

Instanton and the depth of taut foliations

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Abstract

Sutured instanton Floer homology was introduced by Kronheimer and Mrowka in [8]. In this paper, we prove that for a taut balanced sutured manifold with vanishing second homology, the dimension of the sutured instanton Floer homology provides a bound on the minimal depth of all possible taut foliations on that balanced sutured manifold. The same argument can be adapted to the monopole and even the Heegaard Floer settings, which gives a partial answer to one of Juhasz's conjectures in [5]. Using the nature of instanton Floer homology, on knot complements, we can construct a taut foliation with bounded depth, given some information on the representation varieties of the knot fundamental groups. This indicates a mystery relation between the representation varieties and some small depth taut foliations on knot complements, and gives a partial answer to one of Kronheimer and Mrowka's conjecture in [8].

1 Introduction

Sutured manifolds and sutured manifold hierarchies were introduced by Gabai in 1983 in [2] and subsequent papers. They are powerful tools in the study of 3-dimensional topology. A sutured manifold is a compact oriented 3-manifold with boundary, together with an oriented closed 1-submanifold γ on ∂M , which is called the suture. If $S \subset M$ is a properly embedded surface inside M , which satisfies some mild conditions, then we can perform a decomposition of (M, γ) along S and obtain a new sutured manifold (M', γ') . We call this process a sutured manifold decomposition and write

$$(M, \gamma) \xrightarrow{S} (M', \gamma').$$

A balanced sutured manifold (M, γ) is a sutured manifold with some further restrictions on M and γ . It was introduced by Juhász in [4] to accommodate the construction of Heegaard Floer homology on them. Later, when Kronheimer and Mrowka introduced sutured monopole and instanton Floer homologies, they also use the settings of balanced sutured manifolds. So, in this paper, we will only work with balanced sutured manifolds, though it should be understood that Gabai's results were originally proved for all sutured manifolds.

A celebrated theorem proved by Gabai is the following.

Theorem 1.1 (Gabai [2]). *Suppose (M, γ) is a taut balanced sutured manifold, then there exists a finite sequence of sutured manifold decompositions*

$$(M, \gamma) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n, \gamma_n), \quad (1)$$

where (M_i, γ_i) is taut for all i and (M_n, γ_n) is a product sutured manifold, meaning that there is an oriented surface F with non-trivial boundary so that

$$(M_n, \gamma_n) = ([-1, 1] \times F, \{0\} \times \partial F).$$

One original motivation for Gabai to establish Theorem 1.1 is to construct taut foliations on 3-manifolds. In particular, he proved the following theorem.

Theorem 1.2 (Gabai [2]). *Suppose (M, γ) is a taut balanced sutured manifold, then (M, γ) admits a finite depth taut foliation.*

However, Gabai only proved the existence of a taut foliation with finite depth, yet he didn't offer any bounds on how small the depth could be. In [6], Juhász made the following conjecture.

Conjecture 1.3 (Juhász [6]). *Suppose (M, γ) is a taut balanced sutured manifold with $H_2(M) = 0$, and*

$$\mathrm{rk}_{\mathbb{Z}_2}(SFH(M, \gamma)) < 2^{k+1},$$

then (M, γ) admits a taut foliation of depth at most $2k$.

Here, $SFH(M, \gamma)$ is the sutured (Heegaard) Floer homology of (M, γ) , introduced by Juhasz in [4]. It is a finite dimensional vector space over the field $\mathbb{Z}_2$ and is a topological invariant associated to (M, γ) . Following this line, Kronheimer and Mrowka further made the following conjecture.

Conjecture 1.4 (Kronheimer and Mrowka [8]). *Suppose $K \subset S^3$ be a knot, and consider the irreducible homomorphisms*

$$\rho : \pi_1(S^3(K)) \rightarrow SU(2)$$

which maps a chosen meridian m to the element $\mathbf{i} \in SU(2)$. Suppose that these homomorphisms are non-degenerate and that the number of conjugacy classes of such homomorphisms is less than 2^{k+1} . Then, the knot complement $S^3(K)$ admits a taut foliation of depth at most $2k$, transverse to the boundary $\partial S^3(K)$.

