Leveraging difference to promote students' conceptions of graphs as representing relationships between quantities

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By creating different Cartesian graphs to represent the same relationship between quantities, students can expand their conceptions of what graphs represent. Interweaving Marton's variation theory and Thompson's theory of quantitative reasoning, we designed digital task sequences to promote students' conceptions of graphs as representing relationships between quantities. In the tasks, which link animations and dynamic graphs, students created different graphs to represent the same relationship between quantities. We report results of a qualitative study (n=13) investigating secondary students' interactions with the digital tasks in an individual interview setting. We found that the digital tasks were viable for promoting students' creation of graphs to represent relationships between quantities. Our findings have implications for task design. Namely, students should have opportunities to create different graphs to represent the same relationship between quantities.

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The term *representation* can have many meanings (e.g., Kaput, 1998). We use representation as both a noun and a verb. Employing the term representation as a noun, a student can conceive of a graph as a representation of a relationship between quantities. Employing the term representation as a verb, a student can engage in representation by creating a Cartesian graph to represent a relationship between quantities. Interweaving Thompson's theory of quantitative reasoning (Thompson, 1993; 2002) and Marton's variation theory (Kullberg, Kempe, and Marton, 2017; Marton, 2015), we designed digital task sequences linking dynamic animations and graphs to engender secondary students' creation of Cartesian graphs to represent relationships between quantities. In this study, using individual clinical interviews, we investigated what secondary students (n=13) intended to represent when interacting with digital tasks linking Cartesian graphs and dynamic animations.

Theoretical and conceptual framing

Representation and intention

We distinguish a representation from an inscription, the latter of which refers to some observable artefact (e.g., Kaput, 1998). From our perspective, an inscription can only be a representation if that inscription represents something for an individual or group. When we claim that a student is "representing," we interpret that the student did something more than just creating some observable artefact (e.g., a graph). We mean that the student intended to represent some "thing" with that graph, and that we have evidence to support our claim.

We aim to study what students *intend* to represent when working on digital tasks linking Cartesian graphs and dynamic animations. In any task setting, there are tensions between the intentions of the task designers/researchers and the intentions of the students engaging with the tasks (Johnson, Coles,

& Clarke, 2017). Theoretically, we acknowledge that we cannot know the intentions of others. We can only infer those intentions based on observable evidence. Furthermore, we do not assume that students will share a stated task aim with that of the designer/researcher. For example, we intended to provide students multiple opportunities to create different Cartesian graphs to represent the same relationship between quantities, and we did not assume that students interacting with the tasks would share our intentions.

Interweaving Marton's variation theory and Thompson's theory of quantitative reasoning

To frame our study, we interweave two theories: Thompson's theory of quantitative reasoning (Thompson, 1993; 2002) and Marton's variation theory (Kullberg et al., 2017; Marton, 2015). In his theory of quantitative reasoning, Thompson explains students' mathematical thinking in terms of students' conceptions of attributes. Thompson (1993) posited that quantities were something different from units of measure. Rather, quantities depend on students' conceptions. If a student can conceive of the possibility of measuring some attribute, then that attribute is a quantity for the student. For example, a student may view an animation of a "Cannon Man," who is shot vertically into the air, then comes back down to the ground. In this situation, there are many different attributes. Cannon Man's total distance traveled—both up and down—would be a quantity for that student only if she can conceive of the possibility of measuring Cannon Man's total distance.

Difference, rather than sameness, forms the essence of variation theory (e.g., Kullberg et al., 2017; Marton, 2015). With variation theory, Marton and colleagues explain how designers can develop instructional sequences to promote students' discernment of critical aspects of objects of learning. Kullberg et al. (2017, p. 560) link discernment and variation, core components of variation theory, positing "Discernment cannot happen without the learner having experienced variation." Furthermore, the type of variation matters. If a designer intends for students to discern critical aspects of an object of learning, students should have opportunities to experience variation (difference) in those critical aspects (Marton, 2015).

In designing our study, we intended to provide opportunities for students to conceive of a graph as representing a relationship between quantities. Drawing on Marton's variation theory, we argue that students' conceptions of what is possible for Cartesian graphs to represent is inseparable from their experiences with different Cartesian graphs. Drawing on Thompson's theory of quantitative reasoning, we argue that designers' choices of attributes can impact students' opportunities to conceive of Cartesian graphs as representing relationships between quantities.

