A uniformization theorem for Stein spaces

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Abstract

In this paper we establish the Qi-Keng Lu uniformization theorem for Stein spaces with possibly isolated normal singularities.

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1 Introduction

Canonical metrics play a fundamental role in the study of the complex structure of the underlying complex manifolds. They provides a fertile place where complex analysis and complex geometry interact. Here we say a metric ω on a complex manifold M is canonical if it is invariant under biholomorphic transformations. Among important invariant metrics are the Bergman metrics and Kähler-Einstein metrics. Since the classical work including Kobayashi [Ko], and Fefferman [Fe1]-[Fe3], the Bergman kernel and the Bergman metric take a central place in the study of the geometry of Stein manifolds with bounded pseudoconvex boundaries. One expects the underlying manifold to possess special analytic structure if certain geometric conditions are posed on their Bergman metrics. For instance, Cheng conjectured in [CY] that the Bergman metric of a smoothly bounded strongly pseudoconvex domain has a constant Ricci curvature if and only it is biholomorphic to the unit ball. Cheng's conjecture was previously obtained by Fu-Wong [FW] and Nemirovski-Shafikov [NS] in the case of complex dimension two. There are also other closely related versions of the Cheng conjecture in terms of metrics defined by other canonical functions as in the work of Li [L1],[L2] and [L3]. Recently, the authors confirmed Cheng's conjecture in higher dimensional case based on the previous work by many mathematicians. More precisely, one has the following theorem:

Theorem 1.1. *The Bergman metric of a smoothly bounded strongly pseudoconvex domain in* $\mathbb{C}^n (n \geq 2)$ *is Kähler-Einstein if and only if the domain is biholomorphic to the ball.*

A crucial ingredient in the proof of Theorem 1.1 is the following classical uniformization result of Lu [Lu].

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Theorem 1.2. *(Qi-Keng Lu) Let D be a bounded domain in $\mathbb{C}^n$. Then the Bergman metric of D is complete and has constant holomorphic sectional curvature if and only if D is biholomorphic to the unit ball in $\mathbb{C}^n$.*

It is natural to ask whether an analog of Theorem 1.1 can be expected on a Stein space of complex dimension at least two with possibly normal singularities. One of the first steps toward understanding this question is then to generalize the Lu theorem to Stein spaces with isolated normal singularities. And this is the main purpose of this paper. To explain our result, we first recall the notions of the Bergman kernel and the Bergman metric on a Stein space.

Let Ω be a Stein space with possibly isolated singularities and write $\mathrm{Reg}(\Omega)$, or Ω_{reg}, for the collection of the smooth points of Ω. Write $\Lambda^n(\mathrm{Reg}(\Omega))$ for the space of the holomorphic $n-$forms on $\mathrm{Reg}(\Omega)$ and define the Bergman space of Ω to be $A^2(\Omega) = \{\alpha \in \Lambda^n(\mathrm{Reg}(\Omega)) : |\int_{\mathrm{Reg}(\Omega)} \alpha \wedge \overline{\alpha}| < \infty\}$. Assume $A^2(\Omega)$ has a non-zero element. Then $A^2(\Omega)$ is a Hilbert space with inner product:

$$\langle \alpha, \beta \rangle := \frac{1}{i^{n^2}} \int_{\mathrm{Reg}(\Omega)} \alpha \wedge \overline{\beta} \text{ for } \alpha, \beta \in \Lambda^n(\mathrm{Reg}(\Omega)).$$

Let $\{\alpha_j\}_{j=1}^m$ be an orthonormal basis of $A^2(\Omega), m \leq \infty$, and define the Bergman kernel to be $K_\Omega = \sum_{j=1}^m \alpha_j \wedge \overline{\alpha_j}$. In a local holomorphic coordinate chart (U, z) of $\mathrm{Reg}(\Omega)$, we have

$$K_\Omega = k(z, \overline{z}) dz_1 \wedge \cdots dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n} \text{ in } U.$$

Here $k(z, \overline{z}) = \sum_{j=1}^m |a_j(z)|^2$ if $\alpha_j = a_j(z) dz_1 \wedge \cdots \wedge dz_n, j \geq 1$, in the local coordinate chart. Assume K_Ω is nowhere zero on $\mathrm{Reg}(\Omega)$. We define an Hermitian $(1,1)-$form to be $\omega^B = \sqrt{-1}\partial\overline{\partial} \log k(z, \overline{z})$. We call ω^B the Bergman metric if it induces a (positive definite) metric on $\mathrm{Reg}(\Omega)$.

