Tau invariants in monopole and instanton theories

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Abstract

In this paper we study the tau invariants in $\underline{\mathrm{KHM}}^{-}$ and $\underline{\mathrm{KHI}}^{-}$ for knots in S^3, which were defined by the author in [11], and we will prove some basic properties such as concordant invariance. We also use the technique in the current paper to compute the $\underline{\mathrm{KHM}}^{-}$ and $\underline{\mathrm{KHI}}^{-}$ for twisted knots.

Contents

1 Introduction

In [11], the author defined $\underline{\mathrm{KHM}}^{-}$ and $\underline{\mathrm{KHI}}^{-}$, served as the monopole and instanton analogue of HFK^{-}. The author also defined the tau invariants τ_M and τ_I to be the maximal grading of a non-U-torsion elements. In the present paper, we further study the properties of tau invariants for knots inside S^3 and prove that they are concordance invariants. Also we carried out some computations for twisted knots.

To state the results in a unified way, we will use $\underline{\mathrm{KHG}}^{-}$ to denote both $\underline{\mathrm{KHM}}^{-}$ and $\underline{\mathrm{KHI}}^{-}$. The letter G means 'gauge theoritic'. We will write the coefficient ring as $\mathcal{R}$ but it should be understood to be the mod 2 Novikov ring for $\underline{\mathrm{KHG}}^{-}$ and the field of complex numbers $\mathbb{C}$ for $\underline{\mathrm{KHI}}^{-}$. Similarly throughout the paper we will use $\underline{\mathrm{SHG}}$ to denote either $\underline{\mathrm{SHM}}$ or $\underline{\mathrm{SHI}}$ and use τ_G to denote either τ_M or τ_I. The main results of the paper are summarized as follows.

Proposition 1.1. *Suppose $K \subset S^3$ is a knot and $p \in K$ is a fixed base point. Then $\underline{\mathrm{KHG}}^{-}(-S^3, K, p)$ has a unique infinite U-tower.*

Proposition 1.2. *The tau invariants $\tau_G(K)$ is a concordance invariance.*

Proposition 1.3. *Let K_m be the twisted knots as shown in figure 1. Let $p \in K_m$ be a base point.*

(1). If $m>0$, then

$$\underline{\mathrm{KHG}}^-(-S^3,K_m,p)\cong\mathcal{R}[U]_1\oplus(\mathcal{R}_1)^{m-1}\oplus(\mathcal{R}_0)^m$$

and

$$\underline{\mathrm{KHG}}^-(-S^3,\bar{K}_m,p)\cong\mathcal{R}[U]_{-1}\oplus(\mathcal{R}_1)^{m}\oplus(\mathcal{R}_0)^{m-1}.$$

(2). If $m\leqslant 0$, then

$$\underline{\mathrm{KHG}}^-(-S^3,K_m,p)\cong\underline{\mathrm{KHG}}^-(-S^3,\bar{K}_m,p)\cong\mathcal{R}[U]_{-1}\oplus(\mathcal{R}_1)^{m}\oplus(\mathcal{R}_0)^{m-1}.$$

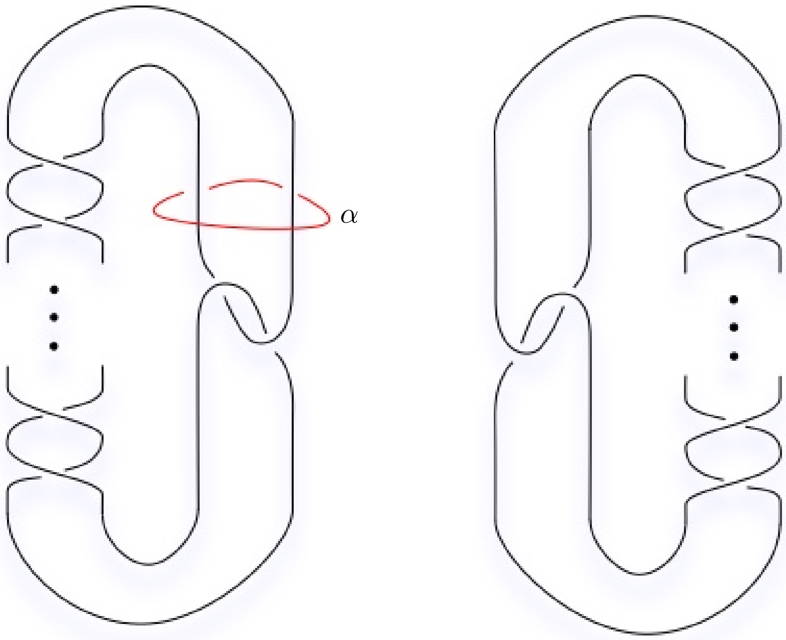

Figure 1: The one handle.

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2 Preliminaries

Suppose $K\subset S^3$ is an oriented knot and $S\subset S^3$ is a (minimal genus) Seifert surface of K. The surface S induces a framing on the boundary of the knot complement $S^3(K)$, so we get longitude curve λ and meridian curve μ. Let Γ_n be the union of two disjoint, parallel, oppositely oriented simple closed curves of slope $-n$ (or equivalently of class $\pm([\lambda] - n[\mu]) \in H_1(\partial S^3(K))$), and let Γ_μ be the union of two meridians. In Kronheimer and Mrowka [8], the suture $\Gamma_\mu \subset \partial S^3(K)$ was used to construct their monopole and instanton knot Floer homologies.

Definition 2.1. Let $p \in K$ be a base point, then we can define

$$\underline{\mathrm{KHM}}(Y, K, p) = \underline{\mathrm{SHM}}(S^3(K), \Gamma_\mu).$$

Here $\underline{\mathrm{SHM}}$ is the projective transitive system for defining sutured monopole Floer homology as in Baldwin and Sivek [2], over the coefficient ring $\mathcal{R}$ which is the mod 2 Novikov ring.

The choice of base point seems not to appear in the above definition but it actually crucial to resolving the extra ambiguity in defining $\underline{\mathrm{KHM}}$, arising from the choices of knot complements, which is not an issue for defining $\underline{\mathrm{SHM}}$. So when we write $S^3(K)$, we actually means a collection of knot complements so that each pair of them is related by a canonical diffeomorphism. Similarly, we can define the following in instanton theory.

Definition 2.2. Let $p \in K$ be a base point, then we can define

$$\underline{\mathrm{KHI}}(Y, K, p) = \underline{\mathrm{SHI}}(S^3(K), \Gamma_\mu).$$

Here $\underline{\mathrm{SHI}}$ is the projective transitive system for defining sutured monopole and instanton Floer homologies as in Baldwin and Sivek [2], over the complex number $\mathbb{C}$.

In the author's previous paper [11], the sutures Γ_n are used to construct a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_n) \arrow[r, "\psi_{n,-}"] \arrow[d, "\psi_{n,+}"] & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n+1}) \arrow[r] \arrow[d, "\psi_{n+1,+}"] & \cdots \\ \cdots \arrow[r] & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n+1}) \arrow[r, "\psi_{n+1,-}"] & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n+2}) \arrow[r] & \cdots \end{tikzcd} \tag{1}$$

Definition 2.3. Define $\underline{\mathrm{KHG}}^-(-S^3, K, p)$ as the direct limit of the horizontal direct system, after a proper grading shifting. The collection of maps $\{\psi_{n,+}\}_{n\in\mathbb{Z}_+}$ defines a map on the direct limit:

$$U : \underline{\mathrm{KHG}}^-(-S^3, K, p) \to \underline{\mathrm{KHG}}^-(-S^3, K, p).$$

In [11] the author proved that a Seifert surface S induces a $\mathbb{Z}$-grading (Alexander grading) on $\underline{\mathrm{KHG}}^-$ and U is of degree -1. Since we only work with knots inside S^3, the gradings induced by different choices of Seifert surfaces are the same, so we will simply write the grading as

$$\underline{\mathrm{KHG}}^-(-S^3, K, p, i),\ i \in \mathbb{Z}.$$

Following the definition of tau invariant in Heegaard Floer homology by Ozsváth and Szabó [13], we could make the following definition.

