A study of chaos and its control in a harvested tri-trophic food chain model with alternative food source and diffusion effect

Binayak Nath [1], Jingjing Lyu[2], Rana D. Parshad[3], Krishna Pada Das [4*], Anuraj Singh [5]

[1]Department of Physics, Mahadevananda Mahavidyalaya
Monirampore, P.O.-Barrackpore,Kolkata 700120, India

[2] Clarkson University,Department of Mathematics
Potsdam, NY, 13699, USA

[3] Iowa State University,Department of Mathematics
Ames, IA, 50011, USA

[4] Department of Mathematics, Mahadevananda Mahavidyalaya
Monirampore, P.O.-Barrackpore, Kolkata 700120, India

[5] ABV-Indian Institute of Information Technology & Management,
Gwalior 474015, India

* *Corresponding Author*. E-mail: krishnaisi@yahoo.co.in

Abstract. The present paper explores a tri-trophic food chain model with alternative food source for prey species and harvesting in prey and intermediate predator. We perform the local dynamical analysis of the model system. We also derive conditions for Turing instability in the diffusive model. To investigate the global dynamics we have performed extensive numerical experiments in the both non-diffusive and diffusive system. We find that harvesting has an important role in preventing chaotic dynamics and sustaining the ecosystem. It is noticed that alternative food also has a stabilizing effect on chaotic dynamics and leads a stable dynamics through chaos, period-doubling and limit cycles. Thus both the harvesting parameter and alternative food may be considered as key biological parameters for controlling chaotic dynamics in three species food chain models. In the diffusive system we observe that the harvesting parameter and alternative food may cause Turing instability.

1 Introduction

The analysis of food chains has been a major area of research in ecology and applied mathematics in recent years. A food chain represents the predator-prey relationships in an ecosystem, between species linked by trophic interaction. The pioneering work on chaos in three species food chains by Hastings and Powell [22] shows chaotic dynamics could exist in natural food webs. Subsequently, a variety of models incorporating different kinds of ecological factors are proposed and analyzed to suppress chaos.

Following the work of Hastings and Powell [22], Ruxton [39] studied the probability of extinction for a system of linked populations and showed that it decreases for weakly linked systems. MaCann and Hastings [27] analyzed the effect of the addition of omnivory to a food chain model and found that it has the ability of suppressing chaos. Xu and Li [47] used intraspecific density dependence (IDD), an important ecological factor, in food chain models to control chaos. Maionchi et al. [28] showed significant changes in the dynamics of the system by introducing local predator-prey interactions. Younghae Do et al. [11] analyzed a three-species food chain system with hybrid type of functional responses and concluded that it can exhibit chaotic dynamics. Gakkhar and Singh [20] showed that four-dimensional models may show stable dynamics in contrast to chaotic dynamics that occurred in three species food chain. Thus the intermediate predator plays an important role in reducing chaos. Zhang et al. [49] considered an experimental marine food chain and showed that feeding selectivity plays an important role in the transfer of energy along marine food chains. Greenhalgh et al.[19] investigated the features of a predator-prey model with disease amongst the prey and ratio-dependent functional response for both infected and susceptible prey.

Now we briefly discuss about the importance of harvested food chain in ecological systems. Harvesting represents population reduction due to hunting or capturing individuals or removal of crops from the fields. It may be considered as a stabilizing factor when there are chaotic population dynamics. Chaudhuri [7, 8] analyzed combined harvesting of two competing fish species and considered the perspective of dynamic optimization of a two competing fish species. Dai and Tang [10] considered a predator-prey model with constant rate of harvesting in two interacting species independently. Bairagi et al.[3] showed harvesting practices may play an important role in a host-parasite systems and it is possible to control the cyclic behavior of the system using impulsive harvesting effort. Madhusudanan et al.[29] studied the effect of harvesting on prey species and showed that it has a strong impact on the population dynamics. Ali khan et al.[1] used threshold harvesting strategy to obtain steady behaviour from a chaotic fish population. Liu and Bai[26] established sufficient and necessary criteria for the existence of optimal harvesting policy.

Alternative food source for predators is also a very important factor that can influence the dynamical behavior of a food chain system significantly. Generally predators do not depend on a single prey species rather they will switch to alternative food when the preferred prey density is low. It is well-known that when prey density drops below a threshold value, optimally foraging predators will switch to alternative food. Werner and Hall [45] considered a model demonstrating optimal prey choice by the bluegill sunfish. Joern[4] analyzed switching among four alternative grasshopper species by the grasshopper sparrow (*Ammodramus savannarun*)through frequency-dependent predation. Colombo and Krivan[5] discussed strategies describing selective feeding and switching for a one-predator two-prey system. Fryxell and Lundberg[13] showed that patterns of partial prey preference are more stabilizing than perfect optimal diet selection. Minus van Baalen et al.[2] showed that if switching to alternative food is unable to stabilize the equilibrium, it may prevent unbounded

oscillations in the system. Sahoo and Poria[42] showed that chaos in a predator-prey system can be controlled by increasing quantity of additional food to top predator.

The effects of diffusion in food chain models also plays a crucial role in the dynamics of the system. Individual species migrate where their population density is perhaps less, in order to get more food. Diffusion also includes ecological interaction such as searching for shelter, mates and absconding from predators, infection risks etc. Su et al.[44] proposed a reaction-diffusion population model with time delay effect and studied the long time dynamical behavior of the system whereas Zhao and Wei [48] considered a reaction-diffusion plankton system with delay and toxic substances effect. Ndam et al.[34] considered a model in an aquatic habitat and studied the effects of cross-diffusion of the intermediate and the top predator. Zuo and Wei [50] studied the stability in a delayed predator-prey model with diffusion effect. Dhar and Jatav [9] considered a delayed stage-structured predator-prey model with impulsive diffusion between the territories of the two predators. Guin and other authors [40, 14, 15, 16, 17] studied the predator-prey system with cross-diffusion effect and stationary pattern formation and discussed the control of pattern formation of the populations and its biological relevance. Rao et al.[38] studied the dynamics of the predator-prey model with constant time delay and diffusion in both prey and predator. They showed that diffusion can either stabilize or destabilize the system, large delay can destabilize the system and the combination of diffusion and delay can intensify the instability of the system.

2 Model formulation

The HP model [22] with pairwise interactions between three species, namely, X, Y, Z, which incorporates a Holling type II functional response in both consumer species, namely Y and Z is as follows:

$$\frac{dX}{dT} = R_0 X \left(1 - \frac{X}{K_0}\right) - \frac{C_1 A_1 Y X}{B_1 + X}$$
$$\frac{dY}{dT} = \frac{A_1 Y X}{B_1 + X} - \frac{A_2 Y Z}{B_2 + Y} - D_1 Y \qquad (2.1)$$
$$\frac{dZ}{dT} = \frac{C_2 A_2 Y Z}{B_2 + Y} - D_2 Z.$$

Here X is the numbers of the species at the lower level of the food chain, Y the numbers of the species that preys upon X, and Z the numbers of the species that preys upon Y. Here T is time. The constant R_0 is the intrinsic growth rate and the constant K_0 is the carrying capacity of the species X. The constants C_1 and C_2 are conversion rates of prey to predator for the species Y and Z respectively; D_1 and D_2 are constant death rates for species Y and Z respectively. The constants A_i and B_i for i=1,2 are maximal predation rate and half saturation constants for Y and Z respectively. Hastings and Powell [22] demonstrated that the dynamic interaction between prey and predators in simple three-species food chain is chaotic in a certain region of parametric space.

Assume that the predator is a generalist one and may switch to another prey when the preferred prey is low in numbers. To keep the model simple and analytically tractable, it is also assumed that the additional food is not dynamic in nature and available at a specific constant level either by nature or by an external agency. This simplification is justified for many arthropod predators[2]. By above assumptions the system(2.1) can be written as the following set of nonlinear ordinary differential equations:

$$\frac{dX}{dT} = R_0 X \left(1 - \frac{X}{K_0}\right) - \frac{C_1 A_1 \Pi Y X}{B_1 + X}$$

$$\frac{dY}{dT} = \frac{A_1 \Pi Y X}{B_1 + X} + (1 - \Pi)Y - \frac{A_2 Y Z}{B_2 + Y} - D_1 Y \tag{2.2}$$

$$\frac{dZ_1}{dT} = \frac{C_2 A_2 Y Z}{B_2 + Y} - D_2 Z$$

Here Π $(0 \leq \Pi \leq 1)$ is a alternative food resource parameter that measures the coupling strength of intermediate predator with its focal prey and alternative food. $\Pi = 1$ implies that intermediate predators feed only on focal prey and the systems (2.1) and (2.2), in this case, are identical. The other extreme case arises when $\Pi = 0$ In this latter case, intermediate predators do not consume its focal prey and their growth solely depends on the alternative prey. We are interested about the situation where $0 < \Pi < 1$. This is the case where intermediate predator consume both its focal prey and alternative prey.

Here we have considered harvesting in prey and intermediate predator to obtain rich dynamics. We use the term harvesting as a removal of species due to some need, perhaps policy driven. Formulating reasonable harvesting policies is indisputably one of the major and interesting problems in ecology and economics and has been well studied. The exploitation of biological resources and harvest of population species are commonly practiced in fishery, forestry, agriculture and wildlife management. Harvesting has sometimes been considered as a stabilizing factor, a destabilizing factor or even oscillation-inducing factor. The problem of predator-prey interactions under constant rate of harvesting or constant quota of harvesting has been studied by many authors.