To further study the Bergman metric, we pause to recall the definition of holomorphic maps from a complex manifold X to the infinite dimensional projective space $\mathbb{P}^\infty$. Let F be a map from X to $\mathbb{P}^\infty := \{[z_1, \cdots, z_m, \cdots] : (z_1, \cdots, z_m, \cdots) \in \ell^2\}$. We say F is holomorphic if for any $p \in X$, there is a local holomorphic coordinate chart (U, z) with $p \in U$ and a set of holomorphic functions $\{f_j\}_{j=1}^\infty$ on U such that

1. The set of functions $\{f_j\}_{j=1}^\infty$ is base point free, i.e., they have no common zeros in U.
2. The infinite sum $\sum_{j=1}^\infty |f_j|^2$ converges uniformly on every compact subsets of U.
3. $F = [f_1, \cdots, f_n, \cdots]$ on U.

If X is equipped with a Kähler metric ω, we further say F is isometric if

$$\omega = \sqrt{-1}\partial\overline{\partial} \log(\sum_{j=1}^\infty |f_j(z)|^2)$$

on U. Let Ω be as above. Always assume that $A^2(\Omega)$ is base-point free over Ω_{reg}. Then it is equipped with the Bergman metric ω^B and let $\{\alpha_j\}_{j=1}^m$ be an orthonormal basis of $A^2(\Omega)$. Then it induces a natural holomorphic map F from $\mathrm{Reg}(\Omega)$ to $\mathbb{P}^\infty$ given by

$$F = [\alpha_1, \cdots, \alpha_j, \cdots].$$

Here if m is finite, we then add zero components to make the target to be $\mathbb{P}^\infty$. The right hand side of the above equation is understood as follows. In a local holomorphic coordinates chart $(U, z) : [\alpha_1, \cdots, \alpha_j, \cdots] = [a_1(z), \cdots, a_j(z), \cdots]$ if $\alpha_j = a_j dz_1 \wedge \cdots \wedge dz_n$. Note this definition is independent of the choices of coordinates. Indeed, if $\alpha_j = b_j dw_1 \wedge \cdots \wedge dw_n$ in another coordinates chart (V, w), then on $U \cap V$ every b_j just differs from a_j by the same non-zero factor, i.e., the Jacobian of the change of coordinates. We also remark that $\{\alpha_j\}_{j=1}^m$ must be base point free by the fact that K_Ω is nowhere zero. We will call F the Bergman-Bochner map from $\mathrm{Reg}(\Omega)$ to $\mathbb{P}^\infty$ and denote it by $\mathcal{B}_\Omega$.

We are now in a position to state our main theorem:

Theorem 1.3. *Let Ω be a normal Stein space with $A^2(\Omega)$ base-point free over Ω_{reg}. Assume its Bergman metric is complete at infinity and the induced Bergman-Bochner map is one-to-one over Ω_{reg}. Then Ω is biholomorphic to the unit ball $\mathbb{B}^n$ if and only if $(\Omega_{\mathrm{Reg}}, \omega^B)$ has constant holomorphic sectional curvature.*

Here we say the metric is complete at infinity if for every real number $R > 0$, the ball centered at $p_0 \in \mathrm{Reg}(\Omega)$ with radius R (with respect to the metric) has compact closure in Ω. To finish off the introduction, we formulate the following generalized Cheng's conjecture, which is affirmatively answered by Theorem 1.3 when the holomorphic sectional curvature is constant:

Conjecture 1.4. *Let Ω be a normal Stein space with a smoothly compact strongly pseudoconvex boundary. Then the Bergman metric of Ω_{reg} has constant Ricci curvature if and only if Ω is biholomorphic to the unit ball in a complex Euclidean space.*

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2 Proof of Theorem 1.3

We give a proof of Theorem 1.3 in this section. The *only-if* part is trivial and we only prove the converse statement. Assume $(\mathrm{Reg}(\Omega), \omega^B)$ has constant holomorphic sectional curvature λ. *A Priori* λ can be of any sign or zero. Write (X_0, ω_{st}) for the space form where the metric is normalized so that the holomorphic sectional curvature equals λ. More precisely,

- If $\lambda < 0$, then we let $X_0 = \mathbb{B}^n$ and ω_{st} the (suitably normalized) Poincaré metric;
- If $\lambda = 0$, then we let $X_0 = \mathbb{C}^n$ and ω_{st} the standard Euclidean metric;
- If $\lambda > 0$, then we let $X_0 = \mathbb{P}^n$ and ω_{st} the (suitably normalized) Fubini-Study metric.

