

New perspectives on covariant quantum error correction

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Covariant codes are quantum codes such that a symmetry transformation on the logical system could be realized by a symmetry transformation on the physical system, usually with limited capability of performing quantum error correction (an important case being the Eastin-Knill theorem). The need for understanding the limits of covariant quantum error correction arises in various realms of physics including fault-tolerant quantum computation, condensed matter physics and quantum gravity. Here, we explore covariant quantum error correction with respect to continuous symmetries from the perspectives of quantum metrology and quantum resource theory, building solid connections between these formerly disparate fields. We prove new and powerful lower bounds on the infidelity of covariant quantum error correction, which not only extend the scope of previous no-go results but also provide a substantial improvement over existing bounds. Explicit lower bounds are derived for both erasure and depolarizing noises. We also present a type of covariant codes which nearly saturate these lower bounds.

Introduction. Quantum error correction (QEC) is a standard approach to protecting quantum systems against noises, which allows the possibility of practical quantum technologies such as quantum computing, and has been a central research topic in quantum information and relevant areas [1–3]. The key idea of QEC is to encode logical quantum information into a subspace of the Hilbert space of a large physical system. The QEC codes need to satisfy certain conditions such that the noise we wish to correct is reversible in the code space [4]. As a result, the structure of the noise must also place restrictions on the allowed logical operations on the QEC codes. This feature was for example beautifully captured by the Eastin-Knill theorem [5] (see also [6–9]), which states that any non-trivial local-error-correcting quantum code does not admit transversal implementations of a universal set of logical gates, ruling out the possibility of realizing fault-tolerant quantum computation using only transversal gates.

In particular, any finite-dimensional local-error-correcting quantum code only admits a finite number of transversal logical operations, which forbids the existence of codes covariant with continuous symmetries (discrete symmetries are allowed though [10, 11]). More generally, quantum codes under symmetry constraints, namely covariant codes, are of great practical and theoretical interest. Besides important implications to fault-tolerant quantum computation, covariant QEC is also closely connected to many other topics in quantum information and physics, such as quantum reference frames [10], quantum clocks [12, 13], symmetries in the AdS/CFT correspondence [14–19] and approximate QEC in condensed matter physics [20]. Although covariant codes cannot be perfectly local-error-correcting, they can still approximately correct errors with infidelity depending on the number of subsystems, the dimension of each subsystem, etc. The quantifications of such infidelity in covariant QEC were explored recently,

leading to an approximate, or robust, version of the Eastin-Knill theorem [11, 13], using complementary channel techniques [21–23]. Note that these existing results only apply to erasure errors.

Here we investigate covariant QEC from the perspectives of quantum metrology and quantum resource theory. Quantum metrology studies the ultimate limit on parameter estimation in quantum systems [24–28], which naturally encompasses covariant QEC—any rotation of the physical system is equivalent to a rotation of the logical system where the angle of the rotation could be estimated with protection against noise. There is a no-go theorem in quantum metrology [29–37], stating that perfect error-correcting codes admitting a non-trivial logical Hamiltonian does not exist if the physical Hamiltonian fall into the Kraus span of the noise channel. It is also a sufficient condition of the non-existence of perfect covariant QEC codes, extending the previous locality restriction in the Eastin-Knill theorem to a generic algebraic relation on Hamiltonians and noises. When the no-go condition is satisfied, we establish a connection between the quantum Fisher information (QFI) of quantum channels [30–32, 38–40] and the performance (or infidelity) of covariant QEC, which gives rise to the desired lower bound. We could also understand covariant QEC in terms of resource theory of asymmetry [41–43] with respect to translations generated by Hamiltonians, where the covariant QEC procedures may naturally be represented by free operations. In quantum resource theory, we also have no-go theorems which dictate that pure resource states cannot be perfectly distilled from generic mixed states [44–46], thereby ruling out the possibility of perfect covariant QEC, leading to a lower bound on the infidelity of covariant QEC, which behaves similarly to the corresponding metrological bound.

Covariant codes. A quantum code is defined by an (usually isometric) encoding channel $\mathcal{E}_{S \leftarrow L}$ from a logical system L to the physical system S . Given logical and physical Hamiltonians H_L and H_S (which generate the unitary representations of the symmetry group), we call a code $\mathcal{E}_{S \leftarrow L}$ *covariant* if

$$\mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L, \theta} = \mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}, \quad \forall \theta \in \mathbb{R}, \quad (1)$$

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where $\mathcal{U}_{L,\theta}(\rho_L) = e^{-iH_L\theta}\rho_L e^{iH_L\theta}$ and $\mathcal{U}_{S,\theta}(\rho_S) = e^{-iH_S\theta}\rho_S e^{iH_S\theta}$ are continuous symmetry transformations on the logical and physical systems, respectively. We assume the dimensions of the physical and logical systems d_S and d_L are both finite and H_L is non-trivial ($H_L \not\propto \mathbb{1}$).

We say a quantum code is *error-correcting* under a noise channel $\mathcal{N}_S$, if $\mathcal{N}_S$ is invertible inside the code subspace, i.e. if there exists a CPTP map $\mathcal{R}_{L \leftarrow S}$ (recovery channel) such that $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L} = \mathbb{1}_L$. We assume the output space of the noise channel $\mathcal{N}_S$ is still S for simplicity, though our results also apply to situations where the output system is different. The error-correcting property of a quantum code is often incompatible with its covariance with respect to continuous symmetries. One representative example is the Eastin-Knill theorem [5, 10], which indicates the non-existence of error-correcting codes which can simultaneously correct local errors and be covariant with respect to a local H_S . However, one could still consider approximate QEC on covariant codes [11, 13, 20]. One natural question to ask is: for a covariant code under a fixed pair of Hamiltonians and noise channel, how good an approximate error-correcting code it can be? Here we use the worst-case entanglement fidelity $f(\Phi_1, \Phi_2)$ [47, 48] defined by [49]

$$f(\Phi_1, \Phi_2) = \min_{\rho} f((\Phi_1 \otimes \mathbb{1}_R)(\rho), (\Phi_2 \otimes \mathbb{1}_R)(\rho)) \quad (2)$$

for two quantum channels Φ_1 and Φ_2 where the fidelity between two states is $f(\rho, \sigma) = \text{Tr}(\sqrt{\rho^{1/2}\sigma\rho^{1/2}})$ [1] and R is a reference system identical to the system $\Phi_{1,2}$ acts on, to characterize the inaccuracy of an approximate error-correcting code. After optimizing over recovery channels $\mathcal{R}_{L \leftarrow S}$, the infidelity of a code $\mathcal{E}_{S \leftarrow L}$ is defined by

$$\varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L}) = 1 - \max_{\mathcal{R}_{L \leftarrow S}} f^2(\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L). \quad (3)$$

We call a code $\mathcal{E}_{S \leftarrow L}$ ε -correctable under $\mathcal{N}_S$ if $\varepsilon \geq \varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L})$ and we always assume $\varepsilon < 1/2$.

Metrological bound. Recently, QEC emerges as a useful tool to enhance the sensitivity in quantum metrology [32, 35, 36, 50–59]. A good (approximately error-correcting) covariant code naturally provides a good quantum sensor to estimate an unknown parameter θ in the symmetry transformation $e^{-iH_S\theta}$. Instead of using the entire system to probe the signal, one could prepare an encoded probe state using covariant codes where H_S is mapped to H_L on the logical system. For covariant codes with low infidelity, the noise will be significantly reduced in the logical system and therefore provide a good sensitivity of the signal.

Since covariant codes provide specific metrological protocols, their performance is subject to theorems and bounds in quantum metrology. No-go theorems in quantum metrology [29–33, 35, 36] prevent the existence of perfectly error-correcting covariant codes in the above scenario. In particular, it was known that given a noise channel $\mathcal{N}_S(\cdot) = \sum_{i=1}^r K_{S,i}(\cdot) K_{S,i}^\dagger$ and a physical Hamiltonian $e^{-iH_S\theta}$, there exist an encoding channel $\mathcal{E}_{S \leftarrow L}$ and a recovery channel $\mathcal{R}_{L \leftarrow S}$ such that $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}$ is a non-trivial unitary channel only if $H_S \notin \text{span}\{K_{S,i}^\dagger K_{S,j}, \forall i, j\}$ [32]. However, the above channel w.r.t. any perfectly error-correcting

covariant code would simply be $\mathcal{U}_{L,\theta}$. Therefore, we conclude that perfectly error-correcting covariant codes does not exist when

$$H_S \in \text{span}\{K_{S,i}^\dagger K_{S,j}, \forall i, j\}, \quad (4)$$

which we call the ‘‘Hamiltonian-in-Kraus-span’’ (HKS) condition [60]. One could check that local Hamiltonians with local errors is a special case of the HKS condition.

Powerful lower bounds on the infidelity of covariant codes could be further derived leveraging tools from entanglement-assisted quantum metrology for quantum channels [30–32, 38]. Given a quantum channel $\mathcal{N}_\theta(\cdot) = \sum_{i=1}^r K_{i,\theta}(\cdot) K_{i,\theta}^\dagger$, one could define its QFI $F(\mathcal{N}_\theta)$ which measures the amount of information it carries about the unknown parameter θ . Here we use the regularized SLD QFI [30–32, 38]:

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) = \begin{cases} 4 \min_{h: \mathbf{K}_\theta^\dagger h \mathbf{K}_\theta = i \mathbf{K}_\theta^\dagger \partial_\theta \mathbf{K}_\theta} \|\partial_\theta \mathbf{K}_\theta + i h \mathbf{K}_\theta\|^2 & \text{(S),} \\ +\infty & \text{otherwise,} \end{cases} \quad (5)$$

(S): $i \mathbf{K}_\theta^\dagger \partial_\theta \mathbf{K}_\theta \in \text{span}\{K_{i,\theta}^\dagger K_{j,\theta}, \forall i, j\},$

where h is Hermitian in $\mathbb{C}^{r \times r}$, $\mathbf{K}_\theta^T = (K_{1,\theta}^T \ K_{2,\theta}^T \ \cdots \ K_{r,\theta}^T)$, $\|\cdot\|$ is the operator norm. $F_{\mathcal{S}}(\rho_\theta)$ is the SLD QFI of quantum states [61, 62] and ‘‘ $\mathbb{1}$ ’’ represents the assistance of an unbounded ancillary system. The regularized SLD QFI is efficiently computable via semidefinite programs [30]. Two key properties of the regularized channel SLD QFI (see the Supplemental Material [63]) we will use are monotonicity: $F_{\mathcal{S}}^{\text{reg}}(\Phi_1 \circ (\mathcal{N}_\theta \otimes \mathbb{1}) \circ \Phi_2) \leq F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta)$ where $\Phi_{1,2}$ are any parameter-independent channels and additivity: $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta \otimes \tilde{\mathcal{N}}_\theta) = F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) + F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_\theta)$ for any $\mathcal{N}_\theta$ and $\tilde{\mathcal{N}}_\theta$.

