Computational Methods in Environmental and Resource Economics

Yongyang Cai*

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Abstract

Computational methods are required to solve problems without closed-form solutions in environmental and resource economics. Efficiency, stability, and accuracy are key elements for computational methods. This review discusses state-of-the-art computational methods applied in environmental and resource economics, including optimal control methods for deterministic models, advances in value function iteration and time iteration for general dynamic stochastic problems, nonlinear certainty equivalent approximation, robust decision making, real option analysis, bilevel optimization, solution methods for continuous-time problems, and so on. This review also clarifies so-called "curse-of-dimensionality", and discusses some computational techniques such as approximation methods without "curse-of-dimensionality" and time-dependent approximation domains. Many existing economic models use simplifying and/or unrealistic assumptions with an excuse of computational feasibility, but these assumptions might be able to be relaxed if we choose an efficient computational method discussed in this review.

*Department of Agricultural, Environmental and Development Economics, The Ohio State University, 2120 Fyffe Road, Columbus, OH, USA 43210. cai.619@osu.edu.

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1 Introduction

In environmental and resource economics, problems often have no closed-form solutions, so we have to rely on computational methods to solve them and then provide economic analysis. While analytic results can be helpful in building intuition, they have to make many strong and simplifying assumptions, so sometimes their closed-form solutions may result in economic analysis that is more confusing than clarifying.

This review focuses on computational methods, particularly general and recent methods, more than substantive modeling and applications, although I provide some references to applied research that focuses on environmental and resource economics. The review also focuses on solution methods for discrete time dynamic stochastic problems, particularly new advances and clarifications in the most popular method – value/policy function iteration. I also review other methods including NLCEQ (NonLinear Certainty Equivalent approximation) (Cai, Judd, and Steinbuks, 2017), robust decision making, and bilevel optimization. For general background, traditional methods, and standard rules in computational methods for economists, see Rust (1996), Judd (1998), Ljungqvist and Sargent (2000), Miranda and Fackler (2002), Bertsekas (2005, 2007), and Cai and Judd (2014) for details. This review will also not discuss agent-based models (see Farmer et al. (2015) for a detailed discussion about agent-based models in climate change economics), nor econometric and statistical methods.[1]

The review is organized as follows. Section 2 discusses optimal control methods for deterministic models, including solving a social planner's problem and finding equilibrium under no uncertainty. Section 3 presents value function iteration, the most popular method for solving discrete time dynamic stochastic programming problems under a social planner's preference. Section

[1] In econometrics, there are many computational issues that may lead to even opposite solutions. For example, Cafiero et al. (2011, 2015) show that using a much finer grid to approximate the equilibrium price function leads to positive evidence for the role of storage arbitrage, contrary to a previous claim of Deaton and Laroque (1995, 1996) using a coarse grid. Guerra et al. (2015) show serious differences in magnitudes of practical interest between using annual price data and using December price data for testing a storage model.

4 presents time iteration, another popular method for finding dynamic general equilibrium of discrete time dynamic stochastic programming problems. While both value function iteration and time iteration are for general dynamic programming problems which may have a finite or infinite time horizon and may be non-stationary, many economic problems study infinite time horizon stationary problems, which Section 5 specifically discusses. Section 6 briefly reviews computational methods for robust decision making problems. Section 7 discusses other computational methods including NLCEQ, approximate dynamic programming, real options pricing, and bilevel optimization for solving principal-agent problems. Section 8 briefly reviews computational methods for continuous time dynamic programming problems. Section 9 provides detailed discussions about the "curse-of-dimensionality", boundedness, Monte Carlo techniques, approximation, and stopping criteria. Section 10 concludes.

2 Optimal Control Methods for Deterministic Models

2.1 Social Planner's Problem

Most of deterministic discrete time dynamic programming (DP) problems in environmental and resource economics can be written as

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} \beta^t u_t(\mathbf{x}_t, \mathbf{a}_t) + \beta^T V_T(\mathbf{x}_T) \qquad (1) \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t),\ t = 0, 1, ..., T-1 \\ & \mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t) \\ & \mathbf{x}_0 \text{ given} \end{aligned}$$

where t is time (period), T is the time horizon (which can be infinite), $\mathbf{x}_t$ is a vector of state variables (e.g., capital or resource stock), $\mathbf{a}_t$ is a vector of decision variables (e.g., consumption), $\mathbf{f}_t$ is a vector of functions representing transition laws of the state variables, $\beta < 1$ is the discount factor, u_t is the social planner's utility function, V_T is the terminal value function (for infinite-horizon problems, V_T is zero everywhere), and $\mathcal{D}_t(\mathbf{x}_t)$ is the feasible set for decision variables at time t.

The optimal control method is the most common method to solve the deterministic problem (1) with a finite horizon. That is, if the functions and variables are continuous, then we view (1) as a non-linear programming (NLP) problem with variables $(\mathbf{x}_1, ..., \mathbf{x}_T, \mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_{T-1})$ and constraints $\mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t)$ and $\mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t)$ for $t = 0, 1, ..., T-1$, and then use an NLP solver to solve it directly;[2] if the functions and variables are discrete, then we can use integer programming to solve (1); and if some but not all of the functions and variables are discrete, then we can use mixed integer nonlinearly constrained optimization (MINLP) solvers to solve (1). In MATLAB, we can use the fmincon solver (1), and in cases where fmincon does not work well, we use an alternative solver such as Knitro. However, GAMS (General Algebraic Modeling System) or AMPL (A Mathematical Programming Language) can provide a more flexible environment than MATLAB, as there are more professional solvers available using GAMS or AMPL (see McCarl et al. (2016) or Fourer, Gay, and Kernighan (2003) for their user guide). For example, CONOPT (Drud, 1994) is often more reliable and efficient than fmincon in solving NLP problems, and SNOPT (Gill, Murray, and Saunders, 2005) is another good alternative. The NEOS server (Czyzyk, Mesnier, and More (1998); `https://neos-server.org/neos/solvers/index.html`) provides a long list of free solvers for various optimization problems (e.g., NLP, MINLP, global optimization) that can run GAMS or AMPL code. The flexibility of GAMS or AMPL is more useful for dealing with challenging optimization problems such as global optimization or problems with a flat objective over some decision variables, as we can try different solvers with the same code written in GAMS or AMPL to solve (1) or even verify accuracy of a solution of (1) obtained from another solver. A good initial guess, scaling, and stopping criteria are also important for solving challenging problems.

[2] Linear programming (LP) is a special case of NLP, and usually is easier to solve with LP solvers such as CPLEX and Gurobi, so we do not discuss LP in this paper.

For deterministic problems (1) with an infinite horizon, usually one can also use the optimal control method after truncating the infinite series at a large finite period T, because $\beta^t \to 0$ as $t \to \infty$ and infinite horizon problems often require a transversality condition. In fact, if the utility function is a power function with a marginal elasticity larger than 1, then it is upper-bounded at zero, so $\beta^t u_t(\mathbf{x}_t, \mathbf{a}_t) \to 0$.

2.2 Find Equilibrium

While DP problems usually focus on intertemporal equilibrium and long-term effects (often with hundreds of periods), another direction is to study short-term effects and spatial or sectoral equilibrium, which typically includes many regions or sectors (up to hundreds of sectors). The Global Trade Analysis Project (GTAP) founded by Thomas Hertel at Purdue University is one representative example (see an overview of GTAP in Hertel (2013)), and many other computable general equilibrium (CGE) models are presented in Dixon and Jorgenson (2013).[3] GTAP applies on the GEMPACK (General Equilibrium Modeling PACKage) platform, which is compared with GAMS and MPSGE (Mathematical Programming System for General Equilibrium (Rutherford, 1999)) in Horridge et al. (2013). GTAP is also combined with other models in the literature. For example, Golub et al. (2009) extend GTAP to GTAP-AEZ-GHG, a general equilibrium framework, to model forest carbon sequestration and land management in agriculture and forestry, and Golub et al. (2013) extend GTAP-AEZ-GHG to study climate policy impacts.