Students' conceptions of what Cartesian graphs can represent

It is useful for students to have opportunities to conceive of graphs as representing relationships between quantities (e.g., Bell & Janvier, 1981; Johnson & McClintock, 2018; Kerslake, 1977; Leinhardt, Zaslavsky, & Stein, 1990; Moore, Silverman, Paoletti, & LaForest, 2014; Thompson, 2002). Yet, researchers have documented secondary students' challenges with creating and interpreting Cartesian graphs (Bell & Janvier, 1981; Johnson & McClintock, 2018; Kerslake, 1977; Leinhardt, Zaslavsky, & Stein, 1990). We identify three key challenges. First, students may interpret graphs as needing to share characteristics with a physical object, such as a hill (Bell & Janvier, 1981; Leinhardt et al., 1990). Second, students may interpret graphs as sharing physical characteristics with the physical path of an object, such as a person's walk from one location to another (Bell & Janvier, 1981; Kerslake, 1977). Third, students may interpret graphs as representing a single varying quantity, rather than as a relationship between quantities (Johnson & McClintock, 2018).

To engender opportunities for students to interpret graphs as relationships between quantities, designers can incorporate variation in individual attributes within a single graph and variation across different graphs incorporating the same attributes. Thompson (2002) argued that students could use their fingers as tools, sliding them along each axis of a graph, to represent change in individual attributes. In a study investigating prospective secondary teachers' reasoning, Moore et al. (2014) incorporated different graphs representing the same attributes on different axes. When making choices about the kinds of variation to incorporate within and across our digital task sequences, we drew inspiration from the work of Thompson and Moore.

The digital task sequences

We report on two digital task sequences: The Cannon Man and the Toy Car. In each digital task sequence, students engaged in five main tasks, shown in Table 1. We drew on Thompson's theory of quantitative reasoning and Marton's variation theory to design and sequence the tasks in each situation. First, students had opportunities to discern and conceive of measuring attributes (Tasks V, A1). Second, students had opportunities to represent change in individual attributes (Task A2). Third, students had opportunities to represent attributes changing together (Task G1). In Task G2, we introduced a difference, against a background of invariance—a different Cartesian graph, with the same attributes represented on different axes. Across the digital task sequences, the Cannon Man and Toy Car situations served as different backgrounds.

Task	Description
V	**View video animation.** Students viewed a video animation of a situation depicting an object in motion, identified attributes in the situation, and discussed how they might measure those attributes.
A1	**Identify task attributes.** Johnson stated attributes on which that task would focus. If students had not already identified those attributes, they then discussed how they might measure the task attributes.
A2	**Represent individual attributes.** Students dragged dynamic segments along the axes of a Cartesian plane to represent change in individual attributes. Then, students viewed a computer-generated video of the dynamic segments changing together.
G1	**Represent attributes changing together.** Students sketched a single Cartesian graph relating both attributes. Students discussed how their graphs showed (or did not show) both attributes at the same time. Then, students viewed a computer-generated graph.
G2	**Re-represent attributes in a new Cartesian plane.** In a new Cartesian plane, with the same attributes represented on different axes, students re-represented individual attributes, then attributes changing together (Repeat tasks A2, G1).

Table 1: Descriptions of tasks in the Cannon Man and Toy Car digital task sequences

Methods

Setting/participants

We implemented the digital task sequences with 13 high school students in a high performing suburban high school in the metropolitan area of a large US city. Five students were in ninth grade (~15 years), and currently enrolled in an Algebra I course. Eight students were in eleventh grade (~17 years), and currently enrolled in an Algebra II course. At the school, 52% of students identified as students of color, and 36% of students qualified for free or reduced lunch (an indicator of low socioeconomic status).

We conducted the study over a 4 week time period near the end of the school year. Students volunteered to participate in the study. Johnson conducted a series of three clinical interviews with individual students (39 interviews). Interviews occurred once or twice per week, with at least one day between interviews. Students who participated in all three interviews received a graphing calculator, which they could use for exams and classwork at their school. We conjecture that students who participated in the study were motivated, in part, by the opportunity to receive a graphing calculator.

Research methods: Clinical interviews, exploratory teaching

During the clinical interviews, Johnson engaged in exploratory teaching (Steffe & Thompson, 2000) to investigate the viability of the digital tasks for promoting students' conceptions of graphs as representing relationships between quantities. In the interview design, Johnson included questions to gather evidence of what students were intending to represent when sketching a graph. These questions included: "What you are trying to graph?"; "Can (How does) your graph show both __ and __ (the task attributes)?"; "Look at a point on your graph, can (how does) this point give you information about __ and __ (the task attributes)?"