By above assumptions the system(2.2) can be written as the following set of nonlinear ordinary differential equations:

$$\frac{dX}{dT} = R_0 X \left(1 - \frac{X}{K_0}\right) - \frac{C_1 A_1 \Pi Y X}{B_1 + X} - E_1 X$$

$$\frac{dY}{dT} = \frac{A_1 \Pi Y X}{B_1 + X} + (1 - \Pi)Y - \frac{A_2 Y Z}{B_2 + Y} - D_1 Y - E_2 Y \tag{2.3}$$

$$\frac{dZ}{dT} = \frac{C_2 A_2 Y Z}{B_2 + Y} - D_2 Z$$

To reduce the number of parameters and to determine which combinations of parameters control the behavior of the system, we dimensionalise the system with the following scaling

$$x = \frac{X}{K_0}, \quad y = \frac{C_1 Y}{K_0}, \quad z = \frac{C_1 Z}{C_2 K_0} \text{ and } t = R_0 T$$

Then the system (2.3) takes the form (after some simplification)

$$\frac{dx}{dt} = x(1 - x) - \frac{a_1 \pi x y}{1 + b_1 x} - e_1 x$$

$$\frac{dy}{dt} = \frac{a_1 \pi x y}{1 + b_1 x} + (1 - \pi)y - \frac{a_2 y z}{1 + b_2 y} - d_1 y - e_2 y \tag{2.4}$$

$$\frac{dz}{dt} = \frac{a_2 y z}{1 + b_2 y} - d_2 z$$

where

$$a_1 = \frac{A_1 K_0}{R_0 B_1}, \quad b_1 = \frac{K_0}{B_1}, a_2 = \frac{C_2 A_2 K_0}{C_1 R_0 B_2}, b_2 = \frac{K_0}{C_1 B_2},$$

$$d_1 = \frac{D_1}{R_0}, \quad d_2 = \frac{D_2}{R_0}, \quad e_1 = \frac{E_1}{R_0}, \quad e_2 = \frac{E_2}{R_0}, \quad \pi = \Pi, 1 - \pi = \frac{1 - \Pi}{R_0}, \quad (0 \leq \pi \leq 1)$$

System (2.4) has to be analyzed with the following initial conditions:

$$x(0) > 0, \ y(0) > 0, \ z(0) > 0.$$

This model gives a system of three ordinary differential equations for the three state variables: lower level prey X, consumer species Y that preys on X and consumer species Z that preys on Y. The nondimensional model is described as follows:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - \frac{a_1 \pi xy}{1 + b_1 x} - e_1 x \\
\frac{dy}{dt} &= \frac{a_1 \pi xy}{1 + b_1 x} + (1 - \pi)y - \frac{a_2 yz}{1 + b_2 y} - d_1 y - e_2 y \\
\frac{dz}{dt} &= \frac{a_2 yz}{1 + b_2 y} - d_2 z
\end{aligned} \tag{2.5}$$

Table 1 Table summarizing the symbol of parameters used in system(3).

Dimentionalised parameters	Biological significance of parameters
$a_1 = \frac{A_1 K_0}{R_0 B_1}$	Maximal predation rate of intermediate predator
$a_2 = \frac{C_2 A_2 K_0}{C_1 R_0 B_2}$	Maximal predation rate of top predator
$b_1 = \frac{K_0}{B_1}$	Half saturation constant in the functional response for intermediate predator
$b_2 = \frac{K_0}{C_1 B_2}$	Half saturation constant in the functional response for top predator
$d_1 = \frac{D_1}{R_0}$	Death rate of intermediate predator
$d_2 = \frac{D_2}{R_0}$	Death rate of top predator
$e_1 = \frac{E_1}{R_0}$	Harvesting rate of the prey population
$e_2 = \frac{E_2}{R_0}$	Harvesting rate of intermediate predator
$\pi = \Pi$	Alternative food for intermediate predator

3 Equilibria and their existence and local and global stability of equilibrium points

3.1 Equilibria

To get the equibria points ,we need to solve the following equations:

$$x(1-x) - \frac{a_1 \pi xy}{1 + b_1 x} - e_1 x = 0 \tag{3.1}$$

$$\frac{a_1\pi xy}{1+b_1x} + (1-\pi)y - \frac{a_2yz}{1+b_2y} - d_1y - e_2y = 0 \tag{3.2}$$

$$\frac{a_2yz}{1+b_2y} - d_2z = 0 \tag{3.3}$$

There are 4 equilibria, Q_1, Q_2, Q_3, Q_4, on the boundary and 1 interior equilibrium, Q.

3.1.1 Boundary equilibria

The equilibria on the boundary are given by

$$
\begin{aligned}
Q_1 &= (0,0,0) \\
Q_2 &= (0, \frac{d_2}{a_2-b_2d_2}, -\frac{\pi+d_1+e_2-1}{a_2-b_2d_2}) \\
Q_3 &= (1-e_1,0,0) \\
Q_4 &= (\frac{\pi+d_1+e_2-1}{\pi a_1-\pi b_1-b_1d_1-b_1e_2+b_1}, \\
&\quad -\frac{\pi(e_1-1)(a_1-b_1)-b_1(d_1+e_2-1)(1-e_1)+\pi+d_1+e_2-1}{(\pi a_1-\pi b_1-b_1d_1-b_1e_2+b_1)^2},0)
\end{aligned}
\tag{3.4}
$$

Q_1 is stable if $1-e_1 < 0$ and $1-\pi-d_1-e_2 < 0$ and unstable if $1-e_1 \geq 0$ or $1-\pi-d_1-e_2 \geq 0$.
Q_2 is unfeasible. Q_3 is unstable.
Q_4 is stable if $\frac{B^2d_2-Cb_2d_2+Ca_2}{B^2-Cb_2} > 0$ and unstable if $\frac{B^2d_2-Cb_2d_2+Ca_2}{B^2-Cb_2} \geq 0$ where

$$
\begin{aligned}
A &= \pi+d_1+e_2-1 \\
B &= \pi a_1 - b_1A \\
C &= \pi a_1(e_1-1) - b_1e_1A + b_1A + A
\end{aligned}
\tag{3.5}
$$

3.1.2 Interior equilibrium

To find the interior equilibrium, $Q = (x^*, y^*, z^*)$, we solve $y^* = \frac{d_2}{a_2-b_2d_2}$ from equation (3.3). Then equations (3.1)-(3.3) can be simplified as solving the following two equations

$$b_1x^2 + (b_1e_1 - b_1 + 1)x + \frac{\pi a_1 d_2}{a_2-b_2d_2} + e_1 - 1 = 0 \tag{3.6}$$

$$(b_2d_2 - a_2)(1+b_1x)z + (1-\pi-d_1e_2)(1+b_1x) + \pi a_1x = 0 \tag{3.7}$$

The solution of (3.6) is given by

$$
\begin{aligned}
x_1 &= \frac{-C_1 + \sqrt{C_1^2 - 4C_0C_2}}{2C_2} \\
x_2 &= \frac{-C_1 - \sqrt{C_1^2 - 4C_0C_2}}{2C_2}
\end{aligned}
\tag{3.8}
$$

where

$$C_2 = b_1$$
$$C_1 = (b_1 e_1 - b_1 + 1)$$
$$C_0 = \frac{\pi a_1 d_2}{a_2 - b_2 d_2} + e_1 - 1 \tag{3.9}$$

The existence of (3.8) implies $C_1^2 - 4C_0 C_2 \geq 0$, to show

$$C_1^2 - 4C_0 C_2 = \frac{(1-e_1)^2 (a_2 - b_2 d_2) b_1^2}{a_2 - b_2 d_2} - \frac{(4\pi a_1 d_2 - 2(1-e_1)(a_2 - b_2 d_2)) b_1}{a_2 - b_2 d_2} + 1 \tag{3.10}$$
$$\geq 0$$

It is enough to show

$$4\pi a_1 d_2 - 2(1-e_1)(a_2 - b_2 d_2) < 0 \tag{3.11}$$

Then we have

$$a_2 - b_2 d_2 > \frac{2\pi a_1 d_2}{1 - e_1} \tag{3.12}$$

To simplify the calculation, we will always assume $0 < e_1 < 1$. Then we will only need to consider $x^* = x_1$ since $x_2 < 0$ due to $C_2 > 0$ and $C_0 < 0$. The condition (3.12) also implies $x_1 > 0$. Define $A = \pi + d_1 + e_2 - 1$. Then z^* is given by

$$z^* = \frac{(\pi a_1 - b_1 A) x^* - A}{(1 + b_1 x^*)(a_2 - b_2 d_2)} \tag{3.13}$$

And $z^* > 0$ if

$$a_1 > \frac{(2 + b_1(1 - e_1))A}{\pi(1 - e_1)} \tag{3.14}$$

Therefore, the interior equilibrium is given by

$$Q = \left(\frac{-C_1 + \sqrt{C_1^2 - 4C_0 C_2}}{2C_2}, \frac{d_2}{a_2 - b_2 d_2}, \frac{(\pi a_1 - b_1 A) x^* - A}{(1 + b_1 x^*)(a_2 - b_2 d_2)} \right) \tag{3.15}$$

And the equilibrium Q is feasible if

$$a_2 - b_2 d_2 > \frac{2\pi a_1 d_2}{1 - e_1}, \quad -b_1 e_1 + b_1 - 1 > \frac{2b_1 A}{\pi a_1 - b_1 A} \tag{3.16}$$

3.2 Stability analysis of interior equilibrium points

The Jocabian matrix of this system about $Q = (x^*, y^*, z^*)$ is

$$J = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & J_{32} & 0 \end{bmatrix}$$

where

$$J_{11} = 1 - 2x^* - \frac{a_1 \pi y^*}{(1 + b_1 x^*)^2} - e_1 \tag{3.17}$$

$$J_{12} = -\frac{a_1 \pi x^*}{1 + b_1 x^*} \tag{3.18}$$

$$J_{21} = \frac{a_1 \pi y^*}{(1 + b_1 x^*)^2} \tag{3.19}$$

$$J_{22} = \frac{b_2 d_2 (a_2 - b_2 d_2) z^*}{a_2} \tag{3.20}$$

$$J_{23} = -\frac{a_2 y^*}{1 + b_2 y^*} \tag{3.21}$$

$$J_{32} = \frac{a_2 z^*}{(1 + b_2 y^*)^2} \tag{3.22}$$

Under the feasibility conditions (3.16), note that $J_{12} < 0, J_{21} > 0, J_{22} > 0, J_{23} < 0$ and $J_{32} > 0$. The sign of J_{11} is undertermined.