Definition 2.4. $\tau_G(K) = \max\{i \in \mathbb{Z} | \exists\ x \in \underline{\mathrm{KHG}}^-(-S^3, K, p, i),\ U^j(x) \neq 0,\ \forall j \in \mathbb{Z}_+\}$.

In the present paper, we can focus on the rank of $\underline{\mathrm{KHG}}^-$. For $\underline{\mathrm{KHI}}^-$, this is clear since it is a vector space over $\mathbb{C}$ and the rank just means the dimension as a complex vector space. For $\underline{\mathrm{KHM}}^-$, we refer to the following lemma.

Lemma 2.5. *For any pair (K, p) and any $i \in \mathbb{Z}$, we have that $\underline{\mathrm{KHG}}^-(-S^3, K, p, i)$ is a finite rank free $\mathcal{R}$ module.*

Proof. From [11] we know that there exists a large enough $n \in \mathbb{Z}$ and integer $j \in \mathbb{Z}$ so that

$$\underline{\mathrm{KHM}}^-(-S^3, K, p, i) \cong \underline{\mathrm{SHM}}(-S^3(K), -\Gamma_n, j).$$

Here i might not be equal to j because we have done a grading shifting in the definition of $\underline{\mathrm{KHM}}^-$. Hence clearly it is of finite rank.

To prove that it is free, recall that from Kronheimer and Mrowka [8], for any balanced sutured manifold (M, γ), we have

$$SHM(M, \gamma; \Gamma_\eta) \cong SHM(M, \gamma; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathcal{R}.$$

This isomorphism respects the grading that can be defined on both sides. The usage of $\mathbb{Z}_2$ coefficients is valid because of Sivek [15]. □

3 Concordance invariance of τ_G.

Throughout the section we have a knot $K \subset S^3$ and the suture Γ_n and Γ_μ on $\partial S^3(K)$ which are described as in the previous section. Fix $n \in \mathbb{Z}_+$. Then on $\partial S^3(K)$, we can pick a meridional curve α (irrelevant to the curve α as in figure 1) so that α intersects the suture Γ_n twice. Let $\partial S^3(K) \times [-1, 0] \subset S^3(K)$ be a collar of $\partial S^3(K)$ inside the knot complement $S^3(K)$ and we can give an $[-1, 0]$-invariant tight contact structure on $\partial S^3(K) \times [-1, 0]$, so that each slice $\partial S^3(K) \times \{t\}$ for $t \in [-1, 0]$ is convex and the dividing set is (isotopic to) Γ_n. By Legendrian realization principle, we can push α into the interior of the collar $\partial S^3(K) \times [-1, 0]$ and get a Legendrian curve β. With respect to the surface framing, the curve β has $tb = -1$. When talking about framings of β, we will always refer to the surface framing with respect to $\partial S^3(K)$.

From Baldwin and Sivek [3], since α intersects the suture (or the dividing set) Γ_n twice, (after making α Legendrian) we can glue a contact 2-handle to $(S^3(K), \Gamma_n)$ along α, and get a new balanced sutured manifold (M, γ). Suppose (Y, R) is a closure of $(S^3(K), \Gamma_n)$ in the sense of Kronheimer and Mrowka [8], so that $g(R)$ is large enough, then from Baldwin and Sivek [1] we know that a closure (Y_0, R) of (M, γ) can be obtained from (Y, R) by performing a 0-Dehn surgery along the curve β. Note inside Y, β is disjoint from R so the surgery can be made disjoint from R and hence the surface R survives in Y_0. Now let (M_{-1}, Γ_n) be the balanced sutured manifold obtained from $(S^3(K), \Gamma_n)$ by performing a (-1)-Dehn surgery. Note β is contained in the interior of $S^3(K)$ so the surgery does not influence the boundary as well as the suture.

Clearly, if we do a (-1)-Dehn surgery along β on Y, we will get a closure (Y_{-1}, R) for the balanced sutured manifold (M_{-1}, Γ_n). Applying the surgery exact triangle, we get the following.

$$\begin{tikzcd} \underline{\mathrm{SHG}}(-M_{-1}, -\Gamma_n) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_n) \arrow[dl, "C_{h,n}"] \\ & \underline{\mathrm{SHG}}(-M, -\gamma) \arrow[ul] & \end{tikzcd}$$

Remark 3.1. Here the surgery exact triangle seems to go in the wrong direction but note the sutured manifolds have been reversed the orientations, so the maps in the surgery exact triangle shall also reverse the directions.

Now let us figure out what are (M, γ) and (M_{-1}, Γ_n). First (M, γ) is obtained from $(S^3(K), \Gamma_n)$ by attaching a contact 2-handle along a meridional curve α, so it is nothing but (D^3, δ), where δ is a connected simple closed curve on ∂D^3. To figure out (M_{-1}, Γ_n), note that we can view β and K inside the 3-sphere S^3 and β is a meridional link around K. So a (-1)-Dehn surgery along β on $S^3(K)$ will result in the same 3-manifold $S^3(K)$ while the framings on it is 1 larger than before performing the surgery (See Rolfsen [14]). Hence we conclude that $(M_{-1}, \Gamma_n) \cong (S^3(K), \Gamma_{n-1})$ (Note the framing of Γ_n is $-n$). Thus the above exact triangle becomes

$$\begin{tikzcd} \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n-1}) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(K), -\Gamma_n) \arrow[dl, "C_{h,n}"] \\ & \underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[ul] & \end{tikzcd} \tag{2}$$

Lemma 3.2. *If $n \geqslant -tb(K)$, then the map $C_{h,n}$ is surjective and hence*

$$rk(\underline{\mathrm{SHG}}(-S^3(K), -\Gamma_n)) = rk(\underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n-1}) + 1.$$

Here $tb(K)$ is the maximal possible Thurston-Bennequin number of a Legendrian representative of the knot class of K, with respect to the standard tight contact structure on S^3. See Ng [12].

