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Quantum limits of superresolution in noisy environment

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We analyze the ultimate quantum limit of resolving two identical sources in a noisy environment. We prove that assuming a generic noise model such as thermal noise, quantum Fisher information of arbitrary states for the separation of the objects, quantifying the resolution, always converges to zero as the separation goes to zero. It contrasts with a noiseless case where it has been shown to be non-zero for a small distance in various circumstances, revealing the superresolution. In addition, we show that for an arbitrary measurement, dark count also makes the classical Fisher information of the measurement converge to zero as the separation goes to zero. Finally, a practically relevant situation, resolving two identical thermal sources, is quantitatively investigated by using quantum Fisher information and classical Fisher information of finite spatial mode multiplexing, showing that the amount of noise poses a limit on the resolution in a noisy system.

Rayleigh criterion poses a limit of resolution of two incoherent objects in classical optics [1, 2]. Recently, inspired by quantum optics and quantum metrology, it has been demonstrated that superresolution overcoming the Rayleigh limit is possible by replacing a conventional direct imaging technique with structured measurement techniques in a weak source regime [3]. Since the breakthrough, the superresolution technique has been generalized to apply to two incoherent thermal sources [4], arbitrary quantum states [5], two-dimensional imaging [6], three-dimensional imaging [7, 8], and an arbitrary number of sources [9–11]. Also, many proof-of-principle experiments have been conducted and demonstrated that elaborately constructed measurements enable surpassing the Rayleigh limit in practice [12–16]. The main idea of revealing the superresolution is to show that quantum Fisher information of the separation of two objects, the inverse of which gives a lower bound of the estimation error of the separation, is still non-zero when the separation gets smaller. This behavior contrasts with a conventional direct imaging method whose classical Fisher information vanishes as the separation drops to zero, which consequently makes the estimation error of the separation to diverge for a small separation.

More recently, the effects of noise on superresolution techniques start to be analyzed, and it has been shown that a signal-to-noise ratio in the system sets a fundamental resolution limit [17–19]. As a result, in the presence of noise, quantum or classical Fisher information has been shown to converge to zero in several specific circumstances such as in a weak source regime with a particular measurement [17], in resolving two incoherent thermal sources [18], and for measurement crosstalk [19].

In this Letter, we consider a more general situation of resolving two identical sources in arbitrary quantum states, assuming a generic noise model that is inevitable in experiments such as thermal noise and dark count. We show that such noises cause quantum and classical Fisher information to vanish for a small separation. We provide a quantitative analysis of noises in resolving two identical incoherent thermal sources and present the effect of noises in terms of quantum and classical Fisher information. Finally, we show that in the presence of thermal noise, finite spatial mode demultiplexing (fin-SPADE) measurement is nearly optimal when a signal-to-noise ratio is large.

The model.— Consider two identical sources with a separation $s > 0$ that emit light described by creation operators $\hat{c}^\dagger_{1,2}$ which are orthogonal each other. The emitted light reaches to the image plane with being attenuated such that $\hat{c}^\dagger_{1,2} \to \sqrt{\eta}\hat{a}^\dagger_{1,2} - \sqrt{1-\eta}\hat{u}^\dagger_{1,2}$ with $\hat{u}^\dagger_{1,2}$ describing the environment and being distorted as

$$\hat{a}^\dagger_1 \equiv \int_{-\infty}^{\infty} dx\psi(x-s/2)\hat{a}^\dagger_x, \quad \hat{a}^\dagger_2 \equiv \int_{-\infty}^{\infty} dx\psi(x+s/2)\hat{a}^\dagger_x, \tag{1}$$

where $\psi(x)$ represents a point-spread function (PSF) on the image plane, assumed to be real for simplicity. Also, the mode operators for different positions satisfy the canonical commutation relation (CCR) $[\hat{a}_x, \hat{a}^\dagger_{x'}] = \delta(x-x')$. In general, the two mode operators do not obey the CCR since two PSFs $\psi(x \pm s/2)$ have a non-zero overlap, i.e., $[\hat{a}_1, \hat{a}^\dagger_2] \neq 0$. Thus, we define symmetric and antisymmetric modes $\hat{a}_\pm$ to orthogonalize them [3–5, 7],

$$\hat{a}_\pm \equiv \frac{\hat{a}_1 \pm \hat{a}_2}{\sqrt{2(1\pm\delta)}}, \quad \delta(s) \equiv \int_{-\infty}^{\infty} dx\psi(x+s/2)\psi(x-s/2), \tag{2}$$

which satisfy the CCR $[\hat{a}_+, \hat{a}_-] = 0$. Now, the overall dynamics can be captured as

$$\hat{c}^\dagger_\pm \equiv \frac{\hat{c}^\dagger_1 \pm \hat{c}^\dagger_2}{\sqrt{2}} \to \sqrt{\eta_\pm}\hat{a}^\dagger_\pm - \sqrt{1-\eta_\pm}\hat{u}^\dagger_\pm, \tag{3}$$

where $\eta_\pm \equiv (1\pm\delta)\eta$ represent the effective attenuation rate, and $\hat{u}_\pm$ represent auxillary modes. Furthermore, the imaging process of estimating the separation s can be described by the following dynamics of the mode operators (see Appendix A) [5],

$$\frac{d\hat{a}_\pm}{ds} = i[\hat{H}^{\text{eff}}_\pm, \hat{a}_\pm], \tag{4}$$

where the effective Hamiltonians are written as

$$\hat{H}^{\text{eff}}_\pm = i\frac{d\theta_\pm}{ds}(\hat{c}^\dagger_\pm \hat{v}_\pm - \hat{c}_\pm \hat{v}^\dagger_\pm) - iB_\pm(\hat{a}_\pm \hat{b}^\dagger_\pm - \hat{a}^\dagger_\pm \hat{b}_\pm), \tag{5}$$

where $\hat{v}_\pm$ are the mode operators of environment before the transformation, $\theta_\pm \equiv \arccos\sqrt{\eta_\pm}$,

$$\hat{b}_\pm \equiv \frac{1}{B_\pm}\frac{\partial \hat{a}_\pm}{\partial s}, \quad \text{and} \quad B_\pm \equiv -\frac{\epsilon_\pm}{2\sqrt{1\pm\delta}}. \tag{6}$$