For multi-regional, multi-sectoral and/or multi-agent static problems, each region, sector, or agent is assumed to optimize their objectives with budget and market-clearing constraints and trade between regions, sectors, and/or agents. The decentralized equilibrium can be solved using a system of first-order conditions (and constraints). That is, the problem is to find a solution

[3] Also see Bergman (2005) for CGE modeling in environmental policy and resource management.

to the system of equations and inequalities

$$\begin{cases} \mathbf{F}(\mathbf{a}) = 0 \\ \mathbf{G}(\mathbf{a}) \geq 0 \end{cases} \tag{2}$$

where $\mathbf{a}$ contains prices, quantities of (intermediate) products, and resource allocations. This may be solved in MATLAB using fsolve or other equation solvers. The system (2) may also be solved as a degenerate optimization problem

$$\begin{aligned} \max_{\mathbf{x}} \quad & 1 \\ \text{s.t.} \quad & \mathbf{F}(\mathbf{a}) = 0 \\ & \mathbf{G}(\mathbf{a}) \geq 0 \end{aligned} \tag{3}$$

using an optimization solver such as CONOPT and SNOPT to find a feasible point using the optimal control method. Sometimes the system (2) may contain complementarity conditions (such as $x_i x_j = 0$ with $x_i, x_j \geq 0$), which often make it challenging to solve. The mixed complementarity problems may be solved using MILES (Rutherford, 1993, 1995) or PATH (Dirkse and Ferris, 1995; Ferris and Munson, 2000).

For deterministic dynamic problems, we can use the same computational methods to solve the following system of equations and inequalities:

$$\begin{cases} \mathbf{F}_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{a}_t, \mathbf{a}_{t+1}) = 0 & \forall t \\ \mathbf{G}_t(\mathbf{x}_t, \mathbf{a}_t) \geq 0 & \forall t \end{cases} \tag{4}$$

where $\mathbf{x}_t$ are state variables, $\mathbf{a}_t$ are other variables including decision variables and prices, $\mathbf{F}_t$ represents transition laws of states, Euler equations, first-order conditions and other equality constraints, and $\mathbf{G}_t$ represents all inequality constraints. For example, Baldwin, Cai, and Kuralbayeva (2018) use the degenerate optimization method (3) but with the constraints (4) to obtain decentralized equilibrium when there is no carbon tax in the dirty energy sector or a subsidy in the renewable energy sector. One disadvantage of this method is that we have to derive the Euler equations and first-order conditions, and the system (4) may be challenging to solve. Cai et al. (2018) propose an alternative method to find competitive equilibrium between two regions by solving a social planner's problem that maximizes the present value of a weighted sum of utilities in the two regions, where the weights are Negishi weights (Negishi, 1972) that can be obtained with an iterative method.

3 Value Function Iteration

Most stochastic discrete time dynamic programming problems in environmental and resource economics can be written as

$$\begin{aligned} \max \quad & \mathbb{E}\left\{\sum_{t=0}^{T-1} \beta^t u_t(\mathbf{x}_t, \mathbf{a}_t) + \beta^T V_T(\mathbf{x}_T)\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t, \epsilon_t),\ t = 0, 1, ..., T-1 \\ & \mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t) \\ & \mathbf{x}_0 \text{ given} \end{aligned} \tag{5}$$

where $\mathbb{E}$ is the expectation operator and ϵ_t is a vector of random variables at time t. Note that some state variables may have deterministic transition laws, e.g., the transition law of the j-th state variable is $x_{t+1,j} = g_{t,j}(\mathbf{x}_t, \mathbf{a}_t)$, but here we simplify our notation by defining $f_{t,j}(\mathbf{x}_t, \mathbf{a}_t, \epsilon_t) = g_{t,j}(\mathbf{x}_t, \mathbf{a}_t) + 0 \cdot \epsilon_t$ so that $x_{t+1,j} = f_{t,j}(\mathbf{x}_t, \mathbf{a}_t, \epsilon_t)$. For simplicity in discussion of computational methods, we assume $\mathbf{x}_t$ are continuous variables (as discrete state variables can be simply added).

Value function iteration (VFI) is the most common method to solve (5). It transforms (5) to the following Bellman equation (Bellman, 1957):

$$\begin{aligned} V_t(\mathbf{x}_t) = \max_{\mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t)} \quad & u_t(\mathbf{x}_t, \mathbf{a}_t) + \beta \mathbb{E}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t, \epsilon_t) \end{aligned} \tag{6}$$

for $t = 0, 1, ..., T - 1$, where $\mathbb{E}_t$ is the expectation operator conditional on the time-t information. With a given terminal value function V_T for finite-horizon problems, it iterates backward over time to get all value functions and policy functions. For infinite-horizon problems, we can truncate it at a finite horizon T and choose

$$V_T(\mathbf{x}_T) \approx \mathbb{E}_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} u_t(\widetilde{\mathbf{x}}_t, \widetilde{\mathbf{a}}_t) \right\}$$

where $\widetilde{\mathbf{a}}_t$ and $\widetilde{\mathbf{x}}_t$ are a series of guessed decisions and states starting from the terminal state $\mathbf{x}_T$. A criterion in setting a terminal value function V_T is to check if a reasonable change in its terminal values (e.g., 10% up or down) will result in a non-negligible change in the solutions at the periods of interest.

Each iteration of (6) contains three main parts: approximation, optimization, and integration. These parts are computed numerically for problems with continuous state variables, continuous random variables, and continuous decision variables. That is, with a given next-period value function approximation $\widehat{V}_{t+1}(\mathbf{x}_{t+1}; \mathbf{b}_{t+1})$, numerical VFI constructs the current-period value function $\widehat{V}_t(\mathbf{x}_t; \mathbf{b}_t)$ by solving

$$\begin{aligned} \widehat{V}_t(\mathbf{x}_t; \mathbf{b}_t) \approx \widehat{\max_{\mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t)}} \quad & u_t(\mathbf{x}_t, \mathbf{a}_t) + \beta \widehat{\mathbb{E}}_t \left\{ \widehat{V}_{t+1}(\mathbf{x}_{t+1}; \mathbf{b}_{t+1}) \right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t, \epsilon_t) \end{aligned} \tag{7}$$

where hatted variables refer to the numerical implementation of approximation, optimization, and integration, and $\mathbf{b}$ is a vector of approximation coefficients. A typical numerical approximation of a value function V is

$$\widehat{V}(\mathbf{x}; \mathbf{b}) = \sum_j b_j \phi_j(\mathbf{x}), \tag{8}$$

where $\{\phi_j(\mathbf{x})\}$ are basis functions (e.g., Chebyshev basis polynomials discussed later in Section 9.4, or ordinary basis polynomials: 1, x, x^2, x^3, ..., for univariate problems) and $\mathbf{b} = \{b_j\}$ are approximation coefficients (see Judd (1998) and Miranda and Fackler (2002) for a detailed discussion). Moreover, it is often associated with a bounded approximation domain in the state space, which could be time-variant. To solve (7), we choose approximation nodes $\{\mathbf{x}_{t,i}\}$, solve the following maximization problem

$$\begin{aligned} v_{t,i} = \widehat{\max_{\mathbf{a}_{t,i}\in\mathcal{D}_t(\mathbf{x}_{t,i})}} \quad & u_t(\mathbf{x}_{t,i},\mathbf{a}_{t,i}) + \beta\widehat{\mathbb{E}}_t\left\{\widehat{V}_{t+1}(\mathbf{x}_{t+1,i};\mathbf{b}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1,i} = \mathbf{f}_t(\mathbf{x}_{t,i},\mathbf{a}_{t,i},\epsilon_t) \end{aligned} \tag{9}$$

for every $\mathbf{x}_{t,i}$, and then find $\mathbf{b}_t$ such that $\widehat{V}_t(\mathbf{x}_{t,i};\mathbf{b}_t) \approx v_{t,i}$ for all i. Numerical optimization solvers include CONOPT, SNOPT, NPSOL (Gill et al., 1994), KNITRO (Byrd, Nocedal, and Waltz, 2006), or fmincon in MATLAB. Numerical integration methods include Gaussian quadrature rules (e.g., Gauss-Hermite quadrature for normal or log-normal distributions and Gauss-Legendre quadrature for uniform distributions), and monomial quadrature rules. See Judd (1998) for a detailed discussion of these.