Proof. Suppose ξ_{st} is the standard tight contact structure on S^3. Since $n > -tb(K)$, we can isotope K so that it is Legendrian with $tb = -n$. We can dig out a standard Legendrian neighborhood of K and then the dividing set on the boundary of the complement is the suture Γ_n. Hence when we glue back a contact 2-handle, we get (D^3, δ) together with the standard tight contact structure on it. From Baldwin and Sivek [3] or [4], we know that the corresponding contact element is a generator of

$$\underline{\mathrm{SHG}}(-D^3, -\delta) \cong \mathcal{R}.$$

Also we know that the contact two handle attaching map $C_{h,n}$ preserves the contact elements so $C_{h,n}$ is surjective and hence we are done. $\square$

Proposition 3.3. *There is a unique infinite U-tower in $\underline{\mathrm{KHG}}^-(S^3, K, p)$ for any knot $K \subset S^3$.*

Proof. From the author's previous paper [11], we know that there are exact triangles

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_n) \arrow[rr, "\psi_{\pm,n}"] & & \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1}) \arrow[dl, "\psi_{\pm,n+1}"] \\
& \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_\mu) \arrow[ul, "\psi_{\pm,\mu}"] &
\end{tikzcd}$$

Suppose S is a minimal genus Seifert surface of K, and choose $n = 2g(S)$. We have graded versions of the exact triangles (see proposition 5.5 in [11]) as follows:

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_n,S_n^+,i) \arrow[r, "\psi_{\pm,n}"] & \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1},S_{n+1},i) \arrow[d, "\psi_{\pm,n+1}"] \\
& \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_\mu,S_\mu^{-n},i) \arrow[ul, "\psi_{\pm,\mu}"]
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_n,S_n^-,i) \arrow[r, "\psi_{\pm,n}"] & \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1},S_{n+1},i) \arrow[d, "\psi_{\pm,n+1}"] \\
& \underline{\mathrm{SHG}}(-S^3(K),-\Gamma_\mu,S_\mu^{+n},i) \arrow[ul, "\psi_{\pm,\mu}"]
\end{tikzcd} \tag{4}$$

Here the notations S_k and S_μ follow from the ones in the author's previous paper [11], i.e. for all k, S_k is an isotopy of S so that ∂S_k intersects the suture Γ_k transversely at $2k$ points. The supscripts in $S_k^{\pm l}$ denote the positive or negative stabilizations as in section 3 of [11]. It's well known (see Kronheimer and Mrowka [8]) that if $i > g(S)$ or $i < -g(S)$, then

$$\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_\mu,S_\mu,i) = 0.$$

From the grading shifting property, proposition 4.3 in [11], we know that

$$\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_\mu,S_\mu^{-n},i) = 0$$

for

$$i < -g(S) + \frac{n}{2} = 0.$$

Hence from (3), we conclude that (recall we have chosen $n = 2g(S)$)

$$\underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_{n+1},i) \cong \underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_n^+,i) \tag{5}$$

for all $i < 0$.

We can apply a similar argument and use (4) to show that

$$\begin{aligned}\underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_{n+1},i) &\cong \underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_n^-,i)\\ &\cong \underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_n^+,i-1)\end{aligned} \tag{6}$$

for all $i > 0$. The last isomorphism also follows from the grading shifting property.

From the construction of the grading in section 3 in the authors previous paper [11], and the adjunction inequalities discussed in Kronheimer and Mrowka [8], we know that

$$\underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S_{n+1},i)=0,\ \underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S^+_n,i)=0$$

for

$$i>g(S)+n=2g(S)\ or\ i<-g(S)-n=-2g(S).$$

From lemma 3.2 and lemma 4.2 of [11], we know that

$$\underline{\mathrm{SHG}}(-S^3(K),\Gamma_{n+1},S^+_n,2g(S))=0.$$

So from (5) and (6), we can fix all the gradings of $\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1})$ by the corresponding term of $\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_n)$, except at the grading 0. Suppose

$$rk(\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1},S_{n+1},0))=x,$$

then we conclude that

$$rk(\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+1})=rk(\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_n)+x.$$

Using the induction and the same argument as above, we can compute the graded $(\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+k})$ for all $k\in\mathbb{Z}_+$ as follows. If k is odd (recall $n=2g(S)$), then we have

$$\begin{aligned}&\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+k},S_{n+k},i)\\&=\begin{cases}0 & i>g(S)+\frac{n+k-1}{2}\\ \underline{\mathrm{SHG}}(-S^3,-\Gamma_n,S^+_n,i-\frac{k+1}{2}) & \frac{k+1}{2}\leqslant i\leqslant g(S)+\frac{n+k-1}{2}\\ \mathcal{R}^x & \frac{1-k}{2}\leqslant i\leqslant\frac{k-1}{2}\\ \underline{\mathrm{SHG}}(-S^3,-\Gamma_n,S^+_n,i+\frac{k-1}{2}) & -g(S)-\frac{n+k-1}{2}\leqslant i\leqslant-\frac{k+1}{2}\\ 0 & i<-g(S)-\frac{n+k-1}{2}\end{cases}\end{aligned}\tag{7}$$

If k is even, then

$$\begin{aligned}&\underline{\mathrm{SHG}}(-S^3(K),-\Gamma_{n+k},S^+_{n+k},i)\\&=\begin{cases}0 & i\geqslant g(S)+\frac{n+k}{2}\\ \underline{\mathrm{SHG}}(-S^3,-\Gamma_n,S^+_n,i-\frac{k}{2}) & \frac{k}{2}\leqslant i\leqslant g(S)+\frac{n+k}{2}-1\\ \mathcal{R}^x & -\frac{k}{2}\leqslant i\leqslant\frac{k}{2}-1\\ \underline{\mathrm{SHG}}(-S^3,-\Gamma_n,S^+_n,i+\frac{k-1}{2}) & -g(S)-\frac{n+k}{2}\leqslant i\leqslant-1-\frac{k}{2}\\ 0 & i<-g(S)-\frac{n+k}{2}\end{cases}\end{aligned}\tag{8}$$

So we conclude that

$$rk(\underline{\mathrm{SHG}}(-S^3(K), -\Gamma_{n+k}) = rk(\underline{\mathrm{SHG}}(-S^3(K), -\Gamma_n) + k \cdot x.$$

Then it follows from lemma 3.2 that $x = 1$. From proposition 5.8 and corollary 5.9 in the author's previous paper [11], we know that this rank 1 will implies that there is a unique infinite U tower in $\underline{\mathrm{KHG}}^-$. □

Lemma 3.4. *The map*

$$C_{h,n} : \underline{\mathrm{SHG}}(-S^3(K), \Gamma_n) \to \underline{\mathrm{SHG}}(-D^3, -\delta)$$

induces a surjective map

$$C_h : \underline{\mathrm{KHG}}^-(-S^3, K, p) \to \underline{\mathrm{SHG}}(-D^3, -\delta).$$

Furthermore, C_h commutes with U.

Proof. The lemma follows from corollary 3.2 and the following commutative diagrams.

$$\begin{tikzcd} \underline{\mathrm{SHG}}(-S^3, \Gamma_n) \arrow[rr, "\psi_{-,n}"] \arrow[dr, "C_{h,n}"'] & & \underline{\mathrm{SHG}}(-S^3, \Gamma_{n+1}) \arrow[dl, "C_{h,n+1}"] \\ & \underline{\mathrm{SHG}}(-D^3, -\delta) & \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{SHG}}(-S^3, \Gamma_n) \arrow[rr, "\psi_{+,n}"] \arrow[dr, "C_{h,n}"'] & & \underline{\mathrm{SHG}}(-S^3, \Gamma_{n+1}) \arrow[dl, "C_{h,n+1}"] \\ & \underline{\mathrm{SHG}}(-D^3, -\delta) & \end{tikzcd}$$

To prove those commutative diagrams, Recall that the maps $\psi_{\pm,n}$ are constructed via by-pass attachments and by-pass attachments can be interpreted as contact handle attachments (see Baldwin and Sivek [3]), and so is $C_{h,n}$. Then the commutativity follows from the observation that the region to attach contact handles for $\psi_{\pm,n}$ and $C_{n,h}$ are disjoint from each other. □

Corollary 3.5. *We can give an alternative definition of $\tau_G(K)$ originally defined in definition 2.4:*

$$\tau_G(K) = \max\{i \in \mathbb{Z} \mid C_h|_{\underline{\mathrm{KHG}}^-(-S^3,K,p,i)} \text{ is surjective}\}$$