The first interview, which served as a preassessment, involved a Ferris wheel situation. In the Ferris wheel situation, students engaged in tasks V, A1, and G1 in the digital task sequence (See Table 1). The second and third interviews, which incorporated the digital task sequences described in this paper, involved a Cannon Man and a Toy Car situation, respectively. In all interviews, students worked on a digital tablet (an iPad).

Ongoing and retrospective data analysis

All interviews were video recorded. During each interview, either McClintock or Gardner wrote field notes. To promote consistency, we used a field note template, which broke down each interview into sub tasks, to write field notes for each student in each interview. Field notes included evidence of students' conceptions of task attributes as possible to measure and capable of varying, as well as evidence of students' intentions to use a graph to represent a relationship between quantities.

We focused retrospective analysis on students' responses to the Cartesian graphing tasks (Tasks G1 and G2, see Table 1). Across the set of three interviews, students had opportunities to sketch five Cartesian graphs. Johnson and McClintock viewed video of each student's work on each of the Cartesian graphing tasks. In the first pass, we described three aspects of students' work: Sketches of (or attempts to sketch) a viable graph; Explanations of their graph (or attempt) in terms of both attributes; Gestures related to their graphs. In the second pass, we made inferences about students' representing. We used different codes to characterize students' representing. Table 2 provides descriptions of each code. In the third pass, we examined students' shifts in representing across the Cartesian graphing tasks in each interview, building from observable evidence to develop explanations to account for students' shifts.

Code	Description of code
Relationships between quantities	Students sketched (or attempted to sketch) a single graph. Students described or showed how a single graph could represent a relationship between two quantities.
Individual quantities	Students sketched two separate graphs. Students described each graph in terms of only one quantity (e.g., the "height" graph).
Motion of objects	Students sketched a graph that showed the motion of an object in the situation (e.g., a toy car moving along a path). Students described the graph in terms of motion.
Iconic/Familiar objects	Student sketched a graph that looked like an iconic object or familiar graph. Students described the graph in terms of its physical characteristics.

Table 2: Descriptions of codes characterizing students' representing

Viewing video alphabetically by students' pseudonyms, we completed the first and second passes for each Cartesian graph task in the Ferris wheel and Cannon Man situations. Next, we analyzed the Toy Car situation, which resulted in refinements to our codes. In particular, we expanded the code for relationships between quantities to include attempts to represent quantities not explicitly represented (e.g., time). In each pass, we first coded individually, then vetted codes as a team.

Although some students demonstrated evidence of more than one form of representing within a task, we elected to use a single code for students' representing within that task. When we coded a form of representing for a task, a student may have shifted to engaging in that form of representing after an "aha" moment or engaged in that form of representing throughout their work on the task. If a student demonstrated partial evidence of one form of representing, but engaged more consistently in another form of representing, we weighed the evidence, then coded the form of representing that we interpreted to best characterize the students' reasoning in that task.

Results

We found that the digital task sequences were viable for promoting students' creation of Cartesian graphs to represent relationships between quantities. Table 3 shows the numbers of students engaging in each type of representing across the Ferris Wheel, Cannon Man, and Toy Car graphing tasks. In the Ferris Wheel preassessment graphing task (Ferris Wheel G1, see Table 3), four students (two ninth grade; two eleventh grade) created graphs to represent relationships between quantities. All four of these students continued to represent relationships between quantities in the Cannon Man and Toy Car graphing tasks (Cannon Man G1 and G2, Toy Car G1 and G2, see Table 3). We share results from the nine students who, in the Ferris wheel preassessment graphing task (Ferris Wheel G1), demonstrated forms of representing other than relationships between quantities. By the end of the Cannon Man and Toy Car task sequences, four of these nine students shifted to representing relationships between quantities (Harun, Aisha, David, and Amanda). Five of these nine students

(Kara, Gemma, Carmen, Keshia, and Eliza) continued to represent (or shifted to representing) the motion of objects.

Code	Ferris Wheel G1	Cannon Man G1	Cannon Man G2	Toy Car G1	Toy Car G2
Relationships between quantities	4	6	9	7	8
Individual quantities	2	2	2	1	0
Motion of Objects	4	5	2	5	5
Iconic objects /familiar graphs	3	0	0	0	0

Table 3: Students engaging in each type of representing across tasks

Shifts to representing relationships between quantities

The two students (Harun, eleventh grade; Aisha, ninth grade) who represented individual quantities on the Ferris Wheel preassessment graphing task (Ferris Wheel G1) represented relationships between quantities by the end of the interview sequence. Harun shifted to representing relationships between quantities on the first Cannon Man graphing task (Cannon Man G1). Across the Cannon Man and Toy Car task sequences, Aisha continued to represent individual quantities for the first graphing tasks (Cannon Man G1, Toy Car G1). It was only on the second graphing task in each sequence (Cannon Man G2, Toy Car G2), that Aisha shifted to representing relationships between quantities.