The Bellman equation (6) applies widely to problems with separate utility functions. There are a large number of applications in the literature, here I provide only several recent examples in environmental and resource economics. Daigneault, Miranda, and Sohngen (2010) use the Bellman equation to find optimal forest management with fire risk and carbon sequestration credits. Cai et al. (2015a) and Lontzek et al. (2015) apply VFI to solve optimal carbon tax with an integrated assessment model with climate tipping risks.

Recently, recursive utility (Epstein and Zin, 1989) has been used in DP. For example, Jensen and Traeger (2014), Cai, Lenton, and Lontzek (2016); Cai, Judd, and Lontzek (2017) and Cai et al. (2018) employ recursive utility in climate change economics for solving dynamic stochastic integrated assessment models (IAMs). They still apply VFI but with the more general Bellman equation

$$\begin{aligned} V_t(\mathbf{x}_t) = \max_{\mathbf{a}_t\in\mathcal{D}_t(\mathbf{x}_t)} \quad & u_t(\mathbf{x}_t,\mathbf{a}_t) + \beta\mathcal{G}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t,\mathbf{a}_t,\epsilon_t) \end{aligned} \tag{10}$$

where

$$\mathcal{G}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \equiv \frac{1}{1-1/\psi}\left(\mathbb{E}_t\left\{\left((1-1/\psi)V_{t+1}(\mathbf{x}_{t+1})\right)^{\frac{1-\gamma}{1-1/\psi}}\right\}\right)^{\frac{1-1/\psi}{1-\gamma}}$$

with ψ and γ as the intertemporal elasticity of substitution and the risk aversion coefficient respectively. Without loss of generality, we will use the Bellman equation (6) for later discussion.

VFI is also used to solve problems with learning. For example, Kelly and Kolstad (1999) employ VFI with neural network approximation to solve climate change economics problems with Bayesian learning. Leach (2007) then extends their pioneering work to the case of learning about two correlated uncertain parameters. Recently, Kelly and Tan (2015) investigate the impact of learning an important uncertain parameter, climate sensitivity (which measures the temperature increase in equilibrium if carbon concentration in the atmosphere doubles), on optimal climate policy under fat-tailed uncertainty about climate change. Rudik (2016) combines Bayesian learning and robust control in the context of optimal carbon taxation.

It is always important to check whether a computational method and code are actually solving problems with the desired accuracy. Cai, Judd, and Lontzek (2017, 2018) present how to verify and measure errors appropriately for numerical solutions from VFI. One way for verification is to use the same code and the same computational methods (including the same approximation domains, and nodes) for a stochastic problem to replicate a solution of its corresponding deterministic model, obtained by another programming language and/or another computational method (e.g., GAMS and the optimal control method). Another way is to check if higher-order approximations with wider approximation domains or higher-order quadrature rules will change results significantly. Normalized Euler equation errors (see e.g., Cai, Judd, and Steinbuks (2017)) and approximation errors are also important for measuring errors (Cai, Judd, and Lontzek, 2018).

4 Time Iteration

Time iteration, another backward iteration method, is also popular. It constructs policy functions with state variables as arguments, while their approximation domains are the same across decision variables. With next-period policy functions, we compute current-period policy functions by solving a system of intertemporal Euler equations and transition laws, and temporal first-order conditions and constraints. That is, at time t, with given next-period policy functions $\mathbf{A}_{t+1}(\mathbf{x}_{t+1})$, instead of solving the maximization problem (6) in VFI we solve the following system of equations and inequalities:

$$\begin{cases} \mathbf{E}_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{A}_t(\mathbf{x}_t), \mathbf{A}_{t+1}(\mathbf{x}_{t+1})) = 0 \\ \mathbf{F}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) = 0 \\ \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t), \epsilon_t) \\ \mathbf{G}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0 \end{cases} \tag{11}$$

where $\mathbf{x}_t$ are state variables, $\mathbf{A}_t(\mathbf{x}_t)$ are policy functions, $\mathbf{E}_t$ represents Euler equations, $\mathbf{F}_t$ represents first-order conditions, $\mathbf{f}_t$ represents transition laws of state variables, and $\mathbf{G}_t$ represents all other constraints (e.g., nonnegativity constraints and complementarity constraints).[4] To solve (11) numerically, we start with a given numerical approximation of next-period policy functions, $\widehat{\mathbf{A}}_{t+1}(\mathbf{x}_{t+1}; \mathbf{B}_{t+1})$, where $\mathbf{B}_{t+1}$ are approximation coefficients for decision variables, choose approximation nodes $\{\mathbf{x}_{t,i}\}$, solve

$$\begin{cases} \mathbf{E}_t(\mathbf{x}_{t,i}, \mathbf{x}_{t+1,i}, \mathbf{a}_{t,i}, \widehat{\mathbf{A}}_{t+1}(\mathbf{x}_{t+1,i}; \mathbf{B}_{t+1})) = 0 \\ \mathbf{F}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}) = 0 \\ \mathbf{x}_{t+1,i} = \mathbf{f}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}, \epsilon_t) \\ \mathbf{G}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}) \geq 0 \end{cases} \tag{12}$$

for every $\mathbf{x}_{t,i}$, and then find $\mathbf{B}_t$ such that $\widehat{\mathbf{A}}_t(\mathbf{x}_{t,i}; \mathbf{B}_t) \approx \mathbf{a}_{t,i}$ for all i.

[4] An equality constraint $g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) = 0$ can be represented as a combination of two inequality constraints: $g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0$ and $-g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0$.

For stationary infinite-horizon problems, time iteration is often faster than VFI. However, most economic problems have monotone and concave value functions, for such problems solving the system (12) may be more challenging than solving the maximization problem in (6) using VFI, as the first-order conditions and Euler equations may lose the convexity of the maximization problem in (6).

Time iteration has been widely used in the literature. For example, recently Judd et al. (2014) incorporate the Smolyak method (Smolyak, 1963) into time iteration, and Brumm and Scheidegger (2017) introduce an adaptive sparse grid method into time iteration, so that time iteration can solve large-dimensional problems.

In particular, time iteration is a typical method for non-stationary DSGE or dynamic stochastic game problems with multiple regions, sectors, and/or agents. That is, at each time, time iteration solves the system (11) to get the optimal resource allocation and investment among regions, sectors, and/or agents, prices, and quantities of goods, and then iterates backward until the initial time (for finite-horizon problems) or until convergence (for infinite-horizon stationary problems).

5 Methods for Stationary Infinite-horizon Problems

For stationary infinite-horizon dynamic problems (i.e., all functions and exogenous parameters are independent of time), the Bellman equation (6) becomes

$$
\begin{aligned}
V(\mathbf{x}) = \max_{\mathbf{a}\in\mathcal{D}(\mathbf{x})} \quad & u(\mathbf{x},\mathbf{a}) + \beta\mathbb{E}\left\{V(\mathbf{x}_+)\right\} \\
\text{s.t.} \quad & \mathbf{x}_+ = \mathbf{f}(\mathbf{x},\mathbf{a},\epsilon)
\end{aligned} \tag{13}
$$

where $\mathbf{x}_+$ is the next-period state transited from current state $\mathbf{x}$, and we can choose an initial guess for the value function V and then iterate until VFI converges.