Proof. This follows directly from proposition 3.3 and corollary 3.5. □

Corollary 3.6. *There is an exact triangle*

$$\begin{tikzcd} \underline{\mathrm{KHG}}^-(-S^3, K, p) \arrow[rr] & & \underline{\mathrm{KHG}}^-(-S^3, K, p) \arrow[dl, "C_h"] \\ & \underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[ul] & \end{tikzcd}$$

Proof. The maps in the exact triangle (2) all commute with the maps $\psi_{-,n}$ in the construction of the direct system, so we can pass them to the direct limit and still get an exact triangle. □

Convention 3.7. In the rest of the paper, we will encounter many different maps. To index them, the subscripts might be a tuple. For example, we will have a map

$$\psi_{0,-,n} : \underline{\mathrm{SHG}}(-S^3(K_0), -\Gamma_n) \to (-S^3(K_0), -\Gamma_{n+1}).$$

The subscripts will be ordered in the following way: the first will indicate which topological object it is associated to (in the above example, it is associated to K_0). The second will indicate how the map is constructed (in the above example it comes from a negative by-pass attachment as in the author's previous paper [11]). The third will indicate the suture. The last will indicate the grading. May be some parts are omitted from the subscript but the rest will still respect this order.

Proposition 3.8. *The invariant $\tau_G(K)$ is a concordance invariant.*

Proof. Suppose K_0 and K_1 are concordant. Then there exists a properly embedded annulus $A \subset S^3 \times [0,1]$ so that

$$(S^3 \times \{0\}, A \cap \partial S^3 \times \{0\}) \cong (S^3, K_0),\ (S^3 \times \{1\}, A \cap \partial S^3 \times \{1\}) \cong (S^3, K_1).$$

The pair $(S^3 \times [0,1], A)$ induces a map

$$F_{A,n} : \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_n) \to \underline{\mathrm{SHG}}(-S^3(K_2), -\Gamma_n)$$

as follows. The pair $(S^3 \times [0,1], A)$ induces a cobordism W_n from $Y_{1,n}$ to $Y_{2,n}$, where $Y_{i,n}$ is a closure of $(-S^3(K_i), -\Gamma_n)$, and this cobordism induces the map $F_{A,n}$. There are two ways to describe W, which are both useful. Though both descriptions can be found in the author's previous paper [10].

First give a parametrization of $A = S^1 \times [0,1]$. Then a tubular neighborhood of $A \subset S^3 \times [0,1]$ can be identified with $A \times D^2 = S^1 \times [0,1] \times D^2$, with

$$A \times D^2 \cap S^3 \times \{0,1\} = S^1 \times \{0,1\} \times D^2.$$

Thus we know that

$$\partial(S^3 \times [0,1] \backslash A \times D^2) = -S^3(K_0) \cup (S^1 \times [0,1] \times \partial D^2) \cup S^3(K_1).$$

So suppose $Y_{0,n}$ is a closure of $(-S^3, -\Gamma_n)$ then let

$$W = -(S^3 \times [0,1] \backslash A \times D^2) \cup [Y_{0,n} \backslash S^3(K_1)] \times [0,1],$$

via a natural identification

$$S^1 \times [0,1] \times \partial D^2 = \partial S^3(K_0) \times [0,1].$$

A second description of W is as follows. Recall that

$$\partial(S^3 \times [0,1] \backslash A \times D^2) = -S^3(K_0) \cup (S^1 \times [0,1] \times \partial D^2) \cup S^3(K_1)$$

and clearly
$$\partial S^3(K_0) = \partial S^3(K_1) = S^1 \times D^2.$$
so as in lemma 3.3 in [10], $S^3 \times [0,1] \backslash A \times D^2$ can be obtained from $S^3(K_1) \times [0,1]$ by attaching a set of 4-dimensional handles $\mathcal{H}$ to the interior of $S^3(K_0) \times \{1\}$. Thus as above if we choose a closure $Y_{0,n}$ of $(S^3(K_0), \Gamma_n)$, we can attach the same set of handles $\mathcal{H}$ to $Y_{0,n} \times \{1\} \subset Y_{0,n} \times [0,1]$ and the result is just the cobordism W.

Claim 1. $F_{A,n}$ gives rise to a map
$$F_A : \underline{\mathrm{KHG}}^-(-S^3, K_0, p_0) \to \underline{\mathrm{KHG}}^-(-S^3, K_1, p_1),$$
where p_0 and p_1 are picked as follows. We pick a point $p \in S^1$ and let $p_i = p \times \{i\}$ in the parametrization $A = S^1 \times [0,1]$.

To prove the claim, it is enough to show that we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K_0), -\Gamma_n) \arrow[r, "F_{A,n}"] \arrow[d, "\psi_{0,-,n}"] & \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_n) \arrow[d, "\psi_{1,-,n}"] \\
\underline{\mathrm{SHG}}(-S^3(K_0), -\Gamma_{n+1}) \arrow[r, "F_{A,n+1}"] & \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_{n+2})
\end{tikzcd}$$

This commutativity follows from the fact that when constructing $F_{A,n}$, we attach handles to $Y_{0,n} \times [0,1]$ to the region $[\mathrm{int}(S^3(K_0))] \times \{1\}$ (see above), while when constructing the map $\psi_{i,-,n}$, we attach handles to $Y_{0,n} \times [0,1]$ to the region $[\partial(S^3(K_0))] \times \{1\}$ (see [11]), so the two set of handles are disjoint from each other and hence the corresponding maps commute.

Claim 2. F_A commutes with the U map on $\underline{\mathrm{KHG}}^-$.

The proof of this claim is completely analogous to one for claim 1.

Claim 3. There is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{KHG}}^-(-S^3, K_0, p_0) \arrow[rr, "F_A"] \arrow[dr, "C_{0,h}"'] & & \underline{\mathrm{KHG}}^-(-S^3, K_1, p_1) \arrow[dl, "C_{1,h}"] \\
& \underline{\mathrm{SHG}}(-D^3, -\delta) &
\end{tikzcd}$$

where C_h is defined as in lemma 3.4.

To prove the claim, it is enough to prove that the following digram commutes for any n:

$$\begin{tikzcd}
\underline{\mathrm{KHG}}^-(-S^3(K_0), -\Gamma_n) \arrow[r, "F_{A,n}"] \arrow[d, "C_{0,h}"] & \underline{\mathrm{KHG}}^-(-S^3(K_1), -\Gamma_n) \arrow[d, "C_{1,h}"] \\
\underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[r, "id"] & \underline{\mathrm{SHG}}(-D^3, -\delta)
\end{tikzcd} \quad (9)$$

