Locally integrating theories to investigate students' transfer of mathematical reasoning

Heather Lynn Johnson[1], Evan McClintock[2], and Amber Gardner[3]

School of Education and Human Development, University of Colorado Denver, USA; [1]heather.johnson@ucdenver.edu

[2]evan.mcclintock@ucdenver.edu

[3]amber.gardner@ucdenver.edu

To theorize the study of students' transfer of mathematical reasoning, we integrate Lobato's theory of actor-oriented transfer, Thompson's theory of quantitative reasoning, and Marton's variation theory. Linking theory and method, we argue for an expansion of design possibilities for researchers' investigations of students' transfer of reasoning. To bolster our argument, we provide empirical data of a secondary student's transfer of a particular form of reasoning—covariational reasoning. Our research has implications for researchers' development of new theoretical and methodological approaches to investigate complex phenomena, such as students' mathematical reasoning.

Keywords: Transfer of learning, reasoning, learning theories, instructional design,

Investigating students' transfer of mathematical reasoning is a complex, multifaceted endeavor. We identify three key elements of this endeavor. The first two elements focus on students' perspectives: students' engagement in mathematical reasoning and students' transfer of their mathematical reasoning. The third element focuses on researchers' design of mathematical task sequences to engender students' mathematical reasoning. We view these elements to be interconnected, and in our theorizing of these elements, we work to address this interconnectedness.

Networking theories can take a variety of forms, depending on the degree of connections between and among theories, as well as researchers' goals with connecting theories (Bikner-Ahsbahs & Prediger, 2010). We are working to locally integrate theories (Bikner-Ahsbahs & Prediger, 2010), in which we extend beyond combining or coordinating theories to explain empirical phenomena, to building new theories. Specifically, we integrate Lobato's (2003) theory of actor-oriented transfer, Thompson's (2011) theory of quantitative reasoning, and Marton's (2015) variation theory to theorize our study of students' transfer of a particular form of reasoning—covariational reasoning.

We aim to extend possibilities for researchers to investigate students' transfer. We argue that researchers can investigate students' transfer of mathematical reasoning by research designs other than pre- and post-tasks. To provide backing for our argument, we share empirical data of a secondary student's transfer of a particular form of reasoning—covariational reasoning (Thompson & Carlson, 2017). We address implications for researchers' development of new theoretical and methodological approaches to investigate complex phenomena, such as students' mathematical reasoning.

Integrating theories to investigate students' transfer of covariational reasoning

Task designers, teachers, and researchers share a goal of promoting students' mathematical reasoning. While mathematical reasoning comprises students' mental activity, in our view, it is impossible to separate students' mental activity from their embodied experiences (English, 2013). Interacting with mathematical tasks can provide students opportunities to engage in mathematical reasoning. By mathematical task, we mean something more than a problem statement. From our perspective, tasks encompass the intentions of a task designer, the implementation of a task by a teacher/researcher, students' interaction with the task, and physical materials associated with the task (Johnson, Coles, & Clarke, 2017). Ideally, when students develop a form of mathematical reasoning, they could engage in that form of reasoning on subsequent tasks and in different situations.

We aim to infer students' reasoning based on their observable behavior. In our view, students are experts in their own mathematical reasoning, and our role is to elicit and explain that reasoning. Our stance on students' reasoning influences our assumptions about the viability of their reasoning. First, we do not assume that students working on a task will share our goals for the task (e.g., Johnson, Coles, & Clarke, 2017). Second, we acknowledge that students may engage in mathematical reasoning that is different from the reasoning we intend. Third, we assume students' different forms of reasoning to be viable and productive. Hence, we do not seek to "fix" students' reasoning. Rather, we seek to understand and engender students' mathematical reasoning, in its myriad forms.