The main obstacle to establishing lower bounds on the infidelity of covariant codes is to relate it to the QFIs of the error-corrected quantum channel. Let us first take a look at an intuitive example: consider N logical qubits each under a unitary evolution $e^{-i\theta H_L}$ with noise rate ε . It is known that the SLD QFI of a noiseless N -qubit GHZ state is $(\Delta H_L)^2 N^2$ [64]. Taking $N = \Theta(1/\varepsilon)$, the total noise rate (roughly $N\varepsilon$) can be bounded by a small constant, and the state SLD QFI per qubit will be $\Theta((\Delta H_L)^2 N) = \Theta((\Delta H_L)^2/\varepsilon)$ which matches the scaling of the channel QFIs. In this Letter, we assume $\mathcal{N}_{\theta,S} = \mathcal{N}_S \circ \mathcal{U}_{\theta,S}$. We construct Φ_1 and Φ_2 such that $\Phi_1 \circ (\mathcal{N}_{\theta,S} \otimes \mathbb{1}) \circ \Phi_2$ is a dephasing channel with noise rate no larger than the code infidelity ε and a Hamiltonian equal to $\frac{1}{2}(\Delta H_L)Z$, where we use ΔH to denote the difference between the maximum and minimum eigenvalues of the operators and Z to denote the Pauli-Z matrix. Dephasing channels have computable QFIs $\Theta((\Delta H_L)^2/\varepsilon)$ (intuitively from the discussion above). It is always smaller than $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta,S})$ due to the monotonicity of the QFIs, leading to an lower bound on the code infidelity of $\Theta((\Delta H_L)^2/F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta,S}))$. Specifically, we obtain the following bounds on ε (we will use ‘‘ $\varepsilon \gtrsim$ ’’ to represent ‘‘ $\varepsilon(\varepsilon) \geq$ ’’ in which $\varepsilon(\varepsilon)$ could be any function of ε satisfying $\lim_{\varepsilon \rightarrow 0^+} \varepsilon(\varepsilon)/\varepsilon = 1$):

Theorem 1. Suppose a covariant code $\mathcal{E}_{S \leftarrow L}$ is ε -correctable under $\mathcal{N}_S(\cdot) = \sum_{i=1}^r K_{S,i}(\cdot) K_{S,i}^\dagger$. If the HKS condition (Eq. (4)) is satisfied, then

$$\varepsilon \gtrsim (\Delta H_L)^2 (4F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S))^{-1}, \quad (6)$$

where $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S)$ is the regularized SLD QFI of $\mathcal{N}_S, \theta$.

More detailed descriptions (including the exact expressions of $\iota(\varepsilon)$ and $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S)$) and proofs of the theorems could be found in the Supplemental Material [63]. Note that the regularized SLD QFI here could be replaced by other types of channel QFI, e.g. the channel RLD QFI, because the former is no larger than the latter. The channel RLD QFI is defined by $F_{\mathcal{R}}(\mathcal{N}_\theta) = \max_\rho F_{\mathcal{R}}((\mathcal{N}_\theta \otimes \mathbb{1})(\rho))$ [39, 40] where $F_{\mathcal{R}}(\rho_\theta)$ is the RLD QFI of quantum states [65]. We also remark that Theorem 1 holds for non-isometric encoding channels, widening the scope of Theorem 1 in [11].

Resource-theoretic bound. Next, we demonstrate how quantum resource theory provides another new pathway towards characterizing the limitations of covariant QEC. More specifically, the covariance property of the allowed operations indicates close connections to the (highly relevant) resource theories of asymmetry, reference frames, coherence, and quantum clocks [41–43, 46, 66]. In our current context, we work with a resource theory of coherence (see e.g. [46] for more discussions on the setting) where the free (incoherent) states are those with density operators commuting with the physical Hamiltonian H_S , and the free operations are covariant operations $\mathcal{C}_{L \leftarrow S}$ from S to L satisfying $\mathcal{C}_{L \leftarrow S} \circ \mathcal{U}_{S,\theta} = \mathcal{U}_{L,\theta} \circ \mathcal{C}_{L \leftarrow S}$, $\forall \theta \in \mathbb{R}$. The free (covariant) operations are completely incoherence-preserving, i.e. map incoherent states to incoherent states even with the assistance of reference systems [46, 67].

It is recently found that, by analyzing suitable resource monotones (functions of states that are nonincreasing under free operations), one can prove strong lower bounds on the infidelity of transforming generic noisy states to pure resource states by any free operation, which underlies the important task of distillation (see [44] for general results that apply to any well-behaved resource theory, and [46] for discussions specific to covariant operations). Here, we use the RLD QFI of quantum states [65], a coherence monotone studied in [46], to derive bounds on the performance of covariant QEC. In particular, the RLD QFI satisfies

$$F_{\mathcal{R}}(\mathcal{C}_{L \leftarrow S}(\rho_S), H_L) \leq F_{\mathcal{R}}(\rho_S, H_S), \quad (7)$$

for all ρ_S and covariant operations $\mathcal{C}_{L \leftarrow S}$ where $F_{\mathcal{R}}(\rho, H) = F_{\mathcal{R}}(e^{-iH\theta} \rho e^{iH\theta}) = \text{Tr}(H \rho^2 H \rho^{-1}) - \text{Tr}(\rho H^2)$ if $\text{supp}(H \rho H) \subseteq \text{supp}(\rho)$ and $+\infty$ otherwise.

We link the resource theory described above and covariant QEC based on the following observation: Suppose $\mathcal{N}_S \circ \mathcal{U}_{S,\theta} = \mathcal{U}_{S,\theta} \circ \mathcal{N}_S$ and $e^{-iH_L\theta}$ and $e^{-iH_S\theta}$ have a same period (a standard assumption in the theory of quantum clocks [13, 46]), we can assume the recovery channel $\mathcal{R}_{S \leftarrow L}$ is covariant without jeopardizing the code fidelity (see the Supplemental Material [63] for the proof). In order to derive the lower bound on the code infidelity ε , we use the fact that $F_{\mathcal{R}}(\rho, H)$ approaches infinity when ρ is close to a pure coherent state [46].

For example, take $\rho_S = \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}(|+_L\rangle \langle +_L|)$ and $\mathcal{C}_{L \leftarrow S} = \mathcal{R}_{L \leftarrow S}$ where $|+_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$ with $|0_L\rangle$ and $|1_L\rangle$ being eigenstates respectively corresponding to the largest and the smallest eigenvalues of H_L . In Eq. (7), $\mathcal{C}_{L \leftarrow S}(\rho_S)$ is the output of the error-corrected quantum channel, which is close to $|+_L\rangle$ when ε is small. In fact, it can be shown that the LHS of Eq. (7) is $\Theta((\Delta H_L)^2/\varepsilon)$. On the other hand, the RHS of Eq. (7) is always finite when ρ_S is a mixed state. It demonstrates that the RLD QFI is a distinguished coherence monotone which can rule out generic noisy-to-pure transformations and further induces lower bounds on ε (detailed proof is provided in the Supplemental Material [63]):

Theorem 2. Suppose a covariant code $\mathcal{E}_{S \leftarrow L}$ is ε -correctable under $\mathcal{N}_S(\cdot) = \sum_{i=1}^r K_{S,i}(\cdot) K_{S,i}^\dagger$ and $\varepsilon < 0.38$. If $\mathcal{N}_S$ commutes with $\mathcal{U}_{S,\theta}$ and $e^{-iH_L\theta}$ and $e^{-iH_S\theta}$ are periodic with a same period,

$$\varepsilon \gtrsim (\Delta H_L)^2 (4F_{\mathcal{R}}((\mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L})(|+_L\rangle \langle +_L|), H_S))^{-1}. \quad (8)$$

Furthermore,

$$\varepsilon \gtrsim (\Delta H_L)^2 (4F_{\mathcal{R}}(\mathcal{N}_S, H_S))^{-1}, \quad (9)$$

where $F_{\mathcal{R}}(\mathcal{N}_S, H_S)$ is the RLD QFI of $\mathcal{N}_S, \theta$.

Here we first obtain a code-dependent lower bound (Eq. (8)), which induces a code-independent version Eq. (9) similar to Theorem 1 using the RLD QFI of quantum channels. It is efficiently computable and has a closed-form expression using the Choi operator and its derivative [39, 40]. As mentioned before, it is always true that $F_{\mathcal{R}}(\mathcal{N}_{S,\theta}) \geq F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S,\theta})$ and therefore Theorem 1 provides a tighter code-independent bound than Theorem 2. However, the code-dependent bound (Eq. (8)) may be of independent interest in determining the infidelity lower bounds for specific types of covariant codes.

Local Hamiltonian and local noise. One of the most common scenarios where covariant codes are considered is when S is an n -partite system, consisting of subsystems $S_1, S_2, \dots, S_n$. The physical Hamiltonian and the noise channel are both local, given by $H_S = \sum_{k=1}^n H_{S_k}$, $\mathcal{N}_S = \bigotimes_{k=1}^n \mathcal{N}_{S_k}$, and $\mathcal{N}_{S_k}(\cdot) = \sum_{i=1}^{r_k} K_{S_k,i}(\cdot) K_{S_k,i}^\dagger$. In general, it takes time exponential in the number of subsystems to solve our lower bounds on the code infidelity. However, when the Hamiltonians and the noises are local, using the additivity of channel QFIs, we could calculate the lower bounds using only computation of the channel QFIs in each subsystem. For ε -correctable codes under $\mathcal{N}_S$, Theorem 1 indicates that when the HKS condition is satisfied for each H_{S_k} and $\mathcal{N}_{S_k}$, $\varepsilon \gtrsim (\Delta H_L)^2 (4 \sum_{k=1}^n F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}, H_{S_k}))^{-1}$.

Instead of finding bounds for local noise channels $\mathcal{N}_S$ with certain noise rates, we sometimes are more interested the capability of a code to correct single errors described by $\mathcal{M}_S = \sum_{k=1}^n q_k \mathcal{M}_{S_k}$, $\sum_{k=1}^n q_k = 1$, where q_k is the probability that an error $\mathcal{M}_{S_k}$ occurs on the k -th subsystem. As shown in the Supplemental Material [63], for ε -correctable codes under single-error noise channels $\mathcal{M}_S$, when the HKS condition is

satisfied for each H_{S_k} and $\mathcal{N}_{S_k}$, we have

$$\varepsilon \gtrsim \liminf_{\delta \rightarrow 0^+} \frac{(\Delta H_L)^2}{4\delta \sum_{k=1}^n F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}(\delta), H_{S_k})}, \quad (10)$$

where $\mathcal{N}_{S_k}(\delta) = (1 - \delta q_k) \mathbb{1} + \delta q_k \mathcal{M}_{S_k}$. Note that discussions here analogously apply to [Theorem 2](#) due to the additivity of the channel RLD QFI [\[39\]](#), although we will only focus on [Theorem 1](#) in the following since it provides the tightest bound.