As above, suppose we have a closure $Y_{0,n}$ for $(-S^3(K_0), \Gamma_n)$. Let $Y_{1,n}$ be the corresponding closure for $(-S^3(K_1), \Gamma_n)$ as in the construction of W above. Recall from the construction of $C_{h,n}$ it is a 2-handle attaching map associated to a 2-handle attached along a meridian curve $\alpha \subset \partial S^3(K_0)$. So we can slightly push it into the interior and get a curve β. Then we get a closure Y_0' for $(-D^3, -\delta)$ by performing a 0-Dehn surgery on $Y_{0,n}$ along β. Note the difference between $S^3(K_0)$ and $S^3(K_1)$ are contained in the interior so we also have the curve $\beta \subset S^3(K_1) \subset Y_{1,n}$. Thus we can get another closure Y_1' for $(-D^3, -\delta)$. We can form a cobordism W' from Y_0' to Y_1' by attaching the set of 4-dimensional handles $\mathcal{H}$ as in the proof of claim 1 to $Y_0' \times \{1\} \subset Y_0' \times [0,1]$, and the attaching region is contained in $\mathrm{int}(S^3(K_0)) \subset Y_0'$. Hence there is a commutative diagram just as in the proof of claim 1:

$$\begin{tikzcd}
\underline{\mathrm{KHG}}^-(-S^3(K_0), -\Gamma_n) \arrow[r, "F_{A,n}"] \arrow[d, "C_{0,h}"] & \underline{\mathrm{KHG}}^-(-S^3(K_1), -\Gamma_n) \arrow[d, "C_{1,h}"] \\
\underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[r, "F_A'"] & \underline{\mathrm{SHG}}(-D^3, -\delta)
\end{tikzcd}$$

where F_A' is the map induced by the cobordism W'. So to prove (9), it is suffice to show that W' is actually a product $Y_0' \times [0,1]$ and hence $F_A' = id$. To do this, recall that W' is obtained from Y_0' by attaching a set of handles $\mathcal{H}$, while the region of attachment is contained in $\mathrm{int}(S^3(K_0)) \subset \mathrm{int}(D^3) \subset Y_0' \times \{1\}$. So this means that we can split W' into two parts

$$W' = W'' \cup (Y_0' \backslash D^3) \times [0,1]$$

and W'' is obtained from $D^3 \times [0,1]$ by attaching the set of handles $\mathcal{H}$. Recall that (D^3, δ) is obtained from $(S^3(K_0), \Gamma_n)$ by attaching the contact 2-handle h so topologically,

$$D^3 = S^3(K_0) \cup B^3.$$

Note the 3-ball B^3 is attached to $S^3(K_0)$ along the boundary, and the set of handles $\mathcal{H}$ is attached to $D^3 \times [0,1]$ within the region $\mathrm{int}(S^3(K_0)) \subset D^3 \times \{1\}$ so the two attaching regions are disjoint. Thus

$$\begin{aligned}
W'' &= D^3 \times [0,1] \cup \mathcal{H} \\
&= ((S^3(K_0) \cup B^3) \times [0,1]) \cup \mathcal{H} \\
&= (S^3(K_0) \times [0,1]) \cup \mathcal{H} \cup B^3 \times [0,1] \\
&= [(S^3 \times [0,1]) \backslash (A \times D^2)] \cup B^3 \times [0,1].
\end{aligned}$$

Note B^3 is glued to $S^3(K_0)$ along an annulus on the boundary (the contact 2-handle attachment), and so $B^3 \times [0,1]$ is glued to $[(S^3 \times [0,1]) \backslash (A \times D^2)]$ along annulus times $[0,1]$. It is then straightforward to check that the resulting manifold (which is W'') is just diffeomorphic to $D^3 \times [0,1]$. Hence we are done.

Claim 4. The map

$$F_A : \underline{\mathrm{KHG}}^-(-S^3, K_0, p_0) \to \underline{\mathrm{KHG}}^-(-S^3, K_1, p_1)$$

is grading preserving.

Note from proposition 5.8 in [11], we know that for any fixed $k \in \mathbb{Z}$, we can pick a large enough odd n, so that for $i = 0, 1$

$$\underline{\mathrm{KHG}}^-(-S^3, K_i, p_i, k) = \underline{\mathrm{SHG}}(-S^3(K_i), -\Gamma_n, S_{n,i}, k + \frac{n-1}{2}).$$

Hence to show that F_A is grading preserving, we only need to show that $F_{A,n}$ is grading preserving. Recall here for $i = 0,1$, $S_{i,n}$ is a minimal genus Seifert surface of K_i which intersects the suture Γ_n exactly n points. Note we can identify the boundaries

$$\partial S^3(K_0) = \partial S^3(K_1)$$

via the parametrization $A = S^1 \times [0,1]$ and we can assume that under the above identification

$$S_{0,n} \cap \partial S^3(K_0) = S_{1,n} \cap \partial S^3(K_1).$$

In section 3 of the author's previous paper [11], we know how to construct the grading on <u>SHG</u> based on a properly embedded surface. Now let $Y_{0,n}$ be a closure of $(-S^3(K_0), -\Gamma_n)$ so that the surface $S_{n,0}$ extends to a closed one $\bar{S}_{n,0}$, as in the construction of grading. Then we have a corresponding closure $Y_{1,n}$ for $(-S^3(K_1), -\Gamma_n)$, inside which $S_{1,n}$ extends to a closed surface $\bar{S}_{1,n}$. To describe this surface, recall that

$$Y_{1,n} = -S^3(K_1) \underset{\partial S^3(K_0)=\partial S^3(K_1)}{\cup} [Y_{0,n} \backslash S^3(K_0)]$$

as in the construction of W at the beginning of the proof. Then we can take

$$\bar{S}_{1,n} = S_{1,n} \cup (\bar{S}_{0,n} \backslash S^3(K_0)).$$

By using the Mayer-Vietoris sequence we know that

$$H_2((S^3 \times [0,1]) \backslash (A \times D^2)) = 0.$$

So the closed surface $-S_1 \cup A \cup S_2 \subset (S^3 \times [0,1]) \backslash (A \times D^2)$ bounds a 3-chain $x \subset (S^3 \times [0,1]) \backslash (A \times D^2)$. Recall we have

$$W = -(S^3 \times [0,1]) \backslash (A \times D^2) \cup [Y_{0,n} \backslash S^3(K_1)] \times [0,1]$$

Now inside W, let

$$y = x \cup \big((\bar{S}_{0,n} \backslash S^3(K_0)) \times [0,1]\big),$$

where the two pieces are glued along

$$A = S^1 \times [0,1] = \partial(\bar{S}_{0,n} \backslash S^3(K_0)) \times [0,1].$$

It is straightforward to check that

$$\partial y = -\bar{S}_{0,n} \cup \bar{S}_{1,n}.$$

Hence we conclude that

$$[\bar{S}_{0,n}] = [\bar{S}_{0,1}] \in H_1(W).$$

Then it follows that $F_{A,n}$ preserves the grading.

Finally, the four claims above, together with corollary 3.5 and the fact that K_0 is concordant to K_1 if and only if K_1 is concordant to K_0 would directly prove the proposition. $\square$

Corollary 3.9. *If A is a Ribbon concordance from K_1 to K_2, then the map*

$$F_A : \underline{\mathrm{KHG}}^-(-S^3, K_0, p_0) \to \underline{\mathrm{KHG}}^-(-S^3, K_1, p_1)$$

as in the claim 1 in the proof of proposition 3.8 is injective.

Proof. From works by Daemi, Lidman, Vela-Vick and Wong [6], for any $n \in \mathbb{Z}_+$, the map

$$F_{A,n} : \underline{\mathrm{SHG}}(-S^3(K_0), -\Gamma_n) \to \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_n)$$

is injective. So when passing to the direct limit, the F_A is also injective. □

4 Some computations

In this section we compute the $\underline{\mathrm{KHG}}^-$ for the family of knots K_m as in figure 1. In particular, K_1 is the right handed trefoil, K_0 is the unknot and K_{-1} is the figure eight.