The characteristic equation is given by

$$\lambda^3 + R_2 \lambda^2 + R_1 \lambda + R_0 = 0 \tag{3.23}$$

where

$$R_2 = -(J_{11} + J_{22}) \tag{3.24}$$

$$R_1 = J_{11} J_{22} - J_{12} J_{21} - J_{23} J_{32} \tag{3.25}$$

$$R_0 = J_{11} J_{23} J_{32} \tag{3.26}$$

By Routh Hurwitz stability criteria, all eigenvalues of (3.23) have negative real part if and only if

$$R_0 > 0, R_2 > 0, R_1 R_2 - R_0 > 0 \tag{3.27}$$

Theorem 3.1. The interior equilibrium Q is unstable if $J_{11} > 0$.

If $J_{11} < 0$, then we can state the following theorem

Theorem 3.2. The interior equilibrium Q is locally stable if $J_{11} J_{22} - J_{12} J_{21} > \frac{J_{22} J_{23} J_{32}}{J_{11} + J_{22}} > 0$ and $J_{11} < -J_{22}$.

Proof. By Routh Hurwitz criteria, $R_0 > 0$ implies $J_{11} < 0$ and $R_2 > 0$ gives $J_{11} < -J_{22}$.

$$\begin{aligned}
R_1 R_2 - R_0 &= -J_{11}^2 J_{22} + J_{11} J_{12} J_{21} - J_{11} J_{22}^2 + J_{12} J_{21} J_{22} + J_{22} J_{23} J_{32} \\
&= -J_{11} J_{22} (J_{11} + J_{22}) + J_{12} J_{21} (J_{11} + J_{22}) + J_{22} J_{23} J_{32} \\
&= -(J_{11} + J_{22})(J_{11} J_{22} - J_{12} J_{21}) + J_{22} J_{23} J_{32}
\end{aligned} \tag{3.28}$$

Theorem 3.3. The interior equilibrium Q is unstable if $J_{11} J_{22} - J_{12} J_{21} < 0$.

4 Hopf bifurcation

Now we will investigate the Hopf bifurcation for the system in terms of parameter e_1. In this paper, we will follow the method developed by Liu [41]. The hopf bifurcation at $e_1 = e_1^*$ can occur iff $R_1(e_1^*), R_0(e_1^*)$, and $\phi(e_1^*) = R_2(e_1^*)R_1(e_1^*) - R_0(e_1^*)$ are smooth functions of e_1 in an open interval of $e_1^* \in \mathbf{R}$ such that:

1. $R_1(e_1^*) > 0, R_0(e_1^*) > 0$, and $\phi(e_1^*) = R_2(e_1^*)R_1(e_1^*) - R_0(e_1^*) = 0$.
2. $\frac{d\phi(e_1)}{de_1}|_{e_1=e_1^*} \neq 0$.

And we will numerically check e_1 exists since it is very hard to solve mathematically.

5 Spatially Explicit Model

Since species disperse in space in search of food, shelter, mates and to avoid predators, spatially dispersing populations are often modeled via partial differential equations (PDE)/spatially explicit models of interacting species, as well as various others, [30, 35, 37, 12, 23]. In this section we shall investigate the effects of diffusion in our earlier described models. We state the form of the spatially explicit model

$$\frac{\partial x}{\partial t} = D_1 \Delta x + x(1-x) - \frac{a_1 \pi xy}{1+b_1 x} - e_1 x \qquad (5.1)$$

$$\frac{\partial y}{\partial t} = D_2 \Delta y + \frac{a_1 \pi xy}{1+b_1 x} + (1-\pi)y - \frac{a_2 yz}{1+b_2 y} - d_1 y - e_2 y$$

$$\frac{\partial z}{\partial t} = D_3 \Delta z + \frac{a_2 yz}{1+b_2 y} - d_2 z$$

We prescribe Neumann boundary conditions and suitable positive initial conditions. Here $x(x^{'},t), y(x^{'},t), z(x^{'},t)$ are the concentrations/population densities of the prey, the juvenile predator and the adult predator, at any given time t respectively.

5.1 Functional preliminaries and local solutions

We now present various function space notations and definitions that will be used frequently. The usual norms in the spaces $\mathbb{L}^p(\Omega)$, $\mathbb{L}^\infty(\Omega)$ and $\mathbb{C}(\overline{\Omega})$ are respectively denoted by

$$||x||_p^p = \frac{1}{|\Omega|} \int_\Omega \left|x(x^{'})\right|^p dx^{'},$$

$$||x||_\infty = \max_{x^{'} \in \Omega} \left|x(x^{'})\right|.$$

It is well known under the "regularizing effect principle" to prove global existence of solutions to (5.1), [24], it suffices to derive uniform estimates on the L^p, norms of the reaction terms, on $[0, T_{\max})$, for some $p > n/2$, where n, is the spatial dimension of the domain Ω, and the reaction terms in our setting are given in (5.1). Here $T_{\max}$ denotes the eventual blowing-up time in $\mathbb{L}^\infty(\Omega)$. The following local existence result is well known [43].

Lemma 5.1. The system (5.1) admits a unique, classical solution (x, y, z) on $[0, T_{\max}) \times \Omega$. If $T_{\max} < \infty$ then

$$\lim_{t \nearrow T_{\max}} \{\|x(t,.)\|_\infty + \|y(t,.)\|_\infty + \|z(t,.)\|_\infty\} = \infty. \tag{5.2}$$

Proof. Since the reaction terms are continuously differentiable in the positive octant, then for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in (1, +\infty)$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator $I_3 (d_1, d_2, d_3)^t \Delta$, where I_3 the three dimensional identity matrix, Δ is the Laplacian operator and $()^t$ denotes the transposition.

5.2 Control of mass and $L^1(\Omega)$ estimates

Note in order to derive global existence via application of [24], we are required to derive uniform in time $L^p(\Omega)$ estimates on the reaction terms in (5.1), for $p > \frac{n}{2}$. We will restrict our selves to the case $n = 1, 2$. We begin by deriving the result in 1d, that is the $n = 1$ case. We proceed by adding up the equations in (5.1), define the grouped variable $W = x + y + z$, and integrate the above equation in space to obtain

$$\frac{d}{dt} \int_\Omega W(x', t) dx'$$
$$= \int_\Omega (x(1-x) - e_1 x + ((1-\pi) - d_1 - e_2)y - d_2 z) dx'$$
$$\leq \max((r\gamma - 1), ((1-\pi) - d_1 - e_2)) \int_\Omega W(x, t) dx' \tag{5.3}$$

Now the uniform in time $L^1(\Omega)$ bound on W, and thus on x, y, z easily follows via applying Gronwall's lemma on the above, on any time interval $[0, T]$, to yield

$$\|x\|_{L^1(\Omega)} \leq \|x_0\|_{L^1(\Omega)} e^{(\max((r\gamma-1), ((1-\pi)-d_1-e_2)))T}, \tag{5.4}$$

$$\|y\|_{L^1(\Omega)} \leq \|y_0\|_{L^1(\Omega)} e^{(\max((r\gamma-1), ((1-\pi)-d_1-e_2)))T}, \tag{5.5}$$

$$\|z\|_{L^1(\Omega)} \leq \|z_0\|_{L^1(\Omega)} e^{(\max((r\gamma-1), ((1-\pi)-d_1-e_2)))T}, \tag{5.6}$$

with the bound depending on only the final time $t = T$. Thus in order to control any of the reaction terms we have to control in essence terms of the form $\frac{a_1 \pi x y}{1 + b_1 x}$

$$\|\frac{a_1 \pi x y}{1 + b_1 x}\|_{L^1(\Omega_T)}$$
$$\leq \|\frac{a_1 \pi x}{1 + b_1 x}\|_{L^\infty(\Omega_T)} \int_{\Omega_T} y dx'$$
$$\leq \frac{a_1 \pi}{b_1} \int_0^T \|y_0\|_{L^1(\Omega)} e^{(\max((r\gamma-1), ((1-\pi)-d_1-e_2)))T} \tag{5.7}$$

This follows from the $L^1(\Omega)$ estimate on y, via (5.4). Thus we have $L^1(\Omega_T)$ control of the reaction terms. Thus we see we can uniformly bound all of the reaction terms in $L^1(\Omega_T)$, and by bootstrap argument then bound them uniformly in L^p to deduce global existence [24].

Now we provide a few details to cover the $n = 2$ case. For this case we need to make uniform estimates on the reaction terms in $L^p(\Omega)$, where $p > \frac{2}{2} = 1$. Thus a uniform (in time estimate, where the estimate can depend on the final time $t = T$) in $L^2(\Omega)$ will suffice.

The estimate on x follows trivially via comparison with the diffusive logistic equation. Thus we have

$$||x||_2^2 \leq C.$$

Here the C is a pure constant and does not depend on time. We now multiply the equation for y and integrate by parts over Ω. Thus we obtain

$$
\begin{aligned}
&\frac{d}{dt}||y||_2^2 + D_2||\nabla y||_2^2 \\
&= \int_\Omega \frac{a_1 \pi x y^2}{1 + b_1 x} dx' + (1 - \pi - e_2 - d_2)||y||_2^2 - \int_\Omega \frac{a_2 \pi z y^2}{1 + b_2 z} dx' \\
&\leq (\frac{a_1 \pi}{b_1} + |1 - \pi - e_2 - d_2|)||y||_2^2
\end{aligned}
$$

$$(5.8)$$

Thus using positivity and Gronwall's equality we obtain, that for any $t \in [0, T]$ we have the estimate

$$||y||_2^2 \leq ||y_0||_2^2 e^{(\frac{a_1 \pi}{b_1} + |1 - \pi - e_2 - d_2|)T}.$$

The uniform bound on z follows similarly. Thus we can now state the following theorem,

Theorem 5.1. Consider the spatially explicit three species model (5.1), where the spatial dimension of the physical domain is $n = 1, 2$. Solutions to this model are classical, that is $(x, y, z) \in C^1(0, T; C^2(\Omega))$, and exist globally in time.