Employing a transfer lens, researchers can investigate how students might engage in different forms of mathematical reasoning across a range of tasks and situations. Yet, researchers' perspectives on transfer impact what researchers may construe as evidence of students' transfer of reasoning (Lobato, 2003; 2014; Marton, 2006). To address the problem of investigating transfer of students' reasoning, we employ an actor-oriented transfer perspective (Lobato, 2003; 2014), because it affords researchers the opportunity to gather evidence of transfer of reasoning beyond forms of reasoning that a researcher may intend.

From an actor-oriented transfer perspective, transfer is generalization, rather than application (Lobato, 2003; 2014). Meaning, students who generalize some form of reasoning from one task to another would engage in transfer, even if they do not apply a particular solution method. Hence, an actor-oriented transfer perspective broadens the scope of what is possible for students to transfer. Furthermore, a researcher's theoretical perspective on transfer impacts methods used to gather evidence of students' transfer. Researchers employing an actor-oriented transfer perspective examine students' constructions of similarities across tasks, rather than students' task performance (Lobato, 2003; 2014). Because students' mathematical reasoning is not the same as task performance, an actor-oriented transfer perspective is a productive theoretical lens to use to investigate students' reasoning.

We focus on students' engagement in and transfer of a particular form of reasoning—covariational reasoning—which is critical for students' development of a conception of function (Thompson and Carlson, 2017). We draw on Thompson's theory of quantitative reasoning (Thompson, 2011; Thompson & Carlson, 2017) to explain our perspective on covariational reasoning. When students engage in covariational reasoning, they form and interpret relationships between attributes they conceive of as capable of varying and possible to measure. For example, consider a situation involving a toy car moving around a track. A student engaging in covariational reasoning could form and interpret relationships between different attributes, such as the toy car's distance traveled around the track and its distance from some stationary object. Because an actor-oriented transfer perspective focuses on students' generalization rather than performance, it is a useful theoretical lens to explain invariance in students' engagement in quantitative (and covariational) reasoning across tasks and situations (Thompson, 2011).

Researchers employing a variation theory perspective (Kullberg, Kempe, & Marton, 2017; Marton, 2006; 2015) consider critical aspects of objects of learning (intended objects of learning), problematize ways in which to provide students opportunities to experience those critical aspects (enacted objects of learning), and acknowledge the viability of learning that students actually demonstrate (lived objects of learning). Although we focus on a particular form of reasoning—covariational reasoning—we acknowledge that students may engage in or transfer forms of reasoning other than what we intend. From a variation theory perspective, students discern the similar (and generalize) because they discern difference, and consequently, designers should design opportunities for students to experience difference. Simply put, variation (in terms of difference) is a necessary condition for students to experience discernment.

Networking theories is useful for addressing complexities inherent in investigations of students' mathematical reasoning. For example, our investigation of students' transfer of covariational reasoning is intertwined with students' creation and interpretation of dynamic graphs in a Cartesian coordinate system. Drawing on Variation Theory, we considered critical aspects to include the types of attributes that students would encounter (e.g., length measures), the motion of objects in physical space (e.g., up/down, left/right, curving paths), and the representation of those attributes in a Cartesian coordinate system. Accordingly, we varied those critical aspects to provide opportunities for students to engage in covariational reasoning.

Networking (coordinating) Thompson's theory of quantitative reasoning and Marton's variation theory of learning, Johnson, McClintock, Hornbein, Gardner, and Grieser (2017) explained how researchers could design learning experiences to engender students' discernment of critical aspects of covariation. We extend this research in two ways. First, we connect the investigation of students' transfer of reasoning, employing an actor-oriented transfer perspective on transfer. Second, we build from coordinating theories to locally integrating theories, to contribute to theory development.

Extending design possibilities for actor-oriented transfer

Lobato (2014) identified four criteria on which researchers should base claims of actor-oriented transfer. We summarize these claims, centering students' mathematical reasoning, which is our focus. First, students should demonstrate a change in reasoning on transfer tasks, from a pre-interview to a post-interview. Second, students should demonstrate the new reasoning during interview tasks occurring between the pre- and post-interviews. Third, researchers should provide evidence that students' reasoning during the interview tasks influenced their reasoning on the post-interview tasks. Fourth, researchers should provide evidence that changes in students' reasoning in a post-interview occurred as a result of their work on the interview tasks, and was not just a spontaneous occurrence.