The three students who represented iconic objects/ familiar graphs on the Ferris Wheel preassessment graphing task (Ferris Wheel G1) were in eleventh grade. Two of those students, David and Amanda, shifted to representing relationships between quantities in the Cannon Man task sequence. Amanda shifted in the first graphing task (Cannon Man G1), relating directions of change in Cannon Man's height and distance. In contrast, in the first graphing task (Cannon Man G1), David represented the path of Cannon Man. It was not until the second graphing task (Cannon Man G2) that David worked to represent relationships between Cannon Man's height and distance. Notably, David was the only student who shifted from representing motion of objects to representing relationships between quantities.

Two of the four students (Kara, eleventh grade; Gemma, ninth grade) who represented motion of objects on the Ferris Wheel preassessment graphing task (Ferris Wheel G1) demonstrated partial evidence of representing relationships between quantities on the Cannon Man graphing tasks (Cannon Man G1, G2). In the second Cannon Man graphing task (Cannon Man G2), both Kara and Gemma explained how their graphs represented Cannon Man's height and distance. Yet, in both Toy Car graphing tasks (Toy Car G1, G2), they attempted to represent the physical path of the toy car.

Shifts to other forms of representing

Two of the four students (Carmen, eleventh grade; Keshia, ninth grade) who represented motion of objects on the Ferris Wheel preassessment graphing task (Ferris Wheel G1) shifted to representing change in individual attributes on at least one of the Cannon Man graphing tasks. In the second

Cannon Man graphing task (Cannon Man G2), both Carmen and Keshia attempted to represent change in Cannon Man's height and distance. Yet, in both Toy Car graphing tasks (Toy Car G1, G2), they attempted to represent the physical path of the toy car. Furthermore, one of the students who represented iconic objects/ familiar graphs (Eliza, eleventh grade) shifted to representing the motion of objects in the first Cannon Man graphing task (Cannon Man G1). Once shifting to representing the physical path of Cannon Man, Eliza continued to represent motion of objects in the rest of the tasks.

Discussion

We aimed to leverage difference to promote students' conceptions of graphs as representing relationships between quantities. By interweaving Marton's variation theory and Thompson's theory of quantitative reasoning, we worked to achieve our aim. Drawing on Marton's variation theory, we incorporated difference within and across task sequences. Within each task sequence, students had opportunities to create different graphs to represent the same relationship between quantities. Across task sequences, we incorporated different backgrounds (Cannon Man/Toy Car), as well as different kinds of graphs (linear/nonlinear). Thompson's theory of quantitative reasoning informed our choices about the kinds of differences that might engender students' quantitative reasoning. Because Cartesian graphs represent relationships between quantities, students had opportunities to vary each quantity represented in a graph. Furthermore, we argue that variation within a graph is insufficient to foster students' conceptions of a graph as some "thing" capable of representing a relationship between quantities. Drawing inspiration in part from the tasks reported in Moore et al. (2014), we incorporated different graphs representing the same relationship between quantities.

We posit that our descriptions of codes indicate increasing levels of sophistication in students' representing with Cartesian graphs; however, we do not claim that each of these levels is a necessary step in a progression of reasoning. When representing iconic objects/familiar graphs or the literal path of an object in motion, students are representing physical phenomena in a situation. When representing quantities, or relationships between quantities, students are mathematizing a situation in terms of attributes they can conceive of as possible to measure. We argue that engendering students' shifts from representing iconic objects/familiar graphs or motion of objects to representing individual quantities and relationships between quantities is important for expanding students' conceptions of what graphs can represent.

Students' conceptions of the direction and nature of the motion of objects in a situation can impact their opportunities for representing relationships between quantities. The Toy Car situation proved useful for investigating the stability of students' representing relationships between quantities, in part because the graphs did not share physical characteristics with the literal path of the toy car (e.g., a graph moving up and down as did the Cannon Man). To promote students' creation of Cartesian graphs to represent relationships between quantities, task designers should incorporate graphs that do not share physical characteristics with motion of the objects in task situations.

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