Let $\widehat{V}_k(\mathbf{x}; \mathbf{b}_k)$ be the value function approximation at the k-th iteration. We assume VFI converges numerically if two consecutive value function approximations are sufficiently close, that is,

$$\left\| \widehat{V}_k - \widehat{V}_{k+1} \right\| < \varepsilon$$

for some functional norm $\|\cdot\|$ and a small positive number ε. The $\mathcal{L}^\infty$ norm is often used, i.e.,

$$\max_{\mathbf{x}} \left| \widehat{V}_k(\mathbf{x}; \mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x}; \mathbf{b}_{k+1}) \right| < \varepsilon.$$

Numerically, we can replace the above formula by

$$\max_{i} \left| \widehat{V}_k(\mathbf{x}_i; \mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x}_i; \mathbf{b}_{k+1}) \right| < \varepsilon$$

over a large-size set of points $\{\mathbf{x}_i\}$ on the state space. For example, Cai, Judd, and Lontzek (2017, 2018) use 1,000 Monte Carlo points in the state space of continuous state variables for every discrete state. We should also pay attention to the magnitude of $\widehat{V}$, as too large a magnitude will make it too challenging or time consuming to stop, and too small a magnitude will make VFI stop too early, with large errors. Thus, typically we first scale utility by a constant such that the value function has a reasonable magnitude. Alternatively we can use the difference of two consecutive policy functions as a substitute for the difference of two consecutive value functions. In addition, the value of the discount factor also matters, because a discount factor close to 1 implies a small time increment. Since one period's utility could then have little contribution to the objective function, which may make VFI stop too early, we often use $\varepsilon/(1-\beta)$ instead of ε.

However, for infinite-horizon stationary DSGE problems, perturbation methods (see e.g., Judd and Guu (1993)) and projection methods (see e.g., Judd (1992)) may be more efficient, though perturbation methods can only provide locally accurate solutions around the non-stochastic steady state, and projection methods may be challenging for high-dimensional problems or problems with strong nonlinearity.[5] Moreover, both perturbation and projection cannot solve problems with kinks in general, except the OccBin method (Guerrieri and Iacoviello, 2015), which can solve some low-dimensional problems with occasionally binding constraints. See Fernandez-Villaverde, Rubio-Ramirez, and Schorfheide (2016) for a detailed discussion about perturbation and projection. Recently, Levintal (2018) proposes an efficient Taylor projection method to solve DSGE models. The algorithm is a hybrid of perturbation and projection, and it can obtain a locally accurate solution around any point on the state space. Fernandez-Villaverde and Levintal (2018) apply the Taylor projection method to solve a DSGE model with Epstein–Zin preferences and rare disasters. Other models include NLCEQ (Cai, Judd, and Steinbuks, 2017), which will be discussed later, and simulation-based methods including GSSA (Judd, Maliar, and Maliar, 2011) and EDS(Maliar and Maliar, 2015), which unfortunately cannot guarantee convergence. In addition, Dynare (Adjemian et al., 2011), a Matlab toolbox, is used for solving DSGE models, particularly for problems in macroeconomics.

6 Robust Decision Making

Experts often provide different models, projected paths, or estimated parameter values, so policymakers have to face Knightian uncertainty, where a particular probability distribution cannot be assigned across the models, projected paths, or parameter values. A typical method to deal with problems involving Knightian uncertainty is sensitivity analysis, and uncertainty quantification is another method (see Harenberg et al. (forthcoming) and Cai, Judd, and Lontzek (2018)). But neither can provide a robust solution for decision makers.

Robust decision making methods help decision makers who face Knightian uncertainty. The max-min method is the most well-known robust decision making method. It tries to maximize the minimal welfare across the uncertain

[5] We can transform non-stationary problems into stationary problem by adding some extra state variables, so that perturbation or projection methods could be applied.

models, projected paths, or estimated parameter values; that is, the max-min method corresponds to the worst case analysis. Thus, the robust decision from the max-min method is often too conservative.

Recently, a min-max regret (MMR) method, a less conservative robust decision making method, has been applied in environmental and resource economics for policy analysis. For an unknown but true model, there is an optimal solution to achieve the maximal welfare under the model. Other models will also propose their corresponding solution. If we implement the proposed decisions from the other models in reality (i.e., the true model), it gives us realized welfare. MMR defines regret to be the difference between the maximal welfare using the optimal decisions under the true model and the realized welfare using the proposed decisions under the other models, and then chooses a robust decision to minimize the maximal regret.

Iverson (2012) implements an iterative approach and applies MMR to climate policy analysis using DICE-2007 (Nordhaus, 2008) under Knightian uncertainty across weights on environmental or growth objectives, climate sensitivity, and the coefficient of the damage function of DICE. Iverson (2013) uses MMR to consider a robust environmental policy decision in the face of Knightian uncertainty about the discount rate. Anthoff and Tol (2014b) also analyze MMR using the "FUND" integrated assessment model (Anthoff and Tol, 2014a).

Cai and Sanstad (2016) introduce an efficient computational method to solve min-max regret (MMR) problems and make robust decisions over Knightian uncertainty, and apply it to the Goulder-Mathai model (Goulder and Mathai, 2000) for studying carbon emissions abatement from the energy sector in the face of model uncertainty about technical change. Cai, Golub, and Hertel (2017) apply the efficient MMR method to study robust decisions of agricultural research and development under uncertainty in population, income and temperature using five Shared Socio-Economic Pathways (O'Neill et al., 2014). Cai, Golub, and Hertel (2016) extend the efficient MMR method to study robust decisions of agricultural research and development under ambiguity over risk of economic growth (i.e., Knightian uncertainty across probability distributions of economic growth), based on a survey of economics about economic growth in the next century(Christensen, Gillingham, and Nordhaus, 2018; Gillingham et al., 2018).

The MMR methods cannot change the level of ambiguity aversion, and have no risk aversion (except in Cai, Golub, and Hertel (2016)). Hansen and Sargent (2008) introduce a robust control framework in the face of both risk and ambiguity (misspecification), with both risk aversion and ambiguity aversion. Athanassoglou and Xepapadeas (2012) implement the robust control framework to consider an analytical pollution control problem, and Rudik (2016) incorporates it numerically in DICE to include learning, and solves his model using VFI with sparse grid approximation. Drouet, Bosetti, and Tavoni (2015) disentangle model uncertainty and risks to economic production due to mitigation costs, climate dynamics, and climate damages. Berger, Emmerling, and Tavoni (2017) apply the robust tools in Cerreia-Vioglio et al. (2013) and Marinacci (2015) to disentangle the role of preferences from the structure of model uncertainty in order to study the impact on optimal mitigation policy.

7 Other Computational Methods

7.1 NLCEQ

It is often challenging to solve dynamic stochastic programming problems with high dimensions or occasionally binding constraints. Cai, Judd, and Steinbuks (2017) introduce a new computational method, called Non-Linear Certainty Equivalent approximation (NLCEQ), to solve these kinds of problems. NLCEQ can solve deterministic infinite-horizon stationary problems accurately, and stochastic infinite-horizon stationary problems with acceptable accuracy, including a social planner's problems and competitive equilibrium. It is simple for coding, naturally parallelizable, and is also very stable, particularly for solving a social planner's problems. For example, NLCEQ can solve a stochastic multi-country optimal growth problem with up to 400 state variables using Smolyak grids and parallelism, a dynamic model of food and clean energy with a stochastic jump process, and a New Keynesian DSGE model with a zero lower bound. See Cai, Judd, and Steinbuks (2017) for a comparison between NLCEQ and perturbation, OccBin, GSSA, and EDS methods.