5.3 Turing Instability Theory

In this section we investigate Turing instability in system (5.1). We uncover both spatio-temporal and spatial patterns, and provide the details of the Turing analysis. We derive conditions where the unique positive interior equilibrium point (x^*, y^*, z^*) is stable with diffusion, and unstable due to the action of diffusion, with a small perturbation to the positive interior equilibrium point. We firstly linearize model (5.1) about the homogeneous steady state, we introduce both space and time-dependent fluctuations around (x^*, y^*, z^*). This is given as

$$x = x^* + \hat{x}(\varepsilon, t), \tag{5.9a}$$
$$y = y^* + \hat{y}(\varepsilon, t), \tag{5.9b}$$
$$z = z^* + \hat{z}(\varepsilon, t), \tag{5.9c}$$

where $|\hat{x}(\varepsilon, t)| \ll x^*, |\hat{y}(\varepsilon, t)| \ll y^*, |\hat{z}(\varepsilon, t)| \ll z^*$. Conventionally, we choose

$$
\begin{bmatrix} \hat{u}(\xi, t) \\ \hat{v}(\xi, t) \\ \hat{r}(\xi, t) \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} e^{\lambda t + ik\varepsilon},
$$

where η_i for $i = 1, 2, 3$ are the corresponding amplitudes, k is the wavenumber, λ is the growth rate of perturbation in time t and ε is the spatial coordinate. Substituting (5.9) into (5.1) and ignoring higher order terms including nonlinear terms, we obtain the characteristic equation is given by

$$(\mathbf{J} - \lambda\mathbf{I} - k^2\mathbf{D}) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = 0, \tag{5.10}$$

where

$$\mathbf{D} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 1 - 2x^* - \frac{a_1\pi y^*}{(1+b_1x^*)^2} - e_1 & -\frac{a_1\pi x^*}{1+b_1x^*} & 0 \\ \frac{a_1\pi y^*}{(1+b_1x^*)^2} & \frac{b_2d_2(a_2-b_2d_2)z^*}{a_2} & -\frac{a_2y^*}{1+b_2y^*} \\ 0 & \frac{a_2z^*}{(1+b_2y^*)^2} & 0 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

and $\mathbf{I}$ is a 3×3 identity matrix.
For the non-trivial solution of (5.10), we require that

$$\begin{vmatrix} J_{11} - \lambda - k^2 D_1 & J_{12} & J_{13} \\ J_{21} & J_{22} - \lambda - k^2 D_2 & J_{23} \\ J_{31} & J_{32} & J_{33} - \lambda - k^2 D_3 \end{vmatrix} = 0,$$

which gives a dispersion relation corresponding to (x^*, y^*, z^*) where $\alpha_* = 1 + b_1x^*$, $\beta_* = 1 + b_2y^*$. To determine the stability domain associated with (x^*, y^*, z^*), we rewrite the dispersion relation as a cubic polynomial function given as

$$P(\lambda(k^2)) = \lambda^3 + \mu_2(k^2)\lambda^2 + \mu_1(k^2)\lambda + \mu_0(k^2), \tag{5.11}$$

with coefficients

$$\mu_2(k^2) = (D_1 + D_2 + D_3)k^2 - (J_{11} + J_{22} + J_{33}),$$
$$\mu_1(k^2) = J_{11}J_{33} + J_{11}J_{22} + J_{22}J_{33} - J_{32}J_{23} - J_{12}J_{21}$$
$$\quad - k^2\big((D_3 + D_1)J_{22} + (D_2 + D_1)J_{33} + (D_2 + D_3)J_{11}\big)$$
$$\quad + k^4(D_2D_3 + D_2D_1 + D_1D_3),$$
$$\mu_0(k^2) = J_{11}J_{32}J_{23} - J_{11}J_{22}J_{33} + J_{12}J_{21}J_{33}$$
$$\quad + k^2\big(D_1(J_{22}J_{33} - J_{32}J_{23}) + D_2J_{11}J_{33} + D_3(J_{22}J_{11} - J_{12}J_{21})\big)$$
$$\quad - k^4\big(D_2D_1J_{33} + D_1D_3J_{22} + D_2D_3J_{11}\big) + k^6D_1D_2D_3.$$

By Routh-Hurwitz criterion for stability, $\mathbb{R}e(\lambda) < 0$ in model (5.1) around equilibrium point (x^*, y^*, z^*) (i.e (x^*, y^*, z^*) is stable) if and only if these conditions hold:

$$\mu_2(k^2) > 0, \mu_1(k^2) > 0, \mu_0(k^2) > 0 \quad \text{and} \quad [\mu_2\mu_1 - \mu_0](k^2) > 0. \tag{5.12}$$

Where as violating either of the above conditions implies instability (i.e $\mathbb{R}e(\lambda) > 0$). We now require conditions where an homogeneous steady state (x^*, y^*, z^*) will be stable to small perturbation in the absence of diffusion and unstable in the present of diffusion with certain k values. Meaning, we require that around the homogeneous steady state (x^*, y^*, z^*)

$$\mathbb{R}e(\lambda(k^2 > 0)) > 0, \text{ for some } k \text{ and } \mathbb{R}e(\lambda(k^2 = 0)) < 0,$$

where we consider k to be real and positive even though k can be complex. This behavior is called *Diffusion driven instability*. Models that exhibits this behavior in 2 or 3 species have been extensively studied in [31, 21, 46], were several different patterns was observed. In order for homogeneous steady state (u^*, v^*, r^*) to be stable (in the absence of diffusion) we need

$$\mu_2(k^2 = 0) > 0, \mu_1(k^2) > 0, \mu_0(k^2 = 0) > 0 \quad \text{and} \quad [\mu_2\mu_1 - \mu_0](k^2 = 0) > 0,$$

whereas with diffusion $(k^2 > 0)$ we look for conditions where we can drive the homogeneous steady state to be unstable, this can be achieved by studying the coefficient of (5.11). In order to achieve this we reverse at least one of the signs in (5.12). For this we have to first study $\mu_2(k^2)$. Irrespective of the value of k^2, $\mu_2(k^2)$ will be positive since $J_{11} + J_{22} + J_{33}$ is always less than zero. Therefore we cannot depend on $\mu_2(k^2)$ for diffusion driven instability to occur. Hence for diffusion driven instability to occur in our case, we only depend on the 2 conditions which are

$$\mu_0(k^2) \quad \text{and} \quad [\mu_2\mu_1 - \mu_0](k^2). \tag{5.13}$$

Both functions are cubic functions of k^2, which has the form as

$$G(k^2) = H_H + k^2 D_D + (k^2)^2 C_C + (k^2)^3 B_B, \text{ with } B_B > 0, \text{ and } H_H > 0.$$

The coefficient of $G(k^2)$ are standard, see [21].
To drive either $\mu_0(k^2)$ or $[\mu_2\mu_1 - \mu_0](k^2)$ to negative for some k, we basically need to find the minimum k^2 referred to as the minimum turing point (k_T^2) such that $G(k^2 = k_T^2) < 0$. This minimum Turing point occurs when

$$\partial G / \partial(k^2) = 0,$$

which when solved for k^2 we obtain

$$k^2 = k_T^2 = \frac{-C_C + \sqrt{C_C^2 - 3B_B D_D}}{3B_B},$$

which ensures k^2 is real and positive such that $\partial^2 G / \partial^2(k^2) > 0$, by which we require either

$$D_D < 0 \quad \text{or} \quad C_C < 0, \tag{5.14}$$

which ensures that

$$C_C^2 - 3B_B D_D > 0.$$

Therefore $G(k^2) < 0$, if at $k^2 = k_T^2$

$$G_{min}(k^2) = 2C_C^3 - 9D_D C_C B_B - 2(C_C^2 - 3D_D B_B)^{3/2} + 27B_B^2 H_H < 0. \tag{5.15}$$