Thus, mode operators $\hat{b}_\pm$ represent the derivative of the spatial modes, $\hat{a}_\pm(s+ds) \approx \hat{a}_\pm(s) + \partial_s \hat{a}_\pm(s) ds$. We have also defined the following parameters:

$$\epsilon^2_\pm \equiv \Delta k^2 \mp \beta - \frac{\gamma^2}{1\pm\delta}, \quad \gamma \equiv \delta'(s), \quad \Delta k^2 \equiv \beta(0), \tag{7}$$

$$\beta(s) \equiv -\delta''(s) = \int_{-\infty}^{\infty} dx \frac{d\psi(x+s/2)}{dx}\frac{d\psi(x-s/2)}{dx}. \tag{8}$$

Here, γ represents the variation of the overlap from the changes of the separation s, Δk^2 accounts for the variance of the momentum operator $-i\partial_x$, and β represents interference between the derivatives of the PSFs. The effective Hamiltonian shows that when the separation s changes, the attenuation to the environment $\hat{u}_\pm$ varies as well as the derivative modes $\hat{b}_\pm$ are excited through the beam-splitter-like Hamiltonian, which is the last term in Eq. (5).

Quantum Fisher information in a noisy system.— In the perspective of quantum metrology, the resolution can be quantified by quantum Fisher information of the separation s [3]. Quantum Fisher information $H(\theta)$ of a quantum state $\hat{\rho}(\theta)$ for an unknown parameter θ gives a lower bound of the estimation error for θ, $\Delta^2\theta \geq 1/MH(\theta)$, which is so-called quantum Cramér-Rao inequality [20–22]. Here, M is the number of independent trials. Note that the quantum Cramér-Rao inequality implies that the estimation error diverges if the quantum Fisher information converges to zero. Now, we present our main result by introducing the following proposition:

Proposition 1. If a quantum state $\hat{\rho}(t)$ satisfies $\partial_t\hat{\rho}(t) \approx it\hat{\sigma}$ with some time-independent Hermitian operator $\hat{\sigma}$, the quantum Fisher information of t converges to zero as $t \to 0$ if the rank of the quantum state does not change around $t = 0$.

Proof. First, recall that quantum Fisher information is written as $H(t) = \text{Tr}[\hat{\rho}(t)\hat{L}(t)^2]$, where $\hat{L}$ is the so-called symmetric logarithmic derivative (SLD) operator satisfying the equation $\partial_t\hat{\rho}(t) = [\hat{\rho}(t)\hat{L}(t) + \hat{L}(t)\hat{\rho}(t)]/2$ [20–22]. Writing the quantum state in a spectral decomposition form $\hat{\rho}(t) = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, the SLD operator can be written as [22]

$$\begin{aligned}\hat{L}(t) &= 2\sum_{i,j:p_i+p_j>0} \frac{\langle\psi_i|\partial_t\hat{\rho}(t)|\psi_j\rangle}{p_i+p_j}|\psi_i\rangle\langle\psi_j| \\ &\approx 2it\sum_{i,j:p_i+p_j>0} \frac{\langle\psi_i|\hat{\sigma}|\psi_j\rangle}{p_i+p_j}|\psi_i\rangle\langle\psi_j| + O(t^2).\end{aligned} \tag{9}$$

The assumption that the rank of the quantum state does not change around $t = 0$ implies that $p_i + p_j > 0$ does not converge to 0 as $t \to 0$; hence, $H(t) = \text{Tr}[\hat{\rho}\hat{L}^2] \propto t^2 \to 0$ as $t \to 0$. □

On the other hand, when the rank of the quantum state changes around $t = 0$, there exists $p_i > 0$ such that $p_i \to 0$ as $t \to 0$. Therefore, the quantum Fisher information may not vanish for small t.

Let us consider quantum Fisher information of the separation s in the imaging problem and apply the proposition to it. First of all, after some algebra, one can find that $d\hat{\rho}/ds \propto is\hat{\sigma}$ with some Hermitian operator $\hat{\sigma}$ if the quantum state of light satisfies $\text{Tr}[\hat{\rho}_{c_+c_-}\hat{c}_-] = 0$ (see Appendix A). An important observation is that for identical objects, this condition is satisfied because $\hat{c}_-$ is an antisymmetric operator between $\hat{c}_1$ and $\hat{c}_2$. Thus, it confirms that the imaging process for identical sources satisfies the first condition of the proposition.

On the other hand, if the system suffers from a thermal noise, a relevant mode transforms as $\hat{a}$, i.e., $\hat{a} \to \sqrt{1-\nu}\hat{a} + \sqrt{\nu}\hat{e}$, with $\hat{e}$ describing the mode of the environment in a thermal state with a non-zero photon number and ν a coupling rate to the environment. As a result, a quantum state of light in mode $\hat{a}$ becomes full-rank. Moreover, since thermal noise may occur any modes in the system, it is natural to assume that the quantum state in modes $\hat{a}_\pm$ and $\hat{b}_\pm$ are full-rank in practice, which together with the proposition consequently shows that the quantum Fisher information of s generally vanishes as $s \to 0$ in a realistic situation. Note that attenuation channel, where the environment $\hat{e}$ is described by the vacuum, does not lead to the same conclusion since it does not transform the state to be a full-rank in general. Thus, in this case, quantum Fisher information can be larger than 0 when $s \to 0$ [5]. We emphasize that the proposition does not rule out the possibility of superresolution overcoming Rayleigh limit but implies that when the objects are very close and the system is noisy, quantum Fisher information of the separation can be extremely small. We supply an important example to analyze the effect of noise in the following section.

Two identical thermal sources.— Let us consider the problem of two incoherent thermal sources with a separation s. When the modes $\hat{a}_1$, $\hat{a}_2$ are occupied by thermal states with the mean photon number N_s at the same temperature, the symmetric and antisymmetric modes $\hat{a}_+$ and $\hat{a}_-$ can also be described by thermal states with

