Third Graders' Interpretations of Subtraction Worked Examples: Matching Number Sentences and Visuals

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This study investigated 37 third graders' explanations of subtraction worked examples shown in number sentence or visual form (ten frame or number line) and their justifications for which visual and numerical worked examples corresponded to the same subtraction strategy. Results showed that third graders gave more detailed explanations in number sentence form than in visual form; whereas, they had higher accuracy in matching number sentences to visuals than vice versa. When matching, they were more likely to reason sufficiently when identifying processes represented in the worked examples as opposed to reasoning about the order of the numbers. When using worked examples, teachers should make use of visuals to help students focus on how the visuals represent the operations.

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Commonly in mathematics instruction, teachers present students with number sentences and then show them visual models (e.g., number lines, ten frames) to assist their understanding (Fosnot & Dolk, 2001). Although visuals represent "a logical extension of their mental procedure" (Baroody, 1984, p. 203), many elementary students struggle with interpreting them (Gersten et al., 2009). Yet, visuals can alleviate the burden of mental processing, helping students learn mathematics (Booth & Siegler, 2008). Understanding the relations between written problems and visuals is an important mathematical practice (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Older elementary students' use of worked examples in algebra enhanced their conceptual understanding (Booth & Koedinger, 2012); however, especially for elementary students whose learning is primarily assisted by visual models and manipulatives (Carpenter & Moser, 1984), how their interpretations of numerical and visual worked examples differ is not clear. The goal of this paper was to investigate the following research question: How did third graders explain and match corresponding numerical and visual worked examples?

Worked Examples

Worked examples consist of a problem statement and solution steps, which can help students to identify important mathematical problem features and solution processes (e.g., Atkinson, Derry, Renkl, & Wortham, 2000; Booth et al., 2015; Carroll, 1994; Sweller, 1988).

Instructional practices involving worked examples can reduce "procedural demands" and enable conceptual learning, which leads to fewer mathematical errors (Booth et al., 2015; Lange, Booth, & Newton, 2014). Worked examples are often combined with other instructional methods to promote learning. One promising method is to require students to respond to prompts about the problem or solution features. Self-explanation can facilitate students' procedural learning, make their prior and current knowledge explicit, and contribute to significant adaptations when solving novel transfer problems (e.g., Atkinson et al., 2000; Booth et al., 2015; Rittle-Johnson, 2006). The design and structure of worked examples (i.e., intra-example features) are essential for effective learning. For example, integrating multiple sources of information (e.g., texts and diagrams) in a unified presentation or highlighting a problem's conceptual subgoals can help students refocus on relevant underlying problems and solution structures (e.g., Atkinson et al., 2000). Many research studies have investigated the effectiveness of worked examples when changing the structural features (e.g., Booth, Lange, Koedinger, & Newton, 2013, presented worked examples with incomplete steps in equations), including other instructional methods (e.g., self-explanation, contrasting correct versus incorrect worked examples), or using multiple sources of information (e.g., Durkin and Rittle-Johnson, 2012, illustrated worked examples using number lines with decimal marks). We add to this literature by detailing third graders' interpretations of visual worked examples and their reasoning about matching two worked examples with numerical versus visual sources of information.

Subtraction Strategies and Visuals

A major goal in early addition and subtraction instruction is to help students make and use groups of ten in their strategies (Fuson et al., 1997). To solve double-digit subtraction problems (e.g., 24 – 12), strategies involving tens include (a) adding tens and ones to the subtrahend to get to the minuend, where the total added is the answer, (b) decomposing the subtrahend into tens and ones and subtracting ten first; and (c) decomposing the subtrahend into a number that will subtract evenly to make a ten (Carpenter & Moser, 1984; Fuson et al., 1997).

For double-digit subtraction, number line and ten-frame visuals can be helpful (Fosnot & Dolk, 2001) to illustrate students' strategies. In terms of the strategies for solving subtraction problems, number lines easily show jump strategies (Wright, Stanger, Stafford, & Martland, 2006); jumps can go up (i.e., adding to do subtraction) or down (i.e., subtracting). On the other hand, ten-frame representations are especially accessible for children at the beginning of learning subtraction with small double-digit numbers and can more effectively show composition and decomposition of numbers (Fuson & Briars, 1990). We explore students' interpretations of these subtraction strategies and visuals within sets of worked examples.

Methods

Participants and Setting

This data came from a larger study on first and third graders' commenting (explaining) and debugging practices in beginning programming and mathematics. Because they should be familiar with many subtraction strategies and visuals, we focus here on the 37 third graders, who came from a midwestern public elementary school with about 46% of students qualifying for free or reduced-price lunch.