In this paper, we prove the following result, constructing a taut foliation whose depth is bounded in terms of the dimension of sutured instanton Floer homology. The sutured instanton Floer homology, denoted by SHI , is another type of Floer homology associated to (M, γ) introduced by Kronheimer and Mrowka in [8]. It is a finite dimensional vector space over $\mathbb{C}$. Though the bound on the depth of taut foliations is not as sharp as the ones in Conjecture 1.3 and Conjecture 1.4, up to the author's knowledge, it is the first of this kind.

Theorem 1.5. *Suppose (M, γ) is a taut balanced sutured manifold with $H_2(M) = 0$, and*

$$\dim_{\mathbb{C}} SHI(M, \gamma) < 2^{k+1}.$$

Then, (M, γ) admits a taut foliation of depth at most 2^{k+6} .

Corollary 1.6. *Conjecture 1.4 holds if we replace the depth $2k$ of the taut foliation in the statement of the conjecture by 2^{k+6} .*

Remark 1.7. It can be reformulated that the minimal depth of all taut foliations is bounded by a multiple of the dimension of the sutured instanton Floer homology, but the original statement in Theorem 1.5 is more convenient for the purpose of carrying out the proof.

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2 Preliminaries

In this paper, all the notations will be kept the same as in Ghosh and Li [3]. So if a term has already been defined in that paper, we will not define it again. We have the following three new definitions.

Definition 2.1 (Gabai [2]). Suppose (M, γ) is a balanced sutured manifold. A surface S is called well-groomed if the following holds.

(1) For each component A of the annular neighborhood $A(\gamma)$, $S \cap A$ consists of either a collection of parallel and coherently oriented non-separating simple arcs, or a collection of parallel simple closed curves each oriented in the same way as $\gamma \cap A$.

(2) For each component V of $R(\gamma)$, $S \cap V$ consists of either a collection of parallel and coherently oriented non-separating simple arcs, or a collection of parallel and coherently oriented non-separating simple closed curves.

Definition 2.2 (Gabai [2]). A transversely oriented co-dimension-one foliation $\mathfrak{F}$ on a balanced sutured manifold (M, γ) is called taut if $\mathfrak{F}$ is transverse to $A(\gamma)$, tangent to $R(\gamma)$ with normal direction pointing inward on $R_-(\gamma)$ and outward on $R_+(\gamma)$, $\mathfrak{F}|_{A(\gamma)}$ has no Reeb components, and each leaf intersects $A(\gamma)$ in transverse curves or properly embedded arcs.

Definition 2.3 (Gabai [2]). Let M be a compact oriented 3-manifold, and $\mathfrak{F}$ a co-dimension-one foliation. We say a leaf L of $\mathfrak{F}$ has *depth 0* if L is compact. Suppose we have defined the depth for $j \leq k$, then say a leaf L of $\mathfrak{F}$ has *depth $k+1$* if $\bar{L} \setminus L$ is a union of leaves of depth at most k and contains at least one leaf of depth k . The foliation $\mathfrak{F}$ is called *depth k* if all its leaves have depth at most k and it admits at least one leaf of depth k . If such a k does not exist, then we say $\mathfrak{F}$ has infinite depth.

The following two Propositions from Ghosh and Li [3] are key ingredients of the proof of Theorem 1.5.

Proposition 2.4. *Suppose (M, γ) is a connected balanced sutured manifold so that $H_2(M) = 0$ and it is horizontally prime, taut, and free of essential product annuli and product disks. If further that (M, γ) is not a product sutured manifold, then, there exists a well-groomed surface $S \subset M$ and a sutured manifold decomposition along S*

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

so that (M', γ') is taut, and

$$\dim_{\mathbb{C}} SHI(M', \gamma') \leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M, \gamma).$$

Proof. This essentially Proposition 5.8 in Ghosh [3]. In the original statement of the proposition, the homology class $\alpha \in H_2(M, \partial M)$ can be chosen freely, so we can choose a well groomed class as guaranteed by Lemma 2.8 in Juhász [6]. Then, Proposition 2.4 follows. $\square$

