

$$f(r,\theta) \equiv 0 \text{ if } \theta = 0 \text{ or } \pi.$$

To put δ into radial temporal gauge, we applied further gauge transformations, so (b_1, b_2) is related to δ by the formulae:

$$b_1(r) = c_1(r) - \frac{1}{r}\int_0^r (\partial_\theta f)(r')dr', \qquad b_2(r) = c_2(r) - \int_0^r (d_\Sigma f)(r')dr'.$$

As a result, $a_2(r,s)|_{s=0} = b_2(r,\theta)|_{\theta=0}$. This equality does not a priori hold for a_1 and b_1, but it is not needed in the proof.

Step 2. The Spinor Component and the Dirac operators. The proof of (15.13) proceeds in the same way as in *Step* 1. We focus on the positivity condition (15.5). Suppose the spinor components of $v_i(r),\ 1 \leqslant i \leqslant 3$ are given respectively by

$$\begin{aligned} v_1|_{\{r\}\times Y} &\rightsquigarrow \Phi_1(r) \in C^\infty(Y, S), & v_2 &\rightsquigarrow \Phi_2(r) \in C^\infty([0,\pi]_\theta \times \Sigma, S),\\ v_3|_{\{r\}\times Y} &\rightsquigarrow \Phi_3(r) \in C^\infty(Y, S). \end{aligned}$$

We focus on sections

$$(\Phi_1(r), \Phi_3(r), \sqrt{r}\Phi_2(r)) \in L^2(Y,S) \oplus L^2(Y,S) \oplus L^2([0,\pi]_\theta \times \Sigma, S)$$

and operators:

$$\frac{1}{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_1^{\mathbb{D}} - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} D_{B_0} & 0 & 0 \\ 0 & -D_{B_0} & 0 \\ 0 & 0 & L_2^{\mathbb{D}} \end{pmatrix}.$$

Unlike Example 15.10, $L_1^{\mathbb{D}}$ is not self-adjoint in this case. In general,

$$2\operatorname{Re}\langle L_1^{\mathbb{D}} v, L_2^{\mathbb{D}} v\rangle_{L^2([0,\pi]_\theta\times\Sigma)} = \int_{\{\theta\}\times\Sigma}\langle v, D_{\check{B}}^{\Sigma} v\rangle\Big|_{\theta=0}^{\theta=\pi} \neq 0,\ v\in L^2([0,\pi]_\theta, S).$$

Let $v=\sqrt{r}\Phi_2(r)$ and follow the proof of Lemma 15.9:

$$\begin{aligned} &\|(L_1^{\mathbb{D}} + rL_2^{\mathbb{D}} + (\alpha-\frac{1}{2}))\sqrt{r}\Phi_2\|^2_{L^2([0,\pi]_\theta\times\Sigma)} - \operatorname{Re}\langle\sqrt{r}\Phi_2, (rL_2^{\mathbb{D}})\sqrt{r}\Phi_2\rangle \\ \geqslant&\frac{2r^2}{2\alpha-1}\operatorname{Re}\langle L_1^{\mathbb{D}}\Phi_2, L_2^{\mathbb{D}}\Phi_2\rangle_{L^2([0,\pi]_\theta\times\Sigma)} \\ =&\frac{r^2}{2\alpha-1}\Big(\int_{\{\pi\}\times\Sigma}\langle\Phi_2, D_{\check{B}}^{\Sigma}\Phi_2\rangle - \int_{\{0\}\times\Sigma}\langle\Phi_2, D_{\check{B}}^{\Sigma}\Phi_2\rangle\Big).\end{aligned}$$

Just as in *Step* 1, sections Φ_1 and Φ_2 have the same boundary value along $\mathcal{Y}_r^{(1)}\cap\mathcal{Y}_r^{(2)}$:

$$\Phi_1(r,s)\big|_{s=0} = \Phi_2(r,\theta)\big|_{\theta=0}.$$

Therefore, it remains to verify the inequality:

$$\|(rD_{B_0}+\alpha)\Phi_1\|^2_{L^2(Y)} - \operatorname{Re}\langle\Phi_1, (rD_{B_0})\Phi_1\rangle \geqslant \frac{r^2}{2\alpha-1}\int_{\{0\}\times\Sigma}\langle\Phi_1, D_{\check{B}}^{\Sigma}\Phi_1\rangle.$$

The left hand side can be rewritten as

$$(1-\frac{1}{2\alpha-1})\|(rD_{B_0}+\frac{(2\alpha-1)^2}{4\alpha-4})\Phi_1\|^2_{L^2(Y)} + \frac{r^2}{2\alpha-1}\|D_{B_0}\Phi_1\|^2_{L^2(Y)} + \frac{4\alpha^2-6\alpha+1}{8\alpha-8}\|\Phi_1\|^2_{L^2(Y)}.$$

Using the Weitzenböck formula [KM07, (4,15)], the last two terms are bounded below by

$$\begin{aligned}\frac{r^2}{2\alpha-1}&\Big(\|\nabla_{B_0}\Phi_1\|^2_{L^2(Y)} + \int_{\{0\}\times\Sigma}\langle\Phi_1, D_{\check{B}}^{\Sigma}\Phi_1\rangle + \int_Y \frac{s}{4}|\Phi_1|^2 + \langle\Phi_1, \frac{1}{2}\rho_3(F_{B_0^t})\Phi_1\rangle\Big) \\ &+\frac{\alpha-1}{2}\|\Phi_1\|^2_{L^2(Y)} \geqslant \frac{r^2}{2\alpha-1}\int_{\{0\}\times\Sigma}\langle\Phi_1, D_{\check{B}}^{\Sigma}\Phi_1\rangle + \frac{\alpha-\alpha_0}{2}\|\Phi_1\|^2_{L^2(Y)}.\end{aligned}$$

Then we take $\alpha > \alpha_0 := e^{100}\max\{\|s\|_\infty, \|F_{B_0^t}\|_\infty, 1\}$.

The common boundary $\mathcal{Y}_r^{(2)}\cap\mathcal{Y}_r^{(3)}$ is dealt with similarly. Hence, the positivity condition (15.5) holds when $\alpha>\alpha_0$. Now we use Proposition 15.6 and 15.7 to complete the proof. □

15.6. **Irreducibility of Spinors.** We accomplish the proof of Theorem 15.3 in this subsection, following the idea above. The spinor part of the equation (15.1) is cast into the form

$$\frac{d}{dt}\Psi(t) + D_{B(t)}\Psi(t) + \mathfrak{q}^1(B(t),\Psi(t)) = 0.$$

where $\mathfrak{q}^1$ is the spinor part of the perturbation $\mathfrak{q} = (\mathfrak{q}^0, \mathfrak{q}^1)$. As $\mathfrak{q}^1(B(t), 0) \equiv 0$, we have

$$\begin{aligned}\|\mathfrak{q}^1(B(t), \Psi(t))\|_2 &= \|\mathfrak{q}^1(B(t), \Psi(t)) - \mathfrak{q}^1(B(t), 0)\|_2 \\ &= \int_0^1 \|\mathcal{D}_{(B(t), r\Psi(t))}\mathfrak{q}^1(\Psi(t))\|_2 dr \leqslant C\|\Psi(t)\|_{L^2(Y)},\end{aligned}$$

for a constant $C > 0$ and any $t \in [t_0 - \epsilon, t_0 + \epsilon]$. Now the proof of Theorem 15.1 can proceed with no difficulty.

15.7. **The Linearized Version.** In this subsection, we accomplish the proof of Theorem 15.2. To some extent, it suffices to "linearize" each step of the proof of Theorem 15.1. Again, assume $I = [-1, 1]_t$ and $t_0 = 0$.

$$\xi^{(1)}(t) = \xi - \int_0^t \delta c(t')dt' \in C^\infty(\widehat{Z}, i\mathbb{R}),$$

and set $V_1 = V - \mathbf{d}_\gamma \xi^{(1)}$. This new section V_1 is smooth, and

$$\begin{aligned}&V_1(t) = (0, \delta b_1(t), \delta\psi_1(t)) \in L^2_k(Z, iT^*Z \oplus S),\\ &V_1(0) = 0 \text{ on } \{0\} \times Y.\end{aligned}$$

As γ solves the non-linear equation (15.1), $\mathbf{d}_\gamma f^{(1)}$ is a solution to the linear equation (15.2), and so is V_1. The equation (13.2) is formally an evolutionary equation on $I \times Y$:

$$\frac{d}{dt}\begin{pmatrix}\delta b_1(t)\\ \delta\psi_1(t)\end{pmatrix} + \begin{pmatrix}*_3 d_Y & 0\\ 0 & D_{B_0}\end{pmatrix}\begin{pmatrix}\delta b(t)\\ \delta\psi(t)\end{pmatrix} = \eta(t)\begin{pmatrix}\delta b(t)\\ \delta\psi(t)\end{pmatrix}, t \in \mathbb{R}. \tag{15.16}$$

where $\eta(t) : L^2(Y) \to L^2(Y)$ is a family of bounded linear operators determined by $\check{\gamma}(t)$.

To borrow the proof of Theorem 15.2, we focus on $\mathcal{Y}^{(2)}_r$. Using polar coordinates, we write

$$V_1(r) = (\delta c_1'(r), \delta b_1'(r), \delta\psi_1'(r)) \in C^\infty(X', i\mathbb{R} \oplus iT^*([0, \pi]_\theta \times \Sigma) \oplus S),$$

on $X' = [0, 1]_r \times [0, \pi]_\theta \times \Sigma \subset \mathbb{H}^2_+ \times \Sigma$. To put $V_1(r)$ into radial temporal gauge, consider the function

$$f^{(2)}(r) = -\int_0^r \delta c_1'(r')dr' \text{ on } X'.$$

Then $f^{(2)}(r, \theta) \equiv 0$ when $\theta = 0, \pi$, and the section $V_1 - \mathbf{d}_\gamma f^{(2)}$ solves the linear equation (15.2) on X. The proof of Theorem 15.1 is now applicable. We conclude that

$$\begin{aligned}V_1(t) &\equiv 0 \text{ on } I \times Y \\ V_1 - \mathbf{d}_\gamma f^{(2)} &\equiv 0 \text{ on } X'.\end{aligned} \tag{15.17}$$

We extend $f^{(2)}$ by zero over the product $I \times Y$. One might worry that $f^{(2)}$ does not form a smooth function on the union

$$(I \times Y) \bigcup X',$$

as we pointed out in *Step* 1 in the proof of Theorem 15.2. However, once the unique continuation property is established, the smoothness of $f^{(2)}$ follows from (15.17) and the smoothness of V_1. As a result,

$$V_1 = \mathbf{d}_\gamma f^{(2)} \text{ on } (I \times Y) \bigcup X'.$$

By induction, we can extend the region where this equality holds. This completes the proof of Theorem 15.2.

16. Transversality

With all machinery developed so far, we are ready to prove the transversality result on the cylinder $\mathbb{R}_t \times \widehat{Y}$ in this section. Here is the main result:

Theorem 16.1. *For any relative spin^c manifold $(\widehat{Y}, \widehat{\mathfrak{s}})$ satisfying constraints in the strict cobordism category* Cob_s*, one can find an admissible perturbation $\mathfrak{q} \in \mathcal{P}(\widehat{Y}, \widehat{\mathfrak{s}})$, in the sense of Definition* 13.3. *Here $\mathcal{P}(\widehat{Y}, \widehat{\mathfrak{s}})$ is the Banach space of tame perturbations constructed Subsection* 8.5.

Pick an admissible perturbation $\mathfrak{q}(\widehat{\mathfrak{s}})$ for each relative *spin*^{c} structure $\widehat{\mathfrak{s}}$ on Y. By putting them altogether, we obtain an object $\mathbb{Y} = (Y, \psi, g_Y, \omega, \mathfrak{q})$ in the category Cob_s: the property (P8) is fulfilled. In this case, the moduli spaces $\mathcal{M}_{[\gamma]}(\mathfrak{a}, \mathfrak{b})$ defined in Section 13 will become a smooth manifold, and the Floer homology of $(\mathbb{Y}, \widehat{\mathfrak{s}})$ will be defined in Part 6.

Theorem 16.1 is a formal consequence of the unique continuation properties, Theorem 15.1-15.3 and the separating properties of cylinder functions, Theorem 8.20. The transversality result for a general morphism $\mathbb{X} : (\mathbb{Y}, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ in the category SCob_s is proved in Subsection 16.3, cf. Theorem 16.5.

16.1. Transversality for the 3-Dimensional Equations.

Consider the Banach space of perturbations $\mathcal{P}$ and a tame perturbation $\mathfrak{q} = \operatorname{grad} f \in \mathcal{P}$. We start with the first condition (E1) in Definition 13.3 which concerns the 3-dimensional equation

$$\operatorname{grad} \pounds_\omega(\mathfrak{a}) = 0,$$

Recall from Definition 12.2 that a critical point $\mathfrak{a} \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ of $\pounds_\omega = \mathcal{L}_\omega + f$ is called non-degenerate if the extended Hessian at $\mathfrak{a}$

$$\widehat{\mathrm{Hess}}_{\mathfrak{q},\mathfrak{a}}$$

is invertible. In fact, this is a generic condition for a perturbation $\mathfrak{q} \in \mathcal{P}$.

Theorem 16.2 (cf. [KM07] Theorem 12.1.12)**.** *There is a residue (and in particular non-empty) subset of $\mathcal{P}$ such that for every $\mathfrak{q}$ in this subset, any critical point $\mathfrak{a} \in \mathrm{Crit}(\pounds_\omega)$ is non-degenerate. For such a perturbation,* $\mathrm{Crit}(\pounds_\omega)$ *comprises a finite collection of gauge orbits.*

Proof. The proof follows the same argument as in [KM07, Section 12.5] with one slight modification, as we explain now. Suppose for some $\mathfrak{q} \in \mathcal{P}$ and $\mathfrak{a} \in \operatorname{Crit} \mathcal{L}_\omega$, the tangent vector $v = (0, \delta b, \delta\psi) \neq 0$ lies in the kernel of $\widehat{\operatorname{Hess}}_{\mathfrak{q},\mathfrak{a}}$:

$$(0, \delta b, \delta\psi) \in \ker \widehat{\operatorname{Hess}}_{\mathfrak{q},\mathfrak{a}}. \tag{16.1}$$

We have to show that v is separated by a cylinder function. To apply Proposition 8.6. we verify that v is not generated by the infinitesimal gauge action on Y. Suppose on the contrary that

$$(\delta b, \delta\psi) = \mathbf{d}_{\mathfrak{a}}\, \xi \text{ on } Y \tag{16.2}$$

for some $\xi \in L^2_{k+1}(\widehat{Y}, i\mathbb{R})$, then by the unique continuation property of tangent vectors, Theorem 15.2, for a possibly different function $\xi' \in L^2_{k+1}(\widehat{Y}, \mathbb{R})$, the equation (16.2) holds on $\widehat{Y}$:

$$(\delta b, \delta\psi) = \mathbf{d}_{\mathfrak{a}}\, \xi'.$$

By (16.1), $\mathbf{d}^*_{\mathfrak{a}}(\delta b, \delta\psi) = 0$, so $(\delta b, \delta\psi)$ is L^2-orthogonal to the subspace $\mathcal{J}_{k,\mathfrak{a}} \subset \mathcal{T}_{k,\mathfrak{a}}$. This implies that $v = 0$, which a contradiction. Alternatively, we may apply the linearized version of [KM07, Theorem 7.2.1] on the 4-manifold

$$S^1 \times \widehat{Y},$$

which possesses a cylindrical end $S^1 \times [0, \infty)_s \times \Sigma$. Now we use Proposition 8.6 to find a cylinder function $f \in \operatorname{Cylin}(Y)$ supported on $Y \subset \widehat{Y}$ such that

$$df(v) \neq 0.$$

The rest of the proof then follows [KM07, Section 12.5]. □

16.2. **Transversality on Cylinders.** Suppose a tame perturbation $\mathfrak{q}_1 = \operatorname{grad} f_1$ in the residue subset of Theorem 16.2 has been chosen. Then the critical set of $\mathcal{L}^1_\omega := \mathcal{L}_\omega + f_1$ consists of a finite collection of gauge orbits; let their representatives be

$$\mathfrak{a}_i,\ 1 \leqslant i \leqslant r.$$

We wish to find a closed Banach subspace $\mathcal{P}'$ of $\mathcal{P}$ such that for any generic $\mathfrak{q}_2 \in \mathcal{P}'$ with $\|\mathcal{P}\| \ll 1$, the sum

$$\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$$

is an admissible perturbation. The Banach subspace $\mathcal{P}'$ that we consider is

$$\mathcal{P}' := \{\mathfrak{q}_2 \in \mathcal{P} : \mathfrak{q}_2(\mathfrak{a}_i) = 0, \mathcal{D}^1_{\mathfrak{a}_i}\mathfrak{q}_2 = 0, \forall i = 1, \cdots, r\}, \tag{16.3}$$

so the perturbation $\mathfrak{q}_2$ vanishes to the first order at each representative $\mathfrak{a}_i$. The subspace $\mathcal{P}'$ is clearly closed inside $\mathcal{P}$. Let us first verify the property (E1) for $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$.

Lemma 16.3 ([KM07] Lemma 15.1.2)**.** *There exists some $\eta > 0$ such that for any $\mathfrak{q}_2 = \operatorname{grad}_2 f_2 \in \mathcal{P}'$ with $\|\mathfrak{q}_2\|_{\mathcal{P}} < \eta$, the critical set of $\mathcal{L}_\omega := \mathcal{L}_\omega + (f_1 + f_2)$ agrees with that of $\mathcal{L}^1_\omega = \mathcal{L}_\omega + f_1$. As a result, the first condition* (E1) *of Definition* 13.3 *continues to hold for the sum $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$.*

In particular, for any $\mathfrak{q}_2 \in \mathcal{P}'$, the critical points of $\mathcal{L}_\omega$ in the quotient configuration space $\mathcal{B}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ are still given by $[\mathfrak{a}_i], 1 \leqslant i \leqslant r$ and

$$\mathcal{D}_{\mathfrak{a}_i} \operatorname{grad}(\mathcal{L}_\omega + f_1) = \mathcal{D}_{\mathfrak{a}_i} \operatorname{grad}(\mathcal{L}_\omega + f_1 + f_2),\ 1 \leqslant i \leqslant r.$$

So each $\mathfrak{a}_i$ is still non-degenerate in the sense of Definition 12.2. Here $[\mathfrak{a}_i]$ is the image of $\mathfrak{a}_i$ in $\mathcal{B}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$.

Proof of Lemma. Suppose on the contrary that there is a sequence of tame perturbations $\mathfrak{q}_2^{(j)} \in \mathcal{P}'$ and a sequence of configurations $\beta_i \in \mathcal{C}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$ such that

$$\|\mathfrak{q}_j\|_{\mathcal{P}} \to 0,\ (\operatorname{grad} \mathcal{L}^1_\omega + \mathfrak{q}_2^{(j)})(\beta_j) = 0$$

and each β_j is not gauge equivalent to any of $\mathfrak{a}_i, 1 \leqslant i \leqslant r$. By Proposition 9.7, a subsequence of $\{\beta_j\}$ converges to some $\mathfrak{a}_i$ up to gauge. Fix $0 < \epsilon \ll 1$ and let $\mathcal{O}_i(\epsilon)$ be the ϵ-neighborhood of $\mathfrak{a}_i$ in $\mathcal{C}_{k+1/2}(\widehat{Y}, \widehat{\mathfrak{s}})$. When $j \gg 1$, each $\beta_j \in \mathcal{O}_i(\epsilon)$, and one may use gauge transformations to put β_j into the Coulomb gauge slice at $\mathfrak{a}_i$, i.e.

$$\mathbf{d}^*_{\mathfrak{a}_i}(\beta - \mathfrak{a}_i) = 0.$$

Then

$$\operatorname{grad} \mathcal{L}^1_\omega(\beta_j) - \operatorname{grad} \mathcal{L}^1_\omega(\mathfrak{a}_i) = -(\mathfrak{q}_2^{(j)}(\beta_j) - \mathfrak{q}_2^{(j)}(\mathfrak{a}_j)). \tag{16.4}$$

As $\mathfrak{a}_i$ is non-degenerate as a critical point of $\mathcal{L}^1_\omega$, the $L^2_{k-1/2}$-norm of the left hand side is bounded below by

$$c\|\beta_j - \mathfrak{a}_i\|_{L^2_{k+1/2,\mathfrak{a}_i}}$$

for some $c > 0$. On the other hand, as $\mathfrak{q}_2^{(j)} \to 0$ in $\mathcal{P}$, the C^2-norm of $\mathfrak{q}$ over the bounded neighborhood $\mathcal{O}_i(\epsilon)$ converges to zero, by Corollary 8.19:

$$\sup_{\gamma \in \mathcal{O}_i(\epsilon)} \|\mathcal{D}^2_\gamma \mathfrak{q}_2^{(j)}\| \to 0 \text{ as } j \to \infty.$$

As a result, the $L^2_{k-1/2}$-norm of the right hand side of (16.4) is bounded above by

$$\|\beta_j - \mathfrak{a}_i\|^2_{L^2_{k-1/2,\mathfrak{a}_i}} \leqslant \|\beta_j - \mathfrak{a}_i\|^2_{L^2_{k+1/2,\mathfrak{a}_i}} \leqslant \epsilon\|\beta_j - \mathfrak{a}_i\|_{L^2_{k+1/2,\mathfrak{a}_i}},$$

when $j \gg 1$, which yields a contradiction if $\epsilon < c$. □

Theorem 16.1 now follows from the strong unique continuation property Theorem 15.1-15.3 together with Lemma 13.5. The proof is modeled on [KM07, Section 15]. In what follows, we will only point out the necessary changes to be made.

Proof of Theorem 16.1. Let $\mathfrak{a}, \mathfrak{b} \in \operatorname{Crit}(\mathcal{L}_\omega)$ be critical points of $\mathcal{L}_\omega$ and $\widehat{Z} = \mathbb{R}_t \times \widehat{Y}$ be the infinite cylinder. Following the scheme of [KM07, Section 15] and notations from Subsection 13.1, it suffice to show for any $\mathfrak{q}_2 \in \mathcal{P}'$ and any solution $\gamma \in \mathcal{C}_k(\mathfrak{a}, \mathfrak{b})$ to the perturbed equation

$$0 = \mathfrak{F}_{\widehat{Z},\mathfrak{q}} := \mathfrak{F}_{\widehat{Z}} + \widehat{\mathfrak{q}},$$

the operator

$$\begin{aligned}(16.5)\qquad \mathcal{P}' \times L^2_k(\widehat{Z}, i\mathbb{R}\oplus iT^*\widehat{Y}\oplus S) &\to L^2_{k-1}(\widehat{Z}, i\mathbb{R}\oplus iT^*\widehat{Y}\oplus S)\\ (\delta\mathfrak{q}, V) &\mapsto \delta\widehat{\mathfrak{q}}(\gamma) + (\mathbf{d}^*_\gamma, \mathcal{D}_\gamma\mathfrak{F}_{\widehat{Z},\mathfrak{q}})(V)\end{aligned}$$

is surjective. The section $\delta\widehat{\mathfrak{q}}(\gamma)$ lies in L^2_{k-1} as the underlying path $\check{\gamma}:\mathbb{R}\to\mathcal{C}_{k-1/2}(\widehat{Y},\widehat{\mathfrak{s}})$ decay exponentially to either $\mathfrak{a}$ or $\mathfrak{b}$ as $t\to\pm\infty$ and $\delta\mathfrak{q}$ vanishes at $\mathfrak{a}$ and $\mathfrak{b}$ to the first order.