Then by assumption, there is a connected open subset U of $\mathrm{Reg}(\Omega)$ and a holomorphic isometric map $f : (U, \omega^B) \to (X_0, \omega_{st})$. By a classical theorem of Calabi [Ca], f extends holomorphically along any path $\gamma \in \mathrm{Reg}(\Omega)$ with $\gamma(0) \in U$. In other words, f extends to a possibly multi-valued map F from $\mathrm{Reg}(\Omega)$ to M_0. We will prove F must be indeed single-valued.

We will need the following two lemmas. Consider the case $\lambda > 0$. Then X_0 is the projective space $\mathbb{P}^n$ and ω_{st} is the normalized Fubini-Study metric. That is, writing $[\eta] = [\eta_0, \eta_1, \cdots, \eta_n]$ for the homogeneous coordinates of $\mathbb{P}^n$, ω_{st} is given by $\omega_{st} = \mu\sqrt{-1}\partial\overline{\partial}\log\|\eta\|^2$ for some constant $\mu > 0$.

Lemma 2.1. *In the case $\lambda > 0$ as above, we must have μ equal to some positive integer m.*

Proof of Lemma 2.1: By shrinking U, we assume U is contained in some holomorphic coordinate chart. And by a holomorphic change of coordinates, we assume $z(p) = 0$ for some $p \in U$. Recall $K(z, \overline{z}) = k(z, \overline{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}$ on U. Without loss of generality, assume $f(0) = [1, 0, \cdots, 0]$. Shrinking U further if necessary, assume $f(U)$ is contained in the Euclidean cell $V_0 = \{[\eta_0, \cdots, \eta_n] : \eta_0 \neq 0\}$ of $\mathbb{P}^n$. Write g for the inverse of f defined from a small neighborhood V of $f(0)$ to U. Note g is also isometric: $g^*(\omega^B) = \omega_0$. This yields if we write $\xi = (\xi_1, \cdots, \xi_n)$ for the inhomogenous coordinates of $\mathbb{P}^n$ on V_0, then

$$\mu\partial\overline{\partial}\log(1 + \|\xi\|^2) = \partial\overline{\partial}\log k(g(\xi), \overline{g(\xi)}) \text{ on } V.$$

Thus the difference of $\log(1 + \|\xi\|^2)^\mu$ and $\log k(g(\xi), \overline{g(\xi)})$ equals to a pluriharmonic function ψ. Shrinking V if necessary, assume $\psi = h + \overline{h}$ for a holomorphic function h on V. We thus conclude

$$k(g(\xi), \overline{g(\xi)})|e^{h(\xi)}|^2 = (1 + \|\xi\|^2)^\mu.$$

We then use the fact that $k(z, \overline{z}) = \sum_{j=1}^m |a_j(z)|^2$ and apply the Taylor expansion of $(1 + x)^\mu$ on $\{x \in \mathbb{R} : |x| < 1\}$ to get

$$\sum_{j=1}^\infty |e^{h(\xi)}(a_j \circ g)(\xi)|^2 = 1 + \sum_{k=1}^\infty \frac{\mu(\mu-1)\cdots(\mu-k+1)}{k!}\|\xi\|^{2k}$$

Fix $l \geq 1$. Apply $\frac{\partial^{2l}}{\partial\xi_1^l\partial\overline{\xi}_1^l}$ to both sides of the equation and evaluate at $\xi = 0$. The left hand side always gives a nonnegative number. But if μ is non-integer, the right hand side will give a negative number for some sufficiently large l, a plain contradiction. Hence μ must be a positive integer. ∎

Lemma 2.2. *The normalized space form (X_0, ω_{st}) can be isometrically embedded into $\mathbb{P}^\infty$ by a holomorphic map $\mathcal{B}$.*