Proposition 2.5. *Suppose (M, γ) is a connected balanced sutured manifold so that $H_2(M) = 0$ and it is horizontally prime, taut, and free of essential product annuli and product disks. Then,*

$$\dim_{\mathbb{C}} SHI(M, \gamma) \geq g(\partial M) + 1.$$

Proof. By Corollary 5.12 in Ghosh and Li [3], we know that

$$\dim_{\mathbb{C}} SHI(M, \gamma) \geq \dim_{\mathbb{R}} H^2(M, \partial M; \mathbb{R}) + 1.$$

Since $H_2(M) = 0$ and M is connected, we know that

$$\dim_{\mathbb{R}} H^2(M, \partial M; \mathbb{R}) = g(\partial M).$$

$\square$

To construct a taut foliation with controlled depth, we also need the follow proposition from Gabai [2].

Theorem 2.6. *Suppose (M, γ) is a balanced sutured manifold. Suppose $S \subset M$ is a well-groomed surface so that we have a decomposition*

$$(M, \gamma) \xrightarrow{S} (M', \gamma').$$

Suppose further that (M', γ') admits a taut foliation of depth k , then (M, γ) admits a taut foliation of depth at most $k + 1$.

3 Constructing taut foliations of bounded depth

Lemma 3.1. *Suppose Σ is a closed connected oriented surface of genus $g \geq 1$, then there are at most $(3g - 2)$ many connected simple closed curves on Σ so that they are each non-separating, pair wise disjoint, and pair wise non-parallel.*

Proof. Suppose γ is a collection of connected simple closed curves on Σ so that they are each non-separating, pair wise disjoint, and pair wise non-parallel. Write $|\gamma|$ be the number of components of γ . We want to show that $|\gamma| \leq 3g - 2$.

First assume $g > 1$. Write $S = \Sigma \setminus N(\gamma)$, where $N(\gamma)$ is an annular neighborhood of γ . By assumption, each component of S has negative Euler

Characteristics. Since $\chi(S) = \chi(\Sigma) = 2g - 2$, we know that $|S|$, i.e., the number of components of S is at most $2g - 2$. Also, we know that

$$|\partial S| - 2 \cdot |S| \leq \chi(S) = 2g - 2,$$

so

$$|\gamma| = \frac{1}{2}|\partial S| \leq 3g - 3.$$

When $g = 1$, clearly $|\gamma| \leq 1$ and we are done. $\square$

Lemma 3.2. *Suppose $d_1, \dots, d_m$ are integers so that each of them is at least 2. Then, we have*

$$d_1 + \dots + d_m \leq d_1 \dots d_m.$$

Proof. It is a straightforward application of the induction. $\square$

Now we are ready to prove the main result of the paper.

Proof of Theorem 1.5. We prove the theorem by an induction on k . When $k = 0$, by Theorem 1.2 in Ghosh and Li [3], we know that (M, γ) is a product sutured manifold and hence it admits a taut foliation of depth 0. Suppose the theorem holds for $k < k_0$. Now we argue for the case $k = k_0$. (We will keep writing k instead of k_0)

Case 1. When (M, γ) is not horizontally prime. Then, we can find a non-boundary-parallel horizontal surface S and perform a decomposition along S :

$$(M, \gamma) \xrightarrow{S} (M_1, \gamma_1),$$

with (M_1, γ_1) being a disjoint union

$$(M_1, \gamma_1) = (M_2, \gamma_2) \sqcup (M_3, \gamma_3).$$

Since S is not boundary parallel, we conclude that

$$\dim_{\mathbb{C}} SHI(M_2, \gamma_2) \leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M, \gamma)$$

and

$$\dim_{\mathbb{C}} SHI(M_3, \gamma_3) \leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M, \gamma).$$

Hence, by the inductive hypothesis, (M_2, γ_2) and (M_3, γ_3) both admits a taut foliation of depth 2^{k+5} . Since S is well-groomed, we are done by Theorem 2.6.