Erasure noise. Now we present our bounds for local erasure noise. For ε -correctable covariant codes under local erasure noise channel $\mathcal{N}_S^e = \bigotimes_{k=1}^n \mathcal{N}_{S_k}^e$ where $\mathcal{N}_{S_k}^e(\rho_{S_k}) = (1 - p_k)\rho_{S_k} + p_k |\text{vac}\rangle \langle \text{vac}|_{S_k}$ (we use the vacuum state $|\text{vac}\rangle$ to represent the state of the erased subsystems), we have $\varepsilon \gtrsim (\Delta H_L)^2 (4 \sum_{k=1}^n \frac{1-p_k}{p_k} (\Delta H_{S_k})^2)^{-1}$. For ε -correctable covariant codes under single-error erasure noise channel $\mathcal{M}_S^e = \sum_{k=1}^n q_k \mathcal{M}_{S_k}^e$ where $\mathcal{M}_{S_k}^e(\rho_{S_k}) = |\text{vac}\rangle \langle \text{vac}|_{S_k}$, we have $\varepsilon \gtrsim (\Delta H_L)^2 (4 \sum_{k=1}^n \frac{1}{q_k} (\Delta H_{S_k})^2)^{-1}$. In particular, when the probability of erasure is uniform on each subsystem, i.e. $q_k = 1/n$, we have $\varepsilon \gtrsim (\Delta H_L)^2 (4n \sum_{k=1}^n (\Delta H_{S_k})^2)^{-1}$. As a comparison, [Theorem 1](#) in [\[11\]](#) showed that $\varepsilon \geq (\Delta H_L)^2 (4n^2 \max_k (\Delta H_{S_k})^2)^{-1}$. Our bound has a clear advantage in the small infidelity limit by improving the maximum of ΔH_{S_k} to their quadratic mean. A direct implication of is an improved approximate Eastin-Knill theorem which establishes the infidelity lower bound for covariant codes with respect to special unitary groups (see [Supplemental Material \[63\]](#)).

Depolarizing noise. Next, we present our bounds for local depolarizing noise. Let $\mathcal{N}_S^d = \bigotimes_{k=1}^n \mathcal{N}_{S_k}^d$ where $\mathcal{N}_{S_k}^d(\rho_{S_k}) = (1 - p_k)\rho_{S_k} + p_k \mathbb{1}_{S_k}/d_k$ and $\mathcal{M}_S^d = \sum_{k=1}^n q_k \mathcal{M}_{S_k}^d$ where $\mathcal{M}_{S_k}^d(\rho_{S_k}) = \mathbb{1}_{S_k}/d_k$. We first consider the qubit case where the subsystem dimension $d_k = 2$ for all k . For ε -correctable covariant codes under $\mathcal{N}_S^d$, we have $\varepsilon \gtrsim (\Delta H_L)^2 (4 \sum_{k=1}^n \frac{2(1-p_k)^2}{p_k(3-2p_k)} (\Delta H_{S_k})^2)^{-1}$. For ε -correctable covariant codes under $\mathcal{M}_S^d$, we have $\varepsilon \gtrsim 3(\Delta H_L)^2 (8 \sum_{k=1}^n \frac{1}{q_k} (\Delta H_{S_k})^2)^{-1}$.

The situation is more complicated when $d_k > 2$, because the regularized SLD QFI may not have a closed-form expression. However, we prove in the [Supplemental Material \[63\]](#) that, $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}^d, H)$ is always upper bounded by $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}^e, H)$ for an arbitrary $d_k \geq 2$. Therefore, all lower bounds derived from for local erasure noise also hold true for local depolarizing noise, regardless of the dimensions of subsystems. The channel RLD QFI $F_{\mathcal{R}}(\mathcal{N}_{S_k}^d, H_{S_k})$ also upper bounds $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}^d, H_{S_k})$ and has a closed-form expression. However, as shown in the [Supplemental Material \[63\]](#), $F_{\mathcal{R}}(\mathcal{N}_{S_k}^d, H_{S_k})$ increases linearly w.r.t. d_k and is close to $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}^d, H_{S_k})$ only for small d_k .

Example: Thermodynamic codes. Finally, we provide an example saturating the lower bound for single-error erasure noise channels in the small infidelity limit and matching the scaling of the lower bound for single-error depolarizing noise channels, while previously only the scaling optimality for erasure channels was demonstrated [\[11\]](#). The two-dimensional

thermodynamic code [\[11, 20, 68, 69\]](#) in an n -qubit system is defined by

$$\mathcal{E}_{S+L}(|0_L/1_L\rangle) = \left(\frac{n}{\frac{n+m}{2}}\right)^{-\frac{1}{2}} \sum_{j: \sum_k j_k = m/-m} |j\rangle, \quad (11)$$

and $j = (j_1, j_2, \dots, j_n) \in \{-1, 1\}^n$. It means the logical subspace is spanned by two Dicke states with different values of the total angular momentum along the z -axis. We also assume $n+m$ is an even number and $3 \leq m \ll N$. It is a covariant code whose physical and logical Hamiltonians are $H_S = \sum_{k=1}^n (\sigma_z)_{S_k}$, $H_L = mZ_L$, where $\sigma_z = |1\rangle\langle 1| - |-1\rangle\langle -1|$. First consider erasure noise $\mathcal{M}_S^e = \frac{1}{n} \sum_{k=1}^n \mathcal{M}_{S_k}^e$ an upper bound of $m^2/4n^2 + O(m^4/n^4)$ on the code infidelity can be derived by providing an explicit recovery channel (see [Supplemental Material \[63\]](#)). Thus the lower bound $\varepsilon \gtrsim m^2/4n^2$ is saturated asymptotically when $m/N \rightarrow 0$. For depolarizing noise $\mathcal{M}_S^d = \frac{1}{n} \sum_{k=1}^n \mathcal{M}_{S_k}^d$, it is in general difficult to write down the optimal recovery map explicitly. Instead, we apply [Corollary 2](#) in [\[21\]](#) to calculate an upper bound on the infidelity of thermodynamic codes in the limit $m/N \rightarrow 0$ and we obtain an upper bound of $3m^2/4n^2 + O(m^3/n^3)$ (see [Supplemental Material \[63\]](#)), which also matches the scaling of our lower bound $\varepsilon \gtrsim 3m^2/8n^2$.

Conclusions and outlook. In this Letter, we established fruitful connections between covariant QEC and quantum metrology as well as quantum resource theory. We first present covariant QEC as a special type of metrological protocol where the sensitivity in parameter estimation could be linked to the code infidelity. We took inspirations from recent developments in quantum channel estimation: a no-go theorem [\[29–33, 35, 36\]](#) on the existence of perfect QEC was discovered based on a relation between sensing Hamiltonians to noise channels (the HKS condition) which leads to an extension of the scope of the Eastin-Knill theorem; computable QFIs of quantum channels were also proposed [\[32, 39, 40\]](#), which leads to computable lower bounds for the code infidelity under generic noise channels. We also studied covariant QEC using resource theory, which is subject to no-go theorems for the distillation of pure coherent states from noisy ones under free operations [\[44, 46\]](#). Our approaches on covariant QEC are innovative and also advantageous compared to previous ones in many ways. In the special case of erasure noise, our lower bound improves the previous results in the small infidelity limit [\[11\]](#), which leads to an improved approximate Eastin-Knill theorem that may be of particular interest in quantum computation. Furthermore, we provide an example of covariant codes that saturates the lower bound for erasure noise and matches the scaling of the lower bound for depolarizing noise, while previous bounds only apply to the erasure noise setting and are not known to be saturable [\[11\]](#).

There are still many open questions and future directions in the study of covariant QEC. First, it is not known, whether the HKS condition, which was shown to be sufficient for the non-existence of perfect covariant QEC codes, is also necessary. There are some examples of perfect covariant QEC codes, such as the [\[\[4,2,2\]\]](#) QEC code under single-qubit erasure noise [\[11, 70\]](#), repetition codes under bit-flip noise [\[50–](#)

[52, 54], but it is not yet clear how to generalize those examples. On the other hand, when the HKS condition is satisfied, it would also be desirable to obtain a systematic procedure to construct covariant codes saturating the infidelity lower bounds, at least in terms of scaling [13]. From the resource theory perspective, it would be interesting to investigate whether different monotones may induce other useful bounds, and whether directly employing channel resource theory [71, 72] techniques may lead to new insights. It would also be important to further explore possible implications of the limitations on covariant QEC for physics, where symmetries naturally play prominent roles in a wide range of scenarios.

Note added. During the completion of this work, an independent work by Kubica and Demkowicz-Dobrzanski [73] appeared on arXiv, where a lower bound on the infidelity of covariant codes was also derived using tools from quantum metrology. Note that we employed different techniques and

obtained lower bounds with a quadratic advantage in terms of scaling over the one in [73].

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Appendix A: Entanglement-assisted quantum metrology for quantum channels

In this section, we provide a comprehensive review of entanglement-assisted quantum metrology for quantum channels [31, 32, 34, 38–40]. We first review the definitions of QFIs of quantum states and then explain their extensions to quantum channels. The QFI is a good measure of the amount of information a quantum state ρ_θ carries about an unknown parameter, characterized by the quantum Cramér-Rao bound [61, 62, 74, 75], $\delta\theta \geq 1/\sqrt{N_{\text{expr}} F(\rho_\theta)}$, where $\delta\theta$ is the standard deviation of any unbiased estimator of θ , N_{expr} is the number of repeated experiments and $F(\rho_\theta)$ is the QFI of ρ_θ . The QFI, as a quantum generalization of the classical Fisher information is not unique due to the noncommutativity of quantum operators. We will restrict our discussions to the two most commonly used QFIs are the symmetric logarithmic derivative (SLD) QFI and the right logarithmic derivative (RLD) QFI, defined by [61, 62, 65],

$$F_{\mathcal{S}}(\rho_\theta) = \text{Tr}(\rho_\theta (L_\theta^{\mathcal{S}})^2), \quad \partial_\theta \rho_\theta = \frac{1}{2}(L_\theta^{\mathcal{S}} \rho_\theta + \rho_\theta L_\theta^{\mathcal{S}}), \quad (\text{A1})$$

$$F_{\mathcal{R}}(\rho_\theta) = \text{Tr}(\rho_\theta L_\theta^{\mathcal{R}} L_\theta^{\mathcal{R}\dagger}), \quad \partial_\theta \rho_\theta = \rho_\theta L_\theta^{\mathcal{R}}, \quad (\text{A2})$$

respectively, where the SLD $L_\theta^{\mathcal{S}}$ is Hermitian and the RLD $L_\theta^{\mathcal{R}}$ is linear. Note that $F_{\mathcal{R}}(\rho_\theta) = +\infty$ if $\text{supp}(\partial_\theta \rho_\theta) \not\subseteq \text{supp}(\rho_\theta)$. The quantum Cramér-Rao bound with respect to the SLD QFI is saturable asymptotically ($N_{\text{expr}} \gg 1$) using maximum likelihood estimators [75–77] and thus we always have $F_{\mathcal{S}}(\rho_\theta) \leq F_{\mathcal{R}}(\rho_\theta)$. The QFIs satisfy many nice information-theoretic properties [40], such as additivity $F(\rho_\theta \otimes \sigma_\theta) = F(\rho_\theta) + F(\sigma_\theta)$ and monotonicity $F(\mathcal{N}(\rho_\theta)) \leq F(\rho_\theta)$ for θ -independent channel $\mathcal{N}$.