Proof of Lemma 2.2: We start with the case when $\lambda > 0$. By Lemma 2.1, we have in this case $X_0 = \mathbb{P}^n$ and $\omega_{st} = m\partial\overline{\partial}\log\|\eta\|^2$ for some positive integer m. We know (M_0, ω_{st}) can be holomorphically isomerically embedded into $(\mathbb{P}^N, \omega_{FS})$ by the m^{th} Veronese embedding for some appropriate N. Here ω_{FS} is the standard Fubini-Study metric on $\mathbb{P}^N$. Furthermore, $(\mathbb{P}^N, \omega_{FS})$ can be canonically embedded into $\mathbb{P}^\infty$ (by adding zero components). This finishes the proof of Lemma 2.2 in the positive λ case.

We next consider the case $\lambda < 0$. In this case, $X_0 = \mathbb{B}^n$ and $\omega_{st} = -\mu\partial\overline{\partial}\log(1 - \|w\|^2)$ for some $\mu > 0$, where $w = (w_1, \cdots, w_n)$ is the coordinates of $\mathbb{B}^n \subset \mathbb{C}^n$. We use the Taylor expansion of $(1-x)^{-\mu}$ about the point $x = 0$ on $\{x \in \mathbb{R} : |x| < 1\}$ to obtain

$$(1 - \|w\|^2)^{-\mu} = 1 + \sum_{k=1}^{\infty} \frac{\mu(\mu+1)\cdots(\mu+k-1)}{k!}\|w\|^{2k}.$$

The right hand side converges uniformly on compact subsets of $\mathbb{B}^n$. This shows that for a certain sequence of monomials $\{P_j\}_{j=1}^\infty$ in w such that $(1 - \|w\|^2)^{-\mu} = 1 + \sum_{j=1}^\infty |P_j(w)|^2$, which converges uniformly on compact subsets of $\mathbb{B}^n$. This leads to a natural isometric map $\mathcal{B} = [1, P_1, \cdots, P_j, \cdots]$ from (X_0, ω_{st}) to $\mathbb{P}^\infty$. It is clear that F is an embedding as the w_j's, $1 \le j \le n$, (with appropriate coefficients) are among the P_j's.

The proof for the case $\lambda = 0$ is similar. Note $\omega_{st} = \partial\overline{\partial}\|w\|^2 = \partial\overline{\partial}\log(e^{\|w\|^2})$ where w is the coordinates of $X_0 = \mathbb{C}^n$, and we use the Taylor expansion of $e^{\|w\|^2}$ at $w = 0$:

$$e^{\|w\|^2} = 1 + \sum_{k=1}^{\infty} \frac{\|w\|^{2k}}{k!}. \tag{1}$$

It converges uniformly on compact subsets of $\mathbb{C}^n$. As in the case $\lambda < 0$, we obtain from (1) an isometric embedding from X_0 to $\mathbb{P}^\infty$. This establishes Lemma 2.2. ■

Now $U \subset \text{Reg}(\Omega)$ can be isometrically embedded into $\mathbb{P}^\infty$ in two ways: (a). $\mathcal{B}_\Omega$ embeds (U, ω^B) isometrically into $\mathbb{P}^\infty$, and (b). (U, ω^B) is isometrically embedded into $\mathbb{P}^\infty$ by $\mathcal{B} \circ f$. By Calabi's theorem [Ca], there is a rigid motion of T of $\mathbb{P}^\infty$ such that

$$T \circ \mathcal{B}_\Omega = \mathcal{B} \circ f \text{ in } U. \tag{2}$$

As mentined above, (f, U) can be extended holomorphically along any path by [Ca]. Let γ_1, γ_2 be two curves connecting $p \in U$ to some point $q \in \text{Reg}(\Omega)$. Write (f_1, V) and (f_2, V) for the two holomorphic branches obtained from holomorphic continuaion of (f, U) along the two curves. By the uniqueness of holomorphic continuation, we know (2) is preserved along the continuation. Thus we have

$$T \circ \mathcal{B}_\Omega = \mathcal{B} \circ f_1 = \mathcal{B} \circ f_2 \text{ in } V.$$

However, since $\mathcal{B}_\Omega, \mathcal{B}$ and T are all embeddings, we have that $f_1 = f_2$ in V. Hence (f, U) extends a well-defined holomorphic map F on $\mathrm{Reg}(\Omega)$, which is local isometric. By (2) again, we have $T \circ \mathcal{B}_\Omega = \mathcal{B} \circ F$ on $\mathrm{Reg}(\Omega)$. This yields that F is an embedding map.