From the Seifert algorithm, we can easily construct a genus 1 Seifert surface for K_m, which we denote by S_m. Hence $g(K_m) = 1$ and also from Rolfsen [14] we know that the (symmetrized) Alexander polynomial of K_m is

$$\Delta_{K_m}(t) = mt + (1 - 2n) + mt^{-1}. \tag{10}$$

First we are going to compute $\underline{\mathrm{KHG}}(-S^3, K_m)$. Suppose $(S_3(K_m), \Gamma_\mu)$ is the balanced sutured manifold obtained by taking meridional sutures on the knot complements. There is a curve $\alpha \subset \mathrm{int}(S_3(K_m))$ as in figure 1 so that we have a surgery exact triangle:

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_\mu) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(K_{m+1}), -\Gamma_\mu) \arrow[dl] \\
& \underline{\mathrm{SHG}}(-M, -\Gamma_\mu) \arrow[ul] &
\end{tikzcd}$$

Here K_m is described as above, and M is obtained from $S^3(K_m)$ by performing a 0-Dehn surgery along α. We can use the surface S_m which intersects the suture Γ_μ twice to construct a grading on the sutured monopole and instanton Floer homologies. Since α is disjoint from S_m, there is a graded version of the exact triangle (since we will always use the surface S_m, we will omit it from the notation):

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_\mu, i) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(K_{m+1}), -\Gamma_\mu, i) \arrow[dl] \\
& \underline{\mathrm{SHG}}(-M, -\Gamma_\mu, i) \arrow[ul] &
\end{tikzcd} \tag{11}$$

Since S_m has genus one and intersects the suture twice, all the graded sutured monopole and instanton Floer homologies in (11) could only possibly be non-trivial for $-1 \leqslant i \leqslant 1$. To understand what is $\underline{\mathrm{SHG}}(-M,-\Gamma_{\mu})$, from Kronheimer and Mrowka [7] and the author's previous paper [9], the surgery exact triangle (3) is just the same as the oriented Skein exact triangle and $\underline{\mathrm{SHG}}(-M,-\Gamma)$ is isomorphic to the monopole or instanton knot Floer homology of the oriented smoothing of K_m, which is a Hopf link. Applying oriented Skein relation again on Hopf links, we can conclude that

$$\mathrm{rk}(\underline{\mathrm{SHG}}(-M,-\Gamma_{\mu})) \leqslant 4. \tag{12}$$

For the monopole and instanton knot Floer homologies of K_1 (trefoil), we could look at the surgery exact triangle along the curve β in figure 2 and argue in the same way as inKronheimer and Mrowka [7] to conclude

$$\mathrm{rk}(\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_{\mu}) \leqslant 3.$$

From the knowledge of Alexander polynomial in (10) and Kronheimer and Mrowka [8, 7], we know that

$$\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_{\mu},i) \cong \mathcal{R} \tag{13}$$

for $i=-1,0,1$.

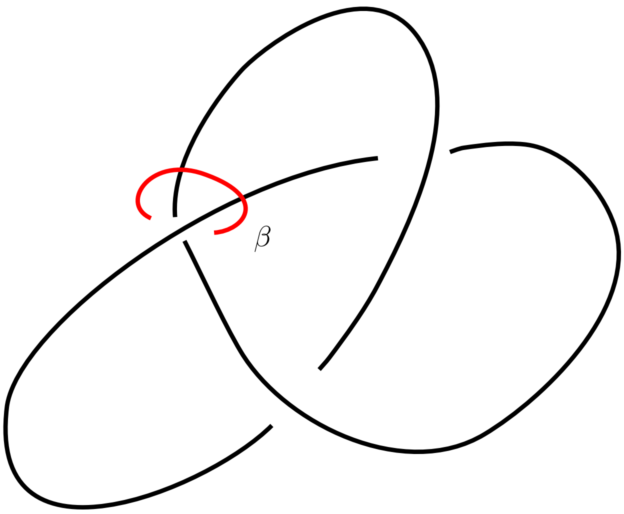

Figure 2: The trefoil and the circle β.

Now let $m=1$ in (11). We know from (10) that

$$\mathrm{rk}(\underline{\mathrm{SHG}}(-S^3(K_2),-\Gamma_{\mu}) \geqslant 7.$$

Then from the exactness, inequality (12) and (13), we know that

$$\mathrm{rk}(\underline{\mathrm{SHG}}(-M,-\Gamma_{\mu})) = 4.$$

To further examine each grade, we know that

$$\underline{\mathrm{SHG}}(-M,-\Gamma_{\mu},i)=\begin{cases}\mathcal{R} & i=1,-1\\ \mathcal{R}^{2} & i=0\\ 0 & \text{others}\end{cases}\tag{14}$$

Thus by using the same argument and induction, we can compute for $m>0$

$$\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{\mu},i)=\begin{cases}\mathcal{R}^{m} & i=1,-1\\ \mathcal{R}^{2m-1} & i=0\\ 0 & \text{others}\end{cases}\tag{15}$$

Since K_0 is the unknot, we can use the same technique to compute for $m\leqslant 0$ that

$$\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{\mu},i)=\begin{cases}\mathcal{R}^{-m} & i=1,-1\\ \mathcal{R}^{1-2m} & i=0\\ 0 & \text{others}\end{cases}\tag{16}$$

Now we are ready to compute the minus version. Recall that the Seifert surface induces a framing on the boundary of the knot complements as well as M. Write Γ_n the suture consists of two curves of slope $-n$. We have graded version of by-pass exact triangle (3) and (4) for even n as well as their cousins

$$\begin{tikzcd}\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_n,S^{+2}_{m,n},i) \arrow[r, "\psi_{+,n}"] & \underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{n+1},S^{+}_{m,n+1},i) \arrow[d, "\psi_{+,n+1}"] \\ & \underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{\mu},S^{-n,+}_{m},i) \arrow[ul, "\psi_{+,\mu}"]\end{tikzcd}\tag{17}$$

and

$$\begin{tikzcd}\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_n,S^{-2}_{m,n},i) \arrow[r, "\psi_{-,n}"] & \underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{n+1},S^{-}_{m,n+1},i) \arrow[d, "\psi_{-,n+1}"] \\ & \underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{\mu},S^{+n,-}_{m},i) \arrow[ul, "\psi_{-,\mu}"]\end{tikzcd}\tag{18}$$

for odd n.

A simple case to analyze is when $m<0$. (For $m=0$ has already been computed in the author's previous paper [10].) For the knot K_m, $m<0$, take $n=1$ in (17), we have table 1.

Here, as in the author's previous paper [11], the top and bottom non-vanishing grading of $\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_n)$ can be computed via sutured manifold decomposition and coincide with the top and bottom non-vanishing grading of $\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_{\mu})$.

From the graded exact triangles on the rows of the table and an extra exact triangle (2), we know that

$$b\geqslant 1-m,\ c\geqslant a+m,\ b+c\leqslant a+1.$$

Hence the only possibility is $b=1-m$, $c=a+m$. Now take $n=2$ in (3), we have table.

	Γ_μ	Γ_1	Γ_2
1	$-m$		$-m$
0	$1-2m$	$-m$	b
-1	$-m$	a	c
-2		$-m$	$-m$

Table 1: The map $\psi_{+,1}$ for K_m. There is a (graded) exact triangle between the three horizontal terms for each row. The leftmost column implies the gradings. The numbers on other columns means the rank of corresponding (graded) homology. If it is a letter (rather than a formula in m), it means that a priori, we don't know what the rank is.