Given a quantum channel $\mathcal{N}_\theta$, one could also define the entanglement-assisted QFI of $\mathcal{N}_\theta$ [38],

$$F(\mathcal{N}_\theta) = \max_{\rho} F((\mathcal{N}_\theta \otimes \mathbb{1}_R)(\rho)), \quad (\text{A3})$$

where R is an unbounded reference system. The RLD QFI of $\mathcal{N}_\theta$ could be calculated efficiently using [39, 40]

$$F_{\mathcal{R}}(\mathcal{N}_\theta) = \begin{cases} \|\text{Tr}_{S(\mathcal{N}_\theta)}((\partial_\theta \Gamma^{\mathcal{N}_\theta})(\Gamma^{\mathcal{N}_\theta})^{-1}(\partial_\theta \Gamma^{\mathcal{N}_\theta}))\| & (\text{R}): \text{span}\{\partial_\theta K_{i,\theta}, \forall i\} \subseteq \text{span}\{K_{i,\theta}, \forall i\}, \\ +\infty & \text{otherwise.} \end{cases} \quad (\text{A4})$$

Here we use the Choi operator of $\mathcal{N}_\theta$: $\Gamma^{\mathcal{N}_\theta} = (\mathcal{N}_\theta \otimes \mathbb{1})(\Gamma)$, where $\Gamma = |\Gamma\rangle\langle\Gamma|$ and $|\Gamma\rangle = \sum_i |i\rangle|i\rangle$. $S(\mathcal{N}_\theta)$ denotes the output system of $\mathcal{N}_\theta$ and $\|\cdot\|$ is the operator norm. While $F_{\mathcal{R}}(\mathcal{N}_\theta)$ is additive [39], $F_{\mathcal{S}}(\mathcal{N}_\theta)$ is in general not and we will use its regularized version [32],

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) = \lim_{N \rightarrow \infty} \frac{F_{\mathcal{S}}(\mathcal{N}_\theta^{\otimes N})}{N} = \begin{cases} 4 \min_{h: \beta_\theta=0} \|\alpha_\theta\| & (\text{S}): i \sum_{j=1}^r K_{i,\theta}^\dagger \partial_\theta K_{j,\theta} \in \text{span}\{K_{i,\theta}^\dagger K_{j,\theta}, \forall i, j\}, \\ +\infty & \text{otherwise,} \end{cases} \quad (\text{A5})$$

where $\mathcal{N}_\theta(\cdot) = \sum_{i=1}^r K_{i,\theta}(\cdot)K_{i,\theta}^\dagger$, h is a Hermitian operator in $\mathbb{C}^{r \times r}$, $\mathbf{K}_\theta^T = (K_{1,\theta}^T \ K_{2,\theta}^T \ \cdots \ K_{r,\theta}^T)$, and

$$\alpha_\theta = (\partial_\theta \mathbf{K}_\theta + i h \mathbf{K}_\theta)^\dagger (\partial_\theta \mathbf{K}_\theta + i h \mathbf{K}_\theta), \quad (\text{A6})$$

$$\beta_\theta = \mathbf{K}_\theta^\dagger h \mathbf{K}_\theta - i \mathbf{K}_\theta^\dagger \partial_\theta \mathbf{K}_\theta. \quad (\text{A7})$$

Note that when (S) is violated, $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) = \infty$ because we will have $F_{\mathcal{S}}(\mathcal{N}_\theta^{\otimes N}) \propto N^2$ [32]. The regularized SLD QFI is additive (proven below) and could be calculated efficiently using semidefinite programs [30]. (R) implies (S) but not vice versa. Note that the QFIs of quantum channels are also monotonic under parameter-independent channels due to the monotonicity of the state QFIs.

Additivity of the regularized SLD QFI

Here we prove the additivity of the regularized SLD QFI:

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta \otimes \tilde{\mathcal{N}}_\theta) = F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) + F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_\theta), \quad (\text{A8})$$

for arbitrary quantum channels $\mathcal{N}_\theta$ and $\tilde{\mathcal{N}}_\theta$.

First, according to the additivity of the state QFI, we must have

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta \otimes \tilde{\mathcal{N}}_\theta) \geq F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_\theta) + F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_\theta). \quad (\text{A9})$$

Thus, we only need to prove

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta} \otimes \tilde{\mathcal{N}}_{\theta}) \leq F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta}) + F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_{\theta}). \quad (\text{A10})$$

We use the following definition of the regularized SLD QFI [30–32] (which is equivalent to Eq. (A5))

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta}) = \begin{cases} 4 \min_{\mathbf{K}': \beta=0} \|\alpha\|, & i \sum_{i=1}^r (\partial_{\theta} K_i)^{\dagger} K_i \in \text{span}\{K_i^{\dagger} K_j, \forall i, j\}, \\ +\infty & \text{otherwise,} \end{cases} \quad (\text{A11})$$

where $\mathbf{K}'$ is any set of Kraus operators representing $\mathcal{N}_{\theta}$, $\alpha = \sum_{i=1}^r (\partial_{\theta} K_i')^{\dagger} (\partial_{\theta} K_i')$ and $\beta = i \sum_{i=1}^r (\partial_{\theta} K_i')^{\dagger} K_i'$. Without loss of generality, assume both $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta})$ and $F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_{\theta})$ are finite, i.e. $i \sum_{i=1}^r (\partial_{\theta} K_i)^{\dagger} K_i \in \text{span}\{K_i^{\dagger} K_j, \forall i, j\}$ and $i \sum_{i=1}^{\tilde{r}} (\partial_{\theta} \tilde{K}_i)^{\dagger} \tilde{K}_i \in \text{span}\{\tilde{K}_i^{\dagger} \tilde{K}_j, \forall i, j\}$. We first note that $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta} \otimes \tilde{\mathcal{N}}_{\theta})$ is also finite, because

$$i \sum_{i=1}^r \sum_{j=1}^{\tilde{r}} (\partial_{\theta} (K_i \otimes \tilde{K}_j))^{\dagger} (K_i \otimes \tilde{K}_j) = i \sum_{i=1}^r (\partial_{\theta} K_i)^{\dagger} K_i \otimes \mathbb{1} + i \sum_{j=1}^{\tilde{r}} \mathbb{1} \otimes (\partial_{\theta} \tilde{K}_j)^{\dagger} \tilde{K}_j \in \text{span}\{\mathbb{1} \otimes K_i^{\dagger} K_j, \tilde{K}_i^{\dagger} \tilde{K}_j \otimes \mathbb{1}, \forall i, j\}. \quad (\text{A12})$$

According to Eq. (A11), there exists $\mathbf{K}'$ and $\tilde{\mathbf{K}}'$ such that $\beta = \tilde{\beta} = 0$ and

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta}) = 4 \|\alpha\|, \quad F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_{\theta}) = 4 \|\tilde{\alpha}\|. \quad (\text{A13})$$

Then $\tilde{K}'_{ij} = K'_i \otimes \tilde{K}'_j$ is a set of Kraus operators representing $\mathcal{N}_{\theta} \otimes \tilde{\mathcal{N}}_{\theta}$.

$$\tilde{\alpha} = \sum_{i=1}^r \sum_{j=1}^{\tilde{r}} \partial_{\theta} (\tilde{K}_{ij})^{\dagger} \partial_{\theta} (\tilde{K}_{ij}) = \alpha \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{\alpha} + 2\beta \otimes \tilde{\beta} = \alpha \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{\alpha}, \quad \tilde{\beta} = \beta \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{\beta} = 0. \quad (\text{A14})$$

Therefore $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta} \otimes \tilde{\mathcal{N}}_{\theta}) \leq 4 \|\tilde{\alpha}\| = 4 \|\alpha\| + 4 \|\tilde{\alpha}\| = F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{\theta}) + F_{\mathcal{S}}^{\text{reg}}(\tilde{\mathcal{N}}_{\theta})$.

Appendix B: Proofs of Theorem 1

In this section, we provide the detailed descriptions and proofs of Theorem 1. The main obstacle to proving Theorem 1 is to relate the infidelity of covariant codes to the QFI of the effective quantum channel in the logical system. Here we overcome this obstacle by employing entanglement-assisted QEC to reduce $\mathcal{N}_{S,\theta}$ to dephasing channels whose regularized SLD QFI has simple mathematical forms and then connecting the noise rate of the dephasing channels to the infidelity of the covariant codes (see Fig. 1).

We define single-qubit dephasing channels to be

$$\mathcal{D}_{p,\phi}(\rho) = (1-p)e^{-i\frac{\phi}{2}Z} \rho e^{i\frac{\phi}{2}Z} + p e^{-i\frac{\phi}{2}Z} Z \rho Z e^{i\frac{\phi}{2}Z}, \quad (\text{B1})$$

where Z is the Pauli-Z operator, $0 < p < 1/2$ and ϕ is real. When ϕ is a function of θ , we could calculate the regularized SLD of $\mathcal{D}_{p,\phi_{\theta}}$ (see Appx. B in [32] or [37]):

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{D}_{p,\phi_{\theta}}) = \frac{(1-2p)^2 (\partial_{\theta} \phi_{\theta})^2}{4p(1-p)}, \quad (\text{B2})$$

which are both inversely proportional to the noise rate p when p is small—a crucial feature in deriving the lower bounds.

Next, we present an entanglement-assisted QEC protocol to reduce $\mathcal{N}_S$ to dephasing channels with a noise rate lower than $\varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L})$. Let $|0_L\rangle$ and $|1_L\rangle$ be eigenstates respectively corresponding to the largest and the smallest eigenvalues of H_L . Consider the following two-dimensional entanglement-assisted code

$$\mathcal{E}_{LA \leftarrow C}^{\text{rep}}(|0_C\rangle) = |0_L 0_A\rangle, \quad \mathcal{E}_{LA \leftarrow C}^{\text{rep}}(|1_C\rangle) = |1_L 1_A\rangle, \quad (\text{B3})$$

where A is a noiseless ancillary qubit and the superscript rep means “repetition”. The encoding channel from the two-level system C to SA will simply be $\mathcal{E}_{SA \leftarrow C} = (\mathcal{E}_{S \leftarrow L} \otimes \mathbb{1}_A) \circ \mathcal{E}_{LA \leftarrow C}^{\text{rep}}$. $\mathcal{E}_{SA \leftarrow C}$ is still a covariant code whose the logical and physical Hamiltonians are

$$H_C = \frac{\Delta H_L}{2} \cdot Z_C, \quad H_{SA} = H_S \otimes \mathbb{1}_A. \quad (\text{B4})$$

The noiseless ancillary qubit will help us suppress off-diagonal noises in the system because any single qubit bit-flip noise on L could be fully corrected by mapping $|i_L j_A\rangle$ to $|j_C\rangle$ for all i, j . In fact, $\mathcal{N}_S$ will be reduced to a dephasing channel, as shown in the following lemma:

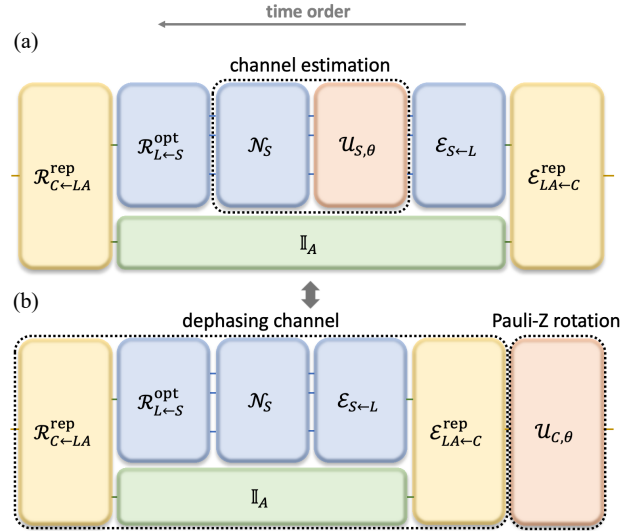


FIG. 1. Reduction of $\mathcal{N}_{S,\theta} = \mathcal{N}_S \circ \mathcal{U}_{S,\theta}$ to dephasing channels using entanglement-assisted QEC. (a) represents the quantum channel $\mathcal{R}_{C \leftarrow SA} \circ (\mathcal{N}_{S,\theta} \otimes \mathbb{1}_A) \circ \mathcal{E}_{SA \leftarrow C}$ with a channel QFI no larger than $F(\mathcal{N}_{S,\theta})$. Because of the covariance of the code, (a) is equivalent to (b) which consists of a Pauli-Z rotation $\mathcal{U}_{C,\theta}$ and a θ -independent dephasing channel $\mathcal{I}_C$ whose noise rate is smaller than $\varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L})$ (see Lemma 1).