7.2 Approximate Dynamic Programming

Simulation-based methods (e.g., GSSA (Judd, Maliar, and Maliar, 2011)) can avoid the so-called "curse-of-dimensionality", based on the property of Monte Carlo simulation methods. Approximate dynamic programming (ADP) (e.g., Powell (2007)) is a simulation-based and nested inner-outer iteration method. It is originally designed for problems with discrete states. It starts with a given initial guess of value functions at all times, $\widehat{V}_t^0(\mathbf{x}_t)$, where $\mathbf{x}_t$ are discrete state variables, then updates them by the outer iteration. In each outer iteration, it generates a new sample path of stochastic variables and then runs an inner forward iteration over time using the Bellman equation. That is, after the $(n-1)$-th outer iteration, we have $\widehat{V}_t^{n-1}(\mathbf{x}_t)$ for all t, then the inner iteration starts with the initial state $\mathbf{x}_0^n$ and a simulated ϵ_0^n, uses the Bellman equation and $\widehat{V}_1^{n-1}(\mathbf{x}_1)$ to compute the optimal decision $\mathbf{a}_0^n$, computes $\widehat{V}_0^n(\mathbf{x}_0^n)$ as a weighted sum of $\widehat{V}_0^{n-1}(\mathbf{x}_0^n)$ and the optimal objective value at $\mathbf{x}_0^n$, lets $\widehat{V}_0^n(\mathbf{x}_0) = \widehat{V}_0^{n-1}(\mathbf{x}_0)$ for all $\mathbf{x}_0 \neq \mathbf{x}_0^n$, and then obtains the next-period state using the transition laws $\mathbf{x}_1^n = \mathbf{f}_0(\mathbf{x}_0^n, \mathbf{a}_0^n, \epsilon_0^n)$. With $\mathbf{x}_1^n$ and a simulated ϵ_1^n, it can similarly obtain $\widehat{V}_1^n(\mathbf{x}_1)$ and $\mathbf{x}_2^n$. Continue this forward iteration until the terminal time and then the inner iteration obtains $\widehat{V}_t^n(\mathbf{x}_t)$ for all t. Thus, in the n-th outer iteration, $\widehat{V}_t^n(\mathbf{x}_t)$ differentiates with $\widehat{V}_t^{n-1}(\mathbf{x}_t)$ at only one visited state $\mathbf{x}_t^n$ at each time t. The inner-outer iteration process stops until the value functions converge, that is, $\widehat{V}_t^n(\mathbf{x}_t)$ and $\widehat{V}_t^{n+1}(\mathbf{x}_t)$ are sufficiently close for all t and all states $\mathbf{x}_t$. Since this is based on Monte Carlo simulation, it requires a large number of outer iterations, otherwise many states may not be visited with enough frequency or even never be visited, limiting the accuracy of ADP.

For problems with continuous state variables $\mathbf{x}_t$, a standard ADP needs to discretize them, but discretization makes it inaccurate for high-dimensional problems.However, ADP can also employ value function approximation $\widehat{V}_t(\mathbf{x}_t; \mathbf{B}_t^n)$, where $\mathbf{B}_t^n$ are approximation coefficients over some basis functions, so the outer iteration updates $\mathbf{B}_t^n$ instead of values at a large discretized state space. Thus, ADP can be fast in some cases, although it may be unstable in other cases. A good choice of basis functions can significantly improve the performance of ADP. Shayegh and Thomas (2015) design a two-step-ahead approximation in ADP to solve problems with continuous state variables, in which they choose utility functions of subsequent states in the next two periods as the basis functions. The two-step-ahead algorithm is then applied in Heutel, Moreno-Cruz, and Shayegh (2016) and its extended four-step-ahead algorithm is applied in Heutel, Moreno-Cruz, and Shayegh (2018) to study the impact of solar geoengineering on climate policy under uncertainty. However, it is important to check errors (e.g., Euler errors), because the multiple-step-ahead approximation may be limited in accuracy for approximating value functions, and ADP with the multiple-step-ahead approximation cannot guarantee that it actually solves the original dynamic stochastic problems even if it converges.

7.3 Real Options Analysis

The costs and benefits of an action in a decision making problem are often uncertain, particularly in a dynamic environment as future management and policy can respond to new information. Real options analysis can take into account uncertainty and also flexibility, so it is often used to value the flexibility in an investment project, including allowances for future deferral, abandonment, or expansion of the project (see e.g., Brennan and Schwartz (1985), Dixit and Pindyck (1994)).

The most common methods for evaluating options are Monte Carlo simulation, decision trees, and partial differential equations (PDEs). For instance, Albers, Fisher, and Hanemann (1996) use a real options approach with a decision tree to discuss the impact of uncertainty and irreversibility on the valuation and management of tropical forests, assuming that there are three types of land use: preservation, an intermediate use, and development. Insley (2002) introduces a real options approach based on a PDE to model the optimal tree harvesting decision, by implementing an implicit finite difference method to discretize a linear complementarity equation for determining the value of the option in a backward iteration. Hansen, Howitt, and Williams (2008) evaluate an annual dry-year option, under which a water agency buys the right to purchase water at a later date with a prespecified strike price, by constructing a distribution of shadow prices that reflect the economic value of water under a simulation–optimization framework. Anda, Golub, and Strukova (2009) apply real options analysis based on Monte Carlo simulation to select a future-flexible climate policy that can be corrected in the future in response to new knowledge. Leroux, Martin, and Goeschl (2009) uses a real options approach based on a PDE and its finite difference solution to find optimal levels of conservation and land development under future stochastic natural damages, with the ecological mechanism of extinction debt as an illustration. Nadolnyak, Miranda, and Sheldon (2011) employ real options to market entry of genetically modified crops using VFI with Chebyshev polynomial approximation, while Monte Carlo simulation can be problematic in econometric estimation for their cases. Linquiti and Vonortas (2012) formulate a Monte Carlo model with real options analysis to test adaptation strategies for defending against sea level rise due to global warming. Ryu et al. (2018) apply real options analysis with a binomial tree to study flood mitigation strategies under uncertainty in global climate change.

7.4 Solving Principal-Agent Models

Baldwin, Cai, and Kuralbayeva (2018) apply the Mathematical Programming with Equilibrium Conditions (MPEC) method to solve a dynamic principal-agent model, where the principal decides dynamic carbon taxes and/or subsidies to maximize social welfare, and the agents maximize their respective utility functions: the representative household maximizes the present value of utilities; the final good firms, fossil fuel firms, and renewable energy firms maximize their present value of profits. MPEC approaches have also been applied in other fields of economics. For example, Su and Judd (2012) apply MPEC in structural estimation to maximize the likelihood subject to equilibrium conditions from a Bellman equation, which is a bilevel optimization problem like the principal-agent structure, and then compare it with the nested fixed-point approach (Rust, 1987). Recently, a polynomial optimization approach has been introduced to solve principal-agent models; see Renner and Schmedders (2015, 2016).

8 Continuous Time Dynamic Programming Problems

Researchers often use continuous time dynamic programming for modeling. Deterministic problems can be formulated as

$$\begin{aligned} \max \quad & \int_0^T e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t), t) dt + e^{-\rho T} W(\mathbf{x}(T)) \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t), t), \end{aligned} \tag{14}$$

where ρ is the discount rate, $\mathbf{a}(t)$ is the vector of decision variables at time t, $\dot{\mathbf{x}}(t)$ is the derivative of the state variables $\mathbf{x}$ over time t, and the terminal time T can be infinite. Under some conditions that economic problems often satisfy, the above problem can be reformulated as the following Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE):

$$\frac{\partial V}{\partial t}(\mathbf{x}, t) - \rho V(\mathbf{x}, t) + \max_{\mathbf{a}} \{\nabla V(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, \mathbf{a}, t) + u(\mathbf{x}, \mathbf{a}, t)\} = 0 \tag{15}$$

subject to the terminal condition $V(\mathbf{x}, T) = W(\mathbf{x})$, where $\nabla V(\mathbf{x}, t)$ denotes the gradient vector of V on $\mathbf{x}$.