Hence (5.14)-(5.15) are necessary and sufficient conditions for (x^*, y^*, z^*) to produce diffusion driven instability, which leads to emergence of patterns. Also to first establish stability when $k = 0$, H_H in each case has to be positive.
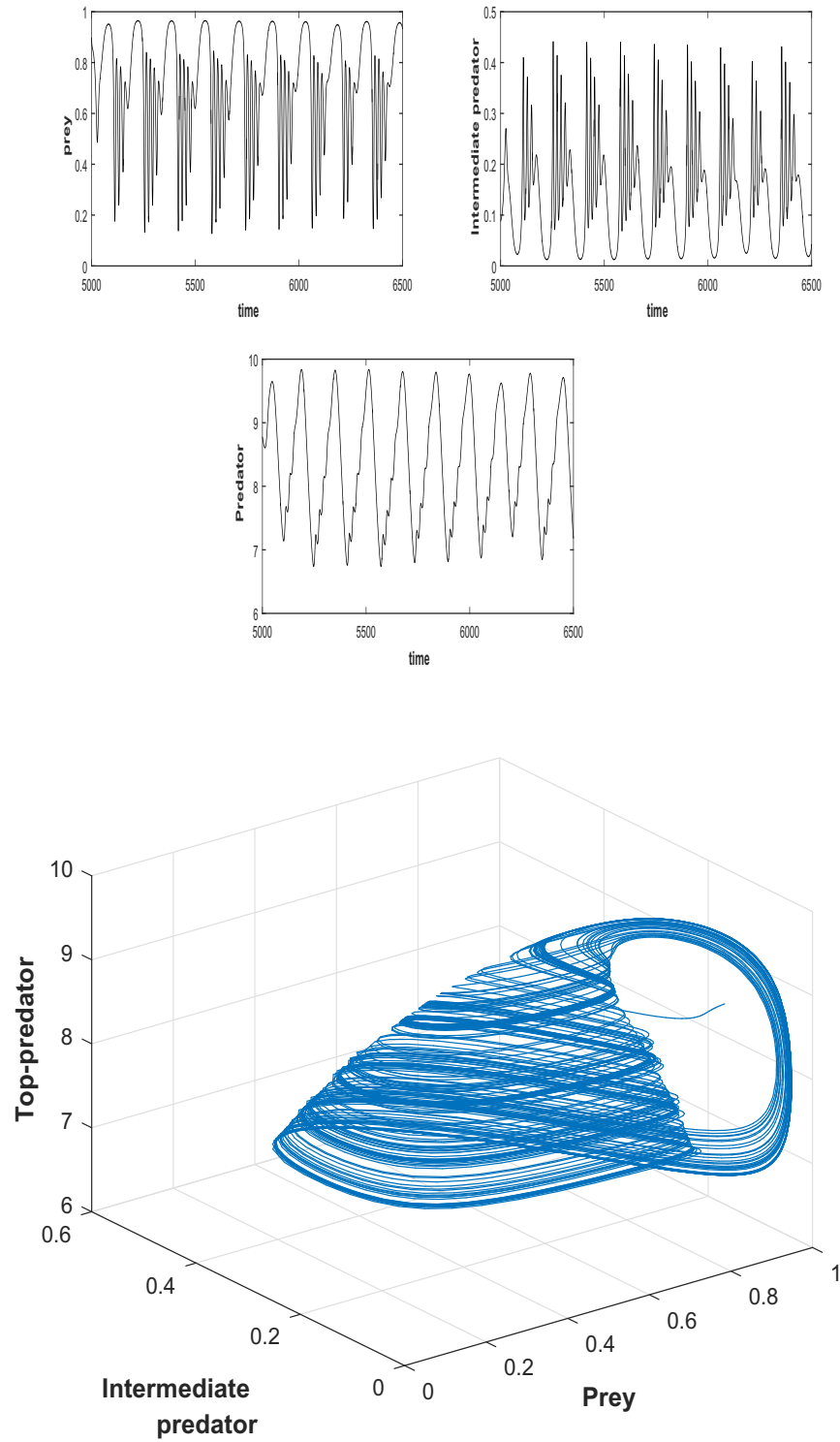
Fig. 1   (a)Figure depicts the time series chaotic oscillation of all three species for $a_1 = 4.8$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.2$, $e_1 = 0.02$, $e_2 = 0.01$, $\pi = 0.01$.
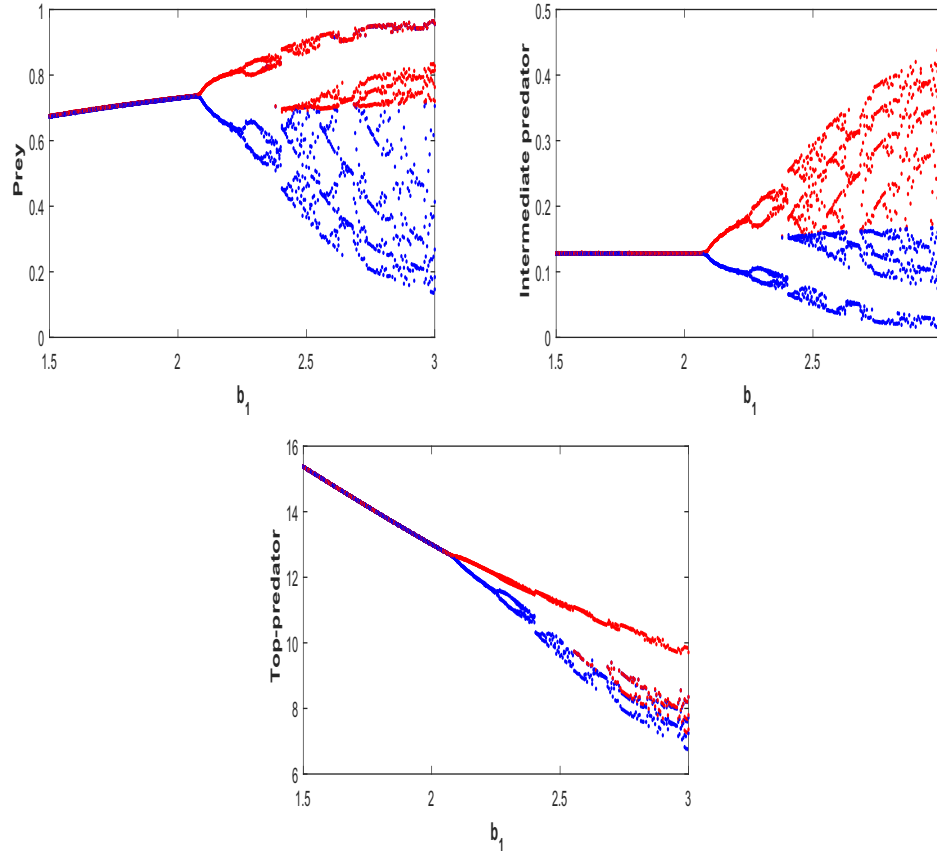
Fig. 2 Figure indicates the bifurcation diagram for $\beta \in [1.5, 3.0]$ and also indicates that system shows stable focus, limit cycle, period-doubling and chaos and other parameter values given in the Figure1.

6 Numerical results and discussion

We have modified the model of Hastings and Powell[22] by introducing harvesting and alternative food. Hastings and Powell[22] observed stable focus, limit cycle, period doubling and chaotic dynamics of the system by changing half saturation constant b_1. Our aim is, first to observe the exchange of states (stability-limit cycle-period-doubling-chaos) in our modified system for different values of b_1 and subsequently to observe the system under the variation of harvesting parameter and alternative food resource parameter. Thus we first observe the dynamics of proposed system for variation of half saturation constant b_1. Here we use Matlab Software for solving our proposed system with the help of Matlab code ode45 and keeping the initial condition $x = 0.89765$, $y = 0.087792$ and $z = 8.7794$ through out the numerical simulations. We observe that the system shows chaotic oscillations for b_1 and we also observe strange chaotic tea-cup attractor (see Figure1). To observe the clear dynamical behaviour of the system, we draw a bifurcation diagram with respect to b_1 and the system shows stable focus, limit cycle, period-doubling and chaos for $1.5 \leq b_1 \leq 3.0$. It is noticed that the system is stable around the interior equilibrium point for $1.5 \leq b_1 \leq 2.1$ (see Figure (2a)) and for $2.1 \leq b_1 < 2.2$ it shows limit cycle oscillations (see Figure (2b)). The period-doubling is observed for $2.2 \leq b_1 < 2.4$
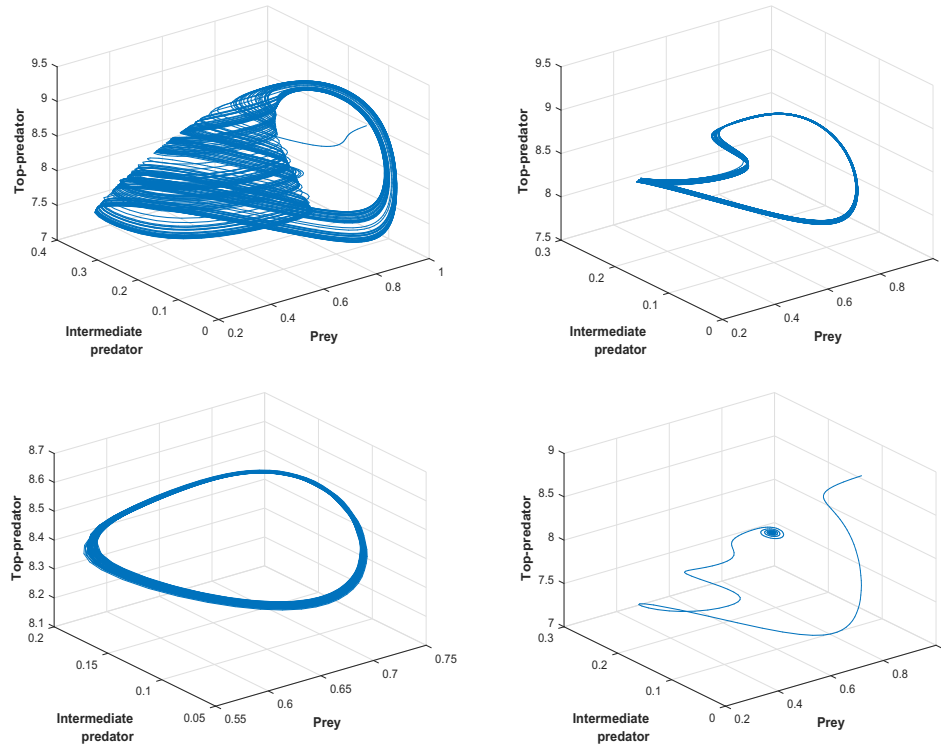
Fig. 3   (a) Figure depicts that strange tea-cup chaotic attractor for $e_1 = 0.05$; (b) Figure shows the period-doubling for $e_2 = 0.086$; (c) Figure depicts the limit cycle periodic solution $e_1 = 0.12$; (d) Figure shows the stable focus for $e_1 = 0.15$ and other parameter values given in Figure 1.

(see Figure (2c)). Finally, chaotic dynamics is observed for $2.4 \leq b_1 \leq 3.0$ (Figure (2d)). However we observe finally the chaotic teacup attractor. The dynamics within the attractor are given roughly as follows. Starting in the "handle" of the "teacup", the system moves to the wide part of the teacup and then spirals along the teacup to the narrow end, entering the handle again. In terms of the species behaviour, the top predator Z crashes, allowing wide swings in the population levels of X and Y. As Z increases in numbers, the swings in X and Y become damped, until Z causes the levels of Y to crash. This leads to a crash in Z and an outbreak in X, starting the process again. The sequence of events in terms of species numbers, always follows the same general pattern. What is unpredictable is the timing. One way to express this is that the time between crashes of species Z varies in an erratic fashion. Also, the number of peaks in species Y between major crashes varies and the population size at the peaks varies.