Case 2. When (M, γ) is horizontally prime. By Proposition 2.16 in Juhasz [6], we know that there is a union A of product annuli, and a sutured manifold decomposition

$$(M, \gamma) \xrightarrow{A} (M', \gamma'),$$

so that (M_1, γ_1) is reduced. Let (M_1, γ_1) be the union of components of (M', γ') that are not product sutured manifolds. There is a smallest union of product annuli $A' \subset A$ so that the decomposition along A' results in

$$(M, \gamma) \xrightarrow{A'} (M_1, \gamma_1) \sqcup (M_2, \gamma_2),$$

where (M_2, γ_2) is a product sutured manifold. Write

$$(M_3, \gamma_3) = (M_1, \gamma_1) \cup (M_2, \gamma_2).$$

Suppose the components of A' are

$$A' = A_1 \cup \dots \cup A_n.$$

Claim 1. We have

$$n < 6 \cdot 2^{k+1}.$$

To prove this claim, first assume that (M_1, γ_1) is connected. By Proposition 2.5, we know that

$$g(\partial M_1) \leq 2^{k+1}.$$

Assume that $|\gamma_1| > 6 \cdot 2^{k+1} - 4$, then by Lemma 3.1 and the pigeon-hole theorem, we know that there are three components of γ_1 that are parallel to each other. Hence, there is obviously a non-trivial product annulus separating the three parallel sutures from the rest, which contradicts the fact that (M_1, γ_1) is reduced. Thus, we conclude that

$$n < |\gamma_1| \leq 6 \cdot 2^{k+1} - 4 < 6 \cdot 2^{k+1}.$$

In general, if (M_1, γ_1) is disconnected, then each component is not a product sutured manifold and its sutured instanton Floer homology has dimension at least 2. Thus, the argument above and Lemma 3.2 apply and we conclude Claim 1.

Claim 2. There is a sequence of sutured manifold decompositions

$$(M, \gamma) \xrightarrow{S_1} (N_1, \delta_1) \dots \xrightarrow{S_n} (N_n, \delta_n),$$

so that the following is true.

- (1) Each S_i is well-groomed.
- (2) Each (N_i, δ_i) is taut.
- (3) Suppose the components of γ_3 are

$$\gamma_3 = \theta_1 \cup \dots \cup \theta_m.$$

Then, for each $i = 1, \dots, m$, there is a compact connected oriented surface-with-boundary F_i satisfying the following properties.

- (a) For $i = 1, \dots, m$, there is an orientation reversing embedding

$$f_i : \theta_i \hookrightarrow \partial F_i.$$

- (b) Write

$$F = F_1 \cup \dots \cup F_m \text{ and } f = f_1 \cup \dots \cup f_m,$$

then we have

$$N_n = M_3 \cup_f [-1, 1] \times F \text{ and } \delta_n = (\gamma_3 \cup \{0\} \times \partial F) \cap \partial(M_3 \cup_f [-1, 1] \times F).$$

To prove this claim, we focus on the case when $n = 1$. The general case follows immediately by induction.

When $n = 1$, we have a sutured manifold decomposition

$$(M, \gamma) \xrightarrow{A_1} (M_3, \gamma_3) = (M_1, \gamma_1) \cup (M_2, \gamma_2).$$

Write $\partial A_1 = \alpha_+ \cup \alpha_-$ so that $\alpha_{\pm} \subset R_{\pm}$. Write $V_{\pm}$ the component of $R_{\pm}$ that contains $\alpha_{\pm}$.

When α_+ and α_- are non-separating in V_+ and V_- , respectively. Then, A_1 has already been well-groomed, and we just take $S_1 = A_1$. Then, we have $(N_1, \delta_1) = (M_3, \gamma_3)$. So, for $i = 1, \dots, m$, we simply pick F_i to be an annulus and identify θ_i with any component of ∂F_i but with orientation reversed.