We posit an extension of design possibilities for researchers employing an actor-oriented transfer perspective. From our perspective, inherent in Lobato's (2014) design criteria is an assumption that researchers design transfer tasks that are separate from interview tasks. We argue that by leveraging Variation Theory in task design, researchers can investigate students' transfer by analyzing students' reasoning on subsequent tasks across a set of interviews. In making this argument, we follow Cobb's (2007) recommendation for theory expansion rather than replacement. In particular, we aim to expand design possibilities, rather than to replace one design type with another design type.

An empirical example: A task sequence and a student's work

To illustrate how we locally integrated theories, we provide an example from the work of a 9th grade student, Aisha, during three individual, task-based interviews, conducted by Johnson. Task sequences in each interview had a different background (first a Ferris Wheel, second a Cannon Man, and finally a Toy Car), given by a computer animation. Aisha worked on a tablet (an iPad), with pencil and paper available. In each task sequence, Aisha sketched and/or interpreted different Cartesian graphs to represent a relationship between attributes in a situation given in an animation.

The Cannon Man and Toy Car task sequences were more involved than the Ferris Wheel task sequence. For the Ferris Wheel, Aisha needed to sketch only one graph. For the Cannon Man and Toy Car, Aisha explored change in individual attributes, manipulating dynamic segments located on the horizontal and vertical axes. Next, Aisha sketched a graph representing a relationship between both attributes. Then, Aisha repeated the process for a second graph with attributes represented on different axes. After sketching each graph, Aisha viewed a computer-generated graph. Transcripts that follow are from each interview. Figure 1 shows graphs that Aisha drew in each interview. The Cannon Man and Toy Car graphs shown in Figure 1 (middle, right) are the second Cartesian graph that Aisha drew during each interview.

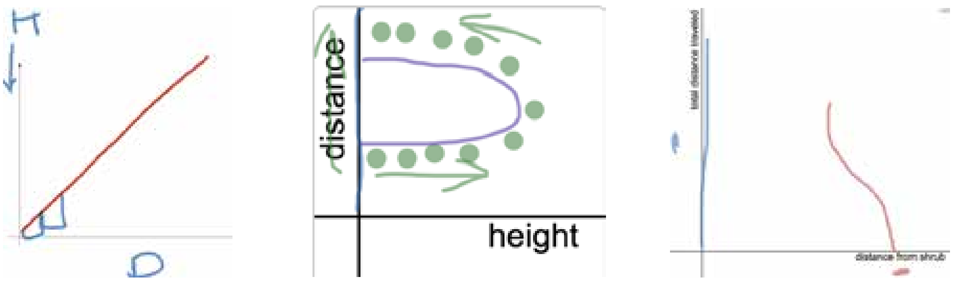

Figure 1: Aisha's Ferris Wheel, Cannon Man, and Toy Car graphs, respectively

Ferris Wheel

Johnson asked Aisha to sketch a graph relating a Ferris wheel cart's height from the ground and total distance traveled, around one revolution of the Ferris wheel. While sketching the graph shown in Figure 1, left, Aisha explained why she drew the graph the way that she did.

Aisha: I feel like the height would be more like the line (sketches a line, Figure 1, left). Distance would be more like the rise and run of the situation (sketches small segments, Figure 1, left). Cause you're using the rise and run to find the line, and you need to use the distance to find the height.

Cannon Man

Johnson asked Aisha to sketch a graph relating a Cannon Man's height from the ground and total distance traveled, with the height on the horizontal axis and the distance on the vertical axis. After

Aisha sketched the graph shown in Figure 1, middle, Johnson asked her to explain how her graph showed Cannon Man's height and distance.