Suppose first that $\delta\widehat{\mathfrak{q}}=0$ in (16.5), then (16.5) becomes a Fredholm operator by Proposition 13.1, and its cokernel is finite dimensional. It remains to show that for any section

$$U = (\delta c'(t), \delta b'(t), \delta\psi'(t)) \in L^2(\widehat{Z}, i\mathbb{R}\oplus iT^*\widehat{Y}\oplus S)$$

that is L^2-orthogonal to the image of $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma\mathfrak{F}_{\widehat{Z},\mathfrak{q}})$, there exist some $\delta\widehat{\mathfrak{q}}\in\mathcal{P}'$ such that

$$(16.6)\qquad \langle\delta\widehat{\mathfrak{q}}(\gamma(t)), U\rangle_{L^2(\mathbb{R}\times\widehat{Y})}\neq 0.$$

We first explain how to achieve (16.6) for a generalized cylinder function $f:\mathcal{C}_{k-1/2}(\widehat{Y},\widehat{\mathfrak{s}})\to\mathbb{R}$:

$$(16.7)\qquad \int_{t\in\mathbb{R}_t}\langle \operatorname{grad} f(\check{\gamma}), U(t)\rangle_{L^2(\widehat{Y})}\neq 0.$$

By the unique continuation properties, Theorem 15.1, 15.2 and 15.3, the underlying path $\check{\gamma}:\mathbb{R}\to\mathcal{C}_{k-1/2}(\widehat{Y},\widehat{\mathfrak{s}})$ satisfies the following properties

- for any $t_1\neq t_2\in\mathbb{R}_t$, $\check{\gamma}(t_1)$ and $\check{\gamma}(t_2)$ are not gauge equivalent over Y;
- for any $t\in\mathbb{R}_t$, $\check{\gamma}(t)$ is not gauge equivalent to $\mathfrak{a}_i$ on Y for any $1\leqslant i\leqslant r$; moreover, $\check{\gamma}(t)$ is irreducible on Y;
- for any $t\in\mathbb{R}_t$, its derivative $\partial_t\check{\gamma}(t)$ is not generated by the infinitesimal gauge action over Y.

As for the section U in the cokernel, by Lemma 13.5, we have

- $\delta c'(t)\equiv 0$;
- for some $t_0\in\mathbb{R}_t$, $U(t_0)=(0,\delta b'(t_0),\delta\psi'(t_0))$ are not generated by the infinitesimal gauge action over Y.

Take a large constant $T>0$ such that $t_0\in[-T,T]$. To apply Theorem 8.20, let the compact subset K be the image of

$$\{\mathfrak{a}_i : 1\leqslant i\leqslant r\}\bigcup\{\check{\gamma}(t): t\in[-T,T]\}$$

in the quotient configuration space $\mathcal{B}^*_{k-1/2}(\widehat{Y},\widehat{\mathfrak{s}})$. Then we can find a finite collection of cylinder functions $\{f_j,\ 1\leqslant j\leqslant l\}$ defined using embeddings $\iota_j: S^1\times D^2\hookrightarrow Y$ such that the map

$$\Xi'_* = (f_1,\cdots,f_l):\mathcal{B}_{k-1/2}(\widehat{Y},\widehat{\mathfrak{s}})\to\mathbb{R}^l$$

gives an embedding of K and $\Xi'(U(t_0))\neq 0$. Choose a smooth function

$$g':\mathbb{R}^l\to\mathbb{R}$$

supported in a small neighborhood $\overline{\Omega}$ of $\Xi'([\check{\gamma}(t_0)])$ with the following additional properties

- $\Xi'([\mathfrak{a}_i]) \notin \Omega,\ \forall 1 \leqslant i \leqslant r$;
- $(\Xi' \circ \check{\gamma})^{-1}(\overline{\Omega})$ is a small connected interval $[t_0 - \epsilon_1, t_0 + \epsilon_2]$ around t_0; to achieve this, we take $T \gg 1$;
- lastly, the integral

$$\int_{\mathbb{R}_t} dg'(\Xi'_*(U(t)))dt \neq 0. \tag{16.8}$$

The last property would be impossible if for some constant $\alpha \in \mathbb{R}$, $\Xi'_*(U(t)) = \alpha \Xi'_*(\partial_t \check{\gamma}_t)$ for any $t \in [t_0 - \epsilon, t_0 + \epsilon]$. However, this cannot hold for the whole real line; otherwise one may draw a contradiction from equations (13.2) and (13.8). Then we can achieve (16.8) by taking a different time slice $t_0 \in \mathbb{R}_t$ and possibly a different Ξ'.

As a result, the inequality (16.7) is achieved for the composition:

$$f := g' \circ \Xi' : \mathcal{B}_{k-1/2}(\widehat{Y}, \widehat{\mathfrak{s}}) \to \mathbb{R},$$

Note that $f \equiv 0$ in some $L^2_{k-1/2}$-neighborhood of $\{[\mathfrak{a}_i] : 1 \leqslant i \leqslant r\}$, so $\operatorname{grad} f$ satisfies the constraints in (16.3). By the density of the Banach space $\mathcal{P}$, we can approximate $\operatorname{grad} f$ by an element $\delta\widehat{\mathfrak{q}}$ in $\mathcal{P}'$ and the inequality (16.6) holds for this approximation.

The rest of the proof follows the same line of argument as in [KM07, Proposition 15.1.3]. □

16.3. **Transversality on 4-Manifolds in General.** Recall the set up from Section 14. For a morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$, the Seiberg-Witten equations $\mathfrak{F}_{\mathcal{X},\mathfrak{p}} = 0$ on the complete Riemannian 4-manifold $\mathcal{X}$ is perturbed by a quadruple

$$\mathfrak{p} = (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \omega_3).$$

While $(\mathfrak{q}_1, \mathfrak{q}_2)$ are encoded in the objects $\mathbb{Y}_1$ and $\mathbb{Y}_2$, the pair

$$(\mathfrak{q}_3, \omega_3) \in \mathcal{P}(Y_3) \times \mathcal{P}_{\mathrm{Form}}$$

is the actual perturbation that allows us to achieve transversality.

Definition 16.4. The quadruple $\mathfrak{p}$ is said to be admissible if

- each $\mathfrak{q}_i \in \mathcal{P}(Y_i), i = 1, 2$ is admissible in the sense of Definition 13.3;
- for any *spin*^{c} cobordism $(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_2, \widehat{\mathfrak{s}}_2)$ (with a prescribed planar metric g_X), the moduli space $\mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ is regular in the sense of Definition 14.2. Here $\mathfrak{a}_i \in \operatorname{Crit}(\mathcal{L}_{\omega_i, \widehat{Y}_i})$ is a critical point of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_{\omega_i, \widehat{Y}_i}$ on $\widehat{Y}_i$, $i = 1, 2$. ◇

Theorem 16.5. *Under above assumptions, for any fixed admissible perturbations* $(\mathfrak{q}_1, \mathfrak{q}_2)$ *on* $\widehat{Y}_1$ *and* $\widehat{Y}_2$ *respectively, there is a residue subset of* $\mathcal{P}(Y_2) \times \mathcal{P}_{\mathrm{Form}}$ *such that for every pair* $(\mathfrak{q}_3, \omega_3)$ *in this subset, the quadruple* $\mathfrak{p}$ *is admissible.*

Proof. Following the proof of Theorem 16.1, it suffices to verify that the operator

$$\begin{aligned}\mathcal{P}(Y_2) \times \mathcal{P}_{\mathrm{Form}} \times L^2_k(\mathcal{X}, iT^*\mathcal{X} \oplus S^+) &\to L^2_{k-1}(\mathcal{X}, i\mathbb{R} \oplus i\mathfrak{su}(S^+) \oplus S-) \\ (\delta\mathfrak{q}_3, \delta\omega_3, V) &\mapsto (\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\mathcal{X},\mathfrak{p}})V + \beta_0(t)\delta\widehat{\mathfrak{q}}_3(\gamma) + \rho_4(\delta\omega_3^+),\end{aligned} \tag{16.9}$$

is surjective, for any solution $\gamma \in \mathcal{C}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ to the perturbed equation $\mathfrak{F}_{\mathcal{X},\mathfrak{p}} = 0$. We begin with $(\delta\mathfrak{q}_3, \delta\omega_3) = 0$, then (16.9) becomes a Fredholm operator by Proposition 14.1. Suppose $U \in L^2(\mathcal{X}, i\mathbb{R} \oplus i\Lambda^+\mathcal{X} \oplus S^-)$ is L^2-orthogonal to the image of $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma\mathfrak{F}_{\mathcal{X},\mathfrak{p}})$. it remains to find $(\delta\widehat{\mathfrak{q}}_3, \delta\omega_3)$ such that

$$\langle U, \beta_0(t)\delta\widehat{\mathfrak{q}}_3(\gamma) + \rho_4(\delta\omega_3^+)\rangle_{L^2} \neq 0. \tag{16.10}$$

Let $I = [1,2]_t$ and write

$$U = (\delta\xi, \delta\omega, \delta\phi) \text{ with } \delta\xi \in L^2(\mathcal{X}, i\mathbb{R}).$$

The same argument as in the proof of Lemma 13.5 implies that $\delta\xi \equiv 0$. The inner product (16.10) is supported on the compact submanifold

$$\widehat{Z} := I \times \widehat{Y}_2,$$

over which the formal adjoint of $(\mathbf{d}^*_\gamma, \mathcal{D}_\gamma\mathfrak{F}_{\mathcal{X},\mathfrak{p}})$ is cast into the form (13.6). If instead we write

$$U(t) = (0, \delta b(t), \delta\psi(t)) \in L^2_1(\widehat{Z}, i\mathbb{R} \oplus iT^*\widehat{Y}_2 \oplus S) \text{ on } I \times \widehat{Y}_2,$$

then we are back to the cylindrical case. Here we have used the bundle map

$$(\rho_3, \rho_4(dt))$$

to identify $iT^*\widehat{Y}_2 \oplus S$ with $i\mathfrak{su}(S^+) \oplus S^-$ over $\widehat{Z}$.

However, Lemma 13.5 does not apply directly here, so we argue as follows. If there exists some $t_0 \in \operatorname{supp}\beta_0 \subset [1,2]$ such that $U(t_0)$ is separated by some cylinder function f, then we set $\delta\omega_3 = 0$ and proceed as in the proof of Theorem 15.1.

If not, then by the proof of Lemma 13.5, for any $t \in [5/4, 7/4]$, there exists some function $\xi(t) \in L^2_1(\widehat{Y}, i\mathbb{R})$ such that

$$(\delta b(t), \delta\psi(t)) = \mathbf{d}_{\check{\gamma}(t)}\,\xi(t) \text{ on } \{t\} \times Y_2.$$

Moreover,

$$\frac{d}{dt} d_{Y_2}\xi(t) \equiv 0 \text{ and } \Delta_{Y_2}\xi(t) + \xi(t)|\Psi(t)|^2 = 0 \text{ on } [5/4, 7/4] \times Y_2. \tag{16.11}$$

Recall that $\delta\omega_3 = -\beta_0(t)dt \wedge d_{Y_2}f_3$ for a compactly supported function $f_3 : I \times Y_2 \to i\mathbb{R}$, so

$$\rho_4(\delta\omega_3^+) = \rho_3(d_{Y_2}(\beta_0(t)f_3)).$$

If U is orthogonal to $\rho_4(\delta\omega_3^+)$ for any $\delta\omega_3 \in \mathcal{P}_{\text{Form}}$, then $\Delta_{Y_2}\xi(t) \equiv 0$. By (16.11), $U(t) \equiv 0$ on $[5/4, 7/4] \times Y_2$. By unique continuation, $U \equiv 0$ on the whole manifold $\mathcal{X}$. $\square$

Part 6. Floer Homology

Let $(\mathbb{Y}, \hat{\mathfrak{s}}) \in \text{SCob}_s$ be an object in the strict *spin*^{c} cobordism category, as defined in Section 3. The underlying 3-manifold Y of $\mathbb{Y}$ is compact connected and oriented, whose boundary is identified with a disjoint union of 2-tori Σ by the diffeomorphism $\psi : \partial Y \to \Sigma$. The quintuple $\mathbb{Y} = (Y, \psi, g_Y, \omega, \{\mathfrak{q}\})$ also dictates a cylindrical metric g_Y and a closed 2-form $\omega \in \Omega^2(Y, i\mathbb{R})$. $\hat{\mathfrak{s}} \in \text{Spin}^c_R(Y)$ is a relative *spin*^{c} structure of the 3-manifold Y.

The primary goal of this part is to define the functor

$$HM_* : \text{SCob}_s \to \mathcal{R}\text{-Mod}$$

which assigns the monopole Floer homology $HM_*(\mathbb{Y}, \hat{\mathfrak{s}})$ for each object $(\mathbb{Y}, \hat{\mathfrak{s}}) \in \text{SCob}_s$, generalizing the construction of Kronheimer-Mrowka for closed 3-manifolds.

So far we have addressed two fundamental problems in order to define the functor HM_*:

- the compactness issue; see Theorem 6.1 for the unperturbed equations and Theorem 9.5 for the perturbed ones;
- the transversality issue; see Theorem 16.1 for the case of cylinders and Theorem 16.5 for morphisms in SCob_s.

Although the proof of the gluing theorem is omitted in this paper, it follows from the standard procedure in [KM07, Section 17-19], as noted in Subsection 13.4.

Now the construction of monopole Floer homology becomes straightforward by following the standard argument. Part 6 is organized as follows. In Section 17, we explain the basic construction using $\mathbb{F}_2$-coefficient. Section 18 is devoted to the canonical grading as well as the canonical mod 2 grading of $HM_*(\mathbb{Y}, \hat{\mathfrak{s}})$.

In Section 19, we address the orientation issue, which allows us to define the monopole Floer homology $HM_*(\mathbb{Y}, \hat{\mathfrak{s}})$ using $\mathbb{Z}$-coefficient. The key ingredient is the notion of relative orientations, which compare the orientations of two Fredholm operators using the excision principle, cf. Theorem 19.2 and Definition B.2. The proof is postponed to Appendix B.

17. The Basic Construction: $\mathbb{F}_2$-coefficient

In this section, we define the monopole Floer homology $HM_*(\mathbb{Y}, \hat{\mathfrak{s}})$ for each object $(\mathbb{Y}, \hat{\mathfrak{s}}) \in \text{SCob}_s$ using $\mathbb{F}_2$-coefficient. For the most general case, we have to use a Novikov ring $\mathcal{R}_2$. To work with the field $\mathbb{F}_2$ of two elements, we will pass to a subcategory of SCob_s in which case a monotonicity condition is required.

17.1. Novikov Rings.

Let us first explain the construction of $HM_*(\mathbb{Y}, \hat{\mathfrak{s}})$ using a Novikov ring

$$\mathcal{R}_2 = \{\sum_{n_i} a_i q^{n_i} : \ a_i \in \mathbb{F}_2, \ n_i \in \mathbb{R}, \ \lim_i n_i = -\infty\},$$

which is a complete topological group. Each element of $\mathcal{R}_2$ is a Laurent series in a formal variable q with possibly infinitely many terms in negative degrees. For any object $(\mathbb{Y}, \hat{\mathfrak{s}}) \in \text{SCob}_s$, the perturbation $\mathfrak{q} = \text{grad} f$ encoded in the quintuple $\mathbb{Y}$ is admissible in the sense of Definition 13.3. Let $\mathfrak{C}(\mathbb{Y}, \hat{\mathfrak{s}})$ be the set of critical points of $\tilde{\mathcal{L}}_\omega = \mathcal{L}_\omega + f$ in the quotient configuration space $\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}})$, then $\mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}})$ is a finite set by Theorem 16.2. Then the chain group $C_*(\mathbb{Y}, \widehat{\mathfrak{s}})$ is freely generated by $\mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}})$ over $\mathcal{R}_2$:

$$C_*(\mathbb{Y}, \widehat{\mathfrak{s}}) = \bigoplus_{[\mathfrak{a}] \in \mathfrak{C}(\widehat{Y}, \widehat{\mathfrak{s}})} \mathcal{R}_2 \cdot [\mathfrak{a}].$$

with differential ∂ defined as

$$\partial[\mathfrak{a}] = \sum_{\substack{z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}]) \\ \dim \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) = 0}} [\mathfrak{b}] \cdot \# \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) \cdot q^{-\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}], [\mathfrak{b}]; z)}. \tag{17.1}$$

The unparameterized moduli space $\breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) := \mathcal{M}_z([\mathfrak{a}], [\mathfrak{b}])/\mathbb{R}_t$ is defined as in (13.5). The topological energy $\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}], [\mathfrak{b}]; z)$ for a homotopy class of paths $z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}])$ equals twice the drop of $\pounds_\omega$ along γ

$$2(\pounds_\omega(\mathfrak{a}) - \pounds_\omega(\mathfrak{b}))$$

if $\gamma : [0, 1] \to \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ is a lift of z with $\gamma(0) = \mathfrak{a}$ and $\gamma(1) = \mathfrak{b}$. This expression is suggested by Proposition 9.1. To ensure the sum in (17.1) is convergent in $\mathcal{R}_2$, we need a finiteness result:

Lemma 17.1. *For any $C > 0$, there are only finitely many homotopy classes of paths $z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}])$ such that $\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}], [\mathfrak{b}], z) < C$ and $\breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}])$ is non-empty. Moreover, each $\breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}])$ is compact if its dimension equals zero.*

To show $\partial^2 = 0$, we follow the standard argument and look at the compactification of moduli spaces $\breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}])$ when dim $= 1$. Readers are referred to [KM07, Section 22] for the details. The monopole Floer homology of $(\mathbb{Y}, \widehat{\mathfrak{s}})$ is then defined as the homology of the chain complex $(C_*(\widehat{Y}, \widehat{\mathfrak{s}}), \partial)$:

$$HM_*(\mathbb{Y}, \widehat{\mathfrak{s}}) := H_*((C_*(\widehat{Y}, \widehat{\mathfrak{s}}), \partial)).$$

To make HM_* into a functor:

$$HM_* : \mathrm{SCob}_s \to \mathcal{R}_2\text{-Mod},$$

we assign for each morphism $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ a chain map:

$$m(\mathbb{X}; g_X, \mathfrak{p}) : (C_*(\mathbb{Y}_1, \widehat{\mathfrak{s}}_1), \partial_1) \to (C_*(\mathbb{Y}_2, \widehat{\mathfrak{s}}_2), \partial_2)$$

which relies on a planar metric g_X of the strict cobordism $X : Y_1 \to Y_2$ and a quadruple

$$\mathfrak{p} = (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \omega_3) \in \mathcal{P}(Y_1) \times \mathcal{P}(Y_2) \times \mathcal{P}(Y_2) \times \mathcal{P}_{\mathrm{Form}}.$$

Here $\mathfrak{p}$ is required to be admissible in the sense of Definition 16.4. While $(\mathfrak{q}_1, \mathfrak{q}_2)$ are encoded in the objects $(\mathbb{Y}_1, \mathbb{Y}_2)$, $(\mathfrak{q}_3, \omega_3)$ are the actual perturbations to the Seiberg-Witten equations on the complete Riemannian 4-manifold $\mathcal{X}$. Now define

$$m(\mathbb{X}; g_X, \mathfrak{q})[\mathfrak{a}_1] = \sum_{\substack{\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2) \\ \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = 0}} [\mathfrak{a}_2] \cdot \# \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) \cdot q^{-\mathcal{E}^{\mathfrak{p}}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)}, \tag{17.2}$$

where $\mathfrak{a}_i$ is a lift of $[\mathfrak{a}_i] \in \mathfrak{C}(\mathbb{Y}_i)$ in $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ for $i = 1, 2$. The moduli space $\mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ is defined as in (14.4) with the admissible quadruple $\mathfrak{q}$ as perturbations. The topological energy is given by the formula

$$\mathcal{E}^{\mathfrak{p}}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) := 2\mathcal{L}_{\omega_1}(\mathfrak{a}_1) - 2\mathcal{L}_{\omega_2}(\mathfrak{a}_2) + C(A_0, \omega_X) \tag{17.3}$$

where A_0 is a background *spin*^{c} connection on $\widehat{X}$ such that the restriction $A_0|_{\widehat{Y}_i}$ is the reference connection on $\widehat{Y}_i$ that defines the Chern-Simons-Dirac functional $\mathcal{L}_{\omega_i}$ for $i = 1, 2$. The constant $C(A_0, \omega_X)$ is given concretely by

$$C(A_0, \omega_X) = \frac{1}{4}\int_{\widehat{X}} F_{A_0^t} \wedge F_{A_0^t} - \int_{\widehat{X}} F_{A_0^t} \wedge \omega_X, \tag{17.4}$$

as suggested by (5.2). To make sense of the expression (17.2), we need another finiteness result:

Lemma 17.2. *For any $C > 0$, any pair of critical points $([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \mathfrak{C}(\mathbb{Y}_1) \times \mathfrak{C}(\mathbb{Y}_2)$ and any admissible quadruple $\mathfrak{p}$, there are only finitely many relative spin^c cobordisms $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ such that $\mathcal{E}^{\mathfrak{p}}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) < C$ and $\mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ is non-empty. Moreover, each moduli space $\mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ is compact if its dimension equals zero.*

Lemma 17.1 and Lemma 17.2 follow from the Compactness Theorem 9.5 and its analogue for a general cobordism. Readers are referred to [KM07, Corollary 31.2.5] for more details; their proofs are omitted here. By analyzing the moduli space $\mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ with $\dim = 1$, we conclude that $m(\mathbb{X}; g_X, \mathfrak{q})$ is a chain map by the standard argument. The chain maps induced from different auxiliary data $(g_X, \mathfrak{q})$ are all chain homotopic to each other, so the resulting maps on the homology are independent of $(g_X, \mathfrak{q})$

$$HM_*(\mathbb{X}) := [m(\mathbb{X}; g_X, \mathfrak{p})] : HM_*(\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to HM_*(\mathbb{Y}_2, \widehat{\mathfrak{s}}_2).$$

To show that HM_* defined this way is a functor and satisfies the composition law, we follow [KM07, Section 26].

17.2. **Monotonicity.** To define the monopole Floer homology using $\mathbb{F}_2$-coefficient, it is necessary to pass to a subcategory of SCob_s, as we explain in this subsection.

Definition 17.3. An object $(\mathbb{Y}, \widehat{\mathfrak{s}}) = (Y, \psi, g_Y, \omega, \mathfrak{q}, \widehat{\mathfrak{s}}) \in \mathrm{SCob}_s$ is called monotone if the period class $[\omega] \in H^2(Y; i\mathbb{R})$ is proportional to the image of $c_1(\widehat{\mathfrak{s}})$ in $\mathrm{Im}(H^2(Y, \partial Y; \mathbb{Z}) \to H^2(Y; \mathbb{R})$:

$$[\frac{\omega}{\pi i}] = \alpha \cdot c_1(\mathfrak{s}) \in H^2(Y; \mathbb{R}) \text{ for some } \alpha \in \mathbb{R}.$$

In addition, $(\widehat{Y}, \widehat{\mathfrak{s}})$ is called

- positively monotone if $\alpha < 1$;
- balanced if $\alpha = 1$;
- negatively monotone if $\alpha > 1$. ◊

In light of Lemma 3.8, under the monotonicity assumption, we have

$$\mathcal{L}_\omega(u \cdot \gamma) - \mathcal{L}_\omega(\gamma) = 2(1 - \alpha)\pi^2[u] \cup c_1(\mathfrak{s}),$$

for any $\gamma \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ and $u \in \mathcal{G}_{k+1}(\widehat{Y})$. In particular, $\mathcal{L}_\omega$ becomes a real valued functional if $(\mathbb{Y}, \widehat{\mathfrak{s}})$ is balanced. One necessary condition of monotonicity is that $\mu = 0$. The construction described below will work in general for any monotone objects, but let us focus on the special case when the period class $[\omega] = 0 \in H^2(Y; i\mathbb{R})$ and the form $\overline{\omega}$ defined in (P6) vanishes, for the sake of simplicity; so

$$\omega = \omega_\lambda = \chi_1(s) ds \wedge \lambda.$$

In this case, $(\widehat{Y}, \widehat{\mathfrak{s}})$ is always positively monotone, since $\alpha = 0$.