When α_+ is separating, while α_- is non-separating. Then, $-\alpha_+$ bounds a sub-surface in V_+ , which we call F_1 . We can glue F_1 to A_1 and push it into the interior of M . Write the resulting surface S_1 , then $\partial S_1 = \alpha_-$ and by assumption it is well-groomed. After the decomposition along A_1 , there is a component of γ_3 corresponding to α_+ , which we write θ_1 . Then, via α_+ , θ_1 is identified with a component of F_1 with orientation reversed. It is straightforward to check that

$$N_1 = M_3 \cup_{\theta_1} [-1, 1] \times F_1 \text{ and } \delta_1 = (\gamma_3 \cup \{0\} \times \partial F_1) \cap \partial(M_3 \cup_{\theta_1} [-1, 1] \times F_1).$$

We can take $F_2, \dots, F_m$ to be annuli just as in the previous case.

When α_+ and α_- are both non-separating, the argument is exactly the same as above. This concludes the proof of Claim 2.

Note, by construction, we have

$$(M_3, \gamma_3) = (M_1, \gamma_1) \cup (M_2, \gamma_2),$$

where (M_1, γ_1) is horizontally prime and reduced, and (M_2, γ_2) is a product. Since disjoint union with a product sutured manifold does not affect sutured instanton Floer homology, and there is a depth 0 taut foliation on any product sutured manifold, we can simply disregard the piece (M_2, γ_2) and assume that (M_3, γ_3) itself is reduced and every component is not a product sutured manifold. Then, by Lemma 2.13 in Juhász [6], it is also free of essential product disks. Thus, Proposition 2.4 applies and there is a well-groomed surface $S \subset (M_3, \gamma_3)$ and a sutured manifold decomposition

$$(M_3, \gamma_3) \xrightarrow{S} (M_4, \gamma_4)$$

so that

$$\dim_{\mathbb{C}} SHI(M_4, \gamma_4) \leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M_3, \gamma_3).$$

Next, we need to modify (N_n, δ_n) a little bit. Suppose for some $i \in \{1, \dots, m\}$, the surface F_i has a connected boundary. Then, ∂F_i is necessarily identified with θ_i , with orientation reversed, by construction.

Claim 3. If ∂F_i is connected, then F_i is not a disk.

Suppose, in the contrary, that F_i is a disk. Then, from the proof of Claim 2, we know that there is a product annulus $A_1 \subset (M, \gamma)$ and a boundary component α of A_j so that $\partial F_i = -\alpha$. Write α' be the other boundary component of A_j , we know that $A_j \cup F_i$ is a disk whose boundary is α' . Since (M, γ) is taut, we know that α' also bounds a disk $D \subset R(\gamma)$. Then, $A_j \cup F_i \cap D$ is a 2-sphere. Since (M, γ) is taut, this 2-sphere bounds a 3-ball,

and, hence, A_j is a trivial product annulus, which contradicts the way we choose A_j .

By Claim 3, if some F_i has connected boundary, then we can pick a non-separating curve β_i on F_i . Then,

$$[-1, 1] \times \beta_i \subset [-1, 1] \times F_i \subset (N_n, \delta_n)$$

is a product annulus, which is also a well-groomed surface. Let A'' be the union of all such product annuli. Note we have assumed that (M_3, γ_3) is reduced, so as in the proof of Claim 1, we know the following.

Claim 4. We have

$$|A''| \leq |\gamma_3| < 6 \cdot 2^{k+1}.$$

We have a sutured manifold decomposition

$$(N_n, \delta_n) \xrightarrow{A''} (N'_n, \delta'_n).$$

By construction, we know that there are connected compact oriented surfaces F'_i , which are either F_i or F_i cut open along β_i , so that the following is true. (Recall θ_i are the components of γ_3 .)

- (a') Each F'_i has at least two boundary components.
- (b') For $i = 1, \dots, m$, there is an orientation reversing embedding

$$f'_i : \theta_i \hookrightarrow \partial F'_i.$$

- (c') Write

$$F' = F'_1 \cup \dots \cup F'_m \text{ and } f' = f'_1 \cup \dots \cup f'_n,$$

then we have

$$N'_n = M_3 \cup_{f'} [-1, 1] \times F' \text{ and } \delta'_n = (\gamma_3 \cup \{0\} \times \partial F') \cap \partial(M_3 \cup_{f'} [-1, 1] \times F').$$