We next prove that the singular set $\mathrm{sing}(\Omega)$ of Ω is empty. Seeking a contradiction, suppose not. Let $p \in \mathrm{sing}(\Omega)$. First note we can assume (Ω, p) is embedded into some $\mathbb{C}^K$ with $z(p) = 0$. For a sufficiently small ϵ, consider the link $N_\epsilon = \{z \in \mathbb{C}^K : \|z\| = \epsilon\} \cap \Omega$ at p and $D_\epsilon = \{z \in \mathbb{C}^K : \|z\| < \epsilon\} \cap \Omega$. Note F is a holomorphic map in $D_\epsilon - \{p\}$ and extends holomorphically across N_ϵ. Recall F is an embedding. Then $\widetilde{N} = F(N_\epsilon)$ is a (connected) closed strongly pseudoconvex hypersurface in X_0 (See Milnor [Mi]) and $F(D_\epsilon - \{p\})$ is contained in some pseudoconvex domain $\widetilde{D}$ in X_0 which has smooth boundary $\widetilde{N}$. We remark that when $X_0 = \mathbb{P}^n$, $\widetilde{D}$ is defined to be the set of points in $\mathbb{P}^n \setminus \widetilde{N}$ that can be path-connected to a small analytic disk attached to $\widetilde{N}$. Clearly $\widetilde{D}$ defined in this way is connected and open. To show $\widetilde{D}$ has $\widetilde{N}$ as its smooth boundary, we observe any point q that is on the pseudoconcave side of $\widetilde{N}$ and close to $\widetilde{N}$ cannot be contained in $\widetilde{D}$. Indeed, suppose $q \in \widetilde{D}$. Then we find a point q' on the pseudoconvex side to $\widetilde{N}$ and close to q. Moreover, pick a short smooth curve γ_1 from q and q' that cuts $\widetilde{N}$ transversally. On the other hand, there is a curve γ_2 in $\widetilde{D}$ from q' to q. Set $\gamma = \gamma_1 + \gamma_2$. By the construction of γ, the intersection number of γ and $\widetilde{N}$ is ± 1. Since intersection number is a homotopic invariant, by the simple connectedness of X_0, this intersection number must be zero which is a plain contradiction. This proves $\widetilde{D}$ is a pseudoconvex domain with smooth boundary $\widetilde{N}$.

Since p is normal, note we can assume F is the restriction of some holomorphic map $\hat{F}$ on $\{z \in \mathbb{C}^K : \|z\| < \epsilon\}$ to D_ϵ. This follows from the property of normal singularities if X_0 is $\mathbb{B}^n$ or $\mathbb{C}^n$, and needs some justification when X_0 is $\mathbb{P}^n$. In this case, we use a theorem of Takeuchi [T] to see $\widetilde{D}$ is Stein, which can be embedded as a submanifold of some $\mathbb{C}^{K'}$. Hence F can be regarded as a map to $\mathbb{C}^{K'}$, and again by the property of normal singularities, F extends continuously across p. Furthermore, we can shrink ϵ to make $\widetilde{D}$ be contained in one Euclidean cell of $\mathbb{P}^n$ and the observation follows readily.

As above by shrinking ϵ if necessary, we can assume the image $F(D_\epsilon)$ is contained in some coordinate chart (U, w) of X_0. Now consider the inverse of $F|_{N_\epsilon}$, which we denote by $H : \widetilde{N} \to N_\epsilon \subset \mathbb{C}^n$. By the Hartogs extension, H extends to a holomorphic function in $\widetilde{D}$, which we still denote by H. By the uniqueness of holomorphic maps and by the maximum principle, $H(\widetilde{D})$ is contained in D_ϵ. Furthermore, we have $\hat{F} \circ H$ equals the identity map on $\overline{\widetilde{D}}$. (In particular, H is embedding). On the other hand, we also have $H \circ F$ equals identity on $\overline{D_\epsilon} - \{p\}$ and by continuity, it also equals identity on $\overline{D_\epsilon}$. This implies H is a biholomorphic map from $\widetilde{D}$ to D_ϵ. Hence we prove $\mathrm{sing}(\Omega)$ is empty.