Here $\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_3)$ can be computed as taking $k=1$ in (7). We know from the author's previous paper [10] that

$$\underline{\mathrm{KHG}}^-(-S^3,K_m,p_m,i)\cong\underline{\mathrm{SHG}}(-S^3(K_m),-\Gamma_3,S_{m,3},i+1)$$

for $i=1,0,-1$ and the U map on $\underline{\mathrm{KHG}}^-(-S^3,K_m,p_m,i)$ for $i=1,2$ coincides with the map $\psi^i_{+,2}$ as in table 2. From the exactness, we know that U map is actually zero at grading 1 and having kernel of rank $-m$ at grading 0. Hence we conclude that

Proposition 4.1. *Suppose $m<0$ and the knot K_m is described as above. Then*

$$\underline{\mathrm{KHG}}^-(-S^3,K_m,p_m)\cong\mathcal{R}[U]_0\oplus(\mathcal{R}_1)^{-m}\oplus(\mathcal{R}_0)^{-m}.$$

Here the subscripts means the grading of the element $1\in\mathcal{R}$ and the formal variable U has degree -1.

From the description it is clear that

$$\tau_G(K_m)=0.$$

To compute $\underline{\mathrm{KHG}}^-$ of K_m for $m>0$, we first deal with the case $m=1$. Now K_1 is a right-handed trefoil. From Ng [12] we know that $tb(K_1)=1$ and hence from corollary 3.2, we know that

$$\mathrm{rk}(\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_1))=\mathrm{rk}(\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0))+1.$$

Now let us compute $\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0)$. Pick S_0 to be a genus 1 Seifert surface of K so that S_0 is disjoint from Γ_0. We can use the surface S_0^-, a negative stabilization of S_0

	Γ_μ	Γ_2		Γ_3
2	$-m$			$-m$
1	$1-2m$	$-m$	$\xrightarrow{\psi^1_{+,2}}$	$1-m$
0	$-m$	$1-m$	$\xrightarrow{\psi^0_{+,2}}$	1
-1		$a+m$		$a+m$
2		$-m$		$-m$

Table 2: The map $\psi_{+,2}$ for K_m. The sup-script means the map at a particular grading

as in the author's previous paper [11] to construct a grading on $\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0)$. From the construction of grading and the adjunction inequality, there could only be three non-vanishing grading $-1,0,1$. For grading 1 part, we can apply lemma 3.2 and lemma 4.2 in [11] and hence look at the balanced sutured manifold (M',γ') obtained from $(-S^3(K),-\Gamma_0)$ by (sutured manifold) decomposing along the surface S_0:

$$\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0,S_0^-,1)\cong\underline{\mathrm{SHG}}(M',\gamma').$$

Since K is a fibred knot, the underlining manifold M' is just a product $S_0\times[-1,1]$. The suture γ' is not just $\partial S\times\{0\}$ but is actually three copies of $\partial S\times\{0\}$ on $\partial S\times[-1,1]$. We can find an annulus $A\subset\partial S\times[-1,1]$ which contains the suture γ'. Then we can push the interior of A into the interior of $S\times[-1,1]$ and get a properly embedded surface. If we further decompose (M',γ') along (the pushed) A, then we get a disjoint union of a product balanced sutured manifold $(S\times[-1,1],\partial S\times\{0\})$ with a solid torus with four longitudes as the suture. The sutured monopole and instanton Floer homologies of the first are both of rank 1 and the second of rank 2, as in Kronheimer and Mrowka [7] and the author's previous paper [9]. Hence we conclude

$$\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0,S_0^-,1)\cong\mathcal{R}^2.$$

For the other two grading, note that from the grading shifting property, proposition 4.3 in [11], we have

$$\begin{aligned}\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0,S_0^-,i)&=\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0,S_0^+,i-1)\\&=\underline{\mathrm{SHG}}(-S^3(K_1),-\Gamma_0,(-S_0)^-,1-i).\end{aligned}$$

The second equality follows from the basic observation that if we reverse the orientation of the surface S_0^+, we just get $(-S_0)^-$. Hence

$$\underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, -1) = \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_0, (-S_0)^-, 2) = 0$$

by the adjunction inequality and

$$\underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, 0) = \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_0, (-S_0)^-, 1) \cong \mathcal{R}^2.$$

by the same argument as above. Thus

$$\underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_1) \cong \mathcal{R}^5.$$

Similarly as above, there are only three possible non-vanishing gradings $-1, 0, 1$. We have already known that the homology at top and bottom gradings are of rank 1 each, so the middle grading has rank 3. Let $n = 1$ in (17) and (18), we have table 3.

	Γ_μ	Γ_1	Γ_2	|	Γ_μ	Γ_1	Γ_2
2				|		1	1
1	1		1	|	1	3	b
0	1	1	b	|	1	1	c
-1	1	3	c	|	1		1
-2		1	1	|			

Table 3: The map $\psi_{+,1}$ (on the left) and $\psi_{-,1}$ (on the right) for K_1.

From the exactness, we know that $b = c = 2$. The rest of the computation is straightforward and we conclude that

$$\underline{\mathrm{KHG}}^-(-S^3, K_1, p_1) \cong \mathcal{R}[U]_1 \oplus \mathcal{R}_0. \tag{19}$$

Now we have the map

$$C_{1,h,1} : \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_1) \to \underline{\mathrm{SHG}}(-D^3, \delta)$$

and by the description of $\underline{\mathrm{KHG}}^-(-S^3, K_1, p_1)$ above, lemma 3.5 and the fact that $C_{1,h,n}$ commutes with $\psi_{-,n}$ (claim 1 in the proof of proposition 3.8), we know that

$$C_{1,h,1} : \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_1, 1) \to \underline{\mathrm{SHG}}(-D^3, -\delta)$$

is surjective and since $\underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_1, 1)$ has rank 1 it is actually an isomorphism. Now we go back to the surgery exact triangle in (11), which corresponds to surgeries on the curve $\alpha \subset \mathrm{int}(S^3(K_m))$. Since α is disjoint from the boundary, (and as above, disjoint from S_m), we have the following exact triangle for any m and n.

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_n, i) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(K_{m+1}), -\Gamma_n, i) \arrow[dl] \\
& \underline{\mathrm{SHG}}(-M, -\Gamma_n, i) \arrow[ul] &
\end{tikzcd} \tag{20}$$

There are contact 2-handle attaching maps

$$C_{m,h,n} : \underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_n) \to \underline{\mathrm{SHG}}(-D^3, -\delta),$$

where the contact 2-handle is attached along a meridional curve on the knot complements. We can attach a contact 2-handle along the same curve on the boundary of M, and the handle attaching maps commute with the maps in the exact triangle (20 and thus we have a diagram:

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_n, i) \arrow[rr] \arrow[dd, "C_{m,h,n}"] & & \underline{\mathrm{SHG}}(-S^3(K_{m+1}), -\Gamma_n, i) \arrow[dl, "\tau_{m,n,i}"] \arrow[dd, "C_{m+1,h,n}"] \\
& \underline{\mathrm{SHG}}(-M, -\Gamma_n, i) \arrow[ul] \arrow[dd, "C_{M,h,n}"] & \\
\underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[rr, "\phi_\infty"] & & \underline{\mathrm{SHG}}(-D^3, -\delta) \arrow[dl, "\phi_1"] \\
& \underline{\mathrm{SHG}}(-S^2 \times S^1(1), -\delta) \arrow[ul, "\phi_0"] &
\end{tikzcd} \tag{21}$$

Here $S^2 \times S^1$ is obtained from S^3 by performing a 0-surgery along an unknot. The balanced sutured manifold $(S^2 \times S^1(1), \delta)$ is obtained from $S^2 \times S^1$ by removing a 3-ball and assigning a connected simple closed curve on the spherical boundary as the suture. Its sutured monopole and instanton Floer homologies are computed in Baldwin and Sivek [4] and the author's previous paper [9] and is of rank 2. Thus the exactness tells us that $\phi_\infty = 0$, ϕ_1 is injective and ϕ_0 is surjective.