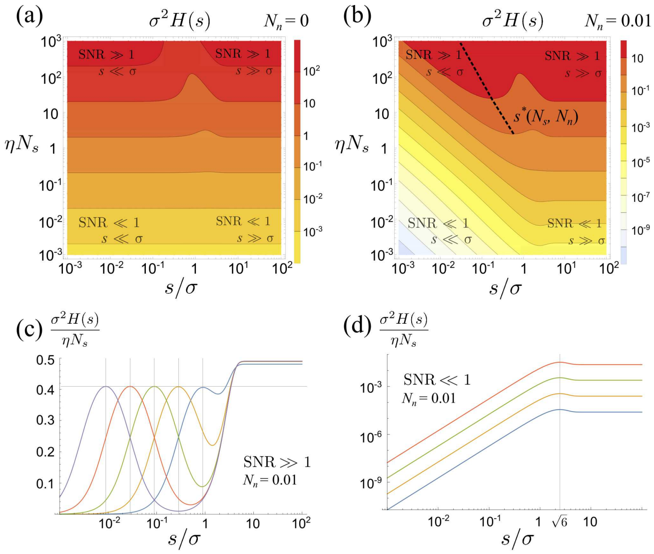

FIG. 1. Quantum Fisher information with respect to s and N_s with (a) $N_n = 0$ (noiseless), (b) $N_n = 0.01$. In the noiseless case, the quantum Fisher information does not decrease as s decreases. However, even with a small amount of noise photons, the quantum Fisher information drops for small s. The dotted line in (b) shows local maxima of quantum Fisher information for fixed ηN_s, $N_n > 0$, and SNR $\gg 1$ as shown in (c). (c) Normalized quantum Fisher information when SNR $\gg 1$ with respect to s with $\eta N_s = 10^4, 10^3, 10^2, 10, 1$ from the left to the right and, $N_n = 0.01$. The horizontal line represents $H(s^*)$ and the vertical lines s^* (see the main text). It captures the non-monotonic behavior of quantum Fisher information. (d) Normalized quantum Fisher information when SNR $\ll 1$ with $\eta N_s = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ from the bottom.

mean photon number $\eta N_s(1+\delta)$ and $\eta N_s(1-\delta)$, respectively [4, 5]. Introducing a thermal noise characterized by the mean photon number N_n onto the relevant modes, the quantum state is written as a product of states of symmetric and antisymmetric modes, $\hat{\rho} = \hat{\rho}_+ \otimes \hat{\rho}_-$, where

$$\hat{\rho}_\pm(s) = \hat{\rho}_{\mathrm{T}}(\eta N_s(1\pm\delta(s)) + N_n) \otimes \hat{\rho}_{\mathrm{T}}(N_n). \tag{10}$$

Here, each mode corresponds to $\hat{a}_\pm, \hat{b}_\pm$, respectively, and $\hat{\rho}_{\mathrm{T}}(N)$ represents a thermal state with a mean photon number N_n. It is worth emphasizing that when $N_n > 0$, the rank of the quantum state does not change as $s \to 0$. On the other hand, when $N_n = 0$, the rank of the state on the antisymmetric mode $\hat{a}_-$ changes as $s \to 0$ because $\delta \to 1$. This observation with the proposition implies that when $N_n > 0$, the quantum Fisher information vanishes as $s \to 0$ while it might not be the case when $N_n = 0$.

More specifically, quantum Fisher information of Gaussian states can be easily calculated [23–29], and we obtained $H(s) = H_+(s) + H_-(s)$ with (see Appendix B for the derivation)

$$H_\pm(s) = \frac{\eta^2 N_s^2 \gamma^2}{(\eta N_s(1\pm\delta) + N_n + 1)(\eta N_s(1\pm\delta) + N_n)} - \frac{2\eta^2 N_s^2[(1\pm\delta)(\delta''(0) \mp \delta''(s)) + \gamma^2]}{(2N_n+1)(2\eta N_s(1\pm\delta) + 2N_n + 1) - 1}. \tag{11}$$

Here, $H_\pm(s)$ represent the quantum Fisher information from symmetric and antisymmetric modes, respectively. Also, the first and second term in quantum Fisher information accounts for the changes of the mean photon number on mode $\hat{a}_\pm$ from the change of effective attenuation factors $\eta_\pm$ and the transformation of the shape of spatial modes $\hat{a}_\pm(s)$ into $\hat{a}_\pm(s+ds) \approx \hat{a}_\pm(s) + ds\partial_s\hat{a}_\pm$, respectively.

First of all, the quantum Fisher information recovers previous results when $N_n = 0$ in Refs. [4, 5]. More importantly, one can verify that the quantum Fisher information vanishes as $s \to 0$ unless $N_n = 0$. Fig. 1 (a) and (b) show the quantum Fisher information $H(s)$ in the ideal case and the noisy case with a Gaussian PSF, $\psi(x) = e^{-x^2/4\sigma^2}/(2\pi\sigma^2)^{1/4}$. A remarkable difference between the two cases is that as $s \to 0$, the quantum Fisher information in the noisy case rapidly drops as expected while it does not change in the ideal case. For example, when the separation s is 0.01σ and the mean signal photons ηN_s is 1, the quantum Fisher information $H(s)$ is $0.5/\sigma^2$ and $6\times 10^{-4}/\sigma^2$ for noiseless case and noisy case with $N_n = 0.01$, respectively, which clearly shows that even a small amount of noise can be critical to the resolution.

Let us first consider the regime where the signal-to-noise ratio (SNR) is large SNR $\equiv \eta N_s/N_n \gg 1$. In this regime, Fig. 1 (c) shows another interesting feature of quantum Fisher information; it is not monotonic with respect to s. For a small separation $s \ll \sigma$ in the regime, the quantum Fisher information in the case of Gaussian PSF can be approximated by

$$H(s) \approx \frac{4\eta^2 N_s^2 s^2}{\eta^2 N_s^2 s^4 + 8\eta N_s s^2\sigma^2 + 64 N_n(N_n+1)\sigma^4}, \tag{12}$$

which has the local maximum

$$\begin{aligned} H(s^*) &\approx \frac{\eta N_s}{2\sigma^2}\frac{\sqrt{N_n^2+N_n}}{(N_n+\sqrt{N_n^2+N_n})(\sqrt{N_n^2+N_n}+N_n+1)} \\ &\overset{N_n \ll 1}{\approx} \frac{\eta N_s}{2\sigma^2}\frac{1}{1+2\sqrt{N_n}} \end{aligned} \tag{13}$$

at $s^* = 2\sqrt{2}(N_n^2+N_n)^{1/4}\sigma/\sqrt{\eta N_s}$, which is shown in Fig. 1 (c). Thus, s^* is a characteristic length scale in this regime, and if $s \ll s^*$, the quantum Fisher information can be further approximated as

$$H(s) \approx \frac{\eta^2 N_s^2}{N_n(N_n+1)}\Delta k^4 s^2 = \frac{\eta^2 N_s^2}{N_n(N_n+1)}\frac{s^2}{16\sigma^4} \overset{N_n \ll 1}{\approx} \frac{\eta^2 N_s^2}{N_n}\frac{s^2}{16\sigma^4} \quad \text{if } \eta N_s \gg N_n \text{ and } s \ll s^*. \tag{14}$$

One can observe that when SNR $\gg 1$, $N_n \ll 1$ and $s \ll s^*$, quantum Fisher information per a signal photon is proportional to the SNR $H(s)/\eta N_s \propto \eta N_s/N_n$, which is consistent with the previous results [17, 18]. Also, quantum Fisher information decreases as s quadratically as $s \to 0$.