Under this assumption, the chain group $C_*(\mathbb{Y}, \widehat{\mathfrak{s}}; \mathbb{F}_2)$ is a finite dimensional $\mathbb{F}_2$-vector space:

$$C_*(\mathbb{Y}, \widehat{\mathfrak{s}}; \mathbb{F}_2) := \bigoplus_{[\mathfrak{a}] \in \mathfrak{C}(\widehat{Y}, \widehat{\mathfrak{s}})} \mathbb{F}_2 \cdot [\mathfrak{a}].$$

with differential defined by

$$\partial[\mathfrak{a}] = \sum_{\substack{z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}]) \\ \dim \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) = 0}} [\mathfrak{b}] \cdot \# \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) \tag{17.5}$$

In light of Lemma 17.1, to make sense of this expression, we need an upper bound on the topological energy $\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}], [\mathfrak{b}]; z)$:

Lemma 17.4. *For any $[\mathfrak{a}], [\mathfrak{b}] \in \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}})$, there exists a constant $C > 0$ such that*

$$\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}], [\mathfrak{b}]; z) < C,$$

for any homotopy classes of paths $z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}), [\mathfrak{a}], [\mathfrak{b}])$ with $\dim \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) = 0$.

As for a morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ with $\omega_1 = \omega_2 = \omega_\lambda$, $\overline{\omega}_X$ is a compactly supported 2-form (see (Q6)) on X. We require that the class defined in (Q7) vanishes: $[\omega_X]_{cpt} = 0 \in H^2(X, \partial X; \mathbb{Z})$. This time the chain map $m(\mathbb{X}; g_X, \mathfrak{q})$ is defined as

$$\begin{aligned} m(\mathbb{X}; g_X, \mathfrak{q}) : C_*(\mathbb{Y}_1, \widehat{\mathfrak{s}}_1; \mathbb{F}_2) &\to C_*(\mathbb{Y}_2, \widehat{\mathfrak{s}}_2; \mathbb{F}_2) \\ [\mathfrak{a}_1] &\mapsto \sum_{\substack{\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2) \\ \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = 0}} [\mathfrak{a}_2] \cdot \# \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2). \end{aligned}$$

Again, we need a upper bound on $\mathcal{E}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ to ensure the sum in the expression above is finite:

Lemma 17.5. *Under above assumptions, for any pair of critical points $([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \mathfrak{C}(\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \times \mathfrak{C}(\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$, any planar metric g_X and any admissible quadruple $\mathfrak{p}$, there is a constant $C > 0$ such that*

$$\mathcal{E}^{\mathfrak{p}}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) < C$$

which holds for any $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ with $\dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = 0$.

Lemma 17.4 and 17.5 follow directly from a general statement relating the dimension with the topological energy $\mathcal{E}_{top}$. In Proposition 17.6 below, we will think of a homotopy class of paths as a relative *spin*^{c} cobordism, following the ideas in Subsection 3.5.

Proposition 17.6. *Under above assumptions, for any relative spin^c cobordism* $\widehat{\mathfrak{s}}_X, \widehat{\mathfrak{s}}'_X \in \mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$*, we have*

$$\mathcal{E}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}'_X, \mathfrak{a}_2) - \mathcal{E}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = -4\pi^2 \big(\dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}'_X, \mathfrak{a}_2) - \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) \big)$$

In particular, the topological energy $\mathcal{E}^{\mathfrak{q}}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ is independent of the choice of $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ if $\dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = 0$.

Proof. Suppose $\widehat{\mathfrak{s}}'_X = \widehat{\mathfrak{s}}_X \otimes L$ for a relative complex line bundle in the class $[L] \in H^2(X, \partial X; \mathbb{Z})$. In terms of (17.3) and (17.4), we compute the difference of the topological energy

$$\begin{aligned} \mathcal{E}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}'_X, \mathfrak{a}_2) - \mathcal{E}_{top}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) &= C(A_0(\widehat{\mathfrak{s}}'_X), \omega_X) - C(A_0(\widehat{\mathfrak{s}}_X), \omega_X) \\ &= -2\pi^2 [L] \cup (c_1(\widehat{\mathfrak{s}}_X) + c_1(\widehat{\mathfrak{s}}'_X))[X, \partial X] \\ &= -4\pi^2 [L] \cup (c_1(\widehat{\mathfrak{s}}_X) + [L])[X, \partial X]. \end{aligned}$$

where $c_1(\widehat{\mathfrak{s}}_X)$ and $c_1(\widehat{\mathfrak{s}}'_X)$ are understood as elements in $H^2(X, [-1, 1] \times \Sigma; \mathbb{Z})$. On the other hand, pick an arbitrary non-vanishing section Φ_0 of

$$S^+|_{\partial X} \to \partial X.$$

Any relative *spin^c* structure $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ dictates an identification of $\widehat{\mathfrak{s}}_X|_{\partial X}$ with a standard *spin^c* structure on the boundary ∂X, so it makes sense to define the relative Euler number $e(\widehat{\mathfrak{s}}_X; \Phi_0)[X, \partial X]$ for any non-vanishing section Φ_0 of the spin bundle $S^+ \to \partial X$. In particular,

$$\big(e(\widehat{\mathfrak{s}}'_X; \Phi_0) - e(\widehat{\mathfrak{s}}_X; \Phi_0)\big)[X, \partial X] = [L] \cup (c_1(\widehat{\mathfrak{s}}_X) + [L])[X, \partial X].$$

In Proposition 18.6 below, we will associate a homotopy class of non-vanishing sections $[\Phi_0(\mathfrak{a}_1, \mathfrak{a}_2)]$ to any pair $(\mathfrak{a}_1, \mathfrak{a}_2)$ such that

$$e(\widehat{\mathfrak{s}}_X; \Phi_0(\mathfrak{a}_1, \mathfrak{a}_2))[X, \partial X] = \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) \tag{17.6}$$

for any $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^{\mathrm{c}}_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$. In fact, (17.6) follows from the Index Axiom (A-I) of the canonical grading of $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$. Another approach is to show

$$\big(e(\widehat{\mathfrak{s}}'_X; \Phi_0) - e(\widehat{\mathfrak{s}}_X; \Phi_0)\big)[X, \partial X] = \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}'_X, \mathfrak{a}_2) - \dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$$

for any non-vanishing section Φ_0 directly using the excision principle. This completes the proof of Proposition 17.6 □

Finally, one has to verify that $m(\mathbb{X}; g_X, \mathfrak{q})$ is a chain map and a generic homotopy of auxiliary data $(g_X, \mathfrak{q})$ gives rise to a chain homotopy of $m(\mathbb{X}; g_X, \mathfrak{q})$. The argument is not different from that of [KM07, Section 25].

18. Canonical Gradings

In this section, we introduce the canonical grading of the monopole Floer homology $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$. It is more natural to think of the grading set of $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$

$$\Xi^{\pi}(\mathbb{Y}, \widehat{\mathfrak{s}})$$

as the space of unit-length relative spinors on $\widehat{Y}$ modulo gauge transformations, identified also as a subset of homotopy classes of oriented relative 2-plane fields on Y. In particular,

$$\Xi^\pi(\mathbb{Y},\widehat{\mathfrak{s}}_1) = \Xi^\pi(\mathbb{Y},\widehat{\mathfrak{s}}_2)$$

if $\widehat{\mathfrak{s}}_1$ and $\widehat{\mathfrak{s}}_2$ come down to the same *spin*^{c} structure on Y.

The main result of this section is Proposition 18.6, which characterizes the canonical grading in terms of the Index Axiom (A-I) and the Normalization Axiom (A-II). They are inspired by the following index computation for a closed Riemannian 4-manifold X:

$$\dim \mathcal{M}(X,\mathfrak{s}_X) = e(\mathfrak{s}_X)[X]$$

where $\mathcal{M}(X,\mathfrak{s}_X)$ is the Seiberg-Witten moduli space and $e(\mathfrak{s}_X)$ is the Euler class of the spin bundle $S_X^+ \to X$. The canonical mod 2 grading will be discussed in Subsection **??**.

18.1. Homotopy Classes of Oriented Relative 2-Plane Fields.

For a closed 3-manifold Y, recall that the three flavors of monopoles Floer homology:

$$\widetilde{HM}_\bullet(Y), \widehat{HM}_\bullet(Y), \overline{HM}_\bullet(Y)$$

defined in the book [KM07] are graded by the homotopy classes of oriented 2-plane fields over Y. The analogous statement continues to hold in our case, using **relative** oriented 2-plane fields instead, as we explain now. The following lemma from [KM07] explains the relationship between 2-plane fields and *spin*^{c} structures:

Lemma 18.1 ([KM07] Lemma 28.1.1)**.** *On an oriented Riemannian 3-manifold Y, there is a bijection between*

(i) oriented 2-plane fields ξ;
(ii) 1-forms θ of length 1; and
(iii) isomorphism classes of pairs $(\mathfrak{s},\Psi)$ comprising a spin^c structure and a unit-length spinor Ψ.

Over the infinite cylinder $\mathbb{R}_s\times\Sigma$, we defined in (2.6) a preferred $\mathbb{R}_s$-translation invariant solution

$$\gamma_* = (B_*, \Psi_*)$$

to the perturbed Seiberg-Witten equations (3.6). The perturbation is provided by a covariantly constant 2-form

$$\omega_* := \mu + ds\wedge\lambda$$

The correspondence in Lemma 18.1 then identifies

$$\text{the unit length 1-form } \theta_* := i *_3 \frac{\omega_*}{|\omega_*|} \leftrightarrow \text{the unit length spinor } \frac{\Psi_*}{|\Psi_*|}, \tag{18.1}$$

Indeed, as γ_* solves the equations (3.6), $(\Psi_*\Psi_*^*)_0 = \rho_3(*_3\omega_*)$, so

$$\mathbb{C}\Psi_* \text{ and } \mathbb{C}(\Psi_*)^\perp$$

are i and $-i$ eigenspaces of $\rho_3(\theta_*)$ respectively. In particular, (18.1) determines a preferred oriented 2-plane fields ξ_* on $\mathbb{R}_s\times\Sigma$ by Lemma 18.1. Now we return to a 3-manifold $\widehat{Y}$ with cylindrical ends and state a relative version of Lemma 18.1.

Definition 18.2. An oriented 2-plane field ξ on $\widehat{Y}$ is called **relative** if ξ agrees with ξ_* over the cylindrical end $[0,\infty)_s \times \Sigma$. Similarly, we define

- **relative** 1-forms and
- **relative** spinors

using θ_* and $\Psi_*/|\Psi_*|$ as the models along the end $[0,\infty)_s \times \Sigma$. ◊

Lemma 18.3. *For any object $\mathbb{Y} \in \mathrm{Cob}_s$, let $\widehat{Y}$ be the extended 3-manifold with cylindrical ends. Then there is a bijection between:*

(i) oriented relative 2-plane fields ξ;
(ii) 1-forms relative θ of length 1; and
(iii) isomorphism classes of pairs $(\mathfrak{s}, \Psi)$ consisting of a spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s})|_\Sigma = 0 \in H^2(\Sigma, \mathbb{Z})$ and a unit-length spinor Ψ that is gauge equivalent to a relative spinor.

Remark 18.4. In the last description, the identification of $\widehat{\mathfrak{s}}|_\Sigma$ is not specified and a gauge transformation does not necessarily lie in the identity component when restricted to Σ. ◊

For each relative *spin*^{c} structure $\widehat{\mathfrak{s}} \in \mathrm{Spin}^c_R(Y)$, let $\Xi(\widehat{Y}, \widehat{\mathfrak{s}})$ be the space of unit-length relative spinors on $\widehat{Y}$. The index set for the monopole Floer homology $HM_*(Y, \widehat{\mathfrak{s}})$ will be

$$\Xi^\pi(Y, \widehat{\mathfrak{s}}) := \pi_0(\Xi(\widehat{Y}, \widehat{\mathfrak{s}}))/H^1(Y, \partial Y; \mathbb{Z}) \tag{18.2}$$

where $H^1(Y, \partial Y; \mathbb{Z}) = \pi_0(\mathcal{G}(\widehat{Y}, \widehat{\mathfrak{s}}))$ acts on $\pi_0(\Xi(\widehat{Y}, \widehat{\mathfrak{s}}))$ by gauge transformations. The last description in Lemma 18.3 suggests that

$$\Xi^\pi(Y, \widehat{\mathfrak{s}}_1) \cong \Xi^\pi(Y, \widehat{\mathfrak{s}}_2)$$

if $\widehat{\mathfrak{s}}_1$ and $\widehat{\mathfrak{s}}_2$ come down to the same *spin*^{c} structure on Y. In this way, $\Xi^\pi(Y, \widehat{\mathfrak{s}})$ is identified with a subset of homotopy classes of oriented relative 2-plane fields.

Now let us introduce the axioms that characterize the canonical grading of $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$.

Definition 18.5. For any configuration $\mathfrak{a} \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ and any tame perturbation $\mathfrak{q} \in \mathcal{P}(Y)$, the pair $\mathfrak{c} = (\mathfrak{a}, \mathfrak{q})$ is called **non-degenerate** if the extended Hessian $\widehat{\mathrm{Hess}}_{\mathfrak{a},\mathfrak{q}}$ is invertible. ◊

For any non-degenerate pair $\mathfrak{c} = (\mathfrak{a}, \mathfrak{q})$, we will assign an element

$$\mathbf{gr}(\mathfrak{c}) \in \pi_0(\Xi(\widehat{Y}, \widehat{\mathfrak{s}})).$$

which descends to a map

$$\mathbf{gr}^\pi : (\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P})/\mathcal{G}_{k+1}(\widehat{Y}) \dashrightarrow \Xi^\pi(\widehat{Y}, \widehat{\mathfrak{s}}), [\mathfrak{c}] \mapsto [\mathbf{gr}(\mathfrak{c})], \tag{18.3}$$

on the "non-degenerate locus" of the quotient space. To state the axioms that characterize the grading function $\mathbf{gr}$, consider a relative *spin*^{c} cobordism

$$(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_1, \widehat{\mathfrak{s}}_2).$$

We defined the moduli space $\mathcal{M}_k(\mathfrak{a}_1, \mathcal{X}, \mathfrak{a}_2)$ in Section 14, when $\mathfrak{a}_i$ is a critical point of $\mathcal{L}_{\omega_i, Y_i}$ for $i = 1, 2$. However, if we are interested only in the linear theory, one may take

$\mathfrak{a}_1$ and $\mathfrak{a}_2$ to be any configurations. Pick a reference configuration γ on $\mathcal{X}$ satisfying conditions (14.1). Then the linearized operator:

$$\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2) := (\mathbf{d}^*_\gamma, \mathcal{D}_\gamma \mathfrak{F}_{\mathcal{X},\mathfrak{p}}) : L^2_1(\mathcal{X}, iT^*\mathcal{X} \oplus S^+) \to L^2(\mathcal{X}, i\mathbb{R} \oplus i\Lambda^+\mathcal{X} \oplus S^-) \quad \text{with } \mathfrak{p} = (\mathfrak{q}_1, \mathfrak{q}_2, 0, 0) \tag{18.4}$$

is Fredholm, by Proposition 14.1, provided that $\mathfrak{c}_i = (\mathfrak{a}_i, \mathfrak{q}_i)$ is non-degenerate for $i = 1, 2$. Any such choices of γ will provide the same operator $\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ up to compact terms, so the underlying path γ is omitted from our notations.

Now we are ready to state the axioms that characterize the grading function **gr**.

(A-I) (Index Axiom) The Fredholm index of $\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ equals the relative Euler number:

$$e(S^+; \Psi_1, \Psi_*/|\Psi_*|, \Psi_2)[X, \partial X] \in \mathbb{Z}.$$

where Ψ_i is a unit-length relative spinor on $\widehat{Y}_i$ representing $\mathbf{gr}(\mathfrak{c}_i)$. Since $\Psi_1, \Psi_*/|\Psi_*|$ and Ψ_2 form a unit-length spinor of S^+ on the boundary

$$\partial X = (-Y_1) \cup [-1, 1]_t \times \Sigma \cup Y_2,$$

the relative Euler class $e(S^+; \Psi_1, \Psi_*/|\Psi_*|, \Psi_2) \in H^4(X, \partial X; \mathbb{Z})$ of this spinor is well-defined.

(A-II) (Normalization Axiom) Suppose $\mathfrak{a} = (B, \Psi) \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ is a configuration such that

(V1) Ψ is nowhere vanishing;
(V2) $\Psi \equiv \Psi_*$ on $[0, +\infty)_s \times \Sigma$, where Ψ_* is the standard spinor on $\mathbb{R}_s \times \Sigma$;
(V3) for any $\tau \geqslant 1$, define the rescaled configuration $\mathfrak{a}(\tau) := (B, \tau\Psi)$; then the extended Hessian $\widehat{\text{Hess}}_{\mathfrak{a}(\tau)}$ at $\mathfrak{a}(\tau)$ is always invertible for any $\tau \geqslant 1$.

We define that

$$\mathbf{gr}(\mathfrak{c}) = [\Psi/|\Psi|] \in \pi_0(\Xi(\widehat{Y}, \widehat{\mathfrak{s}})) \text{ if } \mathfrak{c} = (\mathfrak{a}, 0).$$

Note that $\mathfrak{a}(\tau)$ lies in a different configuration space obtained by rescaling the boundary date (λ, μ).

(A-III) (Equivariance Axiom) The grading function

$$\mathbf{gr} : \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P} \dashrightarrow \pi_0(\Xi(\widehat{Y}, \widehat{\mathfrak{s}}))$$

is equivariant under the action of $\mathcal{G}_{k+1}(\widehat{Y})$ meaning that

$$\mathbf{gr}(u \cdot \mathfrak{a}, \mathfrak{q}) = [u] \cdot \mathbf{gr}(\mathfrak{a}, \mathfrak{q})$$

for any non-generate pair $(\mathfrak{a}, \mathfrak{q})$ and $u \in \mathcal{G}_{k+1}(\widehat{Y})$.

The Index Axiom (A-I) can not determine the grading function **gr** completely. On the other hand, the Equivariance Axiom (A-III) is redundant, since it follows from (A-I)(A-II). It is added to justify the quotient map $\mathbf{gr}^\pi$ in (18.3). Here is the main result of this section:

Proposition 18.6. *There exists a unique grading function*

$$\mathbf{gr} : \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P} \dashrightarrow \Xi(\widehat{Y}, \widehat{\mathfrak{s}})$$

satisfying axioms (A-I)(A-II)(A-III).

The proof of Proposition 18.6 will dominate the rest of this subsection. It relies on two additional lemmas. On the one hand, we have to show the desired configurations in the Normalization Axiom (A-II) exist at least for some special metrics on Y.

Lemma 18.7. *For any 3-manifold Y with $\partial Y \cong \Sigma$, there exists some cylindrical metric g_Y and a configuration $\mathfrak{a} \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ that satisfies all constraints in Axiom* (A-II).

On the other hand, we have to show that Axioms (A-I) and (A-II) are consistent.

Lemma 18.8. *For any relative spin^c cobordism $(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_1, \widehat{\mathfrak{s}}_2)$, suppose non-generate pairs $\mathfrak{c}_i = (\mathfrak{a}_i, 0), i = 1, 2$ are given as in* (A-II)*, then*

$$\operatorname{Ind} \mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2) = e(S^+; \frac{\Psi_1}{|\Psi_1|}, \frac{\Psi_*}{|\Psi_*|}, \frac{\Psi_2}{|\Psi_2|})[X, \partial X],$$

where $\Psi_i \in \Gamma(\widehat{Y}_i, S)$ is the spinor component of $\mathfrak{a}_i \in \mathcal{C}_k(\widehat{Y}_i, \widehat{\mathfrak{s}}_i)$.

Proof of Lemma 18.8. This lemma is in the spirit of [KM97, Theorem 3.3] and we follow the argument therein. When X_1 is a closed Riemannian 4-manifold, the index formula:

$$\dim \mathcal{M}(X_1, \widehat{\mathfrak{s}}_{X_1}) = e(S^+)[X_1]$$

is a consequence of the Atiyah-Singer Index Theorem and [KM07, Lemma 28.2.3]. Using the excision principle, this allows us to reduce Lemma 18.8 to the special case when

$$e(S^+; \Psi_1, \Psi_*, \Psi_2)[X, \partial X] = 0.$$

At this point, choose a reference configuration $\gamma = (A, \Phi)$ on $\mathcal{X}$ such that the spinor Φ is non-vanishing everywhere, and

$$\gamma|_{\mathbb{H}^2_+ \times \Sigma} = (A_*, \Phi_*)$$

is the standard configuration on the planar end. By rescaling the spinor Φ, we define

$$\gamma(\tau) := (A, \tau\Phi).$$

which lies a different configuration space on $\mathcal{X}$. As the pair $\mathfrak{c}_i(\tau) := (\mathfrak{a}_i(\tau), \mathfrak{q}_i = 0), i = 1, 2$ are non-degenerate for any $\tau \geqslant 1$ by assumption (V3), the linearized operator at $\gamma(\tau)$ gives rise to a continuous family of Fredholm operators:

$$\mathcal{Q}(\tau) := \mathcal{Q}(\mathfrak{c}_1(\tau), \widehat{\mathfrak{s}}_X, \mathfrak{c}_2(\tau)).$$

The proof of [KM97, Lemma 3.11 & Corollary 3.12] is valid here, as $\mathfrak{q}_i = 0, i = 1, 2$. As a result, $\mathcal{Q}(\tau)$ is invertible when $\tau \gg 1$; so

$$\operatorname{Ind} \mathcal{Q}(1) = \lim_{\tau \to \infty} \operatorname{Ind} \mathcal{Q}(\tau) = 0. \qquad \square$$

Proof of Lemma 18.7. Following the proof of Lemma 18.8, one can easily show the extended Hessian $\widehat{\operatorname{Hess}}_{\mathfrak{a}(\tau)}$ is invertible when $\tau \gg 1$ for any fixed configuration $\mathfrak{a} = (B, \Psi)$ satisfying properties (V1) and (V2), but we have to pick a good metric on $\widehat{Y}$ so that this range is $[1, +\infty)$.

If Y_1 is a closed 3-manifold, one may instead rescale the metric:

$$Y_1(\tau) = (Y_1, \tau^2 g_{Y_1}).$$

and regard $\mathfrak{a}$ as a configuration on the pull-back *spin*^{c} structure on $Y(\tau)$. The Seiberg-Witten theory does not tell the difference between:

$$(Y(\tau), \mathfrak{a}) \text{ and } (Y, \mathfrak{a}(\tau)),$$

so for $\tau_0 \gg 1$, $(Y(\tau_0), \mathfrak{a})$ satisfies constraints (V1)(V3) in Axiom (A-II).

In our case, instead of rescaling the whole manifold

$$\widehat{Y} = Y \cup [0, \infty)_s \times \Sigma,$$

we rescale the compact region Y and insert a long cylinder:

$$\widehat{Y}(\tau) := Y(\tau) \cup [0, R(\tau)]_s \times \Sigma \cup [0, \infty)_s \times \Sigma.$$

The metric of $[0, R(\tau)]_s \times \Sigma$ interpolates the metrics $\tau^2 g_\Sigma$ and g_Σ at boundary. We make this interpolation mild enough by taking $R(\tau) \gg 1$. The extension of $\mathfrak{a}$ over the cylinder $[0, R(\tau)]_s \times \Sigma$:

$$(B', \Psi')$$

must interpolate (B_*, Ψ_*) at boundary in a mild way. One may use the oriented relative 2-plane field ξ_* and construct the spinor Ψ' using Lemma 18.1. Now [KM97, Lemma 3.11] applies, and all constraints in (A-II) are satisfied by

$$(\widehat{Y}(\tau_0), \tilde{\mathfrak{a}})$$

when $\tau_0 \gg 1$, where $\tilde{\mathfrak{a}}$ is the extension of $\mathfrak{a}$ on $\widehat{Y}(\tau_0)$. □

Proof of Proposition 18.6. The proof is modeled on that of [KM07, Subsection 28.2] which can now proceed with no difficulties. We first deal with the existence of **gr** and divide the proof in six steps.