Data Collection

Our data includes two problem types in which third graders had to explain and match visual and numerical worked examples for a particular subtraction strategy and problem. We focus on two items each from two types of problems: (1) choosing a correct number line (NL) or ten frame (TF) worked example for a given numerical worked example (NS); (2) choosing the correct numerical worked example for a given visual worked example. We administered the two types on separate testing occasions about a month apart. For type 1 (see Figure 1), third graders were asked to "explain how the student solved the math problem" and then identify "which picture, A or B, best shows how the student solved the problem." For type 2, third graders were asked to "explain what math this student was doing in the picture" and then identify "which picture, A or B, matches the math in the picture." After third graders matched, they had to explain how it matched and why the other choice did not match.

[Figure 1]

Analyses

We coded third graders' explanations of the target worked example for whether they read (with or without errors) *all steps* of the strategy, *partial steps*, *invented steps*, or *other*. For the type 1 problems, we also made note of students who identified how the 12 was decomposed (either 10 and 2 or 8 and 4). To analyze their matching, we classified their justifications based on whether they attended to the result (result-matching), the numbers in the problems without mentioning relations among the numbers (number-matching), the order of the numbers in the number sentences and visuals being the same (order-matching), or the process of the operation (process-matching) (see Table 1). Finally, to capture details of how number sentences and visuals worked together, we identified whether third graders' reasoning was sufficient or insufficient to distinguish one choice from the other one.

[Table 1]

Results

Worked Example Explanations

Across all four problems, we found examples of third graders who, when initially explaining how the student solved the numerical problem or what math the student was doing in the picture, captured all steps of the solutions, some of the steps, or invented steps (see Table 2 for examples). For type 1 problems, third graders read all steps for 53% of the NS worked examples on average. The numerical worked examples for type 1 problems showed the subtrahend in the original problem being decomposed (e.g., breaking 12 into 2 and 10) before showing the student's two solution steps. Across all students, ten mentioned the decomposition for problem type 1a and nine for type 1b. Overall, seven instances of students mentioning the decomposition were prompted, but twelve were unprompted.

[Table 2]

For type 2 problems, third graders read all steps for 22% of problems on average; whereas, they read partial steps for 61% on average. On the type 2 problems, the original subtraction problems were not shown, only the steps to solve the problems. Interestingly, only two males mentioned that the overall amount being taken away was 12 for the ten-frame worked example; for the number-line worked example, one female and six males identified that the original problem was 24 - 12.

Matching Numerical and Visual Worked Examples

Overall, students had higher accuracy for type 2 problems, matching the worked examples when they explained the visual as opposed to the numerical worked examples first (see Table 3). Further, within each type, they had higher accuracy when matching worked examples with ten frames as opposed to number lines. The pattern held for females and males with the females having slightly higher performance than the males, although less pronounced for NS to TF and not for TF to NS.

[Table 3]

The results for students' justifications of why they made their matches provide some insight into why they were more accurate for the two cases described above. Overall, process matching was the most frequent (59%) and most accurate (90%) reasoning that third graders used; result matching was the least frequently used reasoning (6%) with the lowest rate of accuracy (22%) (see Table 4). Across all matching problems, focusing on the result was not helpful because the two worked example choices had the same answer. Therefore, the only time students' result justifications were sufficient was when they selected the wrong match and described additional details in that example.

Order matching was mostly used by third graders on the type 1(a) problem. However, their explanations were also largely insufficient (57% of the time) to distinguish between the two visuals; although some students read the number line from right to left as expected when describing subtraction, many students read the number line from left to right. Therefore, justifying, "The student subtracted 2, and then they subtracted 10" (Fox8 for NS to NL) did not help us distinguish between people who were reading choice A from left to right versus choice B from right to left. Number matching became more helpful in these cases because students who used that reasoning tended to pinpoint numbers that were different in the worked examples (e.g., the 14 on the number line for type 1a). Order matching was less frequent but successful for type 2a where the subtrahends in the choices differed and when order was clearer, as in the ten frame problems.

Students' process matching explanations were highly sufficient (92% of the time). Therefore, third graders did best on the ten frame problems because these were the problems on which students were most likely to use process matching (see Table 4). Students tried to match the dots to the numbers in the problems, which led them to go through the steps. For instance, when reasoning about her choice for type 1(b) problem, Ape9 explained, "That would equal 24. Cause this is 4, this is 20 [pointed to B, Step 1]. Now you have 20 [pointed to the uncrossed dots in B, Step 1], so you're taking away 8 [pointed to number sentences], 1, 2, 3, 4, 5, 6, 7, 8 [counted the crossed dots in B, Step 2]. Then you have the rest, that's 12." Ten frames also might have been easier to interpret than the number lines because the steps more clearly progressed from left to right.