Next, we want to extend the well-groomed surface $S \subset (M_3, \gamma_3)$ to a well-groomed one on (N'_n, δ'_n) . To do this, we extend S across all $[-1, 1] \times F'_i$ as follows: For $i \in \{1, \dots, m\}$, if $S \cap \theta_i = \emptyset$, then we are already done. If $S \cap \theta_i \neq \emptyset$, then S being well-groomed implies that $S \cap \theta_i$ is a finite set of points of the same signs. Write σ'_i the component of $\partial F'_i$ so that $\sigma'_i = f'(\theta_i)$. Let $\tau'_i \subset F'_i$ be a disjoint union of parallel arcs so that each component of τ'_i has one end point on σ'_i and the other end point on $\partial F'_i \setminus \sigma'_i$, which is non-empty by condition (a'). We further require that

$$\tau'_i \cap \sigma'_i = f'(S \cap \theta_i).$$

Then, we can glue $[-1, 1] \times \tau'_i$ to S along $[-1, 1] \times (\tau'_i \cap \sigma'_i)$.

Performing this operation for all i , we obtain a surface $S' \subset (N'_n, \delta'_n)$ which is obviously well-groomed. There are two decompositions:

$$(M_3, \gamma_3) \xrightarrow{S} (M_4, \gamma_4) \text{ and } (N'_n, \delta'_n) \xrightarrow{S'} (N_{n+1}, \delta_{n+1}).$$

Claim 5. We have

$$SHI(M_4, \gamma_4) \cong SHI(N_{n+1}, \delta_{n+1}).$$

To prove this claim, for any $i \in \{1, \dots, m\}$, if $S \cap \theta_i = \emptyset$, then θ_i survives in M_4 and is a component of γ_4 . To obtain N_{n+1} from M_4 , we need to glue $[-1, 1] \times F'_i$ to M_4 along θ_i . However, by (an adaption to instanton theory of) Proposition 6.7 in Kronheimer and Mrowka [8], gluing such a product region will not change the sutured instanton Floer homology. Then, we need to deal with the case when $S \cap \theta_i \neq \emptyset$. Suppose

$$S \cap \theta_i = \{p_1, \dots, p_{s_i}\},$$

where $p_1, \dots, p_{s_i}$ are labeled according to the orientation of θ_i . Let θ''_i be the part of θ_i from p_{s_i} to p_1 . Then, θ''_i does not contain any other p_j . Recall there is a collection of arcs $\tau'_i \subset F'_i$. There is a component σ''_i of $\partial F'_i \setminus \tau'_i$ so that $\sigma''_i = f'(\theta''_i) \subset \partial F''_i$. It is straightforward to check that, to obtain N'_{n+1} from M_4 , we need to glue $[-1, 1] \times F''_i$ to M_4 along $[-1, 1] \times \sigma''_i$. Note σ''_i is an arc, so topologically $[-1, 1] \times \sigma''_i$ is a disk which intersects the suture γ_4 along an arc θ''_i , and intersects the suture $\{0\} \times \partial F''_i$ of the product sutured manifold $([-1, 1] \times F''_i, \{0\} \times \partial F''_i)$ along another arc $\{0\} \times \sigma''_i$. Hence, this gluing coincides with the setting of attaching a contact 1-handle to the disjoint union

$$(M_4, \gamma_4) \sqcup ([-1, 1] \times F''_i, \{0\} \times \partial F''_i)$$

in the sense of Baldwin and Sivek [1]. Since both disjoint union with a product sutured manifold and attaching a contact 1-handle do not change sutured instanton Floer homology, we finally conclude that

$$SHI(M_4, \gamma_4) \cong SHI(N_{n+1}, \delta_{n+1}).$$

Finally, we are ready to finish the induction. We have a sequence of decompositions:

$$(M, \gamma) \xrightarrow{S_1} (N_1, \delta_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (N_n, \delta_n) \xrightarrow{A''} (N'_n, \delta'_n) \xrightarrow{S'} (N_{n+1}, \delta_{n+1}). \quad (2)$$