Step 1. Construction. Fix a reference relative *spin*^{c} 3-manifold $(\widehat{Y}_0, \widehat{\mathfrak{s}}_0)$. Let $\mathfrak{c}_0 = (\mathfrak{a}_0, 0)$ be a non-generate pair constructed by Lemma 18.7, then the value $\mathbf{gr}(\mathfrak{c})$ is determined by (A-II). Take Ψ_0 as a unit-length relative spinor on $\widehat{Y}_0$ that represents $\mathbf{gr}(\mathfrak{c})$.

By [KM07, Proposition 28.1.2], any two relative *spin*^{c} manifolds $(\widehat{Y}_0, \widehat{\mathfrak{s}}_0)$ and $(\widehat{Y}_1, \widehat{\mathfrak{s}}_1)$ admit a relative *spin*^{c} cobordism $(\widehat{X}, \widehat{\mathfrak{s}}_X)$

$$(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_0, \widehat{\mathfrak{s}}_0) \to (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \tag{18.5}$$

The Index Axiom (A-I) then determines a unique homotopy class $[\Psi_1]$ of unit-length relative spinors on $\widehat{Y}_1$ such that

$$\operatorname{Ind} \mathcal{Q}(\mathfrak{c}_0, \widehat{\mathfrak{s}}_X, \mathfrak{c}_1) = e(S^+; \Psi_0, \Psi_*/|\Psi_*|, \Psi_1)[X, \partial X].$$

As noted in Remark 3.13, an isomorphism

$$\varphi_1 : (\widehat{X}, \widehat{\mathfrak{s}}_X)|_{\widehat{Y}_i} \cong (\widehat{Y}_1, \widehat{\mathfrak{s}}_1)$$

is always encoded in a relative *spin*^{c} cobordism. Define $\mathbf{gr}(\mathfrak{c}_1) := (\varphi_1)_*[\Psi_1] \in \pi_0(\Xi(\widehat{Y}_1, \widehat{\mathfrak{s}}_1))$.

Step 2. **gr** is well-defined. Suppose there is another relative *spin*^{c} cobordism

$$(\widehat{X}_1, \widehat{\mathfrak{s}}_{X_1}) : (\widehat{Y}_0, \widehat{\mathfrak{s}}_0) \to (\widehat{Y}_1, \widehat{\mathfrak{s}}_1), \tag{18.6}$$

then we reverse the orientation of $(\widehat{X}_1, \hat{\mathfrak{s}}_{X_1})$ and form the composition:

$$(\widehat{X}, \hat{\mathfrak{s}}_X)\#_{(\widehat{Y}_1, \hat{\mathfrak{s}}_1)}((-\widehat{X}_1), \hat{\mathfrak{s}}_{-X_1}) : (\widehat{Y}_0, \hat{\mathfrak{s}}_0) \to (\widehat{Y}_0, \hat{\mathfrak{s}}_0).$$

By Lemma 18.8 and the additivity of Fredholm indices and relative Euler classes, the values of $\mathbf{gr}(\mathfrak{c}_1)$ defined using either (18.5) or (18.6) are equal.

Step 3. Axiom (A-I) holds for $\mathbf{gr}$. The proof is similar to *Step* 2. Instead of (18.6), given any *spin*^{c} cobordism $(\widehat{X}_2, \hat{\mathfrak{s}}_{X_2}) : (\widehat{Y}_1, \hat{\mathfrak{s}}_1) \to (\widehat{Y}_1, \hat{\mathfrak{s}}_2)$, we take the pre-composition with (18.5):

$$(\widehat{X}, \hat{\mathfrak{s}}_X)\#_{(\widehat{Y}_1, \hat{\mathfrak{s}}_1)}(\widehat{X}_2, \hat{\mathfrak{s}}_{X_2}) : (\widehat{Y}_0, \hat{\mathfrak{s}}_0) \to (\widehat{Y}_2, \hat{\mathfrak{s}}_2).$$

The rest of the argument is unchanged.

Step 4. Axiom (A-II) holds for $\mathbf{gr}$. This is by Lemma 18.8.

Step 5. Uniqueness. This is clear from *Step* 1.

Step 6. Axiom (A-III). There are two ways to proceed. In *Step* 1, one may change the isomorphism ϕ_1 by an automorphism of $(\widehat{Y}_1, \hat{\mathfrak{s}}_1)$, i.e a gauge transformation $u \in \mathcal{G}_{k+1}(\widehat{Y})$. As a result, the grading function $\mathbf{gr}$ is gauge equivariant.

In the second approach, we verify the following fact: for the product manifold $X = [-1, 1]_t \times Y$ and $\mathcal{X} = \mathbb{R}_t \times \widehat{Y}$,

$$\text{Ind}\, \mathcal{Q}(\mathfrak{c}, \hat{\mathfrak{s}}, u \cdot \mathfrak{c}) = e(S^+; \Psi, \Psi_*, u \cdot \Psi)[X, \partial X]. \tag{18.7}$$

for any non-generate pair $\mathfrak{c}$ and any gauge transformation $u \in \mathcal{G}_{k+1}(\widehat{Y})$ such that $u \equiv 1$ on $[0, \infty)_s \times \Sigma$. Here Ψ is a relative spinor on $\widehat{Y}$ representing $\mathbf{gr}(\mathfrak{c})$. The identity (18.7) now follows from Lemma 13.16. □

18.2. **Canonical Mod 2 Gradings.** Now we focus a single relative *spin*^{c} 3-manifold $(\widehat{Y}, \hat{\mathfrak{s}})$. In order to define the Euler characteristic of the monopole Floer homology

$$\chi(HM_*(\widehat{Y}, \hat{\mathfrak{s}}))$$

we need a mod 2 reduction of the canonical grading $\mathbf{gr}^\pi$. For each non-generate pair $\mathfrak{c} = (\mathfrak{a}, \mathfrak{q})$, in the sense of Definition 18.5, we will assign a number

$$\mathbf{gr}^{(\mathbf{2})}(\mathfrak{c}) \in \mathbb{Z}/2\mathbb{Z}, \tag{18.8}$$

characterized by the following axioms:

(B-I) (Reduction Axiom) Let $(\widehat{X}, \hat{\mathfrak{s}}_X) = [-1, 1]_t \times (\widehat{Y}, \hat{\mathfrak{s}})$ be the product *spin*^{c} manifold. For any $\mathfrak{c}_1, \mathfrak{c}_2$ non-generate, we have

$$\mathbf{gr}^{(\mathbf{2})}(\mathfrak{c}_1) - \mathbf{gr}^{(\mathbf{2})}(\mathfrak{c}_2) = \text{Ind}\mathcal{Q}(\mathfrak{c}_1, \hat{\mathfrak{s}}_X, \mathfrak{c}_2) \mod 2,$$

(B-II) (Invariance Axiom) The mod 2 grading function

$$\mathbf{gr}^{(\mathbf{2})} : \mathcal{C}_k(\widehat{Y}, \hat{\mathfrak{s}}) \times \mathcal{P} \dashrightarrow \mathbb{Z}/2\mathbb{Z}$$

is invariant under the action of $\mathcal{G}_{k+1}(\widehat{Y})$.

Again, the Invariance Axiom (B-II) is redundant, as it follows from (B-I). One may fix the value $\mathbf{gr}^{(2)}(\mathfrak{c}_1)$ for one particular pair $\mathfrak{c}_1$ and decide the other value $\mathbf{gr}^{(2)}(\mathfrak{c}_2)$ using the Reduction Axiom (B-I), so such a mod 2 grading function $\mathbf{gr}^{(2)}$ clearly exists. It is not unique, as the value of $\mathbf{gr}^{(2)}(\mathfrak{c}_1)$ is arbitrary.

This ambiguity is fixed **simultaneously** for all relative *spin*^{c} structures $\widehat{\mathfrak{s}} \in \mathrm{Spin}^c_R(Y)$, once **a homological orientation** of $(Y, \partial Y)$ is chosen, as explained in [MT96], which is also reminiscent of the case of 4-manifolds as treated in [KM07, Subsection 24.8]. Since this story has been standard nowadays, we only give a brief sketch here.

One may alternatively think of $\mathbf{gr}^{(2)}(\mathfrak{c})$ as an orientation of the extended Hessian

$$\widehat{\mathrm{Hess}}_{\mathfrak{c}}.$$

As $\mathfrak{c}$ is non-generate, an orientation of this invertible operator $\widehat{\mathrm{Hess}}_{\mathfrak{c}}$ is equivalent to a choice of signs in $\{\pm 1\}$. However, this standpoint allows us to extend the domain of $\mathbf{gr}^{(2)}$ to the whose space $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}$. Indeed, $\{\widehat{\mathrm{Hess}}_{\mathfrak{c}}\}$ forms a continuous family of Fredholm operators, and as such gives rise to a determinant line bundle over the base:

$$\begin{array}{ccc} \mathbb{R} \cong \det \widehat{\mathrm{Hess}}_{\mathfrak{c}} & \longrightarrow & L \\ & & \big\downarrow \\ & & \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}. \end{array}$$

The real line bundle L is trivial as $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}$ is contractible. To orient L, it suffices to orient one particular fiber $L_{\mathfrak{c}}$; we choose the one at $\mathfrak{c} = (\mathfrak{a}, 0)$ such that $\mathfrak{a}$ agrees with the standard configuration:

$$(B_*, \Psi_*)$$

on the cylindrical end $[0, \infty)_s \times \Sigma$. As explained in the proof of Proposition 12.1, the extended Hessian $\widehat{\mathrm{Hess}}_{\mathfrak{a}}$ in this case is cast into the form

$$\sigma(\partial_s + \widehat{D}_{\kappa *})$$

on the cylindrical end $[0, \infty)_s \times \Sigma$, where

$$\widehat{D}_{\kappa *} : L^2_1(\Sigma, i\mathbb{R} \oplus i\mathbb{R} \oplus T^*\Sigma \oplus S) \to L^2(\Sigma, i\mathbb{R} \oplus i\mathbb{R} \oplus T^*\Sigma \oplus S) \tag{18.9}$$

is an invertible self-adjoint elliptic operator. For the precise expression, see [Wan20, Subsection 7.4]. Let $H^{\pm}$ be the $(\pm)$-spectral subspaces of $\widehat{D}_{\kappa *}$. Instead of $\widehat{\mathrm{Hess}}_{\mathfrak{a}}$, we consider the operator with a spectral boundary projection:

$$\widehat{\mathrm{Hess}}_{\mathfrak{a}} \oplus \Pi^- \circ r : L^2_k(Y, i\mathbb{R} \oplus iT^*Y \oplus S) \to L^2_{k-1}(Y, i\mathbb{R} \oplus iT^*Y \oplus S) \oplus (H^- \cap L^2_{k-1/2}). \tag{18.10}$$

on the truncated 3-manifold $Y = \{s \leqslant 0\}$. At this point, we can further deform $\mathfrak{a}$ so that $\Psi \equiv 0$, in which case

$$\widehat{\mathrm{Hess}}_{\mathfrak{a}} = \begin{pmatrix} 0 & -d & 0 \\ -d^* & *d & 0 \\ 0 & 0 & D_{B_0} \end{pmatrix} \text{ on } Y$$

for a reference $spin^c$ connection B_0, and

$$\widehat{\mathrm{Hess}}_{\mathfrak{a}} = \sigma(\partial_s + \hat{D}_0) \text{ with } D_0 = \begin{pmatrix} D_{\mathrm{Form}} & 0 \\ 0 & D^{\Sigma}_{\check{B}_*} \end{pmatrix}$$

in the collar $(-1, 0]_s \times \Sigma$. Here

$$D_{\mathrm{Form}} = \begin{pmatrix} 0 & 0 & -*_\Sigma d_\Sigma \\ 0 & 0 & -d^*_\Sigma \\ *_\Sigma d_\Sigma & -d_\Sigma & 0 \end{pmatrix} : L^2_1(\Sigma, i\mathbb{R} \oplus i\mathbb{R} \oplus iT^*\Sigma) \to L^2(\Sigma, i\mathbb{R} \oplus i\mathbb{R} \oplus iT^*\Sigma)$$

is a self-adjoint operator with kernel $H^0(\Sigma, i\mathbb{R}) \oplus H^0(\Sigma, i\mathbb{R}) \oplus H^1(\Sigma, i\mathbb{R})$ and

$$D^{\Sigma}_{\check{B}_*} : L^2_1(\Sigma, S) \to L^2(\Sigma, S)$$

is the Dirac operator on the surface, which is complex linear. Consider the projection map

$$\Pi_{\mathrm{Form}} = \Pi_1 \oplus \Pi^-_{\mathrm{Form}} : L^2(\Sigma) \to H^1(\Sigma, i\mathbb{R}) \oplus H^-_{\mathrm{Form}}.$$

where Π_{Form} is the projection map onto the negative spectral subspace of D_{Form} and Π_1 is the projection onto $H^1(\Sigma, i\mathbb{R}) \subset \ker D_{\mathrm{Form}}$.

Lemma 18.9. *The kernel and the cokernel of the operator:*

$$\begin{pmatrix} 0 & -d \\ -d^* & *d \end{pmatrix} \oplus (\Pi_{\mathrm{Form}} \circ r) : L^2_1(\mathbb{R} \oplus iT^*Y) \to L^2(\mathbb{R} \oplus iT^*Y) \oplus H^1(\Sigma, i\mathbb{R}) \oplus H^-_{\mathrm{Form}}. \tag{18.11}$$

are isomorphic to $H^0(Y; i\mathbb{R}) \oplus H^1(Y, \partial Y; \mathbb{R})$ *and* $H^0(Y, \partial Y; \mathbb{R}) \oplus H^1(Y; i\mathbb{R})$ *respectively. In particular, an orientation of* (18.11) *is equivalent to a homological orientation of* $(Y, \partial Y)$.

Finally, to relate the operator (18.10) with (18.11), we have to deform the boundary projection Π^- in (18.10). Notice that the operator $\widehat{D}_{\kappa_*}$ in (18.9) relies on the standard spinor Ψ_*. The deformation is then made by taking

$$\Psi_* \mapsto \tau \Psi_*, \tau \to 0.$$

In the limit, $\widehat{D}_{\kappa_*}$ will recover $\hat{D}_0$, which is no longer invertible. At this point, one has to examine the deformation of spectral projections very carefully, which is independent of relative $spin^c$ structures. In this way, an orientation of (18.11) gives rise to an orientation of L.

19. Floer Homology with $\mathbb{Z}$-coefficient

Let $\mathcal{R}$ be the Novikov ring of Laurent series with integral coefficients

$$\mathcal{R} = \{\sum_{n_i} a_i q^{n_i} : \; a_i \in \mathbb{Z}, \; n_i \in \mathbb{R}, \; \lim_i n_i = -\infty\}.$$

To define the monopole Floer homology over $\mathcal{R}$, we have to orient moduli spaces in a consistent way. Since the space $\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}})$ does not contain any reducible configurations, the strategy used in [KM07, Section 20] does not work directly here. Moreover, our cobordism maps are induced from oriented 4-manifold with corners. It is not crystal clear what is meant to be a homology orientation in this case.

We will address this problem using an analytic approach. The main result of this section is Theorem 19.2, which leads to the replacement of homology orientations in Definition 19.4. The proof of Theorem 19.2 relies on the notion of relative orientations that compares the determinant line bundles of two Fredholm operators in the excision principle. We will develop the relevant theory in Appendix B and accomplish the proof of Theorem 19.2 in Subsection B.10. The construction of the functor

$$HM_* : \mathrm{SCob}_{s,b} \to \mathcal{R}\text{-Mod}$$

is explained in Subsection 19.3.

19.1. **Determinant Line Bundles and Direct Sums.** To start, let us recall the basic theory of determinant line bundles of Fredholm operators from [KM07, Section 20.2]. Given two real Hilbert spaces E and F, consider a continuous family of Fredholm operators

$$\mathbb{A}_z : E \to F,\ z \in \mathcal{Z},$$

parametrized by a topological space $\mathcal{Z}$. **The determinant line bundle** of this family is a real line bundle over $\mathcal{Z}$

$$\det \mathbb{A} \to \mathcal{Z}$$

such that the fiber $\det \mathbb{A}_z$ at each $z \in \mathcal{Z}$ is identified with

$$\Lambda^{\max} \ker \mathbb{A}_z \otimes (\Lambda^{\max} \operatorname{coker} \mathbb{A}_z)^*.$$

When the determinant line bundle $\det \mathbb{A} \to \mathcal{Z}$ is orientable, denote the 2-element set of orientations by

$$\Lambda(\mathbb{A}) \text{ or } \Lambda(\det \mathbb{A}).$$

Example 19.1. Let $\mathbb{A}_* : E \to F$ be a reference Fredholm operator and $\mathcal{Z}$ be the space of all compact operators:

$$\mathcal{Z} = \{z : E \to F : z \text{ compact}\}.$$

Then the family $\{\mathbb{A}_z = \mathbb{A}_* + z : z \in \mathcal{Z}\}$ is parametrized by a contractible space $\mathcal{Z}$. An orientation of $\mathbb{A}_*$ is meant to be an orientation of this contractible family. Denote the 2-element set of orientations by

$$\Lambda(\mathbb{A}_*) \text{ or } \Lambda(\det \mathbb{A}_*). \qquad \diamondsuit$$

Given two families of operators $\mathbb{A}' \to \mathcal{Z}$ and $\mathbb{A}'' \to \mathcal{Z}$ parametrized by the same space, we form a new family by taking the point-wise direct sum of Fredholm operators

$$\mathbb{A}_z = \mathbb{A}'_z \oplus \mathbb{A}''_z : E' \oplus E'' \to F' \oplus F''.$$

Then there is a natural isomorphism of real line bundles constructed in [KM07, P.379]:

$$q : \det \mathbb{A}' \otimes \det \mathbb{A}'' \to \det \mathbb{A}. \tag{19.1}$$

Suppose α'_z and α''_z are elements in $\Lambda^{\max} \ker \mathbb{A}'_z$ and $\Lambda^{\max} \ker \mathbb{A}''_z$ respectively, while β'_z and β''_z are corresponding elements in $\Lambda^{\max} \operatorname{coker} \mathbb{A}'_z$ and $\Lambda^{\max} \operatorname{coker} \mathbb{A}''_z$. Then the bundle map q is locally defined (up to a positive scalar) by the formula:

$$\big(\alpha'_z \otimes (\beta'_z)^*\big) \otimes \big(\alpha''_z \otimes (\beta''_z)^*\big) \mapsto (-1)^r (\alpha'_z \wedge \alpha''_z) \otimes (\beta'_z \wedge \beta''_z)^* \text{ where}$$
$$r = \dim \operatorname{coker} \mathbb{A}'_z \times \operatorname{Ind}(\mathbb{A}''_z).$$

The sign $(-1)^r$ is added here to ensure that the bundle map q is continuous as the base point z varies in $\mathcal{Z}$. Moreover, the bundle map q becomes associative when we consider the direct sum of three families of operators parametrized by the same space $\mathcal{Z}$.

For any 2-element set Λ, let $\mathbb{Z}/2\mathbb{Z}$ act on Λ by involutions. For any Λ_1 and Λ_2 with $\mathbb{Z}/2\mathbb{Z}$ action, we form their product set

$$\Lambda_1\Lambda_2 := \Lambda_1 \times_{\mathbb{Z}/2\mathbb{Z}} \Lambda_2.$$

As a result, by passing to the 2-element sets of orientations, the bundle map q descends to an associative multiplication, denoted also by q:

$$q : \Lambda(\mathbb{A}') \times \Lambda(\mathbb{A}'') \to \Lambda(\mathbb{A}' \oplus \mathbb{A}''),$$

or an isomorphism preserving the $\mathbb{Z}/2\mathbb{Z}$-action:

$$q : \Lambda(\mathbb{A}')\Lambda(\mathbb{A}'') \xrightarrow{\cong} \Lambda(\mathbb{A}' \oplus \mathbb{A}'').$$

19.2. **Homology Orientations.** Having discussed the abstract properties of determinant line bundles, let us explain now the primary application in gauge theory. Given a morphism $\mathbb{X} : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2)$ in the strict cobordism category SCob_s, consider non-degenerate pairs (in the sense of Definition 18.5)

$$\mathfrak{c}_i = (\mathfrak{a}_i, \mathfrak{q}_i) \in \mathcal{C}_k(\widehat{Y}_i, \widehat{\mathfrak{s}}_i) \times \mathcal{P}(Y_i), i = 1, 2.$$

By looking at the linearized Seiberg-Witten map and the linearized gauge fixing equation on the complete Riemannian 4-manifold $\mathcal{X}$, we obtained in (18.4) a Fredholm operator $\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ for any relative $spin^c$ cobordism $(\widehat{X}, \widehat{\mathfrak{s}}_X) : (\widehat{Y}_1, \widehat{\mathfrak{s}}_1) \to (\widehat{Y}_2, \widehat{\mathfrak{s}}_2)$. Define

$$\Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2) := \Lambda(\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2))$$

for any non-degenerate pairs $\mathfrak{c}_1, \mathfrak{c}_2$ and any $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_R(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$. The 2-element set $\Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ is understood in the sense of Example 19.1. Since the different choices of the reference configuration γ will give rise to the same operator $\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ up to compact terms, $\Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ is independent of the choice of γ.

Our goal is to identify these 2-element sets $\Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ in a canonical way for all relative $spin^c$ cobordisms $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_R(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$. As a result, if the orientation is fixed for one particular $\widehat{\mathfrak{s}}_X$, then it automatically fixes the choice for any other relative $spin^c$ cobordisms.

Recall that $\mathrm{Spin}^c_R(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ is a torsor over $H^2(X, \partial X; \mathbb{Z})$.

Theorem 19.2. *For any isomorphism classe of relative line bundles $[L] \in H^2(X, \partial X; \mathbb{Z})$, there exists a natural bijection*

$$e_L : \Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2) \to \Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X \otimes L, \mathfrak{c}_2),$$

for any $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_R(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$ satisfying the following two properties:

(U1) $e_{L_1} \circ e_{L_2} = e_{L_1 \otimes L_2}$;

(U2) the collection $\{e_L\}$ is compatible with the concatenation map q meaning that the diagram

$$\begin{array}{ccc} \Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{12},\mathfrak{c}_2)\Lambda(\mathfrak{c}_2,\widehat{\mathfrak{s}}_{23},\mathfrak{c}_3) & \xrightarrow{\quad q \quad} & \Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{13},\mathfrak{c}_3) \\ \downarrow{\scriptstyle e_{L_{12}}\otimes e_{L_{23}}} & & \downarrow{\scriptstyle e_{L_{13}}} \\ \Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{12}\otimes L_{12},\mathfrak{c}_2)\Lambda(\mathfrak{c}_2,\widehat{\mathfrak{s}}_{23}\otimes L_{23},\mathfrak{c}_3) & \xrightarrow{\quad q \quad} & \Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{13}\otimes L_{13},\mathfrak{c}_3) \end{array} \tag{19.2}$$

is commutative for any relative spin^c cobordisms:

$$(\widehat{X}_{12},\widehat{\mathfrak{s}}_{12}):(Y_1,\widehat{\mathfrak{s}}_1)\to(Y_2,\widehat{\mathfrak{s}}_2),$$
$$(\widehat{X}_{23},\widehat{\mathfrak{s}}_{23}):(Y_2,\widehat{\mathfrak{s}}_3)\to(Y_3,\widehat{\mathfrak{s}}_3).$$

Here $(\widehat{X}_{13},\widehat{\mathfrak{s}}_{13})=(\widehat{X}_{12}\#\widehat{X}_{23},\widehat{\mathfrak{s}}_{12}\#\widehat{\mathfrak{s}}_{23})$ is the concatenation of relative cobordisms and $L_{13}=L_{12}\#L_{23}$ is the concatenation of relative line bundles.