[Table 4]

Discussion and Implications

The results suggest that starting with visuals aided third graders' performance on the matching problems. Although, they were less likely to explain all steps involved in the visuals, they showed a higher percentage of using process matching reasoning, which suggests that they tended to notice the key elements (e.g., different subtrahends) and operations in the visuals to help them make correct decisions. Thus, mathematics teacher educators could use such examples as guides to help teachers identify benefits of using visuals and how to use visuals with students more effectively, helping them pinpoint key features of multi-step subtraction problems.

When explaining the visual worked examples, students were not likely to talk about the overall problem represented and how the two steps involved a decomposition of the original subtrahend. Therefore, when having younger students evaluate worked examples, an important step is to have them identify what problem the hypothetical student was solving and how that student knew how much to take away in each step. To encourage explanation of each step, we could ask students to describe or evaluate each step. Having students match worked examples often drew their attention to the steps of the problem, so further exploring the benefits of different types of matches could lead to additional strategies for helping students make sense of the structure of problems and visual worked examples.

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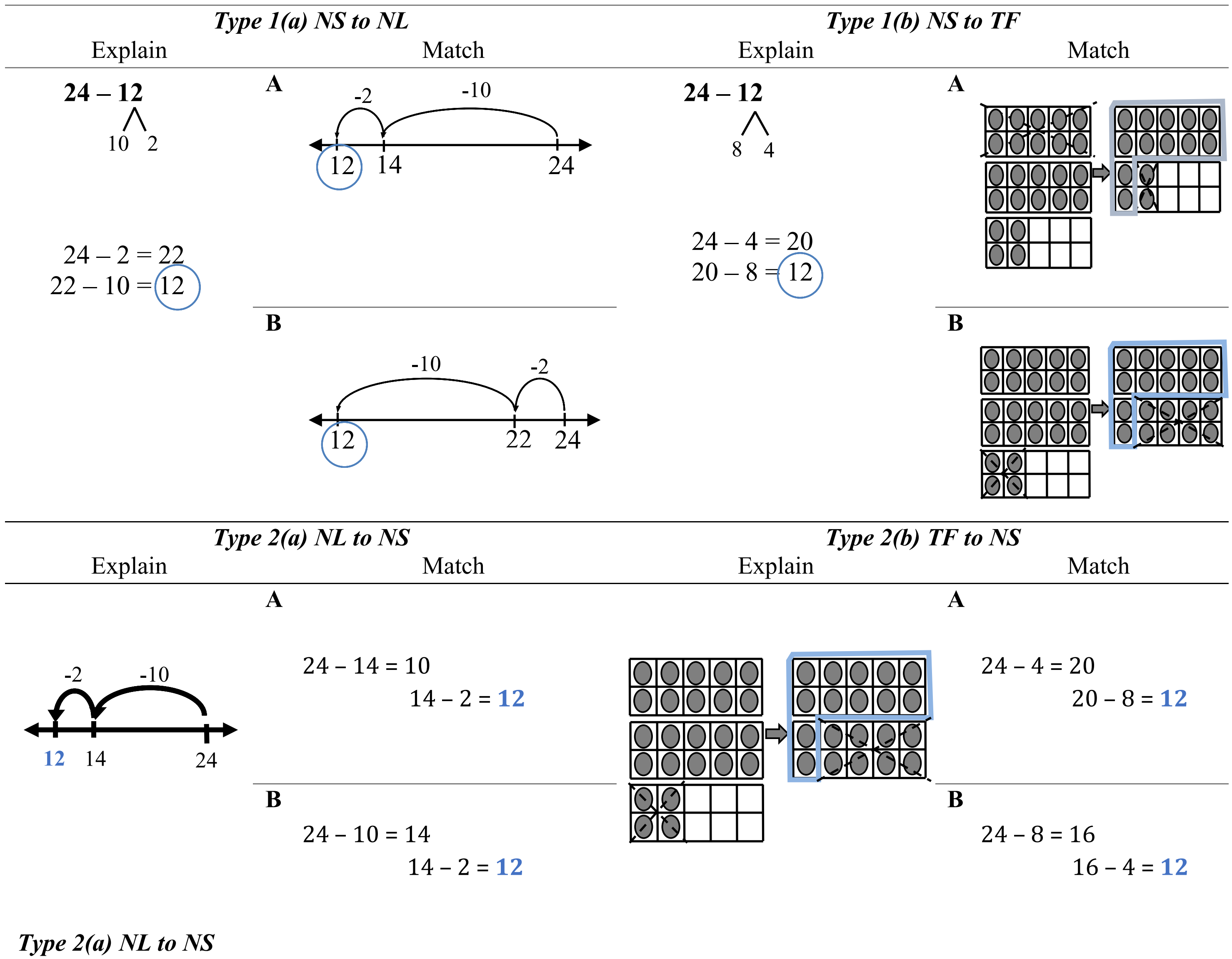