On the other hand, when a SNR is small, i.e., $\eta N_s/N_n \ll 1$, and the separation is small, $s \ll \sqrt{6}\sigma$, the quantum Fisher information is approximated by

$$H(s) \approx \frac{\eta^2 N_s^2}{2N_n(N_n+1)}[3\Delta k^4 + \delta^{(4)}(0)]s^2 = \frac{\eta^2 N_s^2}{N_n(N_n+1)}\frac{3s^2}{16} \quad \text{if } \eta N_s \ll N_n \text{ and } s \ll \sqrt{6}\sigma, \tag{15}$$

which is shown in Fig. 1 (d). Again, when $N_n \ll 1$, quantum Fisher information per a signal photon is proportional to the SNR, $H(s)/\eta N_s \propto \eta N_s/N_n$, and decreases as s quadratically as $s \to 0$.

Finally, for a large separation $s \gg \sigma$, the quantum Fisher information can be approximated as $H(s) \approx 2\eta^2 N_s^2 \Delta k^2/[2N_n^2 + \eta N_s + 2N_n(\eta N_s + 1)]$, which shows that the noise decreases the quantum Fisher information for a large separation as well.

As a remark, we compare the quantum Fisher information in Eq. (11) with the one obtained in Ref. [18] where the same type of noise was considered in the imaging process with two incoherent thermal sources. The discrepancy of the expression is ascribed by the fact that the noise model used in Ref. [18] assumes that noise occurs only on the modes $\hat{a}_\pm$ whereas our noise model assumes the same amount of noise on $\hat{b}_\pm$ modes. Nevertheless, the previous result has also revealed that the quantum Fisher information vanishes as $s \to 0$ because the rank of the quantum state does not change even if we assume $N_n = 0$ for $\hat{b}_\pm$ modes.

Noisy detectors.— As pointed out in Ref. [18], the above thermal noise model might not be appropriate to consider the effect of dark count because quantum Fisher information is a measurement-independent quantity while dark count is a feature of the measurement device. In order to analyze the effect of dark count, we employ classical Fisher information, the inverse of which gives a lower bound of estimation error for a given measurement apparatus, $\Delta^2\theta \geq 1/MF(\theta)$ [31, 32]. By introducing the following proposition, we show that dark count makes the classical Fisher information converge to zero in the same condition of the proposition 1.

Proposition 2. Consider a quantum state that satisfies $\partial_s\hat{\rho} \approx it\hat{\sigma}$ for small t with a time-independent Hermitian operator $\hat{\sigma}$ and a positive-operator-valued-measurement (POVM) $\{\hat{\Pi}_k\}_{k\in K}$ with $\hat{\Pi}_k \geq 0$ and $\sum_{k\in K}\hat{\Pi}_k = \mathbb{1}$. If the support of $p_k = \text{Tr}[\hat{\rho}(t)\hat{\Pi}_k]$, $\{k \in K | p_k > 0\}$, does not change as $t \to 0$, the classical Fisher information converges to zero as $t \to 0$.

Proof. Let us recall that the classical Fisher information of probability distribution $\{p_k\}$ is given by

$$F(t) = \sum_{p_k>0}\frac{1}{p_k}\left(\frac{\partial p_k}{\partial t}\right)^2. \tag{16}$$

The probability of obtaining outcome k by measuring a quantum state $\hat{\rho}(t)$ with POVM $\{\hat{\Pi}_k\}_{k\in K}$ and its derivative with respect to t are given by

$$p_k = \text{Tr}[\hat{\Pi}_k\hat{\rho}(t)], \tag{17}$$

and

$$\frac{\partial p_k}{\partial t} \approx it\text{Tr}[\hat{\Pi}_k\hat{\sigma}]. \tag{18}$$

Therefore, the classical Fisher information of small t is written as

$$F(t) = \sum_{p_k>0}\frac{1}{p_k}\left(\frac{\partial p_k}{\partial t}\right)^2 \approx t^2\sum_{p_k>0}\frac{1}{p_k}\left(i\text{Tr}[\hat{\Pi}_k\hat{\sigma}]\right)^2. \tag{19}$$

Similar to quantum Fisher information, classical Fisher information converges to zero as $t \to 0$ unless there exists p_k such that $p_k \to 0$. □

In realistic situations, dark count rates are generally non-zero in all relevant detectors; thus, it is natural to expect that the classical Fisher information vanishes $F(t) \to 0$ as $t \to 0$ in practice. Moreover, the proposition can be applied to measurement crosstalk [19] which makes all measurement outcomes mixed so that eventually the probability of obtaining each outcome becomes non-zero. Also, the proposition indicates the limitation of direct imaging which always gives non-zero probabilities on any pixels on image plane for generic PSFs even in the noiseless case. As a final remark, proposition 2 does not imply the failure of superresolution; it suggests that dark count rate can pose a limit on the resolution as a thermal noise on quantum Fisher information in the previous section.