Remark 19.3. The proof of Theorem 19.2 is constructive: we will construct each e_L explicitly and verify properties (U1)(U2) by hands. The key ingredient is the notion of relative orientations, which allows us to reduce the problem from a non-compact manifold $\mathcal{X}$ to a closed 4-manifold. In the latter case, we know how to construct e_L, since the Dirac operator and the self-dual operator are now decoupled. The relevant theory is developed in Appendix B. The proof of Theorem 19.2 will be accomplished in Subsection B.10. $\diamondsuit$

The horizontal maps q in the diagram (19.2) require some further explanations. Take non-degenerate pairs $\mathfrak{c}_i$ on $\widehat{Y}_i$ for $1\leqslant i\leqslant 3$. Instead of $\mathcal{Q}$, we look at operators on $\widehat{X}_{ij}$ with spectral projections:

$$\mathcal{Q}'(\mathfrak{c}_i,\widehat{\mathfrak{s}}_{ij},\mathfrak{c}_j):=D_{ij}\oplus(\Pi^+_{\mathbb{A}_i},\Pi^-_{\mathbb{A}_j})\circ(r_i,r_j), 1\leqslant i<j\leqslant 3, \tag{19.3}$$

understood in the sense of Proposition 13.9 and Subsection 13.4 adapted to the case of general cobordisms. In particular, $\Pi^{\pm}_{\mathbb{A}_i}$ are spectral projections of the extended Hessians at $\mathfrak{c}_i$:

$$\widehat{\operatorname{Hess}}_{\mathfrak{c}_i}:L^2_k(\widehat{Y}_i,i\mathbb{R}\oplus iT^*Y_i\oplus S)\to L^2_k(\widehat{Y}_i,i\mathbb{R}\oplus iT^*Y_i\oplus S),\ 1\leqslant i\leqslant 3.$$

The 2-element set $\Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{12},\mathfrak{c}_2)$ can be defined using $\mathcal{Q}'(\mathfrak{c}_i,\widehat{\mathfrak{s}}_{ij},\mathfrak{c}_j)$ instead. As explained in [KM07, P. 384], there is a canonical bundle isomorphism defined using the map (19.1),

$$q:\det\mathcal{Q}'(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{12},\mathfrak{c}_2)\otimes\det\mathcal{Q}'(\mathfrak{c}_2,\widehat{\mathfrak{s}}_{23},\mathfrak{c}_3)\to\det\mathcal{Q}'(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{13},\mathfrak{c}_3). \tag{19.4}$$

which descends to an associative multiplication:

$$q:\Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{12},\mathfrak{c}_2)\Lambda(\mathfrak{c}_2,\widehat{\mathfrak{s}}_{23},\mathfrak{c}_3)\to\Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_{13},\mathfrak{c}_3).$$

Our construction of homology orientations is based upon Theorem 19.2.

Definition 19.4. Following the notations in Theorem 19.2, for any triple $(\mathfrak{c}_1,\mathbb{X},\mathfrak{c}_2)$, define the 2-element set of **homology orientations** as the quotient space

$$\Lambda(\mathfrak{c}_1,\mathbb{X},\mathfrak{c}_2):=\coprod_{\widehat{\mathfrak{s}}_X\in\operatorname{Spin}^c_R(X;\widehat{\mathfrak{s}}_1,\widehat{\mathfrak{s}}_2)}\Lambda(\mathfrak{c}_1,\widehat{\mathfrak{s}}_X,\mathfrak{c}_2)/\{e_L\}_{[L]\in H^2(X,\partial X;\mathbb{Z})},$$

where $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ is any morphism in Cob_s and for $i = 1, 2$, $\mathfrak{c}_i \in \mathcal{C}_k(\widehat{Y}_i, \widehat{\mathfrak{s}}_i) \times \mathcal{P}(Y_i)$ is a non-degenerate pair. By the property (U2) in Theorem 19.2, the concatenation map q descends to an associative multiplication:

$$q : \Lambda(\mathfrak{c}_1, \mathbb{X}_{12}, \mathfrak{c}_2)\Lambda(\mathfrak{c}_2, \mathbb{X}_{23}, \mathfrak{c}_3) \to \Lambda(\mathfrak{c}_1, \mathbb{X}_{13}, \mathfrak{c}_3). \qquad \diamondsuit$$

Remark 19.5. If we replace $\mathcal{X}$ by a closed Riemannian 4-manifold X_1, the construction above will recover the original definition of homology orientations of X_1, i.e. orientations of the real line

$$\Lambda^{\max} H^2_+(X_1, \mathbb{R}) \otimes (\Lambda^{\max} H^1(X_1, \mathbb{R}))^*.$$

Here $H^2_+(X_1, \mathbb{R})$ is any maximal positive subspace of $H^2(X_1, \mathbb{R})$ with respect to the intersection form. $\diamondsuit$

Now let us specialize to the case when $X = [-1, 1] \times Y$ is a product cobordism and $\widehat{\mathfrak{s}}_1 = \widehat{\mathfrak{s}}_2 = \widehat{\mathfrak{s}}$. This is relevant for orienting moduli spaces on the cylinder $\mathbb{R}_t \times \widehat{Y}$. The non-degenerate pairs $\mathfrak{c}_1, \mathfrak{c}_2$ now lie in the same space:

$$\mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}(Y).$$

Definition 19.6. Let $I = [-1, 1]$. Define the 2-element set $\Lambda([\mathfrak{c}_1], [\mathfrak{c}_2])$ to be the homology orientations of $(\mathfrak{c}_1, I \times \mathbb{Y}, \mathfrak{c}_2)$ in the sense of Definition 19.4, where $[\mathfrak{c}_i]$ denotes the class in the quotient configuration space $\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}(Y)$. More concretely, $\Lambda([\mathfrak{c}_1], [\mathfrak{c}_2])$ is realized as the quotient space

$$\coprod_{[L] \in H^2(I \times Y, \partial(I \times Y); \mathbb{Z})} \Lambda(\mathfrak{c}_1, \widehat{\mathfrak{s}} \otimes L, \mathfrak{c}_2)/\{e_L\}. \qquad \diamondsuit$$

When $\mathfrak{c}_1 = \mathfrak{c}_2 \in \mathcal{C}_k(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}$, there is a canonical element $v(\mathfrak{c}_1)$ in $\Lambda([\mathfrak{c}_1], [\mathfrak{c}_1])$ induced from

$$1 \in \Lambda(\mathcal{Q}(\mathfrak{c}_1, \mathbb{R}_t \times (\widehat{Y}, \widehat{\mathfrak{s}}), \mathfrak{c}_1)).$$

In this case, we choose an $\mathbb{R}_t$-invariant configuration γ on $\mathbb{R}_t \times \widehat{Y}$ to define the operator $\mathcal{Q}(\mathfrak{c}_1, I \times (\widehat{Y}, \widehat{\mathfrak{s}}), \mathfrak{c}_1)$. Because $\mathfrak{c}_1$ is non-degenerate, $\mathcal{Q}$ is invertible. The canonical element 1 denotes the positive orientation of this invertible operator.

Remark 19.7. Here we have identified the homotopy classes of paths $\pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{c}_1], [\mathfrak{c}_2])$ with the space of relative *spin*^{c} cobordisms $\text{Spin}^c_R(I \times Y; \widehat{\mathfrak{s}}, \widehat{\mathfrak{s}})$, following the ideas in Subsection 3.5. When $\mathfrak{c} = (\mathfrak{a}, \mathfrak{q})$ is a critical point of the perturbed Chern-Simons-Dirac functional $\mathcal{L}_\omega$, the canonical element $v(\mathfrak{c})$ orients automatically the moduli space $\breve{\mathcal{M}}_z(\mathfrak{c}, \mathfrak{c})$ in (13.5) for any $z \in \pi_1(\mathcal{B}_k(\widehat{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}])$. Moreover, this orientation is compatible with concatenation of paths by the associativity of the concatenation map q. $\diamondsuit$

19.3. **Floer Homology with $\mathbb{Z}$-coefficient.** Having defined homology orientations on cylinders and general cobordisms, let us now explain the construction of $HM_*(\mathbb{Y}, \widehat{\mathfrak{s}})$ using the integral coefficient. In the most general case, we have to use a Novikov ring defined over $\mathbb{Z}$:

$$\mathcal{R} = \{\sum_{n_i} a_i q^{n_i} : \ a_i \in \mathbb{Z}, \ n_i \in \mathbb{R}, \ \lim_i n_i = -\infty\}.$$

To work with $\mathbb{Z}$ directly, we have to assume the monotonicity condition in Definition 17.3 for the object $(\mathbb{Y}, \widehat{\mathfrak{s}})$ and pass to a sub-category of SCob_s.

To better illustrate our construction below, we focus on the first case. Only formal adaptations are actually needed for the second case. At this point, we have to enlarge the strict cobordism category SCob_s slightly to incorporate a base point for each object.

Definition 19.8. An object of *the based strict cobordism category* $\mathrm{SCob}_{s,b}$ is a triple $(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*)$ where $(\mathbb{Y}, \widehat{\mathfrak{s}})$ is an object of SCob_s and $\mathfrak{c}_* = (\mathfrak{a}_*, \mathfrak{q}) \in \mathcal{C}(\widehat{Y}, \widehat{\mathfrak{s}}) \times \mathcal{P}(Y)$ is a non-degenerate pair. We require that the tame perturbation $\mathfrak{q} = \operatorname{grad} f$ is the one encoded in the object $\mathbb{Y} \in \mathrm{Cob}_s$ for the relative *spin*^{c} structure $\widehat{\mathfrak{s}}$. A morphism of $\mathrm{SCob}_{s,b}$ is a pair

$$(\mathbb{X}, o) : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1, \mathfrak{c}_{*,1}) \to (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1, \mathfrak{c}_{*,2}) \tag{19.5}$$

where $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ is a morphism in Cob_s and $o \in \Lambda(\mathfrak{c}_{*,1}, \mathbb{X}, \mathfrak{c}_{*,2})$ is a choice of homology orientations in the sense of Definition 19.4. $\diamondsuit$

The based strict cobordism category $\mathrm{SCob}_{s,b}$ is only a formal enlargement of SCob_s. The base point $\mathfrak{c}_*$ is included here to remove the ambiguity of orientations on the cylinder $\mathbb{R}_t \times \widehat{Y}$. More precisely, for any object $(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*) \in \mathrm{SCob}_{s,b}$ and for any critical point $\mathfrak{a} \in \mathrm{Crit}(\mathcal{L}_\omega)$ of $\mathcal{L}_\omega = \mathcal{L}_\omega + f$, define

$$\Lambda([\mathfrak{a}]) := \Lambda([\mathfrak{c}_*], [(\mathfrak{a}, \mathfrak{q})]),$$

and form the chain group

$$C_*(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*) = \bigoplus_{[\mathfrak{a}] \in \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}})} \mathbb{Z}\Lambda([\mathfrak{a}]) \otimes_{\mathbb{Z}} \mathcal{R}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts non-trivially on $\mathbb{Z}$ and $\mathbb{Z}\Lambda([\mathfrak{a}]) := \mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \Lambda([\mathfrak{a}])$.

Remark 19.9. For closed 3-manifolds, the role of $\mathfrak{c}_*$ is played by a reducible configuration $\mathfrak{c}_*'$ in the blown-up configuration space; see [KM07, Section 20.3]. In that case, the choice of $\mathfrak{c}_*'$ does not matter, since there is a canonical element in

$$\Lambda([\mathfrak{c}_*'], [\mathfrak{c}_*''])$$

when $\mathfrak{c}_*'$ and $\mathfrak{c}_*''$ are both reducible. However, this property does not hold in our case. $\diamondsuit$

In the formula of the differential ∂ below, we take the sum over all possible triples

$$([\mathfrak{a}], [\mathfrak{b}], z) \in \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}}) \times \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}}) \times \pi_1(\mathcal{B}_k(\mathbb{Y}, \widehat{\mathfrak{s}}); [\mathfrak{a}], [\mathfrak{b}])$$

such that $\dim \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}]) = 0$:

$$\partial = \sum_{[\mathfrak{a}]} \sum_{[\mathfrak{b}]} \sum_{z} \sum_{[\gamma] \in \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}])} \Gamma[\gamma] : C_*(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*) \to C_*(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*). \tag{19.6}$$

Since each unparameterized solution $[\gamma] \in \breve{\mathcal{M}}_z([\mathfrak{a}], [\mathfrak{b}])$ is a point, the positive orientation of γ defines an element $v([\gamma])$ in $\Lambda([(\mathfrak{a}, \mathfrak{q})], [(\mathfrak{b}, \mathfrak{q})])$. Combining with the concatenation map q, this provides a homomorphism of abelian groups:

$$\epsilon[\gamma] = \mathrm{Id}_{\mathbb{Z}} \otimes q(\cdot, v[\gamma]) : \mathbb{Z}\Lambda([\mathfrak{a}]) \to \mathbb{Z}\Lambda([\mathfrak{b}]).$$

The $\mathcal{R}$-module homomorphism $\Gamma[\gamma]$ in (19.6) is then defined by taking into account the topological energy $\mathcal{E}_{top}$:

$$\Gamma[\gamma] := \epsilon[\gamma] \otimes q^{-\mathcal{E}^{\mathfrak{q}}_{top}([\mathfrak{a}],[\mathfrak{b}];z)} : \mathbb{Z}\Lambda([\mathfrak{a}]) \otimes \mathcal{R} \to \mathbb{Z}\Lambda([\mathfrak{b}]) \otimes \mathcal{R}.$$

The differential ∂ on $C_*(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*)$ is formed by taking the sum of all $\Gamma[\gamma]$.

Now we come to define HM_* for the morphism sets of $\mathrm{SCob}_{s,b}$. For any morphism $(\mathbb{X}, o) : (\mathbb{Y}_1, \widehat{\mathfrak{s}}_1, \mathfrak{c}_{*,1}) \to (\mathbb{Y}_2, \widehat{\mathfrak{s}}_2, \mathfrak{c}_{*,2})$ of the based cobordism category $\mathrm{SCob}_{s,b}$, pick a planar metric g_X and an admissible quadruple $\mathfrak{p}$ as the perturbation. The chain map is now defined as

(19.7)
$$m(\mathbb{X}, o; g_X, \mathfrak{q}) = \sum_{[\mathfrak{a}_1]} \sum_{[\mathfrak{a}_2]} \sum_{\widehat{\mathfrak{s}}_X} \sum_{[\gamma] \in \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)} \Gamma[o, \gamma] : C_*(\mathbb{Y}_1, \widehat{\mathfrak{s}}_1, \mathfrak{c}_{*,1}) \to C_*(\mathbb{Y}_2, \widehat{\mathfrak{s}}_2, \mathfrak{c}_{*,2}),$$

where the sum is over all possible triples

$$([\mathfrak{a}_1], [\mathfrak{a}_2], \widehat{\mathfrak{s}}_X) \in \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}}_1) \times \mathfrak{C}(\mathbb{Y}, \widehat{\mathfrak{s}}_2) \times \mathrm{Spin}^c_{\mathrm{R}}(X, \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2),$$

such that $\dim \mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2) = 0$. Each solution $[\gamma]$ in $\mathcal{M}(\mathfrak{a}_1, \widehat{\mathfrak{s}}_X, \mathfrak{a}_2)$ is a 0-dimensional manifold, whose positive orientation determines a class $v([\gamma])$ in

$$\Lambda((\mathfrak{a}_1, \mathfrak{q}_1), \mathbb{X}, (\mathfrak{a}_2, \mathfrak{q}_2)).$$

We obtain a morphism

$$\epsilon[o, \gamma] : \mathbb{Z}\Lambda([\mathfrak{a}_1]) \to \mathbb{Z}\Lambda([\mathfrak{a}_2])$$

by chasing around the diagram:

$$\begin{array}{ccc}
\Lambda(\mathfrak{c}_{*,1}, (\mathfrak{a}_1, \mathfrak{q}_1)) & \xrightarrow{\quad q(\cdot,\ v([\gamma])) \quad} & \Lambda(\mathfrak{c}_{*,1}, \mathbb{X}, (\mathfrak{a}_2, \mathfrak{q}_2)) \\
\Big\downarrow{\scriptstyle \epsilon[o,\gamma]} & & \Big\| \\
\Lambda(\mathfrak{c}_{*,2}, (\mathfrak{a}_2, \mathfrak{q}_2)) & \xrightarrow{\quad q(o,\cdot) \quad} & \Lambda(\mathfrak{c}_{*,1}, \mathbb{X}, (\mathfrak{a}_2, \mathfrak{q}_2))
\end{array}$$

Here $o \in \Lambda(\mathfrak{c}_{*,1}, \mathbb{X}, \mathfrak{c}_{*,2})$ is the reference homology orientation that we picked up in the morphism $(\mathbb{X}, o)$. The $\mathcal{R}$-module homomorphism $\Gamma[o, \gamma]$ in (19.7) is defined by the formula

$$\Gamma[o, \gamma] := \epsilon[o, \gamma] \otimes q^{-\mathcal{E}^{\mathfrak{p}}_{top}([\mathfrak{a}_1],[\mathfrak{a}_2];\ \widehat{\mathfrak{s}}_X)} : \mathbb{Z}\Lambda([\mathfrak{a}_1]) \otimes \mathcal{R} \to \mathbb{Z}\Lambda([\mathfrak{a}_2]) \otimes \mathcal{R}.$$

One can verify that each $(C_*(\mathbb{Y}, \widehat{\mathfrak{s}}, \mathfrak{c}_*), \partial)$ is indeed a chain complex and $m(\mathbb{X}, o; g_X, \mathfrak{q})$ gives rise to a chain map by following the standard argument in [KM07, Section 22]. Then the functor

$$HM_* : \mathrm{SCob}_{s,b} \to \mathcal{R}\text{-Mod}$$

is obtained by taking their homology groups.

Appendix A. Harmonic Forms

In this appendix, we summarize some results on the existence of bounded harmonic forms on manifolds with cylindrical ends, which are crucial to the energy equations in Theorem 5.1 and 5.4. Let us point out where these results are used exactly in Section 3:

$$\text{Corollary A.6} \to \text{Lemma 3.2},$$
$$\text{Corollary A.15} \to \text{Lemma 3.5}.$$

Our intention is to sketch a quick proof to the results that we need. No originality is claimed for this appendix, since most of which are already standard nowadays. Unless otherwise specified, all cohomology groups in this appendix are taken over the field $\mathbb{R}$ of real numbers.

A.1. **A Review of Classical Theories.** Let (X^n, Y^{n-1}) be a compact oriented Riemannian manifold with boundary. Suppose that the metric g_X of X is cylindrical near $\partial X = Y$, i.e. g_X is a product metric

$$d^2s + g_Y$$

within a collar neighborhood $(-1, 0]_s \times Y \subset X$ of Y. By attaching a cylindrical end, we obtain a complete Riemannian manifold $\widehat{X}$:

$$\widehat{X} = X \cup [0, \infty)_s \times Y.$$

Let $\mathcal{H}^*(\widehat{X})$ be the space of L^2-harmonic forms on $\widehat{X}$ and $\mathcal{H}^*_b(\widehat{X})$ be the space of bounded harmonic forms. Each element $\omega \in \mathcal{H}^*(\widehat{X})$ decays exponentially as $s \to \infty$, and each $\lambda \in \mathcal{H}^k_b(X)$ converges exponentially to $\lambda_t + ds \wedge \lambda_n$ for some $\lambda_t \in \mathcal{H}^k(Y)$ and $\lambda_n \in \mathcal{H}^{k-1}(Y)$ along the cylindrical end of $\widehat{X}$. Here $\mathcal{H}^*(Y)$ denotes the space of harmonic forms on the boundary (Y, g_Y). Using either the Dirichlet or Neumann boundary condition at infinity, we obtain two subspaces of $\mathcal{H}^*_b(\widehat{X})$:

$$\mathcal{H}^*_D(\widehat{X}) := \{\lambda \in \mathcal{H}^*_b(\widehat{X}) : \lambda_t = 0\},$$
$$\mathcal{H}^*_N(\widehat{X}) := \{\lambda \in \mathcal{H}^*_b(\widehat{X}) : \lambda_n = 0\},$$

By [APS75, Propositon 3.15], each bounded harmonic form $\lambda \in \mathcal{H}^*_b(\widehat{X})$ is both closed and co-closed.

Proposition A.1. [APS75, Proposition 4.9] *The map that associate each L^2-harmonic form to its cohomology class*

$$\alpha : \mathcal{H}^*(\widehat{X}) \to H^*(\widehat{X}) = H^*(X)$$
$$\omega \mapsto [\omega]$$

is a bijection from $\mathcal{H}^*(\widehat{X})$ to the image

$$\mathrm{Im}(H^*(X, Y) \xrightarrow{j^*} H^*(X)),$$

with j^* induced from the inclusion map $j : (X, \varnothing) \to (X, Y)$.

To recover the singular cohomology of X, we have to look at bounded harmonic forms with boundary conditions at infinity: $\mathcal{H}^*_D(\widehat{X})$ and $\mathcal{H}^*_N(\widehat{X})$.

Proposition A.2. *$\mathcal{H}^*_N(\widehat{X}) \cong H^*(X)$ and $\mathcal{H}^*_D(\widehat{X}) \cong H^*(X, Y)$. Moreover, $\mathcal{H}^*_b(\widehat{X}) = \mathcal{H}^*_D(\widehat{X}) + \mathcal{H}^*_N(\widehat{X})$.*

The proof of Proposition A.2 relies on an index computation.

Proposition A.3. *$\dim \mathcal{H}^*_b(\widehat{X})/\mathcal{H}^*(\widehat{X}) = \dim H^*(Y)$.*

Proof of Proposition A.3. This follows from the Atiyah-Patodi-Singer Index theorem [APS75, Theorem 3.10]; see the discussion on [APS75, P.65]. An easier approach is to exploit the symmetry of the Dirac operator $d + d^*$ on the product manifold $(-1, 0]_s \times Y$ and deduce this index computation from some formal properties, using the setup of [Yos91, Section 4]. Proofs are omitted here. ☐

Proof of Proposition A.2. We follow the argument in [APS75, P.65]. For each bounded harmonic form $\lambda \in \mathcal{H}^*_N(\widehat{X})$, we assign its de Rham cohomology class:

$$\alpha(\lambda) = [\lambda] \in H^*(\widehat{X}) = H^*(X).$$

The goal is to show the map $\alpha : \mathcal{H}^*_N(\widehat{X}) \to H^*(X)$ is a bijection. For injectivity, we follow the proof of [APS75, Proposition 4.9].

To prove the surjectivity, note that the restriction of α on the subspace $\mathcal{H}^*(\widehat{X}) \subset \mathcal{H}^*_N(\widehat{X})$ is already surjective onto its image by Proposition A.1. It remains to show that the quotient map

$$\tilde{\alpha} : \mathcal{H}^*_N(\widehat{X})/\mathcal{H}^*(\widehat{X}) \to \operatorname{Coker} j^* \cong \operatorname{Im}(H^*(X) \xrightarrow{i^*} H^*(Y)).$$

is surjective, where i^* is induced by the inclusion map $i : Y \hookrightarrow X$. We prove this by a dimension counting argument. The domain of α extends to the larger space $\mathcal{H}^*_b(\widehat{X})$, and the composition

$$i^* \circ \alpha : \mathcal{H}^*_b(\widehat{X}) \xrightarrow{\alpha} H^*(X) \xrightarrow{i^*} H^*(Y)$$

maps each bounded harmonic form $\lambda \in \mathcal{H}^*_b(\widehat{X})$ to the tangential part of the limit λ_t. Hence, $\ker(i^* \circ \alpha) = \mathcal{H}^*_D(\widehat{X})$. As a result, we obtain that

$$\dim \mathcal{H}^*_N(\widehat{X})/\mathcal{H}^*(\widehat{X}) \leqslant \dim \mathcal{H}^*_b(\widehat{X})/\mathcal{H}^*_D(\widehat{X}) \leqslant \dim \operatorname{Im}(H^*(X) \xrightarrow{i^*} H^*(Y)). \tag{A.1}$$

The Hodge star operator

$$* : \Omega^*(\widehat{X}) \to \Omega^{n-*}(\widehat{X})$$

interchanges $\mathcal{H}^*_D(\widehat{X})$ with $\mathcal{H}^{n-*}_N(\widehat{X})$ and $\mathcal{H}^*(\widehat{X})$ with $\mathcal{H}^{n-*}(\widehat{X})$. The inequality (A.1) together with Poincaré duality then imply that

$$\begin{aligned} \dim \mathcal{H}^*_D(\widehat{X})/\mathcal{H}^*(\widehat{X}) = \dim \mathcal{H}^{n-*}_N(\widehat{X})/\mathcal{H}^{n-*}(\widehat{X}) &\leqslant \dim \operatorname{Im}(H^{n-*}(X) \xrightarrow{i^*} H^{n-*}(Y)) \\ &= \dim \operatorname{Im}(H^{*-1}(Y) \xrightarrow{\delta} H^*(X, Y)), \end{aligned} \tag{A.2}$$

where δ is the co-boundary map. Now, (A.1) and (A.2) imply

$$\sum_{j=0}^{n} \dim \mathcal{H}_b^j(\widehat{X})/\mathcal{H}^j(\widehat{X}) \leqslant \sum_{j=0}^{n-1} \dim H^j(Y).$$

By Proposition A.3, they are actually equal. All equalities in (A.1) and (A.2) are achieved. In particular, $\tilde{\alpha}$ is surjective and $\mathcal{H}_b^*(\widehat{X})$ is spanned by $\mathcal{H}_N^*(\widehat{X})$ and $\mathcal{H}_D^*(\widehat{X})$. The statement on $\mathcal{H}_D^*(\widehat{X})$ is obtained by applying the Hodge star operator on $\mathcal{H}_N^*(\widehat{X})$. □

Remark A.4. The isomorphism $\mathcal{H}_D^*(\widehat{X}) \cong H^*(X, Y)$ is only implicitly constructed using Poincaré duality in our proof. For a direct construction and a proof using Hodge theory on b-manifolds, see [Mel93, Section 6.4] and in particular [Mel93, Proposition 6.18]. ♢

There are two immediate corollaries of Proposition A.2.