Johnson: Can you show me how you see the height increasing and decreasing in this purple graph? (Points to the curved graph Aisha drew, Figure 1, middle)

Aisha: It's (the height's) increasing here, since it's (the graph's) backwards in my opinion (Sketches green dots, beginning on bottom left near the vertical axis, then moving outward, Figure 1, middle). Decreasing here. (Continues to sketch green dots, until getting close to the vertical axis, adding arrows after sketching dots, Figure 1, middle)

Johnson: How is the distance changing?

Aisha: (Turns iPad so that vertical axis is horizontal. Draws arrow parallel to vertical axis, Figure 1, middle.) That way. Continues to get bigger.

Toy Car

Prior to sketching the graph shown in Figure 1, right, Aisha stated, without prompting from Johnson, that distance traveled was the "same as the Cannon Man." Johnson asked Aisha to clarify.

Johnson: So, you said the total distance traveled is like the Cannon Man. Why is that like the Cannon Man again? Cause Cannon Man goes up and down, and this one moves around. How are those things the same?

Aisha: Just because Cannon Man is coming back down, doesn't mean his distance is going down. His distance is still rising.

To explore change in each of the individual attributes, Aisha manipulated dynamic segments located on the horizontal and vertical axes. For the total distance, Aisha began at the origin, continually moving the segment up, along the vertical axis. She explained: "I moved it up. It continuously went up, because the distance doesn't decrease. The total distance traveled doesn't decrease." For the distance from the shrub, Aisha began to the right of the origin, initially moving the segment to the left, and then to the right. She explained: "I moved it (the segment) to the left, because it (the Toy Car) was getting closer to the shrub. Then, when it (the Toy Car) started to turn, I started to moved it (the segment) back up to the right, because it (the Toy Car) was getting closer to the shrub." Next, Aisha sketched the graph shown in Figure 1, right. After viewing the computer-generated graph, Aisha explained what she thought the curved graph represented. Aisha stated: "This (moving her finger from left to right along the horizontal axis) is tracking the distance from the shrub, and this (moving her finger along the curved graph, beginning near the horizontal axis) is also tracking the distance."

Employing individual theories as analytic lenses

We explain how we draw on individual theories to analyze Aisha's transfer of covariational reasoning. While we present the accounts separately, we conceive of these accounts as something more than complementary. In our view, each theoretical lens is like one of the intertwined strands of a braid or cord.

Thompson's theory of quantitative reasoning

In our task design, we made strategic choices about the kinds of attributes that students would represent in graphs. In particular, we selected length attributes (e.g., height, distance) because we anticipated it would be less difficult for students to conceive of measuring length attributes than for other kinds of attributes, such as area or volume. During clinical interviews, Johnson asked students to represent ways in which individual attributes were varying (e.g., increasing and/or decreasing), and to explain how graphs they sketched represented different attributes.

For the Cannon Man and Toy Car, Aisha provided evidence of engaging in an early form of covariational reasoning. In contrast, for the Ferris wheel, Aisha attempted to show how one might obtain one value given another value. For the Cannon Man and Toy Car, Aisha conceived of both attributes as possible to measure and capable of varying. Her graphs (Figure 1, middle, right) represented both attributes, one varying in a single direction (increasing), and another varying in different directions (increasing and decreasing).

Marton's Variation Theory

For the Cannon Man and Toy Car task sequences, we incorporated patterns of variation and invariance to provide opportunities for Aisha to discern critical aspects of covariation (intended object of learning). First, Aisha could vary each attribute individually, then both attributes together. Second, Aisha repeated the process for a new Cartesian graph with the same attributes represented on different axes. We designed each of the first two elements against a background of invariance (e.g., the Cannon Man). Next, we repeated the first two elements against a new background (e.g., the Toy Car). When creating a new background, we varied different elements while keeping other elements the same. In particular, we varied the motion of the object (e.g., the Toy Car moved in a curved line; Cannon Man moved up and down), while keeping the kinds of attributes (length measures) invariant.