Finite spatial mode demultiplexing.— Finally, we analyze the achievable resolution using the method of fin-SPADE. In the noiseless case, Fin-SPADE method employs a photon-counting for each Hermite-Gaussian mode $h_q(x)$ on the image plane, which has been shown to be optimal if an enough number of Hermite-Gaussian modes are accessible in experiment [3, 5]. In general, the analytical expression of the classical Fisher information of fin-SPADE is difficult to obtain due to the statistical correlations between different modes of the measurement. We thus obtain the lower bound of the classical Fisher information using an inequality $F(\theta) \geq \dot{\vec{\mu}}^{\mathrm{T}} C^{-1} \dot{\vec{\mu}}$, where $\vec{\mu}$ and C denote the mean and covariance matrix of the outcome distribution, and $\dot{\vec{\mu}} \equiv \partial_s \vec{\mu}$ [34]. We consider a finite number of Hermite-Gaussian modes h_q with $0 \leq q \leq Q-1$ with $Q = 15$ in the presence of thermal noise in the problem of resolving two incoherent thermal sources. We numerically confirmed that increasing Q larger than 15 does not change the classical Fisher information for $10^{-3} \leq s/\sigma \leq 1$. Fig. 2 shows the ratio of the lower bound of the classical Fisher information of Fin-SPADE to quantum Fisher information (see Appendix C for details). It clearly shows that for the large number of signal photons ηN_s, the ratio converges to the unity, which means that fin-SPADE is optimal in that regime. Even when ηN_s is small, the lower bound of classical Fisher information gives at least 65% of quantum Fisher information. Hence, even in the presence of noise, the performance of fin-SPADE method is not degraded significantly when it is compared to quantum Fisher information. A particular way to improve this further is to directly measure the incoming photon numbers onto the symmetric and antisymmetric modes and their derivative modes $\{\hat{a}_\pm, \hat{b}_\pm\}$ (see Appendix B for details). In general, the implementation of such a measurement requires a prior information, which might be overcome by using adaptive method [33].

Conclusions and discussion.— In this Letter, we have investigated the effect of noise on the resolution of two identical sources with an arbitrary state using quantum and classical Fisher information and shown that the Fisher information generically converges to zero if the system has a non-zero thermal noise or dark count rate. We have shown that in the problem of resolving two incoherent thermal sources with the number of signal photons being larger than that of noise photons, a signal-to-noise

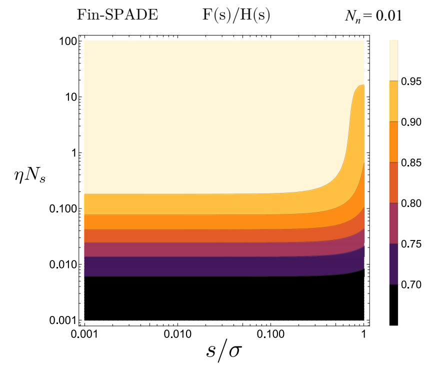

FIG. 2. Relative classical Fisher information to quantum Fisher informaiton of fin-SPADE with respect to different separation s and mean signal photon number ηN_s.

ratio poses a fundamental limit. Finally, we have shown that for a large signal-to-noise ratio, finite spatial demultiplexing measurement is nearly optimal.

Throughout the Letter, we are assuming that the two sources are identical. Thus, the same conclusion might not hold if the sources are not identical [35–38]. It would be interesting to analyze the problem of resolving non-identical sources.

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APPENDIX A: DYNAMICS OF IMAGING PROCESS

In this section, we provide the details of the imaging process and show that the quantum state of imaging process can be linearized by the separation s when $s \ll 1$. As introduced in the main text, two identical sources with separation $s > 0$ emit light that excites modes characterized by $\hat{c}^\dagger_{1,2}$, and the emitted light is attenuated and distorted when it arrives at the image plane such that

$$\hat{a}^\dagger_{1,2} = \sqrt{\eta}\hat{c}^\dagger_{1,2} + \sqrt{1-\eta}\hat{v}^\dagger_{1,2}. \tag{A1}$$

Introducing the symmetric and antisymmetric mode operators,

$$\hat{a}_\pm = \frac{\hat{a}_1 \pm \hat{a}_2}{\sqrt{2(1\pm\delta(s))}}, \qquad \hat{c}_\pm = \frac{\hat{c}_1 \pm \hat{c}_2}{\sqrt{2}}, \tag{A2}$$

and inverting Eq. (3) in the main text, we write

$$\hat{a}_\pm = \sqrt{\eta_\pm}\hat{c}_\pm + \sqrt{1-\eta_\pm}\hat{v}_\pm = e^{i\hat{H}_\pm\theta_\pm}\hat{c}_\pm e^{-i\hat{H}_\pm\theta_\pm}, \tag{A3}$$

with $\eta_\pm \equiv \eta(1\pm\delta)$, $\hat{H}_\pm = i(\hat{c}_\pm^\dagger \hat{v}_\pm - \hat{v}_\pm^\dagger \hat{c}_\pm)$, and $\theta_\pm \equiv \arccos\sqrt{\eta_\pm}$. Thus, when the separation is s, the quantum state on the image plane is written as

$$\hat{\rho}(s) = \text{Tr}_{u_\pm}\left[e^{-i(\hat{H}_+\theta_+ + \hat{H}_-\theta_-)}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_+v_-}\right)e^{i(\hat{H}_+\theta_+ + \hat{H}_-\theta_-)}\right], \tag{A4}$$

where $\hat{\rho}_{c_+c_-}$ represents the quantum state of light emitted by the sources, and $\hat{\sigma}_{v_+v_-}$ represents the quantum state of the environment.

From now on, we analyze the dynamics of the system and show that the derivative of the quantum state with respect to the separation s is linearized in s for a small s limit. When s infinitesimally changes, the quantum state can be written as

$$\hat{\rho}(s+ds) = \text{Tr}_{u_\pm}\left[e^{-i(\hat{H}_+\tilde{\theta}_+ + \hat{H}_-\tilde{\theta}_-)}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_+v_-}\right)e^{i(\hat{H}_+\tilde{\theta}_+ + \hat{H}_-\tilde{\theta}_-)}\right] \tag{A5}$$

Here, $\tilde{\theta} = \theta(s+ds)$. Notice that the quantum state is written in $\hat{a}_\pm(s+ds)$ modes. In order to write the quantum state in terms of $\hat{a}_\pm(s)$ modes as Eq. (A4), we describe the dynamics of the mode operators $\hat{a}_\pm$. Using the Heisenberg equation of motion, we obtain

$$\frac{d\hat{a}_\pm}{ds} = i\frac{d\theta_\pm}{ds}[\hat{H}_\pm,\hat{a}_\pm] + \frac{\partial\hat{a}_\pm}{\partial s} = i\frac{d\theta_\pm}{ds}[\hat{H}_\pm,\hat{a}_\pm] - \frac{\epsilon_\pm}{2\sqrt{1\pm\delta}}\hat{b}_\pm = i\left[\frac{d\theta_\pm}{ds}\hat{H}_\pm + i\frac{\epsilon_\pm}{2\sqrt{1\pm\delta}}(\hat{a}_\pm\hat{b}_\pm^\dagger - \hat{a}_\pm^\dagger\hat{b}_\pm), \hat{a}_\pm\right] \equiv i[\hat{H}_\pm^{\text{eff}},\hat{a}_\pm]. \tag{A6}$$