Corollary A.5. *For a harmonic form $\mu \in \mathcal{H}^j(Y)$, the following three conditions are equivalent:*

(1) $[\mu] \in \operatorname{Im}(H^j(X) \xrightarrow{i_*} H^j(Y))$*;*
(2) There exists a closed form $\omega \in \Omega^j(\widehat{X})$ on $\widehat{X}$ such that $\omega = \mu$ on the cylindrical end $[0, \infty)_s \times Y$;
(3) There exists a bounded harmonic form $\omega_h \in \mathcal{H}_N^j(\widehat{X})$ on $\widehat{X}$ such that $\omega_h \to \mu$ exponentially as $s \to \infty$ on the cylindrical end $[0, \infty)_s \times Y$.

Proof. The equivalence (1) $\Leftrightarrow$ (3) follows from Proposition A.2. The implication (2) $\Rightarrow$ (1) is obvious. It remains to verify (1) $\Rightarrow$ (2).

Let $\omega \in \Omega^j(X)$ be a closed form on X such that $i_*[\omega] = [\mu]$. Write ω as

$$\mu_1(s) + ds \wedge \lambda(s) \text{ in the collar neighborhood } (-1, 0]_s \times Y,$$

then $\mu_1(0) - \mu = d\theta$ for some $\theta \in \Omega^{j-1}(Y)$. Using a cut-off function $\chi : (-1, 0] \to [0, 1]$ with $\operatorname{supp}\chi = [-2/3, 0]$ and $\chi \equiv 1$ on $[-1/3, 0]$, one may replace ω by $\omega - d_X(\chi(s)\theta)$.

This allows us to assume $\mu_1(0) = \mu$ to start with and define

$$\theta_1(s) = \int_s^0 \lambda(s')ds' \in \Omega^{j-1}((-1, 0] \times Y).$$

Replace ω by $\omega + d(\chi(s)\theta_1)$. As the ds-component of ω vanishes on $(-1/3, 0] \times Y$ and $d_X\omega = 0$, $\omega \equiv \mu$ in this collar neighborhood. This completes the proof. □

Corollary A.6. *For a harmonic form $\lambda \in \mathcal{H}^j(Y)$, the following two conditions are equivalent:*

(1) $[*_Y\lambda] \in \operatorname{Im}(H^{n-1-j}(X) \xrightarrow{i_*} H^{n-1-j}(Y))$*;*
(2) There exists a bounded harmonic 2-form $\omega_h \in \mathcal{H}_D^j(\widehat{X})$ on $\widehat{X}$ such that $\omega_h \to ds \wedge \lambda$ exponential as $s \to \infty$ on the cylindrical end $[0, \infty)_s \times Y$.

A.2. **Manifolds with Corners.** Our next step is to generalize Proposition A.2 in the case when X is a manifold with corners and Y has a compact boundary.

Definition A.7. A compact oriented manifold (X, Y_e, Y_b, Z) with corners is a compact space stratified by manifolds:

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} = \varnothing,$$

with the following additional properties:

(1) $X_{n-2} = Z^{n-2}$ is a closed oriented manifold of dimension $(n-2)$;
(2) $X_{n-1} = Y_e \cup Y_b$; both Y_e and Y_b are compact manifolds of dimension $(n-1)$; $Z = Y_e \cap Y_b$ is the common boundary of Y_e and Y_b;
(3) for any $y \in Y_e \backslash Z$, a neighborhood of y in X is diffeomorphic to $(-1, 0]_s \times \mathbb{R}^{n-1}$;
(4) for any $y \in Y_b \backslash Z$, a neighborhood of y in X is diffeomorphic to $(-1, 0]_t \times \mathbb{R}^{n-1}$;
(5) for any $z \in Z$, a neighborhood of z in Z is diffeomorphic to $(-1, 0]_t \times (-1, 0]_s \times \mathbb{R}^{n-2}$ and $s(z) = t(z) = 0$;
(6) The orientation of X is determined locally by $dt \wedge ds \wedge dvol_Z$. ◊

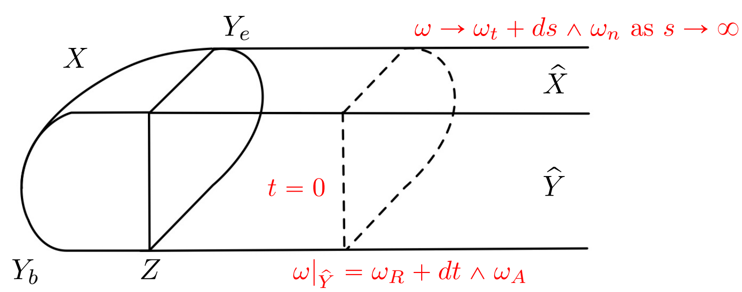


FIGURE 2. A manifold with corners and its completion.

For any such manifold with corners (X, Y_e, Y_b, Z), we can find a "collar neighborhood" U of $\partial X = Y_e \cup Y_b$ such that U is diffeomorphic to

$$(-1, 0]_t \times Y_b \cup (-1, 0]_s \times Y_e$$

and $(-1, 0]_t \times Y_b \cap (-1, 0]_s \times Y_e = (-1, 0]_t \times (-1, 0]_s \times Z$. In particular, when $t = 0$, $(-1, 0]_s \times Z$ is a collar neighborhood of $Z \subset Y_b$. The same holds for (Y_e, Z) when $s = 0$.

Definition A.8. A metric g_X is called planar if g_X restricts to a product metric on a collar neighborhood of $Y_e \cup Y_b$. In particular, within $(-1, 0]_t \times (-1, 0]_s \times Z$,

$$g_X = d^2t + d^2s + g_Z,$$

for a Riemannian metric g_Z on Z. ◊

By attaching a cylindrical end $[0, \infty)_s \times Y_e$ to X, we obtain a manifold with a non-compact boundary:

$$\widehat{X} = X \cup [0, \infty)_s \times Y_e,$$
$$\partial \widehat{X} = \widehat{Y} := Y_b \cup [0, \infty)_s \times Z.$$

The subscript "e" stands for ends, while "b" stands for boundaries. Definition A.8 ensures that the metric $g_{\widehat{X}}$ of the completion $\widehat{X}$ is a product metric in a collar neighborhood $(-1,0]_t \times \widehat{Y}$ of $\widehat{Y}$. For each smooth form $\omega \in \Omega^*(\widehat{X})$, let $\omega_R + dt \wedge \omega_A$ be the restriction of ω on $\widehat{Y}$, with $\omega_R \in \Omega^*(\widehat{Y})$ and $\omega_A \in \Omega^{*-1}(\widehat{Y})$.

Definition A.9. A smooth form $\omega \in \Omega^k(\widehat{X})$ satisfies **the absolute (relative) boundary condition** if $\omega_A = 0$ ($\omega_R = 0$). For either $\alpha = A$ or R, consider the space of L^2 (and bounded) harmonic forms on $\widehat{X}$ with the α-boundary condition:

$$\mathcal{H}^k_\alpha(\widehat{X}) = \{\omega \in \Omega^k(\widehat{X}) : d\omega = d^*\omega = 0, \omega_\alpha = 0, \omega \in L^2(X)\},$$
$$\text{and } \mathcal{H}^k_{\alpha,b}(\widehat{X}) = \{\omega \in \Omega^k(\widehat{X}) : d\omega = d^*\omega = 0, \omega_\alpha = 0, \omega \in L^\infty(X)\}. \qquad \diamondsuit$$

Remark A.10. On manifolds with boundaries, a smooth form ω is called a harmonic field if it is both closed and co-closed. A harmonic form (i.e. $\Delta\omega = 0$) is not necessarily a harmonic field in general. But there is no need to distinguish them in the case we are interested in. ◊

Analogous to Proposition A.1, we have:

Proposition A.11. *By sending each harmonic field ω to its cohomology class, we have*

$$\mathcal{H}^*_A(\widehat{X}) \cong \mathrm{Im}(H^*(X, Y_e) \to H^*(X)),$$
$$\mathcal{H}^*_R(\widehat{X}) \cong \mathrm{Im}(H^*(X, Y_e \cup Y_b) \to H^*(X, Y_b)).$$

The boundary of $\widehat{X}$ at infinity is (Y_e, Z). For either $\alpha = A$ or R, let

$$\mathcal{H}^*_\alpha(Y_e) := \{\omega \in \Omega^*(Y_e) : d\omega = d^*\omega = 0, \omega_\alpha = 0\}.$$

be the space of harmonic fields on Y_e with the α-boundary condition. The classical Hodge theory on compact manifolds with boundary then identifies these spaces with the singular cohomology of Y_e and (Y_e, Z) respectively:

Proposition A.12. [Tay11, Section 5.9] *$\mathcal{H}^*_A(Y_e) \cong H^*(Y_e)$ and $\mathcal{H}^*_R(Y_e) \cong H^*(Y_e, Z)$.*

It is also true that each bounded harmonic field $\omega \in \mathcal{H}^k_{\alpha,b}(\widehat{X})$ converges exponentially to a harmonic field $\omega_t + ds \wedge \omega_n$ as $s \to \infty$ with $\omega_t \in \mathcal{H}^k_\alpha(Y_e)$ and $\omega_n \in \mathcal{H}^{k-1}_\alpha(Y_e)$. Consider the subspaces of $\mathcal{H}^k_{\alpha,b}(\widehat{X})$ satisfying the Dirichlet or Neumann boundary condition at infinity:

$$\mathcal{H}^*_{\alpha,D}(\widehat{X}) := \{\omega \in \mathcal{H}^*_{\alpha,b}(\widehat{X}) : \omega_t = 0\},$$
$$\mathcal{H}^*_{\alpha,N}(\widehat{X}) := \{\omega \in \mathcal{H}^*_{\alpha,b}(\widehat{X}) : \omega_n = 0\}.$$

Proposition A.13. *In analogy with Proposition A.2, we have isomorphisms:*

$$\mathcal{H}^*_{A,D}(\widehat{X}) \cong H^*(X, Y_e), \qquad \mathcal{H}^*_{A,N}(\widehat{X}) \cong H^*(X),$$
$$\mathcal{H}^*_{R,D}(\widehat{X}) \cong H^*(X, Y_e \cup Y_b), \qquad \mathcal{H}^*_{R,N}(\widehat{X}) \cong H^*(X, Y_b).$$

Moreover, for either $\alpha = A$ or R, we have $\mathcal{H}^*_{\alpha,b}(\widehat{X}) = \mathcal{H}^*_{\alpha,D}(\widehat{X}) + \mathcal{H}^*_{\alpha,N}(\widehat{X})$.

As the metric of $\widehat{X}$ is cylindrical along $\widehat{Y}$, Proposition A.11 and A.13 are deduced from Proposition A.1 and A.2 using a doubling trick, as we explain in the next subsection. Their proofs do not require new inputs from analysis.

There are two immediate corollaries of Proposition A.13 that are crucial for the energy equations of the perturbed Seiberg-Witten equations. They are generalizations of Corollary A.5 and A.6 respectively.

Corollary A.14. *For a harmonic form $\mu \in \mathcal{H}^j_A(Y_e)$ satisfying the absolute boundary condition on Y_e, the following two conditions are equivalent:*

(1) $[\mu] \in \operatorname{Im}(H^j(X) \xrightarrow{i_*} H^j(Y_e))$*;*
(2) There exists a bounded harmonic 2-form $\omega_h \in \mathcal{H}^j_{A,N}(\widehat{X})$ on $\widehat{X}$ such that $\omega_h \to \mu$ exponentially as $s \to \infty$ on the cylinder $[0,\infty)_s \times (Y_e, Z)$.

Corollary A.15. *For a harmonic form $\lambda \in \mathcal{H}^{j-1}_A(Y_e)$ satisfying the absolute boundary condition on Y_e, the following two conditions are equivalent:*

(1) $[*_{Y_e}\lambda] \in \operatorname{Im}(H^{n-j}(X, Y_b) \xrightarrow{i_*} H^{n-j}(Y_e, Z))$*;*
(2) There exists a bounded harmonic 2-form $\omega_h \in \mathcal{H}^j_{A,D}(\widehat{X})$ on $\widehat{X}$ such that $\omega_h \to ds \wedge \lambda$ exponentially as $s \to \infty$ on the cylinder $[0,\infty)_s \times (Y_e, Z)$.

A.3. **The Doubling Trick.** Take a CW pair (X, Y). Let X_1 and X_2 be two copies of X with inclusions $j_i : Y \hookrightarrow X_i,\ i = 1, 2$. We form the double of X over Y by gluing X_1 and X_2 along the sub-complex Y:

$$\check{X} = X_1 \coprod_Y X_2 := X_1 \coprod X_2 / j_1(y) \sim j_2(y)\ \forall y \in Y.$$

The double space $\check{X}$ is again a CW complex and carries an involution $\tau : \check{X} \to \check{X}$ interchanging X_1 and X_2. The fixed point set of τ is precisely Y. Let $\pi : \check{X} \to X_1$ be the quotient map. The cohomology group $H^*(\widehat{X}, \mathbb{R})$ is acted on by τ^* with $(\tau^*)^2 = 1$. Let $H^*_\pm$ be the ± 1 eigenspaces of τ^* respectively.

Lemma A.16. *$H^*_+ = \pi^* H^*(X_1)$ and $H^*_- \cong H^*(X_1, Y)$. In particular,*

$$H^*(\check{X}) \cong H^*(X) \oplus H^*(X, Y).$$

Proof. Consider the Mayer-Vietoris sequence associated to the decomposition $\check{X} = X_1 \cup X_2$:

$$\begin{array}{ccccccccc} \longrightarrow & H^*(\check{X}) & \xrightarrow{\iota_1^*\oplus\iota_2^*} & H^*(X_1)\oplus H^*(X_2) & \xrightarrow{j_1^*-j_2^*} & H^*(Y) & \longrightarrow & \cdots \\ & \downarrow{\tau^*} & & \downarrow{\tau^*} & & \downarrow{\tau^*=\mathrm{Id}} & & \\ \longrightarrow & H^*(\check{X}) & \xrightarrow{\iota_1^*\oplus\iota_2^*} & H^*(X_1)\oplus H^*(X_2) & \xrightarrow{j_1^*-j_2^*} & H^*(Y) & \longrightarrow & \cdots \end{array} \tag{A.3}$$

where $\iota_i : X_i \to \check{X}$ is the inclusion map. The involution τ acts on the whole sequence. The middle vertical map is given by

$$(a, b) \mapsto (\tau^*(b), \tau^*(a)).$$

The first square is commutative, while the second one is commutative with a negative sign. The sequence (A.3) decomposes into eigenspaces of τ. For the invariant part of $H^*(\check{X})$, we have

$$\cdots \to H^+ \xrightarrow{\cong} \Delta := \{(a, \tau^*(a)) : a \in H^*(X_1)\} \to \{0\} \to \cdots .$$

Note that Δ being the image of $\pi^* H^*(X_1) \subset H^+$ under $(\iota_1^* \oplus \iota_2^*)$. This proves the first statement. As for H^-, one combines (A.3) with the long exact sequence of the pair (X_1, Y):

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^*(X_1, Y) & \longrightarrow & H^*(X_1) & \longrightarrow & H^*(Y) & \longrightarrow \cdots \\
& \Big\uparrow {\scriptstyle \mathit{excision}} & & \Big\downarrow {\scriptstyle \frac{1}{2}(\mathrm{Id}\,, -\tau^*)} & & \Big\| & \\
& H^*(\widehat{X}, X_2) & & & & & \\
& \Big\downarrow {\scriptstyle \frac{1}{2}(k^* - \tau^* k^*)} & & & & & \\
\cdots \longrightarrow & H^- & \longrightarrow & \{(a, -\tau^*(a)) : a \in H^*(X_1)\} & \longrightarrow & H^*(Y) & \longrightarrow \cdots
\end{array}$$

with k^* induced by the inclusion $k : (\check{X}, \varnothing) \to (\check{X}, X_2)$. This diagram is commutative. The middle and right vertical maps are isomorphisms. By the five-lemma, the left vertical map is also an isomorphism. $\square$

Remark A.17. For the proof of Proposition A.16, it is conceptually easier to think of diagram (A.3) at the co-chain level, where $H^*(\check{X})$ is computed by the sub-complex $C^*(X_1 + X_2)$ of $C^*(\check{X})$, see [Hat02, P.203]. $\diamondsuit$

Lemma A.16 has a relative version. Consider a triple (X, Y, W) with sub-complexes $Y, W \subset X$. We form the double spaces for pairs (X, Y) and $(W, W \cap Y)$:

$$\begin{array}{ccccc}
W \cap Y & \hookrightarrow & W & \Longrightarrow & \widehat{W} = W_1 \coprod_{W \cap Y} W_2 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
Y & \hookrightarrow & X & & \widehat{X} = X_1 \coprod_Y X_2.
\end{array}$$

Lemma A.18. *The involution τ acts on the pair $(\widehat{X}, \widehat{W})$. The ± 1 eigenspaces of τ^* on $H^*(\widehat{X}, \widehat{W})$ are isomorphic respectively to $H^*(X, W)$ and $H^*(X, W \cup Y)$:*

$$H^*_+(\widehat{X}, \widehat{W}) \cong H^*(X, W), \ \ H^*_-(\widehat{X}, \widehat{W}) \cong H^*(X, W \cup Y).$$

Using the relative Mayer-Vietoris sequence in [Hat02, P.204] instead, the proof of Lemma A.18 is identical to that of Lemma A.16.

Proof of Proposition A.13. Consider the isometric double of $\widehat{X}$ over $\widehat{Y}$:

$$\mathcal{X} = \widehat{X}_1 \coprod_{\widehat{Y}} \widehat{X}_2.$$

As oriented manifolds, $\widehat{X}_1 = \widehat{X}$ and $\widehat{X}_2 = -\widehat{X}$ is the orientation reversal of $\widehat{X}$. The isometric double $\mathcal{X}$ inherits a complete Riemannian metric from $\widehat{X}$. Alternatively, we start with the double of X over Y_b:

$$\check{X} = X_1 \coprod_{Y_b} X_2,$$

with $X_1 = X$ and $X_2 = -X$. Then $\check{X}$ is a manifold with a compact boundary $\partial\check{X} = \check{Y}_e = Y_{e,1} \coprod_Z Y_{e,2}$ furnished with a cylindrical metric. $\mathcal{X}$ is obtained from $\check{X}$ by attaching a cylindrical end.

By applying Proposition A.2 to the pair $(\check{X}, \check{Y}_e)$, we obtain that

$$\mathcal{H}^*_D(\mathcal{X}) \cong H^*(\check{X}),\ \mathcal{H}^*_N(\mathcal{X}) \cong H^*(\check{X}, \check{Y}_e). \tag{A.4}$$

Consider the involution τ^* acting on (A.4). By Lemmas A.16 and A.18,

$$H^*_+(\check{X}) \cong H^*(X), \qquad H^*_+(\check{X}, \check{Y}_e) \cong H^*(X, Y_e),$$
$$H^*_-(\check{X}) \cong H^*(X, Y_b), \qquad H^*_-(\check{X}, \check{Y}_e) \cong H^*(X, Y_e \cup Y_b).$$

It suffices to identify the action of τ^* on $\mathcal{H}^*_D(\mathcal{X})$ and $\mathcal{H}^*_N(\mathcal{X})$. We claim that

$$(\mathcal{H}^*_b)_+(\mathcal{X}) \cong \mathcal{H}^*_{A,b}(\widehat{X}),\ (\mathcal{H}^*_b)_-(\mathcal{X}) \cong \mathcal{H}^*_{R,b}(\widehat{X}),$$

and therefore

$$(\mathcal{H}^*_\beta)_+(\mathcal{X}) \cong \mathcal{H}^*_{A,\beta}(\widehat{X}),\ (\mathcal{H}^*_\beta)_-(\mathcal{X}) \cong \mathcal{H}^*_{R,\beta}(\widehat{X}),$$

for any $\beta \in \{N, D\}$. We focus on the absolute boundary condition. By restricting the harmonic field $\omega \in (\mathcal{H}^*_b)_+(\mathcal{X})$ on $\widehat{X}$, we obtain the forward map

$$\phi : (\mathcal{H}^*_b)_+(\mathcal{X}) \to \mathcal{H}^*_{A,b}(\widehat{X}).$$

Its inverse ψ is constructed by the formula:

$$\psi : \mathcal{H}^*_{A,b}(\widehat{X}) \to (\mathcal{H}^*_b)_+(\mathcal{X}),$$
$$\lambda \mapsto \omega = \begin{cases} \lambda & \text{on } \widehat{X}_1 \\ \tau^*\lambda & \text{on } \widehat{X}_2. \end{cases}$$

A priori, $\psi(\lambda)$ is only a bounded harmonic field in $L^2_{1,loc}$. By elliptic regularity, $\psi(\lambda)$ is a smooth harmonic field; so ψ a two-sided inverse of ϕ. □

The proof of Proposition A.11 is similar and omitted here.

Appendix B. Relative Orientations

The primary goal of this appendix is to present the proof of Theorem 19.2, which leads to the notion of homology orientations in Definition 19.4. It allows us to orient the moduli spaces in consistently when the complete Riemannian 4-manifold $\mathcal{X}$ possesses a planar end $\mathbb{H}^2_+ \times \Sigma$.

To do this, we have to develop the theory of **relative orientations** in a systematic way. One possible approach is to use the argument in [KM97, Appendix] in which case a Riemannian 4-manifold with a conic end is considered. The construction that we present here is slightly different. It is self-contained and combinatorial in nature, having the advantage of being very explicit and concrete. It relies on a simple proof of the excision principle of elliptic differential operators, which was due to Mrowka.

The main results are Proposition B.3 and B.12. As an application of this abstract theory, we will prove Theorem 19.2 in Subsection B.10.