Sometimes it is easy to derive an analytical formula for the maximizer in (15), so the HJB equation can be transformed to a standard PDE which can be solved by standard computational methods, such as finite element methods and finite difference methods. Pontryagin's maximum principle can also be implemented to derive a set of equations for us to solve and obtain optimal policy functions. For example, Sohngen and Mendelsohn (1998) uses this method and a shooting algorithm for solving equations to find equilibrium prices and timber harvests under climate change. Sohngen and Mendelsohn (2003) later combine a continuous-time global timber model (Sohngen, Mendelsohn, and Sedjo, 1999) with the discrete-time DICE model (Nordhaus and Boyer, 2000), and implement an iteration method to calculate carbon rental rates and then solve the two models simultaneously.

When there are occasionally binding constraints in (14), HJB equations may be challenging to solve, or the value function $V(\mathbf{x}, t)$ may not even be twice differentiable over the state variables $\mathbf{x}$. Cai, Judd, and Lontzek (2012) apply finite difference methods to solve a continuous time DICE problem where the emission control rate will hit its upper bound after some years. They implement explicit, implicit, or trapezoid finite difference rules to discretize the ordinary differential equation in (14), and employ corresponding numerical integration rules to replace the integration in (14) by summation. They can efficiently solve their problem with weekly time steps, a 600-year horizon, and six continuous state variables. Moreover, their method avoids the kink problems that arise from the transformation to an HJB equation.

The stochastic version of (14) is

$$\begin{aligned} \max \quad & \mathbb{E}\left\{ \int_0^T e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t), t) dt + e^{-\rho T} W(\mathbf{x}(T)) \right\} \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t), \epsilon(t), t), \end{aligned} \tag{16}$$

where $\epsilon(t)$ is a continuous time stochastic process (and can be multi-dimensional). The time discretization method used in Cai, Judd, and Lontzek (2012) can still be applied, and then we can implement NLCEQ, value function iteration, or other computational methods for discrete time problems. The model (16) can also be converted to an HJB PDE equation under some conditions (e.g., $\epsilon(t)$ is normal, log-normal, or binary). For example, in finance, the Black-Scholes equation for pricing a derivative is derived under a number of assumptions, including the assumption that the underlying asset's price follows a geometric

Brownian motion. Polasky, de Zeeuw, and Wagener (2011) build a system of HJB equations for problems with potential regime shifts with exogenous or endogenous probabilities. van der Ploeg and de Zeeuw (2016) use a system of HJB equations to investigate cooperative and non-cooperative responses to climate change with a North-South model of the global economy in the face of stochastic tipping points of productivity. Since the expectation operator disappears in the converted HJB equation, we can again implement standard computational methods for solving PDE on the HJB equations.

The time discretization method may be time consuming if the time horizon is large and time increments used are small, but it is becoming feasible with modern computational power, as shown in Cai, Judd, and Lontzek (2012). However, if there are multiple optimal solutions, it is still challenging to find the global optimizer or all local optimizers.

8.1 Deterministic Infinite-horizon Stationary Problems

In the literature of continuous time dynamic programming problems in environmental and resource economics, many problems are deterministic and stationary assuming an infinite time horizon. For such a deterministic infinite-horizon stationary problem,

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t)) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t)), \end{aligned} \tag{17}$$

its corresponding HJB equation becomes

$$\max_{\mathbf{a}} \{\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{a}) + u(\mathbf{x}, \mathbf{a})\} = \rho V(\mathbf{x}) \tag{18}$$

Instead of solving the HJB equation to get a fixed point for the unknown value function V, we often form a current value Hamiltonian:

$$\mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) = \lambda^\top \mathbf{f}(\mathbf{x}, \mathbf{a}) + u(\mathbf{x}, \mathbf{a})$$

and then implement Pontryagin's maximum principle to obtain the following modified Hamiltonian system of ordinary differential equations (ODEs):

$$\begin{cases} \dot{\lambda} = -\nabla_{\mathbf{x}}\mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) + \rho\lambda \\ 0 = \nabla_{\mathbf{a}}\mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{a}) \end{cases} \tag{19}$$

where λ is the co-state vector. Together with the transversality condition and initial/boundary conditions, we can solve the system of ODEs numerically. For example, the Matlab ODE packages (e.g., ode45, ode15s and bvp4c) or Mathematica's NDSolve routine can be applied to solve ODE problems. In the literature, this deterministic infinite-horizon stationary problem appears in the management of a dynamic ecological system such as lake eutrophication (Carpenter, Ludwig, and Brock (1999), Brock and Starrett (2003), Mäler, Xepapadeas, and de Zeeuw (2003), and Wagener (2003)), and a socioeconomic system of a lake district for fishery (Carpenter and Brock, 2004). Grimsrud and Huffaker (2006) apply singular-perturbation reduction methods to reduce the multidimensional solution space to a lower-dimensional subspace confining longterm dynamics, and then use the Mathematica's NDSolve routine to solve for optimal management of pest resistance to pesticidal crops. Fenichel and Horan (2016) apply numerical function approximation and collocation method, a type of projection methods, to solve their system of ODEs, to show the importance of institutions for managing convex-concave systems with thresholds and tipping points.

When there are multiple agents in the model, it often becomes a differential game. The ODE system (19) provides a solution to the optimal management problem under a social planner's preference, but not under decentralized equilibrium. The literature often discusses two types of differential games for decentralized equilibrium: open-loop Nash equilibrium (OLNE) and feedback Nash equilibrium (FBNE). In OLNE, agents make their decisions ignoring feedback from physical processes and strategies of other economic agents, so that OLNE has no Markov properties. We can solve OLNE via a similar system of ODEs by deriving a modified Hamiltonian system of ODEs for every agent. FBNE takes into account the feedback in the model, so agents' decisions depend on both time and state (i.e., FBNE has Markov properties), and finding a numerical solution becomes more challenging. Kossioris et al. (2008) provide a more detailed discussion about OLNE and FBNE with an application to the eutrophication of lakes, and also introduce a numerical algorithm to solve FBNE, which incorporates the ode15s solver of Matlab. Gopalakrishnan et al. (2017) implement the bvp4c routine of Matlab to find OLNE for the spatial beach nourishment and coastal climate adaptation of two neighboring coastal communities.

Grass (2012) uses information about the long-run behavior of the system to derive appropriate boundary conditions at infinity, and then to reformulate the conditions in a finite time setting. His numerical algorithm is exemplified by a one-dimensional fishery model, using his Matlab package OCMat. Grass, Xepapadeas, and de Zeeuw (2017) apply the algorithm and OCMat to solve for the optimal management of ecosystem services with pollution, using a lake model with fast-slow dynamics, and to find Skiba manifolds and solution paths under full cooperation (i.e., under a social planner's preference) or OLNE.

9 Discussions

9.1 Curse of Dimensionality

For multi-dimensional problems, the "curse-of-dimensionality" is often an excuse to not use VFI or time iteration (see, e.g., Traeger (2014)). However, whether VFI or time iteration has the "curse-of-dimensionality" depends on the methods used. If a simple product rule[6] is used, then VFI or time iteration has the "curse-of-dimensionality", but may not if a non-product rule

[6]For example, if Chebyshev basis functions $\mathcal{T}_k(x_i)$ with $0 \leq k \leq n$ and $1 \leq i \leq d$ are used in (8) for a d-dimensional state space, then the simple product rule uses all of their product, $\mathcal{T}_{\alpha_1}(x_1) \cdots \mathcal{T}_{\alpha_d}(x_d)$, as the basis functions for all $0 \leq \alpha_i \leq n$ and $1 \leq i \leq d$ in (8), so that the number of terms in the approximation is $(n+1)^d$. This method is called tensor-product Chebyshev approximation.

is used. There are three potential levels of the "curse-of-dimensionality". The first two are on the state space, and the last is on the space of random variables.