Defining $\hat{\gamma} \equiv e^{-i(\hat{H}_+\theta_+ + \hat{H}_-\theta_-)}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_+v_-}\right)e^{i(\hat{H}_+\theta_+ + \hat{H}_-\theta_-)}\otimes|0\rangle\langle 0|_{b_+}\otimes|0\rangle\langle 0|_{b_-}$ and $B_\pm \equiv -\epsilon_\pm/(2\sqrt{1\pm\delta})$ and using the equation of motion Eq. (A6) and Eq. (A4), Eq. (A5) can be equivalently written as,

$$\begin{aligned}
\hat{\rho}(s+ds) &\approx \text{Tr}_{u_\pm}\left[e^{-i(\hat{H}_+^{\text{eff}}+\hat{H}_-^{\text{eff}})ds}\hat{\gamma}e^{i(\hat{H}_+^{\text{eff}}+\hat{H}_-^{\text{eff}})ds}\right] \\
&\approx e^{-ds[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-)]}\text{Tr}_{u_\pm}[e^{-i(\hat{H}_+\tilde{\theta}_+ + \hat{H}_-\tilde{\theta}_-)}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_+v_-}\otimes|0\rangle\langle 0|_{b_+}\otimes|0\rangle\langle 0|_{b_-}\right)e^{i(\hat{H}_+\tilde{\theta}_+ + \hat{H}_-\tilde{\theta}_-)}] \\
&\quad\times e^{ds[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-)]} \\
&\approx \left[1 - ds[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-)]\right] \\
&\quad\times \text{Tr}_{u_\pm}[(1 - ids(\hat{H}_+\partial_s\theta_+ + \hat{H}_-\partial_s\theta_-))\hat{\gamma}(1 + ids(\hat{H}_+\partial_s\theta_+ + \hat{H}_-\partial_s\theta_-))] \\
&\quad\times \left[1 + ds[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-)]\right] \\
&= \hat{\rho}(s) - ds[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-), \hat{\rho}(s)] - ids\text{Tr}_{u_\pm}\left([\hat{H}_+\partial_s\theta_+ + \hat{H}_-\partial_s\theta_-, \hat{\gamma}]\right).
\end{aligned} \tag{A7}$$

Thus, the derivative of the quantum state is written as

$$\frac{d\hat{\rho}(s)}{ds} = -[B_+(\hat{a}_+\hat{b}_+^\dagger - \hat{a}_+^\dagger\hat{b}_+) + B_-(\hat{a}_-\hat{b}_-^\dagger - \hat{a}_-^\dagger\hat{b}_-), \hat{\rho}(s)] - i\text{Tr}_{u_\pm}\left(\left[\hat{H}_+\partial_s\theta_+ + \hat{H}_-\partial_s\theta_-, \hat{\gamma}\right]\right). \tag{A8}$$

Now, let us consider a regime where the separation s is small. For small s, we can approximate

$$B_+ \approx -\frac{1}{4}\sqrt{(\delta^{(4)}(0) - \delta''(0)^2)}s + O(s^2) \propto \alpha_+ s + O(s^2) \tag{A9}$$

$$B_- \approx -\sqrt{\frac{1}{12\delta''(0)}\left(\frac{\delta^{(6)}(0)}{5} - \frac{\delta^{(4)}(0)^2}{3\delta''(0)}\right)}s + O(s^2) \propto \alpha_- s + O(s^2), \tag{A10}$$

where $\delta^{(n)}(0) \equiv \partial^n\delta(s)/\partial s^n|_{s=0}$. Thus, the first commutator in Eq. (A8) is linearized in s for small s. Now, let us focus on the second term. Let us assume that $\hat{\sigma}_{v_+v_-} = \hat{\sigma}_{v_+}\otimes\hat{\sigma}_{v_-}$ which is a natural choice as a quantum state for environment. Note that the quantum state is written in $\hat{a}_\pm(s)$ modes. For small s, noting that

$$\frac{d\theta_+}{ds} \approx -\frac{\eta\delta'(s)}{2\sqrt{\eta(1+\delta(s))}\sqrt{1-\eta(1+\delta(s))}} \approx -\frac{\sqrt{\eta}\delta''(s)s}{\sqrt{8(1-2\eta)}} + \mathcal{O}(s^2) \tag{A11}$$

$$\frac{d\theta_-}{ds} \approx \frac{\eta\delta'(s)}{2\sqrt{\eta(1-\delta(s))}\sqrt{1-\eta(1-\delta(s))}} \approx \frac{\sqrt{\eta}\delta'(s)}{2\sqrt{1-\delta(s)}} \approx -\sqrt{-\frac{\eta\delta''(0)}{2}} + \mathcal{O}(s), \tag{A12}$$

we have

$$\mathrm{Tr}_{v_\pm}\left([\hat{H}_+\partial_s\theta_+,\hat{\gamma}]\right)\propto s+O(s^2). \tag{A13}$$

On the other hand, we can expand the remaining term in s around $s=0$ as

$$\begin{aligned}
&\mathrm{Tr}_{u_\pm}\left([\hat{H}_-\partial_s\theta_-,e^{-i(\hat{H}_+\theta_++\hat{H}_-\theta_-)}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_+}\otimes\hat{\sigma}_{v_-}\right)e^{i(\hat{H}_+\theta_++\hat{H}_-\theta_-)}]\right)\\
=&\frac{d\theta_-}{ds}\mathrm{Tr}_{u_+}\left(e^{-i\hat{H}_+\theta_+}\left(\mathrm{Tr}_{u_-}\left[[\hat{H}_-,e^{-i\hat{H}_-\theta_-}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_-}\right)e^{i\hat{H}_-\theta_-}]\right]\otimes\hat{\sigma}_{v_+}\right)e^{i\hat{H}_+\theta_+}\right)\\
\approx&\frac{d\theta_-}{ds}\mathrm{Tr}_{u_+}\left(e^{-i\hat{H}_+\theta_+}\left(\mathrm{Tr}_{u_-}\left[[\hat{H}_-,(1-is\hat{H}_-\partial_s\theta_-)e^{-i\pi\hat{H}_-/2}\left(\hat{\rho}_{c_+c_-}\otimes\hat{\sigma}_{v_-}\right)e^{i\pi\hat{H}_-/2}(1+is\hat{H}_-\partial_s\theta_-)]\right]\otimes\hat{\sigma}_{v_+}\right)e^{i\hat{H}_+\theta_+}\right)\\
\approx&\frac{d\theta_-}{ds}\mathrm{Tr}_{u_+}\left(e^{-i\hat{H}_+\theta_+}\left(\mathrm{Tr}_{u_-}\left[[\hat{H}_-,(1-is\hat{H}_-\partial_s\theta_-)\left(\hat{\rho}_{c_+v_-}\otimes\hat{\sigma}_{c_-}\right)(1+is\hat{H}_-\partial_s\theta_-)]\right]\otimes\hat{\sigma}_{v_+}\right)e^{i\hat{H}_+\theta_+}\right)\\
\approx&\frac{d\theta_-}{ds}\mathrm{Tr}_{u_+}\left(e^{-i\hat{H}_+\theta_+}\left(\mathrm{Tr}_{u_-}\left[[\hat{H}_-,\hat{\rho}_{c_+v_-}\otimes\hat{\sigma}_{c_-}]\right]\otimes\hat{\sigma}_{v_+}\right)e^{i\hat{H}_+\theta_+}\right)+\mathcal{O}(s).
\end{aligned} \tag{A14}$$