When Aisha interacted with the Cannon Man task sequence, she discerned differences between the motion of the Cannon Man (up/down) and the total distance traveled (enacted object of learning). After interacting with both the Cannon Man and Toy Car task sequences, Aisha discerned that the differences in the motion of the Cannon Man and the Toy Car did not impact the total distance traveled, which still increased in both situations (lived object of learning). We argue that the opportunity to experience difference in the motion of objects for the Cannon Man and Toy Car contributed, in part, to Aisha's transfer of covariational reasoning.

Lobato's theory of actor-oriented transfer

We adapted Lobato's (2014) criteria to align with Aisha's work during the task sequence we reported. We analyzed for evidence of the following: (1) Aisha's change in reasoning from the first interview (Ferris Wheel) to the third interview (Toy Car); (2) Aisha's engagement in covariational reasoning during the second interview; (3) Influence of Aisha's reasoning during the second interview on her reasoning during the third interview; and (4) Attribution of Aisha's changed reasoning to the interview tasks, rather than to a spontaneous occurrence.

For the Ferris Wheel, Aisha provided evidence that she was doing something other than engaging in covariational reasoning (1). We interpret that Aisha was conceiving of how she might use a formula or rule to determine one amount (height), given another amount (distance). For the Cannon Man, Aisha demonstrated evidence of engaging in covariational reasoning (2). Rather than attempting to show how one might obtain one value given another value, she conceived of both attributes as possible to measure and capable of varying. Aisha's explanations provide evidence of how she construed similarities in an attribute (total distance), common to the Toy Car and the Cannon Man (3). We interpret Aisha's reasoning for the Toy Car to be influenced by her reasoning for the Cannon Man (4). First, Aisha identified a common attribute across both the Toy Car and the Cannon Man (total distance). Second, Aisha related the toy car's total distance traveled and its distance from the shrub. Third, Aisha's substantial change from the Ferris Wheel to the Toy Car strongly suggests that her reasoning for the Cannon Man influenced her reasoning for the Toy Car.

Locally integrating theories to theorize the "tension of intentions" in task design

In integrating these theories, we aim to theorize the "tension of intentions" when researching task design to account for the student perspective. We aim to position students' perspectives as being just as viable as the perspectives of designers, teachers, and researchers. Accordingly, we aim not to place one theory as hierarchically superior to another, but rather to weave together the theories into a new form. In so doing, we bring together different assumptions: Students' reasoning depends on their conceptions of attributes as being possible to measure and capable of varying (Thompson's theory of quantitative reasoning), students discern difference, rather than sameness (Marton's Variation Theory); and transfer depends on the student perspective (Lobato's theory of actor-oriented transfer).

To claim that we can integrate theories, we need to address their epistemological roots (Bikner-Ahsbahs & Prediger, 2010). Lobato's and Thompson's theories have roots in students' conceptions, which makes integrating those theories more straightforward. In contrast, Marton's theory has roots in students' experience. We argue that both students' conceptions and experiences are central to their learning opportunities. For example, in our research, students' experiences with Cartesian graphs and students' conceptions of attributes were both central to their covariational reasoning. Through centering the student perspective, we integrate these different theories.

Conclusion: Rethinking design possibilities for transfer studies

We view a reflexive relationship between theoretical perspectives and research methods. Employing alternative transfer perspectives, such as an actor-oriented transfer perspective, can result in innovations in methods to investigate transfer. A variation theory lens afforded us novel opportunities to design for and investigate students' transfer of covariational reasoning. Rather than designing a set of pre- and post-tasks, that in our view were structurally similar, we designed task sequences that incorporated difference within and across backgrounds of invariance. Analyzing across students' work on the task sequences, we gathered evidence of students' transfer of reasoning from an actor-oriented transfer perspective. Overall, we acknowledge that the perspectives of designers, teachers, and researchers can impact students' opportunities to engage in mathematical reasoning. In our theorizing, we aim to address this tension of intention, by centering the student perspective.

Acknowledgment. The development of the digital task sequences was supported by a US National Science Foundation Grant (DUE-1709903). Opinions, findings, and conclusions are the authors.

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