The first potential level is in the choice of approximation methods for $\widehat{V}(\mathbf{x};\mathbf{b})$ depending on the state space, as $\widehat{V}(\mathbf{x};\mathbf{b})$ has to be computed on the objective function of the maximization problem (7). A tensor product approximation such as piecewise linear interpolation and cubic spline interpolation will suffer from this level of "curse-of-dimensionality" of the state space, as their number of basis functions $\{\phi_j(\mathbf{x})\}$ (and approximation coefficients $\mathbf{b}$) in $\widehat{V}(\mathbf{x};\mathbf{b})$ grows exponentially in the dimension of the state space. But if we use a non-product approximation method, such as complete Chebyshev polynomials and simplicial complete Chebyshev polynomials described in Section 9.4, or sparse-grid interpolation (Krueger and Kubler, 2004; Malin, Krueger, and Kubler, 2011; Judd et al., 2014; Brumm and Scheidegger, 2017), then this level of "curse-of-dimensionality" can disappear.

The second potential level is in the choice of approximation nodes on the state space. To obtain approximation coefficients, we need to provide data $\{(\mathbf{x}_i, v_i)\}$, where v_i is the objective at the optimal solution of the maximization problem (7) with current-period state $\mathbf{x}_i$. If tensor grids $\{\mathbf{x}_i\}$ in the state space are used, then VFI or time iteration suffers from this level of "curse-of-dimensionality" of the state space, unless parallelism is also used (Cai et al., 2015b). Sparse grids, e.g., Smolyak grids (Smolyak, 1963), or adaptive sparse grids (Brumm and Scheidegger, 2017) can also break this level of "curse-of-dimensionality". Note that if there is the first level of "curse-of-dimensionality", then the second level will also exist, but not vice versa, as the number of approximation coefficients could be less than the number of Lagrange data (but not vice versa) to avoid overfitting.

The last potential level is in the choice of numerical integration methods. If a tensor-product integration rule is used in computing expectations or random variables are discrete, then VFI or time iteration has the "curse-of-dimensionality" on the space of random variables ϵ_t.[7] However, when random

[7] We often let the expectations operate on the space of next states, as the next states are random due to the randomness of ϵ_t. But in many cases the number of random variables

variables are continuous, this level of "curse-of-dimensionality" can be broken by monomial quadrature rules (Stroud, 1971; Judd, 1998), sparse grid integration (Gerstner and Griebel, 1999; Heiss and Winschel, 2008), or Monte Carlo integration methods. Monte Carlo integration methods have to use a large number of simulated points as they only have $O(1/\sqrt{N})$ accuracy with N simulated points, while the optimization solver often needs six-digit accuracy, or even higher accuracy for problems with flat objective functions in the maximization problem (7). Thus, in practice numerical quadrature rules are often more efficient for problems with continuous random variables (Skrainka and Judd, 2011). In addition, even if random variables are discrete, it is still possible to break this level of "curse-of-dimensionality". For example, Doraszelski and Judd (2012) avoid the "curse-of-dimensionality" in discrete-time dynamic stochastic games by transforming into a continuous-time problem.

Thus, we see that all potential levels of "curse-of-dimensionality" may not exist if we choose efficient methods in VFI or time iteration.

9.2 Boundedness

The dismal theorem of Weitzman (2009) shows that the risk premium can be infinite for unboundedly distributed uncertainties. Costello et al. (2010) use a truncation method to get bounded uncertainty and obtain a finite risk premium. Thus, numerical solutions with truncation of unbounded distributions could be qualitatively inconsistent with theoretical results without truncation. However, in the literature, researchers often do not consider this inconsistency issue when they solve dynamic stochastic programming problems with an unbounded distribution. For example, a normal or log-normal distribution is often assumed for Bayesian learning when deriving Bayes' updating rules, and then researchers use a truncation method (or a bounded quadrature or simulation rule) to estimate the integration in the objective function of the

ϵ_t in the same period t is smaller than the dimensionality of state space, for example we may consider only one systematic shock affecting all agents, so it is beneficial to use ϵ_t. Even if the number of random variables ϵ_t is larger than the dimensionality of state space, it is often hard to construct the joint distribution of next state variables from a given joint distribution of ϵ_t.

maximization problem in the Bellman equation (6).

Recently, to avoid the inconsistency between theory and numerical implementation, Cai, Judd, and Lontzek (2017) replace a continuous long-run risk stochastic process by a two-dimensional dense Markov chain in their DSICE model and then solve it numerically using VFI. Cai and Judd (2015) define a bounded distribution that is close to normal and then implement it in their model and numerical methods using Hermite information. In fact, it is often reasonable to assume bounded distributions instead of unbounded distributions in environmental and resource economics. For example, the climate sensitivity parameter is considered to be positive and less than ten (IPCC, 2007, 2013).

9.3 Monte Carlo techniques

The previous discussion assumes stochastic processes, but there are also uncertain parameters which are constant but unknown across time. These parameter values are often estimated from an econometric analysis, so we may know their distributions. Thus, if a distribution can be assigned to an uncertain parameter, then we can solve it using expected welfare maximization, which mimics the expected cost minimization method described in Cai and Sanstad (2016). Some researchers use a Monte Carlo method, which obtains an optimal policy by solving a deterministic welfare maximization problem for each sampled realization of the uncertain parameters under the distributions and then averages over the policies as an approximate solution in the face of the uncertainty. For example, New and Hulme (2000), Nordhaus (2008), Ackerman, Stanton, and Bueno (2010), and Anthoff and Tol (2013) implement this Monte Carlo method to analyze the impact of uncertainty on climate policy. While this Monte Carlo analysis can be helpful in some cases, it does not solve the real problem of a decision maker facing the parameter uncertainty, and it may even lead to the opposite sign for the effect of uncertainty (Crost and Traeger, 2013).[8] Here I use a simple portfolio optimization problem with

[8]Lemoine and Rudik (2017) also discuss this Monte Carlo method in detail.

one stock and one bond to illustrate this point. Assume that the bond has a riskless 3% return, and the stock's return parameter is uncertain but we know that it has a 70% probability of being larger than 3%. The Monte Carlo method will always find it optimal to invest around 70% of wealth on the stock and the remaining 30% on the bond, no matter which risk aversion coefficient or utility function is used in the objective function.

9.4 Approximation

Value function approximation is used in the objective function of the maximization problem (7). For example, Chebyshev basis functions $\mathcal{T}_k(x) = \cos(k\cos^{-1}(Z(x)))$ with

$$Z(x) = \frac{2x - x_{\min} - x_{\max}}{x_{\max} - x_{\min}}$$

are used for Chebyshev approximation on the one-dimensional state space $[x_{\min}, x_{\max}]$, for $k = 0, 1, 2, ...$ Piecewise linear interpolation and cubic spline interpolation are also often used in the literature (see e.g., Judd (1998) and Miranda and Fackler (2002)). Kelly and Kolstad (1999) apply neural networks for approximation. Sometimes value functions have special properties, so special basis functions can be chosen. For example, Hwang (2017) presents a log-linearization method that approximates value functions by a linear combination of the logarithm of state variables, but this method only works for problems with value functions that can be approximated well on a reasonable domain by the log-linearization, while most problems do not have this property.[9]

For multi-dimensional approximation, an efficient approximation method is complete Chebyshev approximation (see e.g., Cai and Judd (2010, 2014)) or

[9] A linearization or log-linearization method may work locally on a narrow domain, but it is often not enough to obtain a globally accurate solution, see Cai, Judd, and Steinbuks (2017) for more discussion. In fact, from a numerical result of Hwang (2017), we can see that his solution from the log-linearization method may lead to an error of around 100% in carbon emission control compared to the solution from the optimal control method, which can be treated as the true solution.

simplicial complete Chebyshev approximation introduced by Cai, Judd, and Lontzek (2018), as both methods have no "curse-of-dimensionality". Moreover, Chebyshev coefficients can be efficiently computed by the Chebyshev regression algorithm (see e.g., Judd (1998) and Cai, Judd, and Lontzek (2018)) if we choose Chebyshev nodes as approximation nodes. See Appendix for detailed discussion.