Now take $m = 0, n = 1, i = 1$, we know that

$$\underline{\mathrm{SHG}}(-M, -\Gamma_1, 1) \cong \underline{\mathrm{SHG}}(-S^3(K_1), -\Gamma_1, 1) \cong \mathcal{R}$$

and $C_{M,h,n}$ is injective. Then take m to be an arbitrary non-negative integer and $n = 1, i = 1$ in (21). From (15) we know that

$$\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_\mu, 1) \cong \mathcal{R}^m.$$

By performing sutured manifold decompositions along S_m and using lemma 4.2 in the author's pervious paper [11], we know that

$$\underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_n, 1) \cong \underline{\mathrm{SHG}}(-S^3(K_m), -\Gamma_\mu, 1) \cong \mathcal{R}^m.$$

Recall from above discussions we have

$$\underline{\mathrm{SHG}}(-M, -\Gamma_1, 1) \cong \mathcal{R},$$

so in the exact triangle in (21), we know that $\tau_{m,1,1}$ is surjective. Then we can use the commutativity part of (21) and conclude that

$$C_{m+1,h,n} : \underline{\mathrm{SHG}}(-S^3(K_{m+1}), -\Gamma_1, 1) \to \underline{\mathrm{SHG}}(-D^3, -\delta)$$

is surjective. From the fact that $\psi_{\pm,n}$ commutes with $C_{h,n}$ as in Claim 1 and 2 in the proof of proposition 3.8, we know that this surjectivity means that the unique U tower in $\underline{\mathrm{KHG}}^-(-S^3, K_m, p_m)$ actually starts at grading 1:

$$\tau_G(K_m) = 1$$

for $m > 0$.

Take $n = 1$ in (17), we have table 4.

	Γ_μ	Γ_1	Γ_2
1	m		m
0	$2m-1$	$m \xrightarrow{\psi^0_{+,1}}$	b
-1	m	a	c
-2		m	m

Table 4: The map $\psi_{+,1}$ for K_m.

The fact that $\tau_G(K_m) = 1$ means that $\psi^0_{+,1} \neq 0$, as $\psi^0_{+,1}$ corresponds to the U map at grading 1 part of $\underline{\mathrm{KHG}}^-(-S^3, K_m, p_m)$. Thus from the exactness we know that

$$b \geqslant m + 1, \ c \geqslant a - m.$$

From the exact triangle (2) we know that

$$b + c \leqslant a + 1$$

and hence $b = m + 1, c = a - m$. Finally we conclude that

Proposition 4.2. *Suppose $m > 0$ and K_m is described as above. Then*

$$\underline{\mathrm{KHG}}^-(-S^3, K_m, p_m) \cong \mathcal{R}[U]_1 \oplus (\mathcal{R}_1)^{m-1} \oplus (\mathcal{R}_0)^m.$$

Furthermore, $\tau_G(K_m) = 1$.

Now we could also compute the $\underline{\mathrm{KHG}}^{-}$ of the knots $\bar{K}_m$, the mirror image of K_m. For $m < 0$, the computation is exactly the same as before, and we conclude that

Proposition 4.3. *Suppose $m < 0$ and the knot $\bar{K}_m$ is described as above. Then*

$$\underline{\mathrm{KHG}}^{-}(-S^3, \bar{K}_m, p_m) \cong \mathcal{R}[U]_0 \oplus (\mathcal{R}_1)^{-m} \oplus (\mathcal{R}_0)^{-m}.$$

From the description it is clear that

$$\tau_G(\bar{K}_m) = 0.$$

For $m > 0$, we have a similar diagram as in 21 as follows.

$$\begin{tikzcd}
\underline{\mathrm{SHG}}(-S^3(\bar{K}_{m+1}), -\Gamma_n, i) \arrow[rr] & & \underline{\mathrm{SHG}}(-S^3(\bar{K}_m), -\Gamma_n, i) \arrow[dl, "\tau_{m,n,i}"] \\
& \underline{\mathrm{SHG}}(-M, -\Gamma_n, i) \arrow[ul] &
\end{tikzcd} \tag{22}$$

Let us first compute the case $m = 1$, when $\bar{K}_m$ is the left handed trefoil. In this case take $n = 1$ in (17), we get table 5.

	Γ_μ	Γ_1	Γ_2
1	1		1
0	1	$1 \xrightarrow{\psi^0_{+,1}}$	b
-1	1	a	c
-2		1	1

Table 5: The map $\psi_{+,1}$ for $\bar{K}_1$.

The left handed trefoil is not right veering in the sense of Baldwin and Sivek [5], so from their discussion we could conclude that $\psi_{+,1} = 0$. (This is how they prove that the second top grading of the instanton knot Floer homology of a not-right-veering knot is non-trivial. Though they only work in the instanton case, the monopole case is exactly the same.) Thus we conclude that $b = 0$.

In (22), let $m = 0, n = 2, i = 0$. Note the grading is induced by $S^{+}_{m,2}$, meaning a Seifert surface of the knot $\bar{K}_m$ which intersects the suture Γ_2 transversely at four points and with a positive stabilization. This is to incorporate with (3). Thus we know that

$$\underline{\mathrm{SHG}}(-S^3(\bar{K}_1), -\Gamma_2, 0) = \mathcal{R}^b = 0,\ \underline{\mathrm{SHG}}(-S^3(\bar{K}_0), -\Gamma_2, 0) \cong \mathcal{R}.$$

Here $\bar{K}_0$ is the unknot and we have computed the $\underline{\mathrm{SHG}}$ of a solid torus with any possible sutures in [11]. Thus we conclude that

$$\underline{\mathrm{SHG}}(-M, -\Gamma_2, 0) \cong \mathcal{R}.$$

Use the exactness and induction, we then have

$$\underline{\mathrm{SHG}}(-S^3(\bar{K}_m), \Gamma_2, 0) \cong \mathcal{R}^{c_m},\ c_m \leqslant m-1.$$

For the knot $\bar{K}_m$, take $n = 2$ in (3), we have table 6. Thus we conclude from

	Γ_μ	Γ_2		Γ_3
2	m			m
1	$2m-1$	m	$\xrightarrow{\psi^1_{+,2}}$	c_m
0	m	c_m	$\xrightarrow{\psi^0_{+,2}}$	1
-1		?		?
2		m		m

Table 6: The map $\psi_{+,2}$ for K_m. The sup-script means the map at a particular grading

the exactness that $c_m = m-1$, $\psi^1_{+,2} = 0$ and $\psi^0_{+,2} = 0$. As above, the two maps $\psi^1_{+,2}$ and $\psi^0_{+,2}$ corresponds to the U maps of $\underline{\mathrm{KHG}}^-(-S^3, \bar{K}_m, p_m)$ at grading 1 and 0 respectively, and hence we conclude

Proposition 4.4. *Suppose $m > 0$ and the knot $\bar{K}_m$ is described as above. Then*

$$\underline{\mathrm{KHG}}^-(-S^3, \bar{K}_m, p_m) \cong \mathcal{R}[U]_{-1} \oplus (\mathcal{R}_1)^m \oplus (\mathcal{R}_0)^{m-1}.$$

From the description it is clear that

$$\tau_G(\bar{K}_m) = 0.$$

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