Lemma 1. Consider a noise channel $\mathcal{N}_{SA} = \mathcal{N}_S \otimes \mathbb{1}_A$. There exists a recovery channel $\mathcal{R}_{C \leftarrow SA}$ such that the effective noise channel $\mathcal{I}_C = \mathcal{R}_{C \leftarrow SA} \circ \mathcal{N}_{SA} \circ \mathcal{E}_{SA \leftarrow C}$ is a dephasing channel, satisfying

$$\mathcal{I}_C = \mathcal{D}_{C,\varepsilon',\phi'}, \quad (\text{B5})$$

where $\varepsilon' \leq \varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L})$.

Proof. Consider the following recovery channel

$$\mathcal{R}_{C \leftarrow SA} = \mathcal{R}_{C \leftarrow LA}^{\text{rep}} \circ (\mathcal{R}_{L \leftarrow S}^{\text{opt}} \otimes \mathbb{1}_A), \quad (\text{B6})$$

where $\mathcal{R}_{C \leftarrow LA}^{\text{rep}}(\rho_{LA}) = \sum_{i=0}^{d_L-1} \sum_{j=0}^1 R_{ij} \rho_{LA} R_{ij}^\dagger$, where $R_{ij} = |j_C\rangle \langle i_L j_A|$. One could check that

$$\mathcal{I}_C(|k_C\rangle \langle j_C|) = \begin{cases} |k_C\rangle \langle j_C|, & k = j, \\ (1 - 2\varepsilon') e^{i\phi'(k-j)} |k_C\rangle \langle j_C|, & k \neq j, \end{cases} \quad (\text{B7})$$

which indicates that $\mathcal{I}_C = \mathcal{D}_{C,\varepsilon',\phi'}$ (Eq. (B5)). Here,

$$\varepsilon' \leq 1 - f^2(\mathcal{I}_C, \mathbb{1}_C) \leq 1 - f^2(\mathcal{I}_L^{\text{opt}}, \mathbb{1}_L) = \varepsilon(\mathcal{N}_S, \mathcal{E}_{S \leftarrow L}). \quad (\text{B8})$$

where the first inequality follows from the worst-case entanglement fidelity for dephasing channels (proven below), and the the second inequality follows from $\mathbb{1}_C = \mathcal{R}_{C \leftarrow LA}^{\text{rep}} \circ \mathcal{E}_{LA \leftarrow C}^{\text{rep}}$ and the monotonicity of the fidelity [1]. $\square$

Lemma 1 shows that $\mathcal{N}_S$ could be reduced to a dephasing channel $\mathcal{I}_C$ through entanglement-assisted QEC. Consider parameter estimation of θ in the quantum channel $\mathcal{N}_{S,\theta} = \mathcal{N}_S \circ \mathcal{U}_{S,\theta}$. We have the error-corrected quantum channel

$$\mathcal{N}_{C,\theta} = \mathcal{R}_{C \leftarrow SA} \circ (\mathcal{N}_{S,\theta} \otimes \mathbb{1}_A) \circ \mathcal{E}_{SA \leftarrow C} = \mathcal{I}_C \circ \mathcal{U}_{C,\theta}, \quad (\text{B9})$$

equal to a dephasing channel with noise rate ε' and phase $\phi_\theta = \phi' + \Delta H_L \theta$. The monotonicity of the regularized channel SLD QFI implies that

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S,\theta}) \geq F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{C,\theta}), \quad (\text{B10})$$

where

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S,\theta}) = \begin{cases} F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S) & H_S \in \text{span}\{K_{S,i}^\dagger K_{S,j}, \forall i, j\}, \\ +\infty & \text{otherwise,} \end{cases} \quad (\text{B11})$$

and

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{C,\theta}) = \frac{(1 - 2\varepsilon')^2 (\Delta H_L)^2}{4\varepsilon'(1 - \varepsilon')}. \quad (\text{B12})$$

Theorem 1 then follow from Eq. (B10) and $\varepsilon' \leq \varepsilon < 1/2$.

Theorem 1. Suppose a covariant code $\mathcal{E}_{S \leftarrow L}$ is ε -correctable under a noise channel $\mathcal{N}_S(\cdot) = \sum_{i=1}^r K_{S,i}(\cdot) K_{S,i}^\dagger$. If the HKS condition is satisfied, i.e.

$$H_S \in \text{span}\{K_{S,i}^\dagger K_{S,j}, \forall i, j\}, \quad (\text{B13})$$

then ε is lower bounded as follows,

$$\varepsilon \cdot \frac{1 - \varepsilon}{(1 - 2\varepsilon)^2} \geq \frac{(\Delta H_L)^2}{4F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S)}, \quad (\text{B14})$$

where $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_S, H_S) = 4 \min_{h: \beta_S=0} \|\alpha_S\|$, h is a Hermitian operator in $\mathbb{C}^{r \times r}$. α_S and β_S are Hermitian operators acting on S defined by

$$\alpha_S = \mathbf{K}_S^\dagger h^2 \mathbf{K}_S - H_S^2, \quad \beta_S = \mathbf{K}_S^\dagger h \mathbf{K}_S - H_S, \quad (\text{B15})$$

where $\mathbf{K}^T = (K_1^T \ K_2^T \ \cdots \ K_r^T)$.

Worst-case entanglement fidelity for dephasing channels

Here we calculate the worst-case entanglement fidelity for dephasing channels (Eq. (B1))

$$\mathcal{D}_{p,\phi}(\rho) = (1 - p)e^{-i\frac{\phi}{2}Z} \rho e^{i\frac{\phi}{2}Z} + p e^{-i\frac{\phi}{2}Z} Z \rho Z e^{i\frac{\phi}{2}Z}. \quad (\text{B16})$$

We use the following formula for the worst-case entanglement fidelity [47]:

$$f^2(\mathcal{D}_{p,\phi}, \mathbb{1}) = \min_{|\psi\rangle} \langle \psi | (\mathcal{D}_{p,\phi} \otimes \mathbb{1}) (|\psi\rangle \langle \psi|) | \psi \rangle. \quad (\text{B17})$$

Let $|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$, then

$$(\mathcal{D}_{p,\phi} \otimes \mathbb{1})(|\psi\rangle \langle \psi|) = \begin{pmatrix} \alpha_{00}\alpha_{00}^* & \alpha_{00}\alpha_{01}^* & (1-2p)e^{-i\phi}\alpha_{00}\alpha_{10}^* & (1-2p)e^{-i\phi}\alpha_{00}\alpha_{11}^* \\ \alpha_{00}\alpha_{01}^* & \alpha_{01}\alpha_{01}^* & (1-2p)e^{-i\phi}\alpha_{01}\alpha_{10}^* & (1-2p)e^{-i\phi}\alpha_{01}\alpha_{11}^* \\ (1-2p)e^{i\phi}\alpha_{10}\alpha_{00}^* & (1-2p)e^{i\phi}\alpha_{10}\alpha_{01}^* & \alpha_{10}\alpha_{10}^* & \alpha_{10}\alpha_{11}^* \\ (1-2p)e^{i\phi}\alpha_{11}\alpha_{00}^* & (1-2p)e^{i\phi}\alpha_{11}\alpha_{01}^* & \alpha_{11}\alpha_{10}^* & \alpha_{11}\alpha_{11}^* \end{pmatrix}. \quad (\text{B18})$$

Then

$$\begin{aligned} 1 - f^2(\mathcal{D}_{p,\phi}, \mathbb{1}) &= \max_{\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}} 2\text{Re}[(1 - (1 - 2p)e^{-i\phi})](|\alpha_{00}|^2 + |\alpha_{01}|^2)(|\alpha_{10}|^2 + |\alpha_{11}|^2) \\ &= \frac{1}{2}(1 - (1 - 2p)\cos\phi) \geq p. \end{aligned} \quad (\text{B19})$$

Appendix C: Proof of Theorem 2

Before we prove **Theorem 2**, we first prove a lemma which shows that the recovery channel $\mathcal{R}_{L \leftarrow S}$ for a covariant code can be covariant under two assumptions: (1) the noise channel and the symmetry transformation commutes (e.g. satisfied by the erasure and depolarizing channels of interest here); (2) $U_S(\theta) = e^{-iH_S\theta}$ and $U_L(\theta) = e^{-iH_L\theta}$ are periodic with a same period τ , i.e. $U_{S,L}(\tau) = \mathbb{1}_{S,L}$.

Lemma 1. Suppose $\mathcal{N}_S \circ \mathcal{U}_{S,\theta} = \mathcal{U}_{S,\theta} \circ \mathcal{N}_S$ and $U_{L,S}(\theta)$ have a period τ . Then any ε -correctable covariant code is still ε -correctable if the recovery channel is restricted to be covariant operations.