B.1. **Statements.** The situation that we have here is similar to the excision principle of elliptic differential operators; we follow its setup. Given a oriented **compact** manifold Y, consider vectors bundles $E, F \to [-1,1]\times Y$ and a reference first-order elliptic differential operator:

$$D : \Gamma([-1,1]\times Y, E) \to \Gamma([-1,1]\times Y, F).$$

We are interested in two classes of elliptic differential operators

$$\mathscr{L} \text{ and } \mathscr{R}.$$

Each element of $\mathscr{L}$ consists of a pair (X_i, L_i) satisfying the following properties:

(J1) X_i is an oriented smooth manifold with boundary Y; moreover, there exists a collar neighborhood $W_i \subset X_i$ of Y and a diffeomorphism

$$\phi_i : (W_i, Y) \to ([-1,1]\times Y, \{1\}\times Y)$$

identifying W_i with the standard cylinder; X_i is not necessarily compact;

(J2) $L_i : \Gamma(X_i, E_i) \to \Gamma(X_i, F_i)$ is a first-order elliptic differential operator where $E_i, F_i \to X_i$ are vector bundles over X_i. The operator L_i is cast into a standard form on the collar neighborhood W_i in the following sense. There exist bundle isomorphisms

$$\phi_i^E : E_i|_{W_i} \to E,\ \phi_i^F : F_i|_{W_i} \to F,$$

covering the diffeomorphism $\phi_i : W_i \to [-1,1]\times Y$ in (J1) such that

$$L_i = (\phi_i^F)^{-1}\circ D\circ \phi_i^E \text{ on } W_i.$$

Similar properties are required for an element (X_j, R_j) of $\mathscr{R}$ with one distinction: the oriented boundary of X_j is $(-Y)$, so under the diffeomorphism ϕ_j, it is mapped to $\{-1\}\times(-Y)$:

$$\phi_j : (W_j, (-Y)) \to ([-1,1]\times Y, \{-1\}\times(-Y))$$

For any operators $(X_i, L_i)\in\mathscr{L}$ and $(X_j, R_j)\in\mathscr{R}$, we first glue up their underlying manifolds and obtain a manifold without boundary:

$$X_i\# X_j : X_i \coprod X_j/\sim_{ij} \text{ where } \phi_i(x_i)\sim_{ij}\phi_j(x_j) \text{ if } x_i\in W_i, x_j\in W_j.$$

Similarly we glue vector bundles and obtain $E_i \# E_j,\ F_i \# F_j \to X_i \# X_j$ using (ϕ_i^E, ϕ_j^E) and (ϕ_i^F, ϕ_j^F) instead. Finally, we glue operators and obtain

$$L_i \# R_j : \Gamma(E_i \# E_j) \to \Gamma(F_i \# F_j).$$

Assumption B.1. *The first-order elliptic differential operator*

$$L_i \# R_j : L^2_1(E_i \# E_j) \to L^2(F_i \# F_j)$$

is assumed to be Fredholm for any elements $(X_i, L_i) \in \mathscr{L}$ *and* $(X_j, R_j) \in \mathscr{R}$*.*

In terms of Example 19.1, define

$$\Lambda(L_i \# R_j)$$

to be the 2-element set of orientations of this Fredholm operator $L_i \# R_j$.

From now on, we will omit the underlying manifolds when it is clear from the context. For any operators $L_1, L_2 \in \mathscr{R}$, we wish to define a 2-element set $\Lambda(L_1, L_2)$ such that any element $x \in \Lambda(L_1, L_2)$ defines a preferred $\mathbb{Z}/2\mathbb{Z}$-equivariant map

$$\Lambda(L_1 \# R_3) \to \Lambda(L_2 \# R_3)$$

for any $R_3 \in \mathscr{R}$. We will proceed in the opposite order and first define

$$\Lambda(L_1, L_2; R_3) := \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\Lambda(L_1 \# R_3), \Lambda(L_2 \# R_3)).$$

Then the goal is to construct natural bijections:

$$p(R_3, R_4) : \Lambda(L_1, L_2; R_3) \to \Lambda(L_1, L_2; R_4) \tag{B.1}$$

for any operators $L_1, L_2 \in \mathscr{L}$ and $L_3, L_4 \in \mathscr{R}$ such that the following axioms are satisfied:

(C-I) p is associative meaning that for any three operators $R_j \in \mathscr{R}, 3 \leqslant j \leqslant 5$, we have

$$p(R_4, R_5) \circ p(R_3, R_4) = p(R_3, R_5) : \Lambda(L_1, L_2; R_3) \to \Lambda(L_1, L_2; R_5);$$

(C-II) p is reflexive meaning that $p(R_3, R_3) = \mathrm{Id}$;

(C-III) When $L_1 = L_2$, p preserves the identity element:

$$p : 1 \in \Lambda(L_1, L_1; R_3) \mapsto 1 \in \Lambda(L_1, L_1; R_4);$$

(C-IV) p commutes with compositions of Hom-sets, i.e. for any three operators $L_i \in \mathscr{L}, 0 \leqslant i \leqslant 2$, the following diagram is commutative:

$$\begin{array}{ccc} \Lambda(L_0, L_1; R_3) \times \Lambda(L_1, L_2; R_3) & \xrightarrow{m} & \Lambda(L_0, L_2; R_3) \\ \downarrow{\scriptstyle (p,p)} & & \downarrow{\scriptstyle p} \\ \Lambda(L_0, L_1; R_4) \times \Lambda(L_1, L_2; R_4) & \xrightarrow{m} & \Lambda(L_0, L_2; R_4), \end{array}$$

where horizontal arrows m are given by compositions of maps.

Definition B.2. For any classes $\mathscr{L}$ and $\mathscr{R}$, a collection of bijections $\{p\}$ satisfying axioms (C-I)-(C-IV) defines an equivalence relation on the disjoint union:

$$\coprod_{R_j \in \mathscr{R}} \Lambda(L_1, L_2; R_j).$$

Let $\Lambda(L_1, L_2)$ be the quotient space, then the composition map m descends to an associative multiplication:

$$\bar{m} : \Lambda(L_0, L_1) \times \Lambda(L_1, L_2) \to \Lambda(L_0, L_2),$$

which admits a unit in each $\Lambda(L_i, L_i)$. An element of $\Lambda(L_1, L_2)$ is called **a relative orientation** of L_1 and L_2. ♢

Here is the main result of this appendix.

Proposition B.3. *There exists a collection of bijections* $\{p(R_3, R_4)\}$ *satisfying* (C-I) – (C-IV) *for any classes of operators* $\mathscr{L}$ *and* $\mathscr{R}$ *such that Assumption* B.1 *holds.*

One can prove that the collection $\{p(R_3, R_4)\}$ is unique in a suitable sense:

Proposition B.4. *Under the assumptions of Proposition* B.3*, suppose that there are two collections of bijections* $\{p\}$ *and* $\{p'\}$ *satisfying axioms* (C-I) – (C-IV)*, then one can find a function:*

$$\iota : \mathscr{L} \times \mathscr{R} \to \mathbb{Z}/2\mathbb{Z}$$

such that

$$p(L_1, L_2; R_3, R_4) = (-1)^{\eta} p'(L_1, L_2; R_3, R_4) : \Lambda(L_1, L_2; R_3) \to \Lambda(L_1, L_2; R_4)$$

with $\eta = \iota(L_1, R_3) + \iota(L_1, R_4) + \iota(L_2, R_3) + \iota(L_2, R_4)$*. In other words,* p' *is obtained from* p *by applying the automorphism*

$$\iota(L_i, R_j) : \Lambda(L_i \# R_j) \to \Lambda(L_i \# R_j)$$

for each pair $(L_i, R_j) \in \mathscr{L} \times \mathscr{R}$*.*

Remark B.5. The proof of Proposition B.3 is constructive; see Proposition B.12 below for a refined statement. In particular, we will pick up a preferred collection $\{p\}$ for our primary applications in gauge theory. Axioms (C-II) and (C-III) are redundant, since they follow from the other two axioms. ♢

B.2. **Compatibility with Direct Sums.** Proposition B.3 will guarantee the first property (U1) in Theorem 19.2, but (U2) will require an additional property of the collection $\{p(R_3, R_4)\}$, as we explain now.

The class $\mathscr{R}$ can be extended slightly to incorporate more operators. Denote this new class by $\widehat{\mathscr{R}}$. Each element of $\widehat{\mathscr{R}}$ is a triple $\widehat{R}_j := (P_j, R_j, Q_j)$ where

- $R_j \in \mathscr{R}$;
- $P_j : H_j^a \to H_j^b$ and $Q_j : H_j^c \to H_j^d$ are arbitrary Fredholm operators; here $H_j^a \sim H_j^d$ are arbitrary Hilbert spaces.

Now instead of $L_i \# R_j$, we look at

$$L_i \# \widehat{R}_j := P_j \oplus (L_i \# R_j) \oplus Q_j : H_j^a \oplus L_1^2(E_i \# E_j) \oplus H_j^c \to H_j^b \oplus L^2(F_i \# F_j) \oplus H_j^d.$$

To extend Proposition B.3 for this new class of operators $\widehat{\mathscr{R}}$, we impose a convenient condition on the first class $\mathscr{L}$.

Definition B.6. The class of operators $\mathscr{L}$ is called even if for any $L_1, L_2 \in \mathscr{L}$,

$$\text{(B.2)} \qquad \operatorname{Ind} L_1 \# R_3 - \operatorname{Ind} L_2 \# R_3 \equiv 0 \mod 2, \ \forall R_3 \in \mathscr{R}. \qquad ♢$$

Also, we look for a normalization property on the map

$$p(R_3, \widehat{R}_3) : \Lambda(L_1, L_2; R_3) \to \Lambda(L_1, L_2; \widehat{R}_3).$$

Proposition B.7. *Suppose $\mathscr{L}$ is an even family of operators and Assumption* B.1 *holds for $(\mathscr{L}, \mathscr{R})$, then there exists a collection of bijections $\{p(\widehat{R}_3, \widehat{R}_4)\}$ satisfying axioms* (C-I)-(C-IV) *for the class $\mathscr{L}$ and $\widehat{\mathscr{R}}$. This collection satisfies the following additional property: for any $\widehat{R}_3 = (P_3, R_3, Q_3) \in \widehat{\mathscr{R}}$, the following diagram fis commutative:*

$$\begin{array}{ccc} \Lambda(P_3)\Lambda(L_1\#R_3)\Lambda(Q_3) & \xrightarrow{\mathrm{Id}\otimes x\otimes \mathrm{Id}} & \Lambda(P_3)\Lambda(L_2\#R_3)\Lambda(Q_3) \\ \downarrow q & & \downarrow q \\ \Lambda(L_1\#\widehat{R}_3) & \xrightarrow{p(R_3,\widehat{R}_3)(x)} & \Lambda(L_2\#\widehat{R}_3) \end{array} \tag{B.3}$$

for any $x \in \Lambda(L_1, L_2; L_3)$. The vertical maps are induced from (19.1)*.*

Proposition B.7 will be proved in Subsection B.9.

B.3. **Construction of Bijections.** Our construction of bijections $\{p\}$ is motivated by a simple proof of the excision principle which states that

$$\mathrm{Ind}(L_1\#R_3) + \mathrm{Ind}(L_2\#R_4) = \mathrm{Ind}(L_1\#R_4) + \mathrm{Ind}(L_2\#R_3) \tag{B.4}$$

for any $L_1, L_2 \in \mathscr{L}$ and $R_3, R_4 \in \mathscr{R}$. The author learned this elegant proof of excision principle in a graduate course at MIT, taught by Prof. Mrowka, who has kindly agreed to present his proof here.

Consider a cut-off function $\theta : [-1, 1] \to \mathbb{R}$ such that

$$\theta(t) \equiv 0 \text{ if } t \in [-1, -\frac{1}{2}];\ \theta(t) \equiv \frac{\pi}{2} \text{ if } t \in [\frac{1}{2}, 1].$$

Over each manifold $X_i\#X_j$, θ extends to a global function by setting $\theta \equiv 0$ on $X_i\backslash W_i$ and $\theta \equiv 1$ on $X_j\backslash W_j$. Consider functions $\phi_L := \cos\theta$ and $\phi_R := \sin\theta$, then the matrix

$$U = \begin{pmatrix} \phi_L & -\phi_R \\ \phi_R & \phi_L \end{pmatrix} \text{ with inverse } U^{-1} = \begin{pmatrix} \phi_L & \phi_R \\ -\phi_R & \phi_L \end{pmatrix}$$

defines an invertible operator between Hilbert spaces:

$$L^2_k(E_1\#E_3) \oplus L^2_k(E_2\#E_4) \to L^2_k(E_1\#E_4) \oplus L^2_k(E_2\#E_3)$$

for any $k \in \{0, 1\}$. The same statement holds if we use bundles F_i instead. In what follows, we write E_{ij} for $E_i\#E_j$, F_{ij} for $F_i\#F_j$ and D_{ij} for $L_i\#R_j$.

Lemma B.8. *The following diagram is commutative up to a compact operator:*

$$\begin{array}{ccc} L^2_1(E_{13}) \oplus L^2_1(E_{24}) & \xrightarrow{U} & L^2_1(E_{14}) \oplus L^2_1(E_{13}) \\ \downarrow D_{13}\oplus D_{24} & & \downarrow D_{14}\oplus D_{23} \\ L^2(F_{13}) \oplus L^2(F_{24}) & \xrightarrow{U} & L^2(F_{14}) \oplus L^2(F_{13}) \end{array} \tag{B.5}$$

Proof. Note that the inclusion $L^2_1([-1, 1] \times Y) \to L^2([-1, 1] \times Y)$ is compact, since Y is compact. □

Apparently, the excision principle (B.4) is an immediate corollary of Lemma B.8. On the other hand, the digram (B.5) also gives rise to an identification of orientations:

$$U_* : \Lambda(D_{13} \oplus D_{24}) \to \Lambda(D_{14} \oplus D_{23}) \tag{B.6}$$

understood in the sense of Example 19.1. Let us make a more precise statement:

Lemma B.9. *Suppose $\{\mathbb{A}_z : H_1 \to H_2\}_{z\in\mathcal{Z}}$ is a family of Fredholme operators parametrized by a topological space $\mathcal{Z}$. In addition, let $\{U_z : H_0 \to H_1\}_{z\in\mathcal{Z}}$ and $\{V_z : H_2 \to H_3\}_{z\in\mathcal{Z}}$ be families of invertible operators parametrized by the same space. Form the new family $\{U_z \circ \mathbb{A}_z \circ V_z : H_0 \to H_3\}_{z\in\mathcal{Z}}$, then there is continuous bundle map:*

$$(U, V)_* : \det \mathbb{A} \to \det(U \circ \mathbb{A} \circ V), \tag{B.7}$$

whose restriction at each fiber is given by

$$\alpha_z \otimes \beta_z^* \mapsto U_z^{-1}(\alpha_z) \otimes (V_z(\beta_z))^*$$

if α_z and β_z are elements in $\Lambda^{max} \ker \mathbb{A}_z$ and $\Lambda^{max} \operatorname{coker} \mathbb{A}_z$ respectively.

Proof. One has to go back to the definition of determinant line bundles in [KM07, Section 20.2] to verify that $(U, V)_*$ is continuous, using the fact that U and V are families of invertible operators. □

Remark B.10. It is clear that this construction is functorial with respect to compositions of families of invertible operators. ♢

Lemma B.11. *The bundle map* (B.7) *is functorial with respect to direct sums of operators in the following sense. Suppose $\{\mathbb{A}'_z : H'_1 \to H'_2\}_{z\in\mathcal{Z}}$ and $\{\mathbb{A}''_z : H''_1 \to H''_2\}_{z\in\mathcal{Z}}$ are two families of Fredholm operators, and similarly we have families of invertible operators:*

$$\{U'_z\}, \{U''_z\}, \{V'_z\}, \{V''_z\}.$$

as in Lemma B.9 *parametrized by the same topological space $\mathcal{Z}$.Then we have a commutative diagram:*

$$\begin{array}{ccc} \det \mathbb{A}' \otimes \det \mathbb{A}'' & \xrightarrow{(U',V')_*\otimes(U'',V'')_*} & \det(U' \circ \mathbb{A}' \circ V') \otimes \det(U'' \circ \mathbb{A}'' \circ V'') \\ \downarrow q & & \downarrow q \\ \det(\mathbb{A}' \oplus \mathbb{A}'') & \xrightarrow{(U'\oplus U'',V'\oplus V'')_*} & \det(U' \oplus U'') \circ (\mathbb{A}' \oplus \mathbb{A}'') \circ (V' \oplus V''), \end{array} \tag{B.8}$$

where vertical maps are induced from (19.1)*.*

In our primary applications, $\mathcal{Z}$ is always a contractible space; see Example 19.1. In light of Lemma B.9, the identification in (B.6) is in fact $(U^{-1}, U)_*$, but we will keep using the notation U_* for convenience. Now consider the following diagram:

$$\begin{array}{ccc} \Lambda(D_{13})\Lambda(D_{24}) & \xrightarrow{\bar{p}(R_3,R_4)} & \Lambda(D_{14})\Lambda(D_{23}) \\ \downarrow q_{13;24} & & \downarrow q_{14;23} \\ \Lambda(D_{13} \oplus D_{24}) & \xrightarrow{(-1)^r U_*} & \Lambda(D_{14} \oplus D_{23}) \end{array} \tag{B.9}$$

where vertical maps are induced from (19.1). The top horizontal arrow $\bar{p}(R_3, R_4)$ is equivalent to a map:

$$p(R_3, R_4) : \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\Lambda(D_{13}), \Lambda(D_{23})) \to \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\Lambda(D_{14}), \Lambda(D_{24}))$$

for which we are aiming in (B.1). One may define $\bar{p}(D_3, D_4)$ by making the diagram (B.9) commutative, but there is a choice of freedom for the sign $(-1)^r$. In fact, there is no reason to believe that the identification map U_* in (B.6) is just the natural one, as there are different ways to set up the excision picture.

Proposition B.12. *Suppose Assumption* B.1 *holds for the families of operators* $(\mathscr{L}, \mathscr{R})$. *We construct the bijection in* (B.1) *by declaring the diagram* (B.9) *to be commutative with*

$$r(L_1, L_2; R_3, R_4) = \mathrm{Ind}\, D_{23} \cdot \mathrm{Ind}\, D_{24} + \mathrm{Ind}\, D_{24}. \tag{B.10}$$

Then the collection of bijections $\{p(R_3, R_4)\}$ *satisfies Axioms* (C-I)-(C-IV).

The proof of Proposition B.12 will dominate Subsections B.4-B.8.

B.4. **A Toy Model.** To convince ourselves that the formula (B.10) indeed provides the correct convention, let us verify a degenerate case when $Y = \emptyset$. In this case, we assume that every L_i and R_j are Fredholm operators themselves, so

$$D_{ij} = L_i \oplus R_j,$$

and (B.9) fits into a larger diagram:

$$\begin{array}{ccc} \Lambda(L_1)\Lambda(R_3)\Lambda(L_2)\Lambda(R_4) & \xrightarrow{\tilde{p}(R_3,R_4)=\mathrm{Id}} & \Lambda(L_1)\Lambda(R_4)\Lambda(L_2)\Lambda(R_2) \\ \downarrow q_{1;3}\otimes q_{2;4} & & \downarrow q_{1;4}\otimes q_{2;3} \\ \Lambda(D_{13})\Lambda(D_{24}) & \xrightarrow{\bar{p}(R_3,R_4)} & \Lambda(D_{14})\Lambda(D_{23}) \\ \downarrow q_{13;24} & & \downarrow q_{14;23} \\ \Lambda(D_{13}\oplus D_{24}) & \xrightarrow{(-1)^r U_*} & \Lambda(D_{14}\oplus D_{23}) \end{array} . \tag{B.11}$$

If we declare the top horizontal map $\tilde{p}(R_3, R_4)$ to be the identity map, then the resulting collection $\{p(R_3, R_4)\}$ will satisfy all axioms we want. Hence, we can determine the sign $(-1)^r$ on the bottom if the digram (B.11) is commutative. In this case, the matrix U is a 4×4 matrix:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : L^2_k(E_1 \oplus E_3 \oplus E_2 \oplus E_4) \to L^2_k(E_1 \oplus E_4 \oplus E_2 \oplus E_3)$$

for $k \in \{0, 1\}$ (which is also true for F_i). To compute the sign induced from U, let us record two lemmas:

Lemma B.13. *Given* $\{\mathbb{A}'_z : H'_1 \to H'_2\}_{z\in\mathcal{Z}}$ *and* $\{\mathbb{A}''_z : H''_1 \to H''_2\}_{z\in\mathcal{Z}}$ *two families of Fredholm operators parametrized by the same topological space* $\mathcal{Z}$*, consider the permutation operator:*

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : H'_1 \oplus H'_2 \to H'_2 \oplus H'_1 \text{ and } H''_1 \oplus H''_2 \to H''_2 \oplus H''_1.$$

Then the following digram is commutative with $r_1 = \operatorname{Ind} \mathbb{A}' \cdot \operatorname{Ind} \mathbb{A}''$*:*

$$\begin{array}{ccc} \Lambda(\mathbb{A}')\Lambda(\mathbb{A}'') & \xrightarrow{\quad \mathrm{Id} \quad} & \Lambda(\mathbb{A}'')\Lambda(\mathbb{A}') \\ \big\downarrow q(\mathbb{A}',\mathbb{A}'') & & \big\downarrow q(\mathbb{A}'',\mathbb{A}') \\ \Lambda(\mathbb{A}' \oplus \mathbb{A}'') & \xrightarrow{(-1)^{r_1}(\tau^{-1},\tau)_*} & \Lambda(\mathbb{A}'' \oplus \mathbb{A}'). \end{array}$$

where vertical maps are induced from (19.1)*.*

Lemma B.14. *Given a family of Fredholm operators* $\{\mathbb{A}_z : H_1 \to H_2\}_{z\in\mathcal{Z}}$*, consider the operator*

$$\sigma = -\operatorname{Id} : H_1 \to H_1 \text{ and } H_2 \to H_2.$$

Then the map $(\sigma^{-1}, \sigma)_*$ *defined by Lemma* B.9 *equals*

$$(-1)^{\operatorname{Ind} \mathbb{A}} : \Lambda(\mathbb{A}) \to \Lambda(\mathbb{A}).$$

By Remark B.10, we decompose U into a composition of permutations and σ, so

$$\begin{aligned} r &= \operatorname{Ind} L_2(\operatorname{Ind} L_3 + \operatorname{Ind} L_4) + \operatorname{Ind} L_3 \operatorname{Ind} L_4 + \operatorname{Ind} L_4 \\ &= \operatorname{Ind} D_{23} \cdot \operatorname{Ind} D_{24} + \operatorname{Ind} D_{24}, \end{aligned}$$

by Lemma B.13 and B.14.

B.5. **Verification of Axiom (C-III).** The toy model above can partially justify the choice of r in (B.10). Let us give another reason by verifying Axiom (C-III), in which case $L_1 = L_2$. Consider the family of operators parametrized by $\tau \in [0, 1]$:

$$U_\tau = \begin{pmatrix} \cos\theta_\tau & -\sin\theta_\tau \\ \sin\theta_\tau & \cos\theta_\tau \end{pmatrix} : L^2_k(E_{23}) \oplus L^2_k(E_{24}) \to L^2_k(E_{24}) \oplus L^2_k(E_{23}),\ k \in \{0, 1\} \tag{B.12}$$

with $\theta_\tau = \theta + \tau(\pi/2 - \theta) : X_{ij} \to \mathbb{R}$, so $U_0 = U$ and

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have to verify the top horizontal map $\bar{p}(R_3, R_4)$ in (B.9) is the identity map. The diagram (B.9) remains commutative if we carry out the homotopy $\{U_\tau\}_{\tau\in[0,1]}$:

$$\begin{array}{ccc} \Lambda(D_{23})\Lambda(D_{24}) & \xrightarrow{\bar{p}(R_3,R_4)} & \Lambda(D_{24})\Lambda(D_{23}) \\ \big\downarrow q_{23;24} & & \big\downarrow q_{24;23} \\ \Lambda(D_{23} \oplus D_{24}) & \xrightarrow{(-1)^r(U_\tau)_*} & \Lambda(D_{24} \oplus D_{23}) \end{array}. \tag{B.13}$$

When $\tau = 1$, by Lemma B.13 and B.14, $\bar{p}(R_3, R_4) = \operatorname{Id}$ if we define r by (B.10).