9.4.1 Approximation Domains

Since most approximation methods are defined on hyperrectangles, we often have to truncate an unbounded or too-large state space into a bounded hyperrectangle $[\mathbf{x}_{\min}, \mathbf{x}_{\max}]$ that is wide enough to contain all necessary states. If the hyperrectangle is too narrow, then it may lead to a bad approximation for points outside the hyperrectangle as extrapolation often does not work well. Thus, solutions using VFI with narrow hyperrectangles may not be reliable. If the hyperrectangle is too wide, then it requires more approximation nodes and higher degree of approximation to achieve the necessary accuracy, so the problem may be too time-consuming or even infeasible to run with a modern computer. Since an initial state $\mathbf{x}_0$ is given in dynamic stochastic programming problems (5), we can choose a very narrow initial state space and then expand it gradually in the next periods to contain all states originated from any states in the initial state space and reasonable decisions. Cai and Judd (2012) use time-dependent state spaces to solve a simple dynamic portfolio example, where next period's state space is chosen to contain all possible states transited from any states in current-period state space, defined as an interval of wealth.

Cai et al. (2015a); Cai, Lenton, and Lontzek (2016); Cai, Judd, and Lontzek (2017, 2018); and Lontzek et al. (2015) choose a series of approximation domains by setting consumption-output ratios and emission control rates in reasonable ranges, e.g., the optimal states and decisions of their corresponding deterministic dynamic programming models should be well inside the approximation domains and the ranges of decisions at each time; and simulated paths of states for the stochastic model should be well within the approximation domains. They then efficiently solve large-dimensional (from 7 to 15 dimensions) dynamic stochastic integrated assessment models (IAMs) based on the framework of DSICE (Dynamic Stochastic Integration of Climate and Economy), in which its corresponding deterministic IAM is the annual analog of the Dynamic Integrated Climate-Economy (DICE) model (Nordhaus, 2008). The largest example in Cai, Judd, and Lontzek (2017) has six continuous state variables (corresponding to DICE) and three dense discrete stochastic state variables. Its horizon is 600 years and it uses annual time steps. It is solved it in less than eight hours using 110,688 cores in parallel on the Blue Waters supercomputer. The smallest example in Cai et al. (2015a) has six continuous state variables (corresponding to DICE) and one binary stochastic state variable indicating whether a tipping event happens or not, and it has 600 annual time steps, but it took only minutes to get an accurate solution on a laptop. Cai, Lenton, and Lontzek (2016) solve DSICE with five interacting tipping elements in the climate system, which has ten continuous state variables and five binary stochastic state variables. Although the problems in Cai, Lenton, and Lontzek (2016) have more state variables than in Cai, Judd, and Lontzek (2017), they can use narrower approximation domains and then lower-degree Chebyshev approximation methods as well as a much smaller number of discrete states, so they can be solved in about three hours using 10,560 cores of the BlueWaters supercomputer. All of these problems have 0.1-1% estimated errors for policy functions, and 0.01-0.1% for the value functions.

The stationary problems (13) require only one hyperrectangle for all iterations until VFI converges. Some researchers transform the non-stationary problems (6) into the following problem

$$\begin{aligned} V(\mathbf{x},\tau) = \max_{\mathbf{a}_\tau \in \mathcal{D}_\tau(\mathbf{x})} \quad & u_\tau(\mathbf{x},\mathbf{a}_\tau) + \beta \mathbb{E}_\tau \left\{ V(\mathbf{x}_+,\tau_+) \right\} \\ \text{s.t.} \quad & \mathbf{x}_+ = \mathbf{f}_\tau(\mathbf{x},\mathbf{a}_\tau,\epsilon_\tau) \\ & \tau_+ = g(\tau) \end{aligned} \tag{20}$$

by adding τ as an extra continuous state variable in the value function V, where τ is bounded and has a one-to-one monotonic map to time t. For example, Lemoine and Traeger (2014) apply this trick to solve a four-dimensional dynamic stochastic IAM based on a reduced system of DICE. However, this trick increases one dimension and also has to expand its approximation domain significantly because it has to contain the minimal and maximal states along time, while states could increase significantly along time. Thus, the approximation domain would be significantly much wider than the largest domain using the time-dependent state spaces. Therefore, this trick makes VFI much more time consuming. In addition, Lemoine and Traeger (2014) implement tensor-product Chebyshev approximation and MATLAB. These reasons explain why their run took days using a laptop.

9.5 Stopping Criterion

Infinite horizon stationary dynamic programming problems (as in 13) require a stopping criterion for value function iteration or time iteration. If researchers do not pay careful attention to the choice of stopping criteria, it may lead to large numerical errors, even if the value function has a proper magnitude and the discount factor β is not very close to 1. For example, Lemoine and Traeger (2014) use the following stopping criterion

$$\max_{j=1,\ldots,N} |b_{k,j} - b_{k+1,j}| \leq 10^{-4},$$

where $\{b_{i,j} : j = 1, \ldots, N\}$ are value function approximation coefficients of tensor-product Chebyshev polynomials at the k-th iteration. However,

$$\begin{aligned} \left|\widehat{V}_k(\mathbf{x}; \mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x}; \mathbf{b}_{k+1})\right| &= \left|\sum_{j=1}^{N} b_{k,j}\phi_j(\mathbf{x}) - \sum_{j=1}^{N} b_{k+1,j}\phi_j(\mathbf{x})\right| \\ &\leq N \max_{j=1,\ldots,N} |b_{k,j} - b_{k+1,j}| \leq 10^{-4}N, \end{aligned}$$

where $\phi_j(\mathbf{x})$ are Chebyshev basis functions with 1 as the maximal value. Thus the upper bound of $\left|\widehat{V}_i(\mathbf{x}; \mathbf{b}_i) - \widehat{V}_{i+1}(\mathbf{x}; \mathbf{b}_{i+1})\right|$ is $10^{-4}N$, not 10^{-4}. If N is huge, then the errors could be huge too. Lemoine and Traeger (2014) uses $N = 10,000$, so the upper bound of $\left|\widehat{V}_i(\mathbf{x}; \mathbf{b}_i) - \widehat{V}_{i+1}(\mathbf{x}; \mathbf{b}_{i+1})\right|$ could be 1.

10 Conclusion

I have reviewed various state-of-the-art computational methods and their application in environmental and resource economics. Each computational method has its advantages and disadvantages. For example, VFI and time iteration are quite general, but require more complicated computational techniques such as efficient approximation, appropriate approximation domains, efficient numerical integration, and/or a suitable stopping criterion. NLCEQ is relatively simple and robust, but is limited when obtaining very high accuracy for stochastic problems. Researchers should choose a proper algorithm for their specific problem, and it will also be important to verify and check accuracy of the solution, because it is often hard to guarantee that the numerical solution found via computational methods is actually close to the true solution (for reasons such as nonlinearity, multiplicity of local optimizers, numerical errors, or bugs in code).

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