Proof. Let $\mathcal{R}_{L \leftarrow S}$ be a recovery channel for an ε -correctable code $\mathcal{E}_{S \leftarrow L}$ such that $1 - f^2(\mathcal{R}_{S \leftarrow L} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L) \leq \varepsilon$. Consider the following recovery channel:

$$\mathcal{R}_{L \leftarrow S}^{\text{cov}} = \frac{1}{\tau} \int_0^\tau d\theta \mathcal{U}_{L,\theta} \circ \mathcal{R}_{L \leftarrow S} \circ \mathcal{U}_{S,\theta}^\dagger. \quad (\text{C1})$$

We first observe that $\mathcal{R}_{L \leftarrow S}^{\text{cov}}$ is covariant:

$$\mathcal{R}_{L \leftarrow S}^{\text{cov}} \circ \mathcal{U}_{S,\theta'} = \frac{1}{\tau} \int_0^\tau d\theta \mathcal{U}_{L,\theta} \circ \mathcal{R}_{L \leftarrow S} \circ \mathcal{U}_{S,\theta-\theta'}^\dagger = \mathcal{U}_{L,\theta'} \circ \mathcal{R}_{L \leftarrow S}^{\text{cov}}. \quad (\text{C2})$$

Furthermore,

$$\mathcal{R}_{L \leftarrow S}^{\text{cov}} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L} = \frac{1}{\tau} \int_0^\tau d\theta \mathcal{U}_{L,\theta} \circ \mathcal{I}_L \circ \mathcal{U}_{L,\theta}^\dagger. \quad (\text{C3})$$

Using the concavity of $f^2(\Phi, \mathbb{1})$ [47], we have $1 - f^2(\mathcal{R}_{S \leftarrow L}^{\text{cov}} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L) \leq \varepsilon$. $\square$

This lemma allows us to understand covariant QEC as a resource conversion task, the aim of which is to transform noisy physical states to logical states by covariant operations. We are now ready to provide the proof of [Theorem 2](#):

Theorem 2. Suppose a covariant code $\mathcal{E}_{S \leftarrow L}$ is ε -correctable under a noise channel $\mathcal{N}_S(\cdot) = \sum_{i=1}^r K_{S,i}(\cdot) K_{S,i}^\dagger$ and $\varepsilon < 0.38$. If $\mathcal{N}_S$ commutes with $\mathcal{U}_{S,\theta}$, $\mathcal{U}_{L,S}(\theta)$ are periodic with a same period is satisfied, then ε is lower bounded as follows,

$$\varepsilon \cdot \frac{1}{1 - 3\varepsilon + \varepsilon^2} \geq \frac{(\Delta H_L)^2}{4F_{\mathcal{R}}((\mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L})(|+_L\rangle \langle +_L|), H_S)}, \quad (\text{C4})$$

where $|+_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$. Furthermore,

$$\varepsilon \cdot \frac{1}{1 - 3\varepsilon + \varepsilon^2} \geq \frac{(\Delta H_L)^2}{4F_{\mathcal{R}}(\mathcal{N}_S, H_S)}. \quad (\text{C5})$$

Note that $F_{\mathcal{R}}(\mathcal{N}_S, H_S)$ is finite only when $\text{span}\{K_{S,i}H_S, \forall i\} \subseteq \text{span}\{K_{S,i}, \forall i\}$, a sufficient condition of the HKS condition.

Proof. According to [Lemma 1](#), there exists a covariant recovery channel $\mathcal{R}_{L \leftarrow S}^{\text{cov}}$ such that

$$1 - \langle +_L | \rho_L | +_L \rangle \leq \varepsilon, \quad (\text{C6})$$

where $\rho_L = (\mathcal{R}_{L \leftarrow S}^{\text{cov}} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L})(|+_L\rangle \langle +_L|)$ and $\mathcal{R}_{L \leftarrow S}^{\text{cov}}$ is a covariant operation. According to Supplementary Note 3 in [46],

$$F_{\mathcal{R}}(\rho_L, H_L) \geq \frac{1 - 3\varepsilon + \varepsilon^2}{\varepsilon} \cdot V_{H_L}(|+_L\rangle), \quad (\text{C7})$$

where the variance $V_{H_L}(|+_L\rangle) = \langle +_L | H_L^2 | +_L \rangle - \langle +_L | H_L | +_L \rangle^2 = \frac{(\Delta H_L)^2}{4}$. $\varepsilon < 0.38$ guarantees the RHS is positive. On the other hand, using $F_{\mathcal{R}}(\mathcal{R}_{L \leftarrow S}^{\text{cov}}(\rho_S), H_L) \leq F_{\mathcal{R}}(\rho_S, H_S)$,

$$F_{\mathcal{R}}(\rho_L, H_L) \leq F_{\mathcal{R}}(\rho_S, H_S) = F_{\mathcal{R}}(\mathcal{N}_{S,\theta}(\mathcal{E}_{S \leftarrow L}(|+_L\rangle \langle +_L|))) \leq F_{\mathcal{R}}(\mathcal{N}_{S,\theta}), \quad (\text{C8})$$

where $\rho_S = (\mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L})(|+_L\rangle \langle +_L|)$. [Eq. \(C5\)](#) is then proven using the channel RLD QFI $F_{\mathcal{R}}(\mathcal{N}_{S,\theta})$ ([Eq. \(A4\)](#)). $\square$

Although, [Eq. \(B14\)](#) is tighter than [Eq. \(C5\)](#), from the proof of [Theorem 2](#), there might be a further improvement of [Eq. \(C5\)](#) by replacing the entanglement-assisted RLD QFI with the one without entanglement assistance: $\max_\rho F_{\mathcal{R}}(\mathcal{N}_\theta(\rho))$.

Appendix D: Lower bounds on the code infidelity for single-error noise channels

In order to obtain lower bounds on the code infidelity under noise channels $\mathcal{M}_S = \sum_{k=1}^n q_k \mathcal{M}_{S_k}$, we consider the following local noise channel

$$\mathcal{N}_S(\delta) = \bigotimes_{k=1}^n \mathcal{N}_{S_k}(\delta) = \bigotimes_{k=1}^n ((1 - \delta q_k) \mathbb{1} + \delta q_k \mathcal{M}_{S_k}) = (1 - \delta) \mathbb{1} + \delta \sum_{k=1}^n q_k \mathcal{M}_{S_k} + O(\delta^2), \quad (\text{D1})$$

whose local noise rates are proportional to a small positive parameter δ . Using the concavity of $f^2(\Phi, \mathbb{1})$, we have

$$f^2(\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S(\delta) \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L) \geq (1 - \delta) + \delta f^2(\mathcal{R}_{L \leftarrow S} \circ \mathcal{M}_S \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L) + O(\delta^2). \quad (\text{D2})$$

Taking the limit $\delta \rightarrow 0^+$, we must have $\varepsilon(\mathcal{M}_S, \mathcal{E}_{S \leftarrow L}) \geq \liminf_{\delta \rightarrow 0^+} \frac{1}{\delta} \cdot \varepsilon(\mathcal{N}_S(\delta), \mathcal{E}_{S \leftarrow L})$. Therefore, for ε -correctable codes under single-error noise channels $\mathcal{M}_S$, [Theorem 1](#) indicates that when the HKS condition is satisfied for each subsystem,

$$\varepsilon \cdot \frac{1 - \varepsilon}{(1 - 2\varepsilon)^2} \geq \liminf_{\delta \rightarrow 0^+} \frac{(\Delta H_L)^2}{4\delta \sum_{k=1}^n F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}_{S_k}(\delta), H_{S_k})}. \quad (\text{D3})$$

Note that the treatment where we first calculate the channel QFIs for $\mathcal{N}_{S_k}(\delta)$ and then take the limit $\delta \rightarrow 0^+$ in [Eq. \(D3\)](#) is crucial not only in order to simplify the computation of the bounds, but also because [Theorem 1](#) cannot be directly applied to single-error noise channels $\mathcal{M}_S$, as the conditions [Eq. \(B13\)](#) in [Theorem 1](#) may not be satisfied for single-error noise channels $\mathcal{M}_S$, even when they are satisfied for local noise channels $\mathcal{N}_S(\delta)$. In other words, [Eq. \(D3\)](#) provide extensions of the ranges of applications of [Theorem 1](#).

Appendix E: Lower bounds on the code infidelity for erasure noise

Here we calculate the regularized SLD QFI for erasure noise channels, which leads to lower bounds on the code infidelity for erasure noise. We calculate $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^e, H)$ where $\mathcal{N}^e = (1 - p)\rho + p|\text{vac}\rangle\langle\text{vac}|$. Using the Kraus operators

$$K_1 = \sqrt{1 - p}\mathbb{1}, \quad K_{i+1} = \sqrt{p}|\text{vac}\rangle\langle i|, \quad \forall 1 \leq i \leq d, \quad (\text{E1})$$

We have

$$\beta = \mathbf{K}^\dagger h \mathbf{K} - H \Leftrightarrow h = \begin{pmatrix} \frac{h_{11}}{1-p} & 0 \\ 0 & \frac{H - h_{11}\mathbb{1}}{p} \end{pmatrix}. \quad (\text{E2})$$

Then

$$\alpha = \mathbf{K}^\dagger h^2 \mathbf{K} - H^2 = \frac{h_{11}^2}{1-p} + \frac{(H - h_{11}\mathbb{1})^2}{p} - H^2 = \frac{1-p}{p} H^2 - \frac{2h_{11}}{p} H + \frac{h_{11}^2}{p(1-p)}, \quad (\text{E3})$$

$$\begin{aligned} F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^e, H) &= 4 \min_{h_{11}} \|\alpha\| = 4 \max_{\rho} \min_{h_{11}} \text{Tr}(\rho\alpha) \\ &= 4 \max_{\rho} \frac{1-p}{p} (\text{Tr}(H^2\rho) - \text{Tr}(\rho H)^2) = \frac{1-p}{p} (\Delta H)^2, \end{aligned} \quad (\text{E4})$$

where we use the minimax theorem [\[78, 79\]](#) in the second step.

Appendix F: Lower bounds on the code infidelity for depolarizing noise

Here we calculate the regularized SLD QFI for depolarizing noise channels, which leads to lower bounds on the code infidelity for depolarizing noise. We assume $\text{Tr}(H) = 0$ in this section.

When $d = 2$, we use the formula in Sec. VII(A) in [\[32\]](#) to calculate the regularized SLD QFI $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H)$ for single-qubit depolarizing channels $\mathcal{N}^d(\rho) = (1 - p)\rho + p\frac{\mathbb{1}}{2}$:

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H) = (\Delta H)^2 \frac{1 - w}{w}, \quad (\text{F1})$$

where $w = 4 \left(\frac{y^2}{2y} + \frac{xy}{x+y} \right)$ with $x = 1 - \frac{3}{4}p$ and $y = \frac{p}{4}$. Then $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H) = (\Delta H)^2 \frac{2(1-p)^2}{p(3-2p)}$.

When $d > 2$, we prove an upper bound on $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H)$ for general depolarizing channels $\mathcal{N}^d(\rho) = (1 - p)\rho + p\frac{\mathbb{1}}{d}$ with the Kraus operators

$$K_1 = \sqrt{x}\mathbb{1}, \quad K_i = \sqrt{y}U_{i-1}, \quad \forall 2 \leq i \leq d^2, \quad (\text{F2})$$

where we define $x = 1 - \frac{d^2-1}{d^2}p$, $y = \frac{1}{d^2}p$.