Remark B.15. By Proposition B.4, there exist other choices of signs $(-1)^r$ in Proposition B.12 that also fulfill Axioms (C-I)-(C-IV), but (B.10) seems to be the preferred one by what we have discussed so far. In fact, the toy model in Subsection B.4 may provide a normalization axiom that removes the ambiguity in Proposition B.4. ♢

B.6. **Verification of Axiom (C-II).** In this case, $R_3 = R_4$. Analogous to (C-III), we consider the family of operators parametrized by $\tau \in [0,1]$:

$$U_\tau = \begin{pmatrix} \cos\theta'_\tau & -\sin\theta'_\tau \\ \sin\theta'_\tau & \cos\theta'_\tau \end{pmatrix},$$

with $\theta'_\tau = (1-\tau)\theta$. Then $U_0 = U$ and $U_1 = \mathrm{Id}$. In this case, $r \equiv 0 \mod 2$.

B.7. **Verification of Axiom (C-I).** For operators $R_j \in \mathscr{R}$, $j \in \{3,4,5\}$, we have to verify that

$$\bar{p}(R_3,R_4) \otimes \bar{p}(R_4,R_5) = \mathrm{Id} \otimes \bar{p}(R_3,R_5)$$

as maps:

$$\Lambda(D_{13})\Lambda(D_{24})\Lambda(D_{14})\Lambda(D_{25}) \to \Lambda(D_{14})\Lambda(D_{23})\Lambda(D_{15})\Lambda(D_{24}).$$

To do this, we introduce a huge diagram and explain the construction of each piece in 5 steps:

$$\begin{array}{ccc}
\Lambda(D_{13})\Lambda(D_{24})\Lambda(D_{14})\Lambda(D_{25}) & \xrightarrow{\bar{p}(R_3,R_4)\otimes\bar{p}(R_4,R_5)} & \Lambda(D_{14})\Lambda(D_{23})\Lambda(D_{15})\Lambda(D_{24}) \\
\downarrow{\scriptstyle q_{13,24}\otimes q_{14;25}} & \mathbb{W}_1 & \downarrow{\scriptstyle q_{14,23}\otimes q_{15;24}} \\
\Lambda(D_{13}\oplus D_{24})\Lambda(D_{14}\oplus D_{25}) & \xrightarrow{U_*\otimes U_*} & \Lambda(D_{14}\oplus D_{23})\Lambda(D_{15}\oplus D_{24}) \\
\downarrow{\scriptstyle q_{1342;1425}} & \mathbb{W}_2 & \downarrow{\scriptstyle q_{1423;1524}} \\
\Lambda(D_{13}\oplus D_{24}\oplus D_{14}\oplus D_{25}) & \xrightarrow{(V_1)_*} & \Lambda(D_{14}\oplus D_{23}\oplus D_{13}\oplus D_{24}) \\
{\scriptstyle (V_3)_*}\uparrow & \mathbb{W}_3 & {\scriptstyle (V_4)_*}\uparrow \\
\Lambda(D_{14}\oplus D_{24}\oplus D_{13}\oplus D_{25}) & \xrightarrow{(V_2)_*} & \Lambda(D_{14}\oplus D_{24}\oplus D_{15}\oplus D_{23}) \\
{\scriptstyle q_{1424;1325}}\uparrow & \mathbb{W}_4 & {\scriptstyle q_{1424;1523}}\uparrow \\
\Lambda(D_{14}\oplus D_{24})\Lambda(D_{13}\oplus D_{25}) & \xrightarrow{U_*\otimes U_*} & \Lambda(D_{14}\oplus D_{24})\Lambda(D_{15}\oplus D_{23}) \\
{\scriptstyle q_{14,24}\otimes q_{13;25}}\uparrow & \mathbb{W}_5 & {\scriptstyle q_{14,24}\otimes q_{15;23}}\uparrow \\
\Lambda(D_{14})\Lambda(D_{24})\Lambda(D_{13})\Lambda(D_{25}) & \xrightarrow{\mathrm{Id}\otimes\bar{p}(R_3,R_5)} & \Lambda(D_{14})\Lambda(D_{24})\Lambda(D_{15})\Lambda(D_{23})
\end{array} \tag{B.14}$$

Step 1. The first square $(\mathbb{W}_1)$ is the tensor of two diagrams of the form (B.9), for operators $(L_1, L_2; R_3, R_4)$ and $(L_1, L_2; R_4, R_5)$. $(\mathbb{W}_1)$ is commutative if we correct it by $(-1)^{a_1}$ where

$$a_1 := r_{12;34} + r_{12;45} \text{ with } r_{ij;kl} := r(L_i, L_j; R_k, R_l) \text{ defined by (B.10).}$$

Step 2. Similarly, the last square $(\mathbb{W}_5)$ is the tensor of two diagrams of the form (B.9), for operators $(L_1, L_2; R_4, R_4)$ and $(L_1, L_2; R_3, R_5)$. $(\mathbb{W}_5)$ is commutative if we correct it by $(-1)^{a_5}$ with

$$a_5 := r_{12;35}.$$

Step 3. In the second square $(\mathbb{W}_2)$, the bottom horizontal arrow is induced by the diagonal matrix

$$V_1 = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

The square $(\mathbb{W}_2)$ is constructed by Lemma B.11, and as such is commutative.

Step 4. In the fourth square $(\mathbb{W}_4)$, the top horizontal arrow is induced by the same matrix

$$V_2 = V_1 = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

Similarly, the square $(\mathbb{W}_2)$ is commutative by Lemma B.11.

Step 5. In the third square $(\mathbb{W}_3)$, the two vertical maps are induced respectively by

$$V_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, V_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{B.15}$$

The commutativity of $(\mathbb{W}_3)$ follows from the next lemma:

Lemma B.16. *The matrix V_2 is homotopic to the composition $V_4^{-1} \circ V_1 \circ V_3$ by a path of invertible operators:*

$$V_4^{-1} \circ V_1 \circ V_3 = \begin{pmatrix} 0 & -\phi_R & -\phi_L & 0 \\ \phi_R & 0 & 0 & \phi_L \\ \phi_L & 0 & 0 & -\phi_R \\ 0 & -\phi_L & \phi_R & 0 \end{pmatrix} : L^2_k(E_{14} \oplus E_{24} \oplus E_{14} \oplus E_{25}) \to L^2_k(E_{14} \oplus E_{24} \oplus E_{15} \oplus E_{23}), \tag{B.16}$$

for any $k \in \{0, 1\}$. The same statement holds for bundles F_{ij}.

Proof of Lemma B.16. We construct the homotopy in 2 steps. If we compare V_2 with (B.16), only positions of ϕ_L are different. It suffices to move them around by homotopy.

Step 1. Take $\tau \in [0, 1]$ and define:

$$V_2(\tau) = \begin{pmatrix} \phi_L \cos\frac{\pi\tau}{2} & -\phi_R & -\phi_L \sin\frac{\pi\tau}{2} & 0 \\ \phi_R & \phi_L & 0 & 0 \\ \phi_L \sin\frac{\pi\tau}{2} & 0 & \phi_L \cos\frac{\pi\tau}{2} & -\phi_R \\ 0 & 0 & \phi_R & \phi_L \end{pmatrix}, \ \det V_2(\tau) = \phi_L^4 + \phi_R^4 + 2\phi_L^2\phi_R^2 \cos\frac{\pi\tau}{2}.$$

Step 2. Take $\tau \in [1, 2]$ and define:

$$V_2(\tau) = \begin{pmatrix} 0 & -\phi_R & -\phi_L & 0 \\ \phi_R & \phi_L \sin\frac{\pi\tau}{2} & 0 & \phi_L \cos\frac{\pi\tau}{2} \\ \phi_L & 0 & \phi_L & -\phi_R \\ 0 & -\phi_L \cos\frac{\pi\tau}{2} & \phi_R & \phi_L \sin\frac{\pi\tau}{2} \end{pmatrix}, \ \det V_2(\tau) = \phi_L^4 + \phi_R^4 - 2\phi_L^2\phi_R^2 \cos\frac{\pi\tau}{2}.$$

Then $V_2(0) = V_0$ and $V_2(2) = V_4^{-1} \circ V_1 \circ V_3$. □

Back to the proof of (C-I). To figure out the overall sign involved in the diagram (B.14), we have to compute the compositions of all left vertical maps and all right vertical maps using Lemma B.13 and B.14. They are induced by V_3 and V_4 respectively, so the outcomes are

$$\begin{aligned} a_l &= \operatorname{Ind} D_{13} \operatorname{Ind} D_{14} + (\operatorname{Ind} D_{13} + \operatorname{Ind} D_{14}) \operatorname{Ind} D_{24} + \operatorname{Ind} D_{13}, \\ a_r &= \operatorname{Ind} D_{23} \operatorname{Ind} D_{24} + (\operatorname{Ind} D_{23} + \operatorname{Ind} D_{24}) \operatorname{Ind} D_{15} + \operatorname{Ind} D_{23}. \end{aligned}$$

We have to verify that

$$a_1 + a_5 + a_l + a_r \equiv 0 \mod 2, \tag{B.17}$$

which is the sum of 14 terms. In the computation below, we use the excision principle (B.6) and set

$$b = \operatorname{Ind} D_{1j} - \operatorname{Ind} D_{2j},\ 3 \leqslant j \leqslant 5,$$

so

$$\begin{aligned} a_1 + a_5 + a_l + a_r = {} & 2 \operatorname{Ind} D_{23} \operatorname{Ind} D_{24} + 2 \operatorname{Ind} D_{25} + (\operatorname{Ind} D_{13} + \operatorname{Ind} D_{23}) \\ & + (\operatorname{Ind} D_{23} + \operatorname{Ind} D_{24})(\operatorname{Ind} D_{15} + \operatorname{Ind} D_{25}) \\ & + \operatorname{Ind} D_{24}(1 + \operatorname{Ind} D_{14}) + \operatorname{Ind} D_{13}(\operatorname{Ind} D_{14} + \operatorname{Ind} D_{24}) \\ \equiv {} & b + (\operatorname{Ind} D_{23} + \operatorname{Ind} D_{24}) \cdot b \\ & + \operatorname{Ind} D_{24}(1 + \operatorname{Ind} D_{24}) + \operatorname{Ind} D_{24} \cdot b + \operatorname{Ind} D_{13} \cdot b \\ \equiv {} & b + b^2 \equiv 0 \mod 2. \end{aligned}$$

This completes the proof of (C-I).

Remark B.17. The computation above is not enlightening at all. However, once we know the sum (B.17) admits an expression that involves indices of D_{ij} only, one may refer to the case when $Y = \varnothing$ in Subsection B.4, as the computation does not see the difference. In that case, there is much easier to see why $\{q(R_3, R_4)\}$ are associative. ◇

B.8. **Verification of Axiom (C-IV).** We have formulated the problem in a way that is asymmetric in $\mathscr{L}$ and $\mathscr{R}$. But Axiom (C-IV) is identical to Axiom (C-III) if one interchanges the roles of $\mathscr{L}$ and $\mathscr{R}$. The proof (C-IV) follows the same line of arguments as above. For any operators $L_i \in \mathscr{L}, 0 \leqslant i \leqslant 2$, we have to verify that

$$\bar{p}(L_0, L_1; R_3, R_4) \otimes \bar{p}(L_1, L_2; R_3, R_4) = \operatorname{Id} \otimes \bar{p}(L_0, L_2; R_3, R_4)$$

as maps:

$$\Lambda(D_{03})\Lambda(D_{14})\Lambda(D_{13})\Lambda(D_{24}) \to \Lambda(D_{04})\Lambda(D_{13})\Lambda(D_{14})\Lambda(D_{23}),$$

and the corresponding diagram is:

$$
\text{(B.18)}\quad
\begin{array}{ccc}
\Lambda(D_{03})\Lambda(D_{14})\Lambda(D_{13})\Lambda(D_{24}) & \xrightarrow{\bar{p}(L_0,L_1)\otimes\bar{p}(L_1,L_2)} & \Lambda(D_{04})\Lambda(D_{13})\Lambda(D_{14})\Lambda(D_{23}) \\
\downarrow q_{03,14}\otimes q_{13;24} & & \downarrow q_{04,13}\otimes q_{14;23} \\
\Lambda(D_{03}\oplus D_{14})\Lambda(D_{13}\oplus D_{24}) & \xrightarrow{U_*\otimes U_*} & \Lambda(D_{04}\oplus D_{13})\Lambda(D_{14}\oplus D_{23}) \\
\downarrow q_{0314;1324} & & \downarrow q_{0413;1423} \\
\Lambda(D_{03}\oplus D_{14}\oplus D_{13}\oplus D_{24}) & \xrightarrow{(V_1)_*} & \Lambda(D_{04}\oplus D_{13}\oplus D_{14}\oplus D_{23}) \\
(V_3)_*\uparrow & & (V_3)_*\uparrow \\
\Lambda(D_{13}\oplus D_{14}\oplus D_{03}\oplus D_{24}) & \xrightarrow{(V_2)_*} & \Lambda(D_{14}\oplus D_{13}\oplus D_{04}\oplus D_{23}) \\
q_{1413;0324}\uparrow & & q_{1314;0423}\uparrow \\
\Lambda(D_{13}\oplus D_{14})\Lambda(D_{03}\oplus D_{24}) & \xrightarrow{U_*\otimes U_*} & \Lambda(D_{13}\oplus D_{14})\Lambda(D_{04}\oplus D_{23}) \\
q_{13,14}\otimes q_{03;24}\uparrow & & q_{14,13}\otimes q_{04;23}\uparrow \\
\Lambda(D_{14})\Lambda(D_{13})\Lambda(D_{03})\Lambda(D_{24}) & \xrightarrow{\mathrm{Id}\otimes\bar{p}(R_3,R_5)} & \Lambda(D_{14})\Lambda(D_{13})\Lambda(D_{04})\Lambda(D_{23})
\end{array}.
$$

with V_3 defined as in (B.15). Again we have to verify the sum

$$a'_1 + a'_5 + a'_l + a'_r \equiv 0 \mod 2$$

where

$$
\begin{aligned}
a'_1 &= r_{01;34} + r_{12;34} = r_{11;34} + r_{02;34} = a'_5,\\
a'_l &= \mathrm{Ind}\, D_{13}\,\mathrm{Ind}\, D_{14} + (\mathrm{Ind}\, D_{13} + \mathrm{Ind}\, D_{14})\,\mathrm{Ind}\, D_{03} + \mathrm{Ind}\, D_{03},\\
a'_r &= \mathrm{Ind}\, D_{13}\,\mathrm{Ind}\, D_{14} + (\mathrm{Ind}\, D_{13} + \mathrm{Ind}\, D_{14})\,\mathrm{Ind}\, D_{04} + \mathrm{Ind}\, D_{04}.
\end{aligned}
$$

If we set $c = \mathrm{Ind}\, D_{i3} - \mathrm{Ind}\, D_{i4},\ i \in \{0,1,2\}$, then

$$a'_1 + a'_5 + a'_l + a'_r \equiv c^2 + c \equiv 0 \mod 2.$$

The last step is to show that the matrix V_2 is homotopic to

$$
V_3^{-1}\circ V_1 \circ V_3 = \begin{pmatrix} \phi_L & 0 & 0 & -\phi_R \\ 0 & \phi_L & -\phi_R & 0 \\ 0 & \phi_R & \phi_L & 0 \\ \phi_R & 0 & 0 & \phi_L \end{pmatrix} : L^2_k(E_{13}\oplus E_{14}\oplus E_{03}\oplus E_{24}) \to L^2_k(E_{14}\oplus E_{13}\oplus E_{04}\oplus E_{23}).
$$

The homotopy is again constructed by "rotating" the four entries colored red and the other four entries colored blue. The proofs of Proposition B.3 and B.12 are now completed.

B.9. **Proof of Proposition B.7.** We claim that the construction in Proposition B.12 still works in this general setup, if $\mathscr{L}$ is an even class of operators in the sense of Definition B.6. If we stick to operators $\widehat{R} = (P, R, Q) \in \widehat{\mathscr{R}}$ with $P = \varnothing$, then the proof of Proposition B.12 remains valid, since it does not see the difference.

In the general case, let $\widehat{R}_j = (P_j, R_j, Q_j) \in \widehat{\mathscr{R}}$, $j = 3, 4$. We wish to compare $p(\widehat{R}_3, \widehat{R}_4)$ with $p(R_3, R_4)$. To illustrate, we focus on the special case when $P_3 = Q_3 = \varnothing$ and verify the following digram is commutative:

$$\begin{array}{ccc} \Lambda(D_{13})\Lambda(P_4)\Lambda(D_{24})\Lambda(Q_4) & \xrightarrow{\mathrm{Id}\otimes\bar{p}(R_3,R_4)} & \Lambda(P_4)\Lambda(D_{13})\Lambda(Q_4)\Lambda(D_{23}) \\ \downarrow{\scriptstyle \mathrm{Id}\otimes q} & & \downarrow{\scriptstyle q\otimes \mathrm{Id}} \\ \Lambda(D_{13})\Lambda(P_3\oplus D_{24}\oplus Q_4) & \xrightarrow{\bar{p}(R_3,\hat{R}_4)} & \Lambda(P_4\oplus D_{14}\oplus Q_4)\Lambda(D_{23}) \\ \downarrow{\scriptstyle q_{13;24}} & & \downarrow{\scriptstyle q_{14;23}} \\ \Lambda(D_{13}\oplus P_4\oplus D_{24}\oplus Q_4) & \xrightarrow{(-1)^r U_*} & \Lambda(P_4\oplus D_{14}\oplus Q_4\oplus D_{23}). \end{array} \tag{B.19}$$

The second square comes from the digram (B.9) with R_4 replaced by $\widehat{R}_4$; so

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \phi_L & 0 & -\phi_R & 0 \\ 0 & 0 & 0 & -1 \\ \phi_R & 0 & \phi_L & 0 \end{pmatrix}$$

and

$$r = (1 + \operatorname{Ind} D_{23}) \operatorname{Ind}(L_2 \# \widehat{R}_4) = (1 + \operatorname{Ind} D_{23})(\operatorname{Ind} P_4 + \operatorname{Ind} D_{23} + \operatorname{Ind} Q_3).$$

One may verify that the first square of (B.19) also is commutative, using diagrams like (B.14) and (B.18). The computation boils down to

$$\operatorname{Ind} P_4 \cdot (\operatorname{Ind} D_{13} + \operatorname{Ind} D_{23}) \equiv 0 \mod 2,$$

so the assumption that $\mathscr{L}$ is even is crucially here. In general, one has to verify that a digram like (B.19) commutes for arbitrary $\widehat{R}_3, \widehat{R}_4 \in \widehat{\mathscr{R}}$. This reduces the problem from $\widehat{\mathscr{R}}$ to the smaller family $\mathscr{R}$: it suffices to verify axiom (C-I)-(C-IV) for $(\mathscr{L}, \mathscr{R})$, but this is done in Proposition B.12. Details are left for the readers.

Finally, to verify the additional property (B.3), we set

$$\widehat{R}_3 = (\varnothing, R_3, \varnothing),\ \widehat{R}_4 = (P_3, R_3, Q_3),$$

in the diagram (B.19). Then we use the fact that the top arrow $\bar{p}(R_3, R_3) = \mathrm{Id}$ to conclude.

B.10. **Proof of Theorem 19.2.** Having developed the abstract theory of relative orientations, let us explain its application in gauge theory and prove Theorem 19.2. Consider a strict cobordism $X : Y_1 \to Y_2$, let

$$Y = \partial X = (-Y_1) \cup ([-1, 1] \times \Sigma) \cup Y_2.$$

We regard Y as a compact oriented 3-manifold by smoothing the corners.

For any relative *spin*^{c} cobordism $\widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)$, let the operator

$$L_{\widehat{\mathfrak{s}}_X}$$

be the restriction of the Fredholm operator $\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2)$ on X and R_* be the restriction on the complement $\mathcal{X}\backslash X$, then

$$\mathcal{Q}(\mathfrak{c}_1, \widehat{\mathfrak{s}}_X, \mathfrak{c}_2) = L_{\widehat{\mathfrak{s}}_X} \# R_*.$$

Let $\mathscr{L} = \{L_{\widehat{\mathfrak{s}}_X} : \widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)\}$ be the space of all such operators. The underlying manifold of $L_{\widehat{\mathfrak{s}}_X}$ is always the compact 4-manifold X. As for the other class $\mathscr{R}$, let X_3 be any smooth 4-manifold with boundary $(-Y)$ such that $X \# X_3$ is a closed oriented manifold and $\widehat{\mathfrak{s}}_X|_{\partial X}$ extends to a *spin*^{c} structure $\widehat{\mathfrak{s}}_3$ on $X \# X_3$. Define R_3 to be the linearized Seiberg-Witten map together with the linearized gauge fixing equation on $X \# X_3$ restricted on X_3. As a result

$$L_{\widehat{\mathfrak{s}}_X} \# R_3$$

is the linearized operator at some configuration for the closed *spin*^{c} manifold $(X \# X_3, \widehat{\mathfrak{s}}_X \# \widehat{\mathfrak{s}}_3)$. Set $\mathscr{R} = \{R_*\} \cup \{\text{all possible } (X_3, R_3)\}$. Our goal is to construct the natural bijection

$$e_E : \Lambda(L_{\widehat{\mathfrak{s}}_X} \# R_*) \to \Lambda(L_{\widehat{\mathfrak{s}}_X \otimes E} \# R_*),$$

for each relative line bundle $[E] \in H^2(X, \partial X; \mathbb{Z})$. (Here we changed the notation for a line bundle to avoid confusion). Using the set of bijections $\{p(R_3, R_4)\}$ in Proposition B.3 or B.12, we can define e_E using any compact piece (X_3, R_3) instead:

$$e_E : \Lambda(L_{\widehat{\mathfrak{s}}_X} \# R_3) \to \Lambda(L_{\widehat{\mathfrak{s}}_X \otimes E} \# R_3).$$

It is constructed as follows. The linearized operator at a reducible configuration on $X_3 \# R_3$ is

$$(d^* \oplus d^+) \oplus D_A^+$$

The second operator is complex linear, while the first one is independent of the line bundle $[E] \in H^2(X, \partial X; \mathbb{Z})$, so e_E is defined by the commutative digram

$$\begin{array}{ccc} \Lambda(d^* \oplus d^+)\Lambda(D_A^+) & \xrightarrow{\mathrm{Id} \otimes h} & \Lambda(d^* \oplus d^+)\Lambda(D_{A'}^+) \\ \big\downarrow q & & \big\downarrow q \\ \Lambda(L_{\widehat{\mathfrak{s}}_X} \# R_3) & \xrightarrow{\quad e_E \quad} & \Lambda(L_{\widehat{\mathfrak{s}}_X \otimes L} \# R_3), \end{array}$$

where $h : \Lambda(D_A^+) \to \Lambda(D_{A'}^+)$ preserves the complex orientations. Notice that $\{e_E\}$ is independent of the compact piece (X_3, R_3) by our construction of $\{p(R_3, R_4)\}$.

Now the first property (U1) of Theorem 19.2 follows from Axiom (C-IV).

As for (U2), it suffices to address the special case when either $[E_{12}] = 0$ or $[E_{23}] = 0$. Technically, we have to work with the operators $\mathcal{Q}'$ defined in (19.3), which involve manifolds with boundary and spectral projections. We can enlarge the family $\mathscr{R}$ to incorporate such operators, so it is not a problem.

At this point, we conclude using the additional property (B.3) in Proposition B.7 by setting either $P_3 = \varnothing$ or $Q_3 = \varnothing$. The assumption is verified by the next lemma.

Lemma B.18. *The class of operators* $\mathscr{L} := \{L_{\widehat{\mathfrak{s}}_X} : \widehat{\mathfrak{s}}_X \in \mathrm{Spin}^c_{\mathrm{R}}(X; \widehat{\mathfrak{s}}_1, \widehat{\mathfrak{s}}_2)\}$ *is even in the sense of Definition* B.6*.*

Proof of Lemma. By the excision principle, it suffices to verify the condition (B.2) for a special operator $(X_3, R_3) \in \mathscr{R}$. In particular, we take (X_3, R_3) to be a compact piece. □

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