We observe that system enters into chaotic dynamics from stable focus from the Figure (2). Food chain model suggests that chaotic behavior may be much more common in natural systems with interacting producers and consumers (Predators and prey) than Hassel et al.[25] predicted on the basis of one species discrete time models. It is our interpretation that chaos ultimately arises in this food chain model because of the tendency for predator-prey systems to oscillate. One predator-prey subsystem, for concreteness say X and Y oscillate at one frequency, while another, Y and Z oscillate at a different frequency; the frequencies are determined by the model parameters. In particular, the interaction at the higher trophic levels has a longer natural period because the average lifetime of the top predators is longer than the average lifetime of the consumers at the lower trophic levels. The two systems are
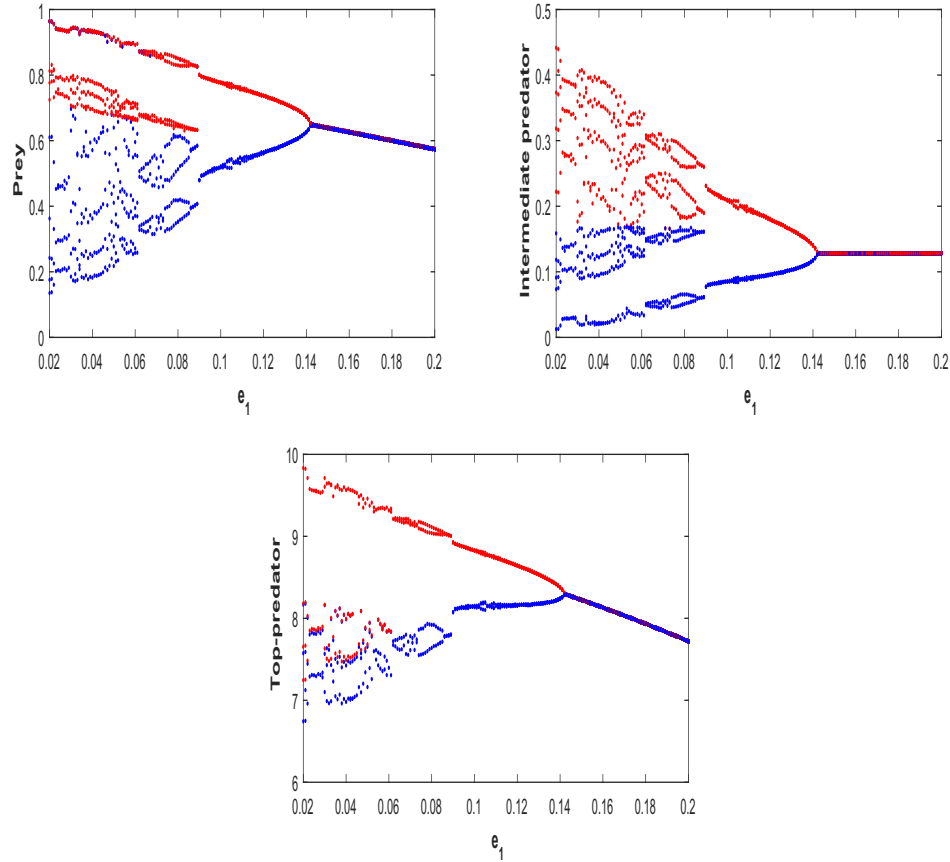
Fig. 4 Figure indicates the bifurcation diagram for $e_1 \in [0.02, 0.2]$ and depicts the chaos, period-double, limit cycle and stable focus and other parameter values given in the Figure1.

coupled through species Y because the predator in one is the prey in the other. We conjecture that when the period of one oscillation is not some multiple of the other frequency (i.e. the frequencies are incommensurate) chaos arises in a manner similar to the occurrence of chaos in periodically forced oscillations. Food webs depict a complex net coupled producer-consumer interactions; accordingly one should not be surprised to find many subsystems within a food web going through the oscillations that are a common feature of all predator-prey systems. It seems unlikely that the frequencies of all the linked subsystems will be commensurate. Accordingly, one expects that at least some, perhaps most of the linked predator-prey subsystems may be simultaneously undergoing chaotic oscillations. It is instructive to compare this view with the three species study (a predator consuming two competing prey) of Gilpin [18], in which chaotic behavior was identified. Typically, one does not associate oscillations with competition models based on results from two species Lotka-Volterra competition [32]. However, once three or more species are allowed, oscillations can occur [33, 36]. We conjecture that food webs containing a number of competing species can also go chaotic in the same way as our consumer-producer food chain does.

To observe the effect of harvesting and alternative food in our proposed system, we first increase the harvesting parameters, keeping the half-saturation constant b_1 fixed at $b_1 = 3.0$. In our model
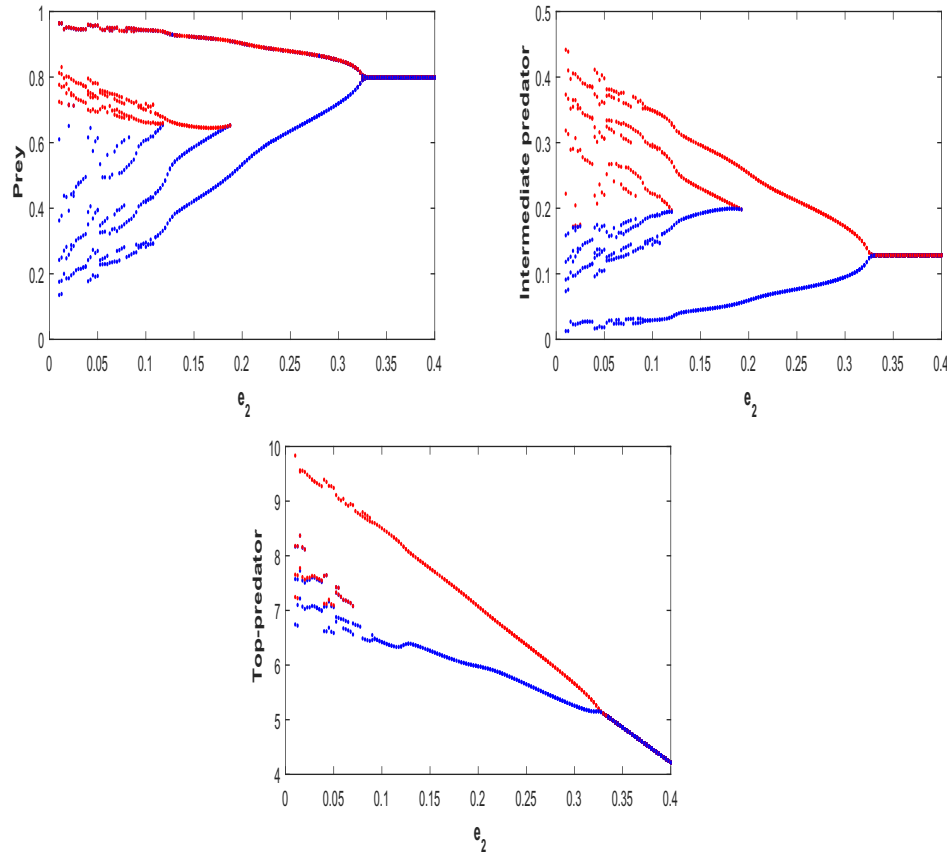
Fig. 5 Figure indicates that system shows chaos, period-doubling, limit cycle and stable focus for variation of $e_2 \in [0.01, 0.4]$ and other parameter values given in the Figure1.

harvesting parameters e_1, e_2 and alternative food resource parameter π are important in controlling chaotic dynamics. We first vary e_1 and observe the clear dynamics such as chaos, period-double, limit cycle and stable focus. It is noticed that system shows strange chaotic attractor for $e_1 = 0.05$ (see Figure(3)a). If we increase the value of e_1 from 0.05 to 0.086 system shows period-doubling (see Figure(3)b). From Figure(3c), it is observed that system shows periodic limit cycle solution for $e_1 = 0.12$. Finally we have observed that system enters into stable focus for $e_1 = 0.15$ (see Figure(3d)). To observe clear picture we draw bifurcation diagram with respect to e_1 (see Figure(4)) and from this figure we observe that system enters into period-doubling from chaotic dynamics, limit cycle from period-doubling, finally stable focus from limit cycle oscillations. It is very interesting that harvesting in prey species stabilizes the chaotic dynamics. Now we explain the stabilization mechanism. When values of harvesting is low the chaotic dynamics remains same in the system but when the values of harvesting is high system settle down into stable focus. When harvesting of prey species is increasing population level of prey species decreases and intermediate and top predator also decrease due to scarcity of food. Such type of decreasing of population level stabilizes the chaotic oscillations. We also observe the dynamics of the system for variation of harvesting parameter e_2. We observe from Figure(5) that system shows chaos when the values of e_2 is low but chaos reduces to stable focus
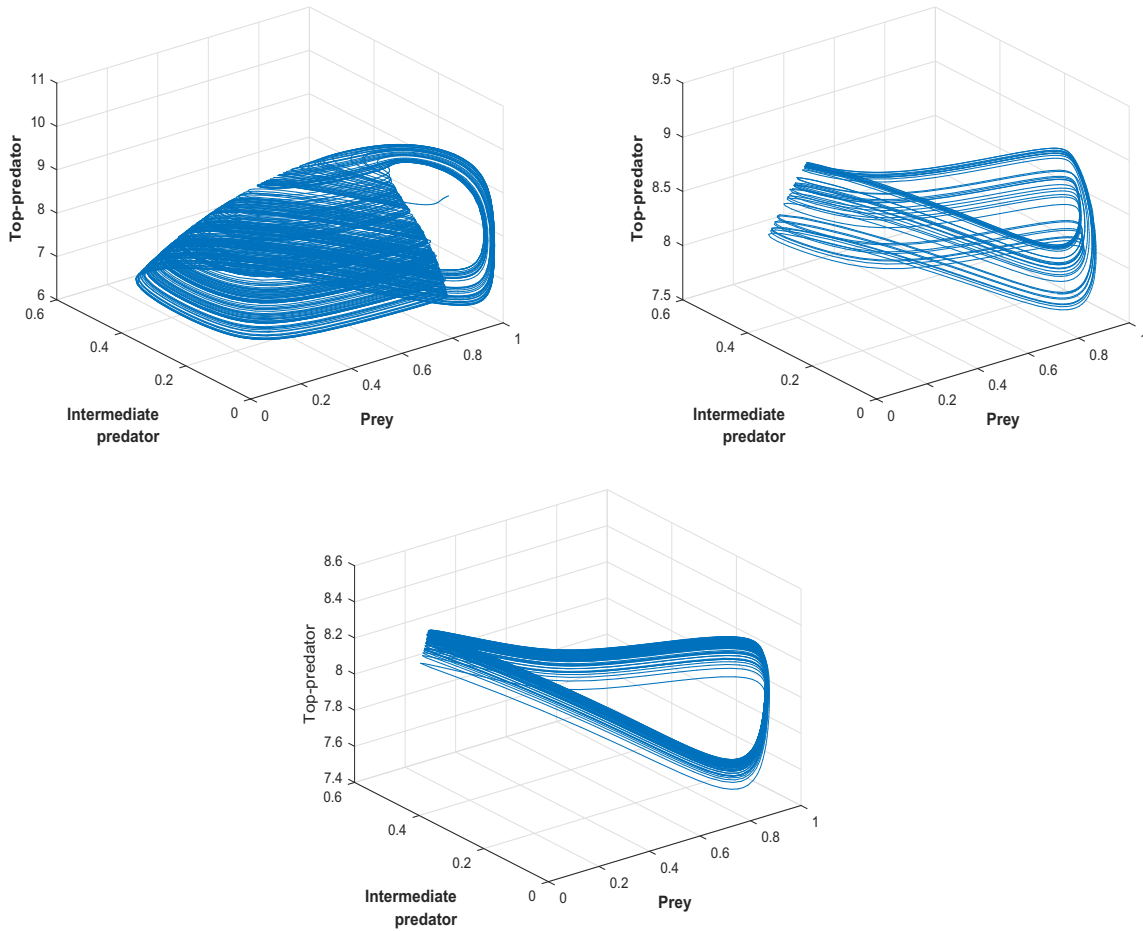
Fig. 6 (a)Figure shows the strange tea-cup chaotic attractor for $\pi = 0.95$; (b) Figure shows the multiple periodic solution for $\pi = 0.83$; (c) Figure shows periodic limit cycle solution for $\pi = 0.8$ and other parameter values given in Figure 1.

for higher values of e_1. We explain the biological reason for reducing chaos into stable focus for increasing e_2. When e_2 is increased the population level of intermediate predator decreases and prey population increases but top-predator decreases due to scarcity of food. This type of competition reduces chaos.