We have used the fact that $\theta_-\to\pi/2$ as $s\to 0$ to expand the unitary operator $e^{-i\hat{H}_-\theta_-}$. Thus, the zeroth order of s becomes zero if $\mathrm{Tr}_{u_-}\left([\hat{H}_-,\hat{\rho}_{c_+u_-}\otimes\hat{\sigma}_{c_-}]\right)=0$. The condition becomes

$$\begin{aligned}
&\mathrm{Tr}_{u_-}\left([\hat{H}_-,\hat{\rho}_{c_+u_-}\otimes\hat{\sigma}_{c_-}]\right)=i\mathrm{Tr}_{u_-}\left([\hat{c}_-^\dagger\hat{v}_--\hat{v}_-^\dagger\hat{c}_-,\hat{\rho}_{c_+u_-}\otimes\hat{\sigma}_{c_-}]\right)\\
=&i\mathrm{Tr}_{u_-}(\hat{\rho}_{c_+v_-}\hat{v}_-)[\hat{c}_-^\dagger,\hat{\sigma}_{c_-}]+i\mathrm{Tr}_{u_-}(\hat{\rho}_{c_+v_-}\hat{v}_-^\dagger)[\hat{\sigma}_{c_-}^\dagger,\hat{c}_-]=0.
\end{aligned} \tag{A15}$$

Thus, this condition is satisfied if $\mathrm{Tr}_{u_-}(\hat{\rho}_{c_+v_-}\hat{v}_-)=0$. Since we assume two identical objects, the quantum state $\hat{\rho}_{c_+c_-}$ satisfies

$$\mathrm{Tr}_{c_-}[\hat{\rho}_{c_+c_-}\hat{c}_-]=\mathrm{Tr}_{c_-}[\hat{\rho}_{c_+c_-}\frac{\hat{c}_1-\hat{c}_2}{\sqrt{2}}]=\mathrm{Tr}_{c_-}[\hat{\rho}_{c_+c_-}\frac{\hat{c}_2-\hat{c}_1}{\sqrt{2}}]. \tag{A16}$$

Thus, $\mathrm{Tr}_{u_-}[\hat{\rho}_{c_+v_-}\hat{v}_-]=0$. Hence, we have shown that $\partial\hat{\rho}/\partial s\propto s+O(s^2)$.

APPENDIX B: QUANTUM FISHER INFORMATION FOR TWO INCOHERENT THERMAL SOURCES

In this section, we derive quantum Fisher information of the separation of two identical thermal sources and obtain the optimal measurement corresponding to the quantum Fisher information. Quantum Fisher information of n-mode Gaussian states is well-known, which is given by [23–29]

$$H(s)=-\mathrm{Tr}[G\frac{\partial V(s)}{\partial s}], \tag{B1}$$

where $V(s)$ is the $2n\times 2n$ covariance matrix of the Gaussian state $\hat{\rho}$, $V_{ij}=\mathrm{Tr}[\hat{\rho}\{\hat{Q}_i-\langle\hat{Q}_i\rangle,\hat{Q}_j-\langle\hat{Q}_j\rangle\}]/2$, $\hat{Q}=(\hat{x}_1,\hat{p}_1,\cdots,\hat{x}_n,\hat{p}_n)^{\mathrm{T}}$ and Ω a skew symmetric matrix giving the canonical commutation relation,

$$\Omega=\mathbb{1}_n\otimes\omega,\quad \omega=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{B2}$$

and G is a $2n\times 2n$ real symmetric matrix satisfying

$$4V(s)GV(s)+\Omega G\Omega+2\frac{\partial V(s)}{\partial s}=0. \tag{B3}$$

Here, $\mathbb{1}_n$ denotes the $n\times n$ identity matrix.

When two incoherent sources of a distance s are in thermal states with a same temperature characterized by the mean photon number N_s, the quantum state can be written as a product form of states in modes $\hat{c}_\pm$, $\hat{\rho}_{\mathrm{T}}(N_s)\otimes\hat{\rho}_{\mathrm{T}}(N_s)$. When the light arrived at the image plane, the quantum state is described in symmetric and antisymmetric modes as,

$$\begin{aligned}
\hat{\rho}(s)&=\mathrm{Tr}_{v_+v_-}\left[e^{-i(\hat{H}_+\theta_++\hat{H}_-\theta_-)}\left[\hat{\rho}_{\mathrm{T}}(N_s)\otimes\hat{\rho}_{\mathrm{T}}(N_s)\otimes|0\rangle\langle 0|\otimes|0\rangle\langle 0|\right]_{c_+c_-v_+v_-}e^{i(\hat{H}_+\theta_++\hat{H}_-\theta_-)}\otimes|0\rangle\langle 0|_{b_+}\otimes|0\rangle\langle 0|_{b_-}\right]\\
&=[\hat{\rho}_{\mathrm{T}}(\eta_+N_s)\otimes\hat{\rho}_{\mathrm{T}}(\eta_-N_s)]_{a_+a_-}\otimes|0\rangle\langle 0|_{b_+}\otimes|0\rangle\langle 0|_{b_-}\\
&=[\hat{\rho}_{\mathrm{T}}(\eta N_s(1+\delta))\otimes\hat{\rho}_{\mathrm{T}}(\eta N_s(1-\delta))]_{a_+a_-}\otimes|0\rangle\langle 0|_{b_+}\otimes|0\rangle\langle 0|_{b_-}.
\end{aligned} \tag{B4}$$