Any $\tilde{h}$ satisfying $\tilde{\beta} = \mathbf{K}^\dagger \tilde{h} \mathbf{K} - H = 0$ provides an upper bound on $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H)$ through

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H) = 4 \min_{h: \beta=0} \|\alpha\| \leq 4 \|\alpha\|_{h=\tilde{h}}. \quad (\text{F3})$$

To find a suitable $\tilde{h}$ which provides a good upper bound on $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H)$, we use $\tilde{h}$ which is the solution of

$$4 \min_{h: \beta=0} \text{Tr}(\alpha). \quad (\text{F4})$$

The solution of Eq. (F4) is

$$\tilde{h} = \frac{1}{2zd} \begin{pmatrix} 0 & \frac{\sqrt{xy}}{x+y} \text{Tr}(HU_1^\dagger U_0) & \cdots & \frac{\sqrt{xy}}{x+y} \text{Tr}(HU_{d^2-1}^\dagger U_0) \\ \frac{\sqrt{xy}}{x+y} \text{Tr}(HU_0^\dagger U_1) & 0 & \cdots & \frac{1}{2} \text{Tr}(HU_{d^2-1}^\dagger U_1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{xy}}{x+y} \text{Tr}(HU_0^\dagger U_{d^2-1}) & \frac{1}{2} \text{Tr}(HU_1^\dagger U_{d^2-1}) & \cdots & 0 \end{pmatrix}, \quad (\text{F5})$$

where $z = \frac{xy}{x+y} + \frac{y(d^2-2)}{4}$ and we used the assumption $\text{Tr}(H) = 0$ and

$$\mathbf{K}^\dagger \tilde{h}^2 \mathbf{K} = \left(\frac{1}{4z} - \frac{y}{4z^2} \left(\frac{1}{4} - \frac{xy}{(x+y)^2} \right) - 1 \right) H^2 + \frac{y}{4z^2 d} \left(\frac{x}{x+y} - \frac{1}{2} \right)^2 \text{Tr}(H^2) \mathbb{1}. \quad (\text{F6})$$

Using $\|H^2\| = \frac{(\Delta H)^2}{4}$ and $\text{Tr}(H^2) \leq \frac{d}{4}(\Delta H)^2$,

$$F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H) \leq 4 \|\alpha\| \leq (\Delta H)^2 \left(\frac{1}{4z} - 1 \right) = (\Delta H)^2 \frac{d^2(1-p)^2}{p(d^2(1-p)+2)} \leq (\Delta H)^2 \left(\frac{1-p}{p} \right), \quad (\text{F7})$$

upper bounded by the $F_{\mathcal{S}}^{\text{reg}}(\mathcal{N}^d, H)$ for erasure channels.

We can also calculate the RLD QFI $F_{\mathcal{R}}(\mathcal{N}^d, H)$ for depolarizing channels as an upper bound of the regularized SLD QFI:

$$F_{\mathcal{R}}(\mathcal{N}^d, H) = \left\| \text{Tr}_{S(\mathcal{N}^d)} (\Gamma^{\mathcal{N}^d, H} (\Gamma^{\mathcal{N}^d})^{-1} \Gamma^{\mathcal{N}^d, H}) \right\|, \quad (\text{F8})$$

where $\Gamma^{\mathcal{N}^d} = (\mathcal{N}^d \otimes \mathbb{1}) \Gamma$, $\Gamma^{\mathcal{N}^d, H} = (\mathcal{N}^d \otimes \mathbb{1}) (|H\rangle \langle \Gamma| - |\Gamma\rangle \langle H|)$ and $|H\rangle = (H \otimes \mathbb{1}) |\Gamma\rangle$. Then

$$F_{\mathcal{R}}(\mathcal{N}^d, H) = \frac{(1-p)^2}{4(1 - \frac{d^2-1}{d^2} p)} (\Delta H)^2 + \frac{d(1-p)^2}{p} \text{Tr}(H^2), \quad (\text{F9})$$

where the second term grows linearly w.r.t. d and does not perform as well as Eq. (F7).

Appendix G: Improved approximate Eastin-Knill theorem

Here we derive specific lower bounds on the infidelity of codes covariant with respect to unitary groups which lead to new approximate Eastin-Knill theorems, following the discussion in [11].

$SU(d_L)$ -covariant codes in an n -partite system S are defined by the encoding channels $\mathcal{E}_{S \leftarrow L}$ which satisfy

$$\mathcal{E}_{S \leftarrow L}(U_L(g)(\cdot)U_L^\dagger(g)) = \left(\bigotimes_{k=1}^n U_{S_k}(g) \right) \mathcal{E}_{S \leftarrow L}(\cdot) \left(\bigotimes_{k=1}^n U_{S_k}^\dagger(g) \right), \quad \forall g \in SU(d_L), \quad (\text{G1})$$

where $U_{S_k}(g)$ and $U_L(g)$ are unitary representations of $SU(d_L)$. It was shown in Theorem 18 in the Supplemental Material of [11] that fixing $H_L = \text{diag}(1, 0, \dots, -1)$ and letting H_{S_k} be the corresponding generator acting on the subsystem k , we have

$$d_k \geq \binom{d_L - 1 + \lceil \|H_{S_k}\| \rceil}{d_L - 1}, \quad (\text{G2})$$

where $\lceil \|H_{S_k}\| \rceil$ denotes the closest integer no smaller than $\|H_{S_k}\|$. Using the inequality $\binom{a+b}{a} \geq (1 + \frac{b}{a})^a$,

$$d_k \geq \left(\frac{d_L - 1 + \lceil \|H_{S_k}\| \rceil}{d_L - 1} \right)^{d_L - 1}, \quad \Rightarrow \quad \left(\exp \left(\frac{\ln d_k}{d_L - 1} \right) - 1 \right) (d_L - 1) \geq \|H_{S_k}\|, \quad (\text{G3})$$

$$\Rightarrow \sum_{k=1}^n \left(\exp \left(\frac{\ln d_k}{d_L - 1} \right) - 1 \right)^2 (d_L - 1)^2 \geq \frac{1}{4} \sum_k (\Delta H_{S_k})^2. \quad (\text{G4})$$

Then using the lower bound for single-error erasure noise,

$$\varepsilon \cdot \frac{1 - \varepsilon}{(1 - 2\varepsilon)^2} \geq \frac{(\Delta H_L)^2}{4n \sum_{k=1}^n (\Delta H_{S_k})^2}, \quad (\text{G5})$$

we have for any $\varepsilon \geq \varepsilon(\mathcal{M}_S, \mathcal{E}_{S \leftarrow L})$,

$$\varepsilon \cdot \frac{1 - \varepsilon}{(1 - 2\varepsilon)^2} \geq \frac{1}{4n \sum_{k=1}^n \left(\exp\left(\frac{\ln d_k}{d_L - 1}\right) - 1 \right)^2 (d_L - 1)^2}. \quad (\text{G6})$$

For large d_L ,

$$\varepsilon \cdot \frac{1 - \varepsilon}{(1 - 2\varepsilon)^2} \geq \frac{1}{4n \sum_{k=1}^n (\ln d_k)^2} + O\left(\frac{1}{n^2 d_L}\right). \quad (\text{G7})$$

Compared to Theorem 4 in [11]:

$$\varepsilon \geq \left(\frac{1}{2n \max_k \ln d_k} + O\left(\frac{1}{n d_L}\right) \right)^2, \quad (\text{G8})$$

our bound improves the maximum of $\ln d_k$ in the denominator to their quadratic mean. Moreover, it works for not only single-error erasure noise channel $\mathcal{M}_S = \sum_{k=1}^n \frac{1}{n} \mathcal{M}_{S_k}$ where $\mathcal{M}_{S_k}(\cdot) = |\text{vac}\rangle \langle \text{vac}|_{S_k}$, but also single-error depolarizing noise channel $\mathcal{M}_S = \sum_{k=1}^n \frac{1}{n} \mathcal{M}_{S_k}$ where $\mathcal{M}_{S_k}(\cdot) = \frac{\mathbb{1}}{d_k}$.

Appendix H: Upper bounds on the code infidelity of thermodynamic codes

1. Erasure noise

Here we provide an explicit recovery channel for thermodynamic codes under erasure noise which leads to an upper bound on the code infidelity. Let

$$\mathcal{E}_{S \leftarrow L}(|0_L\rangle) = |g_0\rangle = |m_n\rangle, \quad \mathcal{E}_{S \leftarrow L}(|1_L\rangle) = |g_1\rangle = |(-m)_n\rangle, \quad (\text{H1})$$

where

$$|(\pm m)_n\rangle = \left(\frac{n}{\frac{n \pm m}{2}} \right)^{-\frac{1}{2}} \sum_{j: \sum_k j_k = \pm m} |j\rangle, \quad (\text{H2})$$

and

$$|g_{0,\pm 1}^{(k)}\rangle = |(m \pm 1)_{n-1}\rangle_{S \setminus S_k} |\text{vac}\rangle_{S_k}, \quad |g_{1,\pm 1}^{(k)}\rangle = |(-m \pm 1)_{n-1}\rangle_{S \setminus S_k} |\text{vac}\rangle_{S_k}, \quad (\text{H3})$$

which represent the logical states after an erasure error occurs on S_k , and $\Pi^\perp$ be the projector onto the orthogonal subspace of $\text{span}\{|g_{0,\pm 1}^{(k)}\rangle, |g_{1,\pm 1}^{(k)}\rangle, \forall k\}$. Consider the single-error erasure noise channel $\mathcal{M}_S^e = \frac{1}{n} \sum_{k=1}^n \mathcal{M}_{S_k}^e$ where $\mathcal{M}_{S_k}^e(\rho_{S_k}) = |\text{vac}\rangle \langle \text{vac}|_{S_k}$ and the recovery channel

$$\mathcal{R}_{L \leftarrow S}(\rho_S) = \sum_{k=1}^n \sum_{i,i'=0}^1 \sum_{j=\pm 1} |g_i\rangle \langle g_{i',j}^{(k)}| \rho_S |g_{i',j}^{(k)}\rangle \langle g_{i'}| + \text{Tr}(\Pi^\perp \rho_S \Pi^\perp) |g_0\rangle \langle g_0|, \quad (\text{H4})$$

which maps the state $|g_{i,\pm 1}^{(k)}\rangle$ to $|g_i\rangle$ for all k . Then we could verify that

$$\mathcal{R}_{L \leftarrow S} \circ \mathcal{M}_S^e \circ \mathcal{E}_{S \leftarrow L} = \mathcal{D}_{p,0}, \quad (\text{H5})$$

with $p = \frac{1}{2} \left(1 - \sqrt{1 - \frac{m^2}{n^2}} \right)$. Using the relation between the noise rate p and the worst-case entanglement fidelity of a dephasing channel (see [Appx. B](#)), we must have

$$\varepsilon(\mathcal{M}_S^e, \mathcal{E}_{S \leftarrow L}) \leq 1 - f^2(\mathcal{R}_{L \leftarrow S} \circ \mathcal{M}_S^e \circ \mathcal{E}_{S \leftarrow L}, \mathbb{1}_L) = \frac{1}{2} \left(1 - \sqrt{1 - \frac{m^2}{n^2}} \right) = \frac{m^2}{4n^2} + O\left(\frac{m^4}{n^4}\right). \quad (\text{H6})$$

2. Depolarizing noise

Here we use Corollary 2 from [21] to calculate the infidelity of thermodynamic codes under depolarizing noise channels in the limit $m/N \rightarrow 0$:

Lemma 2 ([21]). *A code defined by its projector P is ε -correctable under a noise channel $\mathcal{M}(\cdot) = \sum_{i=1}^r K_i(\cdot)K_i^\dagger$ if and only if $PK_i^\dagger K_j P = A_{ij}P + P\delta A_{ij}P$ for some A_{ij} and δA_{ij} where A_{ij} are the components of a density operator, and $1 - f^2(\mathcal{A} + \delta\mathcal{A}, \mathcal{A}) \leq \varepsilon$ where $\mathcal{A}(\rho) = \sum_{ij} A_{ij} \text{Tr}(\rho) |i\rangle\langle j|$ and $(\mathcal{A} + \delta\mathcal{A})(\rho) = \mathcal{A}(\rho) + \sum_{ij} \text{Tr}(\rho\delta A_{ij}) |i\rangle\langle j|$.*

Let $P = |g_0\rangle\langle g_0| + |g_1\rangle\langle g_1|$, $\mathcal{M} = \mathcal{M}_S$ with Kraus operators

$$K_{k,i} = \frac{1}{2\sqrt{n}}(U_i)_{S_k}, \quad i = 0, 1, 2, 3, \quad (\text{H7})$$

where U_0, U_1, U_2, U_3 are respectively $\mathbb{1}, \sigma_x = |1\rangle\langle -1| + |-1\rangle\langle 1|, \sigma_y = -i|1\rangle\langle -1| + i|-1\rangle\langle 1|$, and $\sigma_z = |1\rangle\langle 1| - |-1\rangle\langle -1|$.