We now observe the consequences of alternative food on the system dynamics. First we fix the value of b_1 at 3.0 so that the system exhibits chaos and then we gradually decrease the value of π to observe the changed dynamics. We have plotted solution trajectories (Figure(6)) for different values of π, where $0 < \pi < 1$. These figures show that the irregular behaviour of the system becomes regular as the intermediate predator shifts to alternative food. A bifurcation diagram (Figure(7)) with π as the bifurcation parameter has also been presented to show the variation in the system dynamics when π is smoothly varied. This figure demonstrates that the system returns to regular oscillations from chaotic oscillations through period-halving bifurcations as π is gradually decreased from its maximum value.

Here we demonstrate Turing patterns that form in 1D. The initial condition used is a small perturbation around the positive homogeneous steady state given as
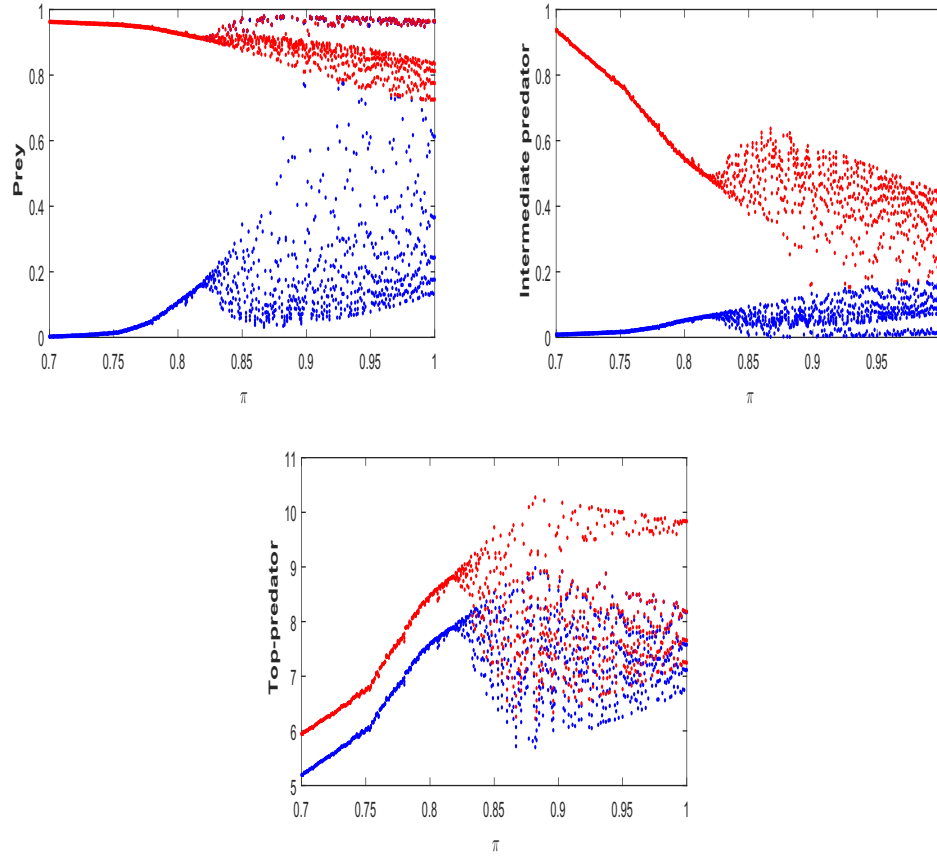
Fig. 7 Figure indicates that the system shows different dynamical behaviour such as chaos, multiple periodic solution and limit cycle for variation of $\pi \in [0.7, 1.0]$ and other parameter values given in the Figure1.
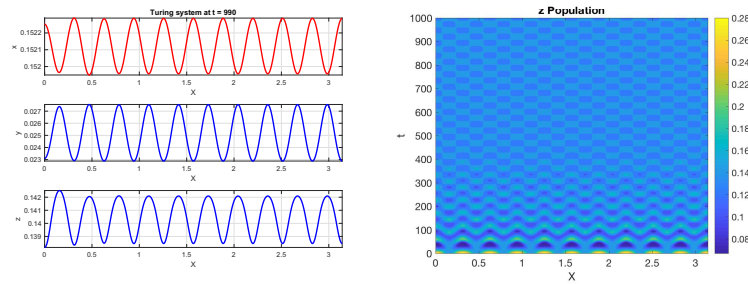

Fig. 8 Here we demonstrate that alternative food can indeed induce Turing instability. The densities of the three species are shown as contour plots in the x-t plane (1 dimensional in space). The long-time simulation yields spatial patterns.

$$x = x^* + \eta_1 cos^2(nr)(r > 0)(r < \pi),$$
$$y = y^* + \eta_2 cos^2(nr)(r > 0)(r < \pi),$$
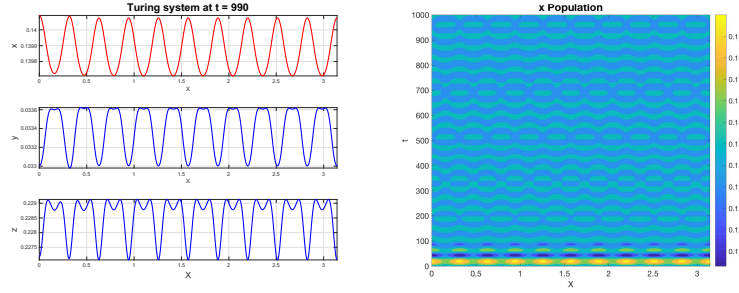$$z = z^* + \eta_3 cos^2(nr)(r > 0)(r < \pi),$$

Fig. 9 The densities of the three species are shown as contour plots in the x-t plane (1 dimensional in space). The long-time simulation yields spatial patterns.
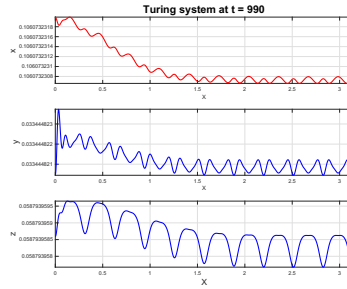

Fig. 10 Here we found no Turing as $\pi = 1$.

where $\eta_i = 0.005 \; \forall i$.

We choose parameters $a_1 = 4.2, a_2 = 2, b_1 = 5, b_2 = 0.1, e_1 = 0.8, e_2 = 0.2, \pi = 0.8, d_1 = 0.01, d_2 = 0.05$, and simulate the PDE system (5.1) to obtain spatial patterns as seen in Figure 8.

We next choose parameter values as $a_1 = 4, a_2 = 1.5, b_1 = 4, b_2 = 0.1, e_1 = 0.8, e_2 = 0.1, \pi = 0.7, d_1 = 0.01, d_2 = 0.05$, and $x^* = 0.1400, y^* = 0.0334 z^* = 0.2283$, then simulate system (5.1) to obtain spatial patterns as seen in Figure 9.

Remark 6.1. Next we want to investigate the effect of removing the alternative food, on the Turing instability in the system. We numerically investigate the situation without alternative food, that is, $\pi = 1$, we found no Turing here as Figure. 10. We use the parameter set as Figure.9 and replace $\pi = 1$. This shows a certain amount of alternative food can induce Turing instability.

In the present paper we modified the HP model by introducing harvesting in populations. We worked out the conditions for which the system is stable around different equilibria. Hastings and Powell [22] observed stability, limit cycle, period-doubling and finally chaos in simple tri-trophic food chain model by increasing half saturation constant b_1. However, the introduction of harvesting in the populations change the dynamics of the original model. From our numerical simulations we observe that by increasing the harvesting parameters the system enters into period-doubling from chaos, limit cycle solution from period-doubling and finally settles down to a steady state solution, a stable situation when other parameters are fixed. Thus increase in the harvesting rates not only prevent the system from entering into chaotic region, but also stabilizes the coexistence steady state.

We observe that chaotic oscillations in a tri-trophic food chain model is common, and control of such oscillation is utmost important from ecological as well as economic viewpoint. From our analysis it is clear that harvesting free system enters into chaos from stable focus for increasing b_1. It is also

observed that chaotic behaviour due to enhancement of half saturation constant can be prevented by increasing the harvesting effort on population. Thus we may conclude that harvesting in tri-trophic food chain model may be used as biological control for stabilizing chaotic dynamics, if any. There are various practical directions herein. Cropper and DiResta [6] studied the effects of harvesting on Florida commercial sponge population. They concluded that unregulated harvesting might lead to a decline in the sponge population and affects the benthic community composition.