We know from Claim 5 that

$$\begin{aligned} \dim_{\mathbb{C}} SHI(N_{n+1}, \delta_{n+1}) &= \dim_{\mathbb{C}} SHI(M_4, \gamma_4) \\ &\leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M_3, \gamma_3) \\ &\leq \frac{1}{2} \dim_{\mathbb{C}} SHI(M, \gamma) \\ &< 2^k. \end{aligned}$$

Thus, the inductive hypothesis applies on (N_{n+1}, δ_{n+1}) , and there is a taut foliation $\mathfrak{F}'$ of depth at most 2^{k+5} on (N_{n+1}, δ_{n+1}) . We now go through decomposition (2) to construct a taut foliation on (M, γ) . First, each S_i is well groomed by Claim 2, and $n < 6 \cdot 2^{k+1}$ by Claim 1. Second, each component of A'' is well-groomed and, since the components of A'' are contained in sufficiently disjoint regions of N_n , decomposing along some subset of A'' will keep the rest components of A'' being well-groomed. By Claim 4, $|A''| < 6 \cdot 2^{k+1}$. Finally, there is one last decomposition along the well-groomed surface S' , so by Theorem 2.6, there is a taut foliation $\mathfrak{F}$ on (M, γ) of depth at most

$$6 \cdot 2^{k+1} + 6 \cdot 2^{k+1} + 1 + 2^{k+5} < 2^{k+6}.$$

Hence, the inductive step is completed and we finish the proof of Theorem 1.5. $\square$

Corollary 3.3. *Suppose (M, γ) is a taut balanced sutured manifold with $H_2(M) = 0$, and*

$$\mathrm{rk}_{\mathbb{Z}_2}(SFH(M, \gamma)) < 2^{k+1},$$

then (M, γ) admits a taut foliation of depth at most 2^{k+6} .

Proof. The proof of Theorem 1.5 applies verbatim. $\square$

The above corollary can be used to prove the following, which gives a partial answer to question 9.14 in Juhász [5].

Corollary 3.4. *Let K be a knot in a rational homology 3-sphere Y and suppose that k is a positive integer so that*

$$\mathrm{rk}_{\mathbb{Z}_2}\widehat{HFK}(Y, K, g(K)) < 2^k.$$

Then, $Y \setminus N(K)$ admits a taut foliation of depth at most 2^{k+5} transverse to the boundary of $N(K)$.

Proof. We have a sutured manifold $Y \setminus N(K)$, with toroidal suture. Pick S being a minimal genus rational Seifert surface of K , then we have a sutured manifold decomposition

$$(Y \setminus N(K)) \xrightarrow{S} (M, \gamma), \quad (3)$$

and we know that

$$SFH(M, \gamma) \cong \widehat{HFK}(Y, K, g(K)).$$

Thus, Corollary 3.3 applies and we obtain a taut foliation on (M, γ) of depth at most 2^{k+5} . We can further glue it along the decomposition (3), which will not increase the depth of the taut foliation. Hence, we are done. $\square$

Proof of Corollary 1.6. On the knot complement $S^3(K) = S^3 \setminus N(K)$, we can pick Γ_μ to be a suture consisting of two meridians. From Corollary 4.2 in Kronheimer and Mrowka [7], we know that

$$\dim_{\mathbb{C}} SHI(S^3(K), \Gamma_\mu) < 1 + 2^{k+2}.$$

Pick a minimal genus Seifert surface S of K , we know that there is a decomposition

$$(S^3(K), \Gamma_\mu) \xrightarrow{S} (M, \gamma),$$

and by (adaptions to instanton theory of) Lemma 5.7 and Proposition 5.11 in Kronheimer and Mrowka [8], we know that

$$\dim_{\mathbb{C}} SHI(M, \gamma) < 2^{k+1}.$$

Hence, we can apply Theorem 1.5, and there is a taut foliation on (M, γ) of depth at most 2^{k+6} . Note we can also regard $S^3(K)$ as a sutured manifold with toroidal sutures, and decomposing $S^3(K)$ along S also gives rise to (M, γ) . So, we can glue the just obtained taut foliation on (M, γ) along this later decomposition to conclude the proof of Corollary 1.6. $\square$

References

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