Once we know Ω has no singularities and thus it is a complete Kähler manifold equipped with the Bergman metric, a standard argument will show that Ω must be biholomorphic to the ball. Indeed, we now have a holomorphic isometric immersion F from Ω to X_0. On the other hand, we consider the local inverse (g, V) of (f, U), where $V = f(U)$. It is a local isometric embedding from $f(U)$ to Ω. Since now (Ω, ω^B) is complete, $(g, f(U))$ extends holomorphically along any path in X_0. (See Proposition 11.3, 11.4 in [He]). As X_0 is simply connected, we thus obtain a holomorphic map G from X_0 to Ω. Note $f \circ g$ equals the identity map on V and $g \circ f$ equals the identity map on U. By the uniqueness of holomorphic functions, $F \circ G$ and $G \circ F$ equal to the identity map on X_0 and Ω, respectively. We thus conclude Ω is biholomorphic to X_0. Finally, it follows from the assumption on Ω and ω^B that X_0 can only be the complex unit ball. Furthermore, as the holomorphic sectional curvature of the Bergman metric is a biholomorphic invariant, we have $\lambda = -\frac{2}{n+1}$. ■

References

[Ca] E. Calabi, Isomeric imbedding of complex manifolds, *Ann. of Math.*, 58 (1953), 1-23.

[Ch] S.-Y. Cheng, Open Problems, Conference on Nonlinear Problems in Geometry held in Katata, Sep. 3-8, 1979, p.2, Tohoku University, Department of Math., Sendai, 1979.

[CY] S.-Y. Cheng and S.-T. Yau, on the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation, Comm. Pure Appl. Math. 33 (1980), 507-544.

[Fe1] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math. 26 (1974), 1-65.

[Fe2] C. Fefferman, Monge-Ampère equations, the Bergman kernel, and the geometry of pseudoconvex domains, *Ann. of Math.* 103 (1976), 395-416.

[Fe3] C. Fefferman, Parabolic invariants theory in complex analysis, Adv. in Math. 31 (1979), 131-262.

[FW] S. Fu and B. Wong, On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics, *Math. Res. Lett.* 4 (1997), 697-703.

[He] S. Helgason, Differential geometry and symmetric spaces, Pure and Applied Math., Vol. 12, Academic Press, New York, 1962.

[HX] X. Huang and M. Xiao, Bergman-Eistein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries, submitted, arxiv: 1604. 07065.

[Ko] S. Kobayashi, Geometry of bounded domains, Trans. Am. Math. Soc. 92 (1959), 267-290.

[L1] Song-Ying Li, Characterization for Balls by Potential Function of Kähler-Einstein Metrics for domains in $\mathbb{C}^n$, *Comm. in Anal. and Geom.,* 13(2005), 461–478.

[L2] Song-Ying Li, Characterization for a class of pseudoconvex domains whose boundaries having positive constant pseudo scalar curvature, *Comm. in Anal. and Geom.*, 17(2009), 17–35.

[L3] Song-Ying Li, On plurisubharmonicity of the solution of the Fefferman equation and its applications to estimate the bottom of the spectrum of Laplace-Beltrami operators, Bull. Math. Sci. 6(2016), no 2, 287–309.

[Lu] Q.-K. Lu, On Kähler manifolds with constant curvature, *Acta Math. Sinica* 16 (1966), 269-281 (Chinese). (transl. in Chinese Math.-Acta 8 (1966), 283-298.)

[Mi] J. Milnor, Singular Points of Complex Hypersurfaces. (AM-61) Annals of Mathematics Studies, vol. 61, Princeton Univ. Press, 1968.

[Mo1] Mok, N.: Metric rigidity theorems on Hermitian locally symmetric manifolds, Series in Pure Mathematics. 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989. xiv+278 pp.

[NS] S. Nemirovski and R. Shafikov, Conjectures of Cheng and Ramadanov, *Russian Mathematical Surveys*, 61 (2006), 780.

[T] Takeuchi, A. , Domains pseudoconvexes infinis et la métrique riemannienne dans un espace projectif, J. Math. Soc. Japan 16 (1964) 159-181.

[Y] S.-T. Yau, Problem Section, Seminar on Differential geomtry (S.-T. Yau, eds.), Ann. of Math. Studies, vol. 102, Princeton Univ. Press, 1982, pp. 669-706.

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