Figure 1: Worked Example Problems Students had to Explain and Match

Table1.
Examples of Each Mathematical Matching Reasoning Type

Reasoning	Definition	Example
Order matching	Mentioning sequence of numbers	They subtracted 4 first [pointed to the crossed dots in Step 1 in Picture B and then pointed to Step 1 in Picture A], and then they subtracted 8. (Fox10, NS to TF)
Result matching	Mentioning resulting numbers as key	Because there is 12 right here [pointed to Step 2 in A] and 12 right here [pointed to 12 in the second number sentence]. (Ape3, NS to TF)
Process matching	Mentioning something about relations between two numbers or operations.	It shows us subtracting four from 20, which they are doing in that one [pointed to the number sentence], and then 20 minus eight, so there must be 20 on this one [pointed to all dots in Step 2 in Picture B], then subtracting eight. (Ape7, NS to TF)
Number matching	Mentioning discrete numbers without the relations with other numbers or operations.	Because [counted crossed dots in Step 2 in B silently, only the head nodding was seen] it is 8 [pointed to number sentence], it's eight [pointed to crossed dots in Step 2 in B]. (Yak2, NS to TF)

Table 2.
Percent of Third Graders' Explanation Types and Example Explanations for Each Problem

Problem	All steps	Partial steps	Invented steps/Other
Type 1(a) NS to NL	43% *They do 24 minus 2 equals 22, 22 minus 10 equals 12.*	38% *24 minus 2, 22 minus 10.*	19% *This 24 minus 12, and this is 10 and 2. so he got, the answer is 12. He probably like put these on top of each other [pointed to 24-12], he subtract 4 minus 2, 2. 2 minus 1 equals 1, so he probably does - that's how he got his answer.*
Type 1(b) NS to TF	62% *He did 24 then minus 4 to get 20. He did 20 minus 8 to get 12.*	22% *They did minus. Minus 24 minus 4 equals 20.*	16% *They added to 12, they showed 8 and then 20...if 24 minus 4 is 20, they should put 24 minus 20 is 4.*
Type 2(a) NL to NS	22% *Subtraction because there is minus sign. 24 minus 10 equals 14, then 14 minus 2 equals 12.*	56% *It's subtraction. It's subtracting by 10, subtracting by 2.*	22% *He is adding then subtracting. He is going from 2, he is counting by 2, then he is counting by 10.*
Type 2(b) TF to NS	22% *She had 24, she took away the 4, and she had 20, and she took away 8, and she had 12.*	65% *Subtraction. 24 taking 4 away. Subtraction, taking away 4 from 20.*	13% *So 20, 30, that's 30, OK. That doesn't count [pointed to the crossed dots in Step 1]. 30, 31, 32 [pointed to the uncrossed dots in the second ten frame in Step 2]. So it equals 32.*

Table 3.

Percent of Third Graders Who Correctly Matched the Worked Examples When the Numerical Versus Visual Worked Examples Were Presented First

	Numerical worked example first		Visual worked example first	
3rd Graders	NS to NL	NS to TF	NL to NS	TF to NS
Females (n=15)	67%	73%	87%	93%
Males (n=22)	55%	68%	77%	95%
Total (N=37)	59%	70%	81%	95%

Table 4.

Percent of Third Graders' (N=37) Who Used Different Types of Reasoning to Match Examples and Correctness by Reasoning Type

	Order Matching	Result Matching	Process Matching	Number Matching
Problem	Frequency (% Correct)	Frequency (% Correct)	Frequency (% Correct)	Frequency (% Correct)
NS to NL[1]	62% (48%)	--- ---	22% (75%)	14% (100%)
NS to TF	8% (100%)	22% (25%)	62% (87%)	8% (67%)
NL to NS	5% (100%)	--- ---	81% (87%)	14% (40%)
TF to NS	16% (100%)	3% (0%)	70% (100%)	11% (75%)
Total	23% (65%)	6% (22%)	59% (90%)	12% (71%)

[1]Ape11 said, "I don't know. I just guess." Because he does not fall under any of the categories, the percentages for NS to NL do not add to 100%.