7 Conclusion

The present study explores a mathematical model of a tri-trophic food chain with alternative food and harvesting. Here we consider alternative food for prey species and harvesting in prey and intermediate predator. We have rigorosly shown the local stability of different equilibrium points. These theorems highlight some biological threshold conditions which determine the stability of the different equilibrium points. We pay attention to the Hopf-bifurcation and observed that e_1 has the critical value above which system exhibits Hopf-bifurcation. We also study Turing instability by considering diffusion in our proposed model, and have derived the conditions for Turing instability. To get insight into the global dynamics of our proposed model we have performed extensive numerical simulation in the ODE system as well as the diffusive system. Firstly we have observed chaotic dynamics via the standard period-doubling route, by increasing the value of b_1. Secondly we have tried to control such chaotic dynamics by using harvesting parameter as well as alternative food. We have found that if we increase the value of e_1, the system settles into a stable focus through different dynamics such as chaos, period-doubling and limit cycle. Similar dynamics are observed for variation of e_2. So we may conclude that both harvesting parameters play an important role to get a stable and sustainable ecosystem by preventing chaotic dynamics. We also find that alternative food has a stabilizing effect on chaotic dynamics. By varying the alternative food we find that chaos will disappear. Our results show that the harvesting parameter and alternative food are the two biological parameters which play an important role to prevent chaotic dynamics and lead to a stable ecosystem where the concerned three species persist, in the ODE case. However, in the diffusive system we have observed that harvesting parameter and alternative food cause instability of the system. Thus the effect of alternative food seems to be different in the ODE and PDE cases, stabilizing in the former, but destabilizing in the latter. This is highly interesting and warrants further study. Such directions will be the subject of our future investigations.

References

[1] M. Ali Khan, J. Ghosh, B. Sahoo (2015). Controlling Chaos in a Food Chain Model through Threshold Harvesting.Fisheries and Aquaculture Journal.

[2] M. v. Baalen, V. Krivan, P.C.J.v. Rijn, M. W. Sabelis (2001). Alternative Food, Switching Predators, and the Persistence of Predator-Prey Systems. The American Naturalist Volume 157, Number 5

[3] N. Bairagi, S. Chaudhuri, J. Chattopadhyay (2009). Harvesting as a disease control measure in an eco-epidemiological system -A theoretical study.Mathematical Biosciences 217:134-74

[4] J. Anthony (1988). Foraging Behavior and Switching by the Grasshopper Sparrow Ammodramus savannarum Searching for Multiple Prey in a Heterogeneous Environment.The American Midland Naturalist Vol. 119, No. 2,pp. 225-234

[5] R. Colombo, V. Krivan (1993). Selective strategies in food webs. IMA Journal of Mathematics Applied in Medicine and Biology 10:281-291

[6] W.P.Jr. Cropper, D. DiResta (1999). Simulation of a Biscayne Bay, Florida commercial sponge population: effects of harvesting after Hurricane Andrew. Ecol. Model.,118, 1-15

[7] K.S. Chaudhuri (1986). A bioeconomic model of harvesting a multispecies fishery. Ecol. Model., 32: 267-279

[8] K.S. Chaudhuri (1988). Dynamic optimization of combined harvesting of a two-species fishery. Ecol. Model., 41: 17-25

[9] J. Dhar, K.S. Jatav (2013). Mathematical analysis of a delayed stage-structured predatorprey model with impulsive diffusion between two predators territories. Ecol. Complex. 16, 59-67

[10] G. Dai, M.Tang (1998). Coexistence region and global dynamics of a harvested predatorprey system SIAM J. Appl. Math., 58(1), pp. 193-210

[11] Y. Do, H. Baek, Y. Lim, D. Lim (2011) A Three-Species Food Chain System with Two Types of Functional Responses.Abstract and Applied Analysis Volume 2011 Article ID 934569

[12] R. Eftimie (2012).*Hyperbolic and kinetic models for self-organized biological aggregations and movement: a brief review.* Journal of mathematical biology, 65,35-75.

[13] J. M. Fryxell, P. Lundberg (1994). Diet choice and predator-prey dynamics. Evol. Ecol. 8, 407-421.

[14] L. N. Guin (2014). Existence of spatial patterns in a predator-prey model with self- and cross-diffusion, Applied Mathematics and Computation, Vol. 226, pp. 320-335.

[15] L. N. Guin , P. K. Mandal (2014). Effect of prey refuge on spatiotemporal dynamics of reaction-diffusion system, Computers and Mathematics with Applications, Vol 68, No.10,pp. 1325-1340.

[16] L. N. Guin (2015). Spatial patterns through Turing instability in a reaction-diffusion predator-prey model, Mathematics and Computation in Simulation, Vol 109,pp. 174-185.

[17] L. N. Guin , S. Acharya (2017). Dynamic behaviour of a reaction-diffusion predator-prey model with both refuge and harvesting, Nolinear Dynamics ,Vol 88,No.2,pp.1501-1533.

[18] M. E. Gilpin (1979). Spiral chaos in a predator-prey model. Am. Natural. 113, 306-308. doi: 10.1086/283389

[19] D. Greenhalgh, Q.J.A. Khan, J. S. Pettigrew (2016). An eco-epidemiological predator-prey model where predators distinguish between susceptible and infected prey.Mathematical methods in the applied sciences 40(1):146-166

[20] S. Gakkhar, A. Singh (2012). Control of chaos due to additional predator in the Hastings-Powell food chain model. Journal of Mathematical Analysis and Applications Volume 385, Issue 1, 423-438

[21] K.A.J. White, C. A. Gilligan (1998).*Spatial heterogeneity in three species, plant-parasite-hyperparasite, systems*, Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences 353.1368 , 543-557.

[22] A. Hastings, T. Powell (1991). Chaos in three-species food chain. Ecology 72(3), 896-903

[23] T. Hillen, K. Painter (2009).*A users guide to PDE models for chemotaxis*, Journal of Mathematical Biology, 57,183-217.

[24] D. Henry (1984).*Geometric Theory of Semi-linear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer-Verlag, New-York.

[25] M. Hassell, J. Lawton, R. M. May (1976). Pattern of dynamical behavior in single-species populations. Journal of Animal Ecology 45: 471-486

[26] M. Liu, C. Bai (2016). Optimal harvesting of a stochastic mutualism model with jumps. Appl. Math. Comput. 276, 301-309

[27] K. McCann, A. Hastings (1997). Re-evaluating the omnivory stability relationship in food webs. Proceedings of the Royal Society B Biological Sciences.

[28] D.O. Maionchi, S.F.d. Reis, M.A.M.d. Aguiar (2006). Chaos and pattern formation in a spatial tritrophic food chain. Ecol. Model., 191, 291-303

[29] V. Madhusudanan, S. Vijaya, M. Gunasekaran, (2014). Impact of Harvesting in Three Species Food Web Model With Two Distinct Functional Responses.International Journal of Innovative Research in Science, Engineering and Technology Vol. 3, Issue 2.

[30] J. D. Murray (1993). Mathematical Biology, Springer-Verlag, Berlin.

[31] J. D. Murray (1990). Discussion: Turing's theory of morphogenesisis influence on modeling biological pattern and form, Bull. Math. Biol. 52,117-152.

[32] R. M. MAY (1973). Stability and Complexity in Model Ecosystems, Princeton Univ. Press, Princeton, N. J.

[33] R. M. May, J. W. Leonard (1975). Nonlinear Aspects of Competition Between Three Species. SIAM Journal on Applied Mathematics,Vol 29,Issue-2 243-253

[34] N. J. Ndam, J. P. Chollom, T. G. Kassem (2011). A Mathematical Model of Three-Species Interactions in an Aquatic Habitat.ISRN Applied Mathematics Vol 2012, Article ID 391547, 11 pages

[35] A. Okubo, S. A. Levin (2001).Diffusion and ecological problems: modern perspectives, Springer.

[36] T. Powell, P. J. Richerson (1985). Temporal Variation, Spatial Heterogeneity, and Competition for Resources in Plankton Systems: A Theoretical Model. The American Naturalist 125(3)

[37] V. Rai (2013). *Spatial Ecology: Patterns and Processes* Bentham Science Publishers.

[38] F. Rao , C. Castillo-Chavez, Y. Kang (2018). Dynamics of a diffusion reaction preypredator model with delay in prey: Effects of delay and spatial components. J. Math. Anal. Appl. 461 (2018) 1177-1214.

[39] G. D. Ruxton (1994). Low levels of immigration between chaotic populations can reduce system extinctions by inducing asynchronous cycles Proc. R. Soc. Lond. B 256, 189-193

[40] M. Sambath, K. Balachandran , L. N. Guin (2018).Spatiotemporal patterns in a predator-prey model with cross-diffusion effect, International Journal Bifurcation and Chaos,Vol 28,No.2, 1830004.

[41] W. M. Liu (1994). *Criterion of Hopf Bifurcations without Using Eigenvalues*. Journal of Mathematical Analysis and Applications, 182:250-256.

[42] B. Sahoo, S. Poria (2014).The chaos and control of a food chain model supplying additional food to top-predator.Chaos, Solitons and Fractals.Volume 58, 52-64

[43] J. Smoller (1983). Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York.

[44] Y. Su, J.J. Wei, J.P. Shi (2009). Hopf bifurcations in a reactiondiffusion population model with delay effect, J. Differential Equations 247(4) 1156-1184.

[45] E. E. Werner, D. J. Hall (1974). Optimal foraging and the size selection of prey by the bluegill sunfish (Lepomis macrochirus). Ecology 55:1042-52.

[46] Z. Xie (2012).*Cross diffusion induced Turing instability for a three species food chain model*, Journal of Mathematical Analysis and Applications, vol 388,539-547.

[47] C. Xu, Z. Li (2002). Influence of intraspecific density dependence on a three-species food chain with and without external stochastic disturbances. Ecol. Model. 155, 71-83.

[48] J. T. Zhao, J. J. Wei (2015). Dynamics in a diffusive plankton system with delay and toxic substances effect, Nonlinear Anal. Real World Appl. 22 66-83.

[49] P. Zhang, J. Sun, J. Chen, J. Wei, W. Zhao, Q. Liu, H. Sun (2013). Effect of feeding selectivity on the transfer of methylmercury through experimental marine food chains. Mar Environ Res 89:39-44

[50] W. J. Zuo, J. J. Wei (2011). Stability and Hopf bifurcation in a diffusive predatorprey system with delay effect, Nonlinear Anal. Real World Appl. 12(4) 1998-2011.