For $m \geq 3$, $\langle g_0|E|g_1\rangle = 0$ for any operator E acting on at most two qubits. Here we consider $\delta A_{ij} \propto (|g_0\rangle\langle g_0| - |g_1\rangle\langle g_1|)$. That is, let $\delta A_{ij} = B_{ij}(|g_0\rangle\langle g_0| - |g_1\rangle\langle g_1|)$. A and B are $4n \times 4n$ matrices

$$A = \begin{pmatrix} A^{(0,0)} & A^{(0,1)} & A^{(0,2)} & A^{(0,3)} \\ A^{(1,0)} & A^{(1,1)} & A^{(1,2)} & A^{(1,3)} \\ A^{(2,0)} & A^{(2,1)} & A^{(2,2)} & A^{(2,3)} \\ A^{(3,0)} & A^{(3,1)} & A^{(3,2)} & A^{(3,3)} \end{pmatrix}, \quad B = \begin{pmatrix} B^{(0,0)} & B^{(0,1)} & B^{(0,2)} & B^{(0,3)} \\ B^{(1,0)} & B^{(1,1)} & B^{(1,2)} & B^{(1,3)} \\ B^{(2,0)} & B^{(2,1)} & B^{(2,2)} & B^{(2,3)} \\ B^{(3,0)} & B^{(3,1)} & B^{(3,2)} & B^{(3,3)} \end{pmatrix}, \quad (\text{H8})$$

where

$$A_{kk'}^{(i,j)} = \frac{1}{2}(\langle g_0|K_{k,i}^\dagger K_{k',j}|g_0\rangle + \langle g_1|K_{k,i}^\dagger K_{k',j}|g_1\rangle), \quad B_{kk'}^{(i,j)} = \frac{1}{2}(\langle g_0|K_{k,i}^\dagger K_{k',j}|g_0\rangle - \langle g_1|K_{k,i}^\dagger K_{k',j}|g_1\rangle), \quad (\text{H9})$$

so that $PK_i^\dagger K_j P = A_{ij}P + P\delta A_{ij}P$ holds. A detailed calculation shows that $A^{(i,j)} = 0$ when $i \neq j$, $B^{(i,j)} = 0$ when $i + j \neq 3$, and

$$A^{(0,0)} = \frac{1}{4n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad (\text{H10})$$

$$A^{(1,1)} = A^{(2,2)} = \frac{1}{4n} \begin{pmatrix} 1 & \frac{n^2-m^2}{2n(n-1)} & \cdots & \frac{n^2-m^2}{2n(n-1)} \\ \frac{n^2-m^2}{2n(n-1)} & 1 & \cdots & \frac{n^2-m^2}{2n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n^2-m^2}{2n(n-1)} & \frac{n^2-m^2}{2n(n-1)} & \cdots & 1 \end{pmatrix}, \quad (\text{H11})$$

$$A^{(3,3)} = \frac{1}{4n} \begin{pmatrix} 1 & \frac{m^2-n}{n(n-1)} & \cdots & \frac{m^2-n}{n(n-1)} \\ \frac{m^2-n}{n(n-1)} & 1 & \cdots & \frac{m^2-n}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m^2-n}{n(n-1)} & \frac{m^2-n}{n(n-1)} & \cdots & 1 \end{pmatrix}, \quad (\text{H12})$$

$$B^{(0,3)} = B^{(3,0)} = \frac{m}{4n^2} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad B^{(1,2)} = -B^{(2,1)} = i \frac{m}{4n^2} \mathbb{1}. \quad (\text{H13})$$

Next we note that

$$\begin{aligned} f(\mathcal{A}, \mathcal{A} + \delta\mathcal{A}) &= \min_{|\psi\rangle} f((\mathcal{A} \otimes \mathbb{1}_R)(|\psi\rangle\langle\psi|), ((\mathcal{A} + \delta\mathcal{A}) \otimes \mathbb{1}_R)(|\psi\rangle\langle\psi|)) \\ &= \min_{p_i, \rho_i, i=0,1} f(A \otimes (p_0\rho_0 + p_1\rho_1), p_0(A + B) \otimes \rho_0 + p_1(A - B) \otimes \rho_1) \\ &\geq \min_{p_i, \rho_i, i=0,1} p_0 f(A, A + B) + p_1 f(A, A - B) = f(A, A + B), \end{aligned} \quad (\text{H14})$$

where in the second step we define $\langle g_i | \psi \rangle \langle \psi | g_i \rangle = p_i \rho_i$ for $i = 0, 1$, and in the third step we use the joint concavity of fidelity and in the last step we use $f(A + B) = f(A - B)$. Therefore we must have

$$f(\mathcal{A}, \mathcal{A} + \delta\mathcal{A}) = f(A, A + B), \quad (\text{H15})$$

by noticing that $f(\mathcal{A}(|g_0\rangle\langle g_0|), (\mathcal{A} + \delta\mathcal{A})(|g_0\rangle\langle g_0|)) = f(A, A + B)$. First note that $A^{(i,i)}$ and $B^{(i,j)}$ could be diagonalized in the following way:

$$A^{(0,0)} = \frac{1}{4n}(n|\psi_1\rangle\langle\psi_1|), \quad B^{(0,3)} = B^{(3,0)} = \frac{m}{4n}|\psi_1\rangle\langle\psi_1|, \quad (\text{H16})$$

$$A^{(1,1)} = A^{(2,2)} = \frac{1}{4n} \left(\frac{n^2 + 2n - m^2}{2n} |\psi_1\rangle\langle\psi_1| + \frac{n^2 - 2n + m^2}{2n(n-1)} \sum_{k=2}^n |\psi_k\rangle\langle\psi_k| \right), \quad (\text{H17})$$

$$A^{(3,3)} = \frac{1}{4n} \left(\frac{m^2}{n} |\psi_1\rangle\langle\psi_1| + \frac{n^2 - m^2}{n(n-1)} \sum_{k=2}^n |\psi_k\rangle\langle\psi_k| \right), \quad (\text{H18})$$

where $|\psi_1\rangle = \frac{1}{\sqrt{n}}(1 \ 1 \ \cdots \ 1)$ and $\{|\psi_k\rangle\}_{k>1}$ is an arbitrary orthonormal basis of the orthogonal subspace of $|\psi_1\rangle$. Since $A^{(i,j)} = A^{(j,i)} = B^{(i,j)} = B^{(j,i)} = 0$ when $i \in \{1, 2\}$ and $j \in \{0, 3\}$, we have

$$f(A, A + B) = f(A^{(0)}, A^{(0)} + B^{(0)}) + f(A^{(1)}, A^{(1)} + B^{(1)}), \quad (\text{H19})$$

where

$$(\cdot)^{(0)} = \begin{pmatrix} (\cdot)^{(0,0)} & (\cdot)^{(0,3)} \\ (\cdot)^{(3,0)} & (\cdot)^{(3,3)} \end{pmatrix}, \quad (\cdot)^{(1)} = \begin{pmatrix} (\cdot)^{(1,1)} & (\cdot)^{(1,2)} \\ (\cdot)^{(2,1)} & (\cdot)^{(2,2)} \end{pmatrix}. \quad (\text{H20})$$

We first calculate $f(A^{(0)}, A^{(0)} + B^{(0)})$. We have

$$(A^{(0)})^{1/2}(A^{(0)} + B^{(0)})(A^{(0)})^{1/2} = \begin{pmatrix} \frac{1}{4} & \frac{m^2}{4n^2} \\ \frac{m^2}{4n^2} & \frac{n^2 - m^2}{4n^2(n-1)} \end{pmatrix} \otimes |\psi_1\rangle\langle\psi_1| + \begin{pmatrix} 0 & 0 \\ 0 & \frac{n^2 - m^2}{4n^2(n-1)} \end{pmatrix} \otimes \sum_{k=2}^n |\psi_k\rangle\langle\psi_k|. \quad (\text{H21})$$

Then

$$\begin{aligned} f(A^{(0)}, A^{(0)} + B^{(0)}) &= \text{Tr} \left(((A^{(0)})^{1/2}(A^{(0)} + B^{(0)})(A^{(0)})^{1/2})^{1/2} \right) \\ &= \sqrt{\frac{1}{4^2} + \left(\frac{m^2}{4n^2}\right)^2} + \frac{n^2 - m^2}{4n^2} = \frac{1}{2} - \frac{m^2}{4n^2} + O\left(\frac{m^4}{n^4}\right). \end{aligned} \quad (\text{H22})$$

In order to calculate $f(A^{(1)}, A^{(1)} + B^{(1)})$, we first note that

$$(A^{(1)})^{1/2}(A^{(1)} + B^{(1)})(A^{(1)})^{1/2} = \begin{pmatrix} (A^{(1,1)})^2 & 0 \\ 0 & (A^{(1,1)})^2 \end{pmatrix} + \begin{pmatrix} 0 & i\frac{m}{4n^2}A^{(1,1)} \\ -i\frac{m}{4n^2}A^{(1,1)} & 0 \end{pmatrix}. \quad (\text{H23})$$

Then we use the Taylor expansion formula for square root of positive matrices: $\sqrt{\Lambda^2 + Y} = \Lambda + \chi[Y] - \chi(\chi[Y]^2) + O(Y^3)$ for any positive diagonal matrix Λ and small Y [80], where

$$\chi[(\cdot)]_{ij} = \frac{(\cdot)_{ij}}{\Lambda_i + \Lambda_j}. \quad (\text{H24})$$

Let $A^{(1)} = \Lambda$ such that $\Lambda_1 = \frac{n^2 + 2n - m^2}{8n^2}$ and $\Lambda_k = \frac{n^2 - 2n + m^2}{8n^2(n-1)}$ for $k > 1$, we find that

$$f(A^{(1)}, A^{(1)} + B^{(1)}) = \frac{1}{2} - \left(\frac{m}{4n^2}\right)^2 \sum_{k=1}^n \frac{1}{4\Lambda_k} + O\left(\frac{m^3}{n^3}\right) = \frac{1}{2} - \frac{m^2}{8n^2} + O\left(\frac{m^3}{n^3}\right). \quad (\text{H25})$$

Therefore

$$1 - f(\mathcal{A}, \mathcal{A} + \delta\mathcal{A})^2 = 1 - f(A, A + B)^2 = \frac{3m^2}{4n^2} + O\left(\frac{m^3}{n^3}\right), \quad (\text{H26})$$

which serves as an upper bound on the infidelity of thermodynamic codes under depolarizing noise due to [Lemma 2](#).