Let us introduce a thermal noise assuming that the thermal photon number of the noise is the same on the relevant modes. Thus, the thermal photon numbers on each mode increase as

$$\hat{\rho}(s)=[\hat{\rho}_{\rm T}(\eta N_s(1+\delta)+N_n)\otimes\hat{\rho}_{\rm T}(\eta N_s(1-\delta)+N_n)]_{a_+a_-}\otimes[\hat{\rho}_{\rm T}(N_n)\otimes\hat{\rho}_{\rm T}(N_n)]_{b_+b_-}. \tag{B5}$$

Let us first focus on the symmetric modes $\hat{a}_+,\hat{b}_+$. For infinitesimal change of s, The quantum state in the symmetric modes can be written as

$$\hat{\rho}_\pm(s)=\hat{\rho}_{\rm T}(\eta N_s(1\pm\delta)+N_n)\otimes\hat{\rho}_{\rm T}(N_n). \tag{B6}$$

The quantum state with an infinitesimal change ds of s is given by

$$\begin{aligned}\hat{\rho}_+(s+ds)&\approx {\rm Tr}_{u_+u_-}\left[e^{-i\hat{H}_+^{\rm eff}ds}e^{-i\hat{H}_+\theta_+}\left[\hat{\rho}_{\rm T}(N_s)\otimes|0\rangle\langle 0|\right]_{c_+v_+}e^{i\hat{H}_+\theta_+}e^{i\hat{H}_+^{\rm eff}ds}\right]\\ &\approx e^{-B_+ds(\hat{a}_+^\dagger\hat{b}_+-\hat{a}_+\hat{b}_+^\dagger)}\left[\hat{\rho}_{\rm T}(\tilde{\eta}_+N_s)\otimes|0\rangle\langle 0|\right]_{a_+b_+}e^{B_+ds(\hat{a}_+^\dagger\hat{b}_+-\hat{a}_+\hat{b}_+^\dagger)},\end{aligned} \tag{B7}$$

where $\tilde{\eta}=\eta[1+\delta(s+ds)]$. Again, introducing the thermal noise, the state becomes

$$\hat{\rho}_+(s+ds)\approx e^{-B_+ds(\hat{a}_+\hat{b}_+^\dagger-\hat{a}_+^\dagger\hat{b}_+)}\left[\hat{\rho}_{\rm T}(\eta N_s[1+\delta(s+ds)]+N_n)\otimes\hat{\rho}_{\rm T}(N_n)\right]_{a_+b_+}e^{B_+ds(\hat{a}_+\hat{b}_+^\dagger-\hat{a}_+^\dagger\hat{b}_+)}. \tag{B8}$$

Thus, the covariance matrix of the symmetric modes can be written as

$$V_+(s+ds)=SV_+(s)S^{\rm T}=\begin{pmatrix}\mu_+^2v_1+(1-\mu_+^2)v_2 & -\mu_+\sqrt{1-\mu_+^2}(v_2-v_1)\\ -\mu_+\sqrt{1-\mu_+^2}(v_2-v_1) & \mu_+^2v_2+(1-\mu_+^2)v_1\end{pmatrix}\otimes\mathbb{1}_2, \tag{B9}$$

$$V_+(s)={\rm diag}(v_1,v_1,v_2,v_2),\quad S=\begin{pmatrix}\mu_+ & \sqrt{1-\mu_+^2}\\ -\sqrt{1-\mu_+^2} & \mu_+\end{pmatrix}\otimes\mathbb{1},\quad \mu_+=\cos B_+ds. \tag{B10}$$

Here, the first (second) row and column of the first matrix represents the mode $\hat{a}_+$ ($\hat{b}_+$), $v_1=(1+\delta(s+ds))\eta N_s+N_n+1/2$, and $v_2=N_n+1/2$, and μ_+ transmittance of the beam splitter unitary operator. Noting that

$$\mu_+\simeq 1-\frac{1}{2}B_+(s)^2ds^2=1+ds^2\left(\frac{\delta''(0)-\delta''(s)}{8(1+\delta)}+\frac{\delta'(s)^2}{8(1+\delta)^2}\right), \tag{B11}$$

the derivative of the covariance matrix with respect to s is written as

$$\frac{\partial V_+(s)}{\partial s}=\left[-(v_2-v_1)\sqrt{-\frac{\partial^2\mu_+}{\partial s^2}}\sigma_x+\eta N_s\delta'(s)|0\rangle\langle 0|\right]\otimes\mathbb{1}, \tag{B12}$$

where σ_x is the Pauli-x matrix. One can readily find the solution of Eq. (B3) for G which is given by

$$G=\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\otimes\mathbb{1} \tag{B13}$$

with

$$g_{11}=\frac{-2\eta N_s\delta'(s)}{4v_1^2-1},\quad g_{12}=g_{21}=\frac{-2(v_2-v_1)}{4v_1v_2-1}\sqrt{-\frac{\partial^2\mu_+}{\partial s^2}} \tag{B14}$$

$$g_{22}=0 \text{ if } v_2>1/2,\quad g_{22} \text{ is arbitrary if } v_2=1/2. \tag{B15}$$

Thus,

$$H_+(s)=2\left[\frac{2\eta^2N_s^2\delta'(s)^2}{4v_1^2-1}+\frac{4(v_2-v_1)^2}{4v_1v_2-1}\left(-\frac{\partial^2\mu_+}{\partial s^2}\right)\right] \tag{B16}$$

After some simplification of the expression, we obtain the quantum Fisher information from the symmetric mode, which is given by

$$H_+(s)=\frac{\eta^2N_s^2\delta'(s)^2}{(\eta N_s(1+\delta)+N_n+1)(\eta N_s(1+\delta)+N_n)}-\frac{2\eta^2N_s^2[(1+\delta)(\delta''(0)-\delta''(s))+\delta'(s)^2]}{(2N_n+1)(2\eta N_s(1+\delta)+2N_n+1)-1}. \tag{B17}$$

Similarly, one can easily find that

$$H_-(s)=\frac{\eta^2N_s^2\delta'(s)^2}{(\eta N_s(1-\delta)+N_n+1)(\eta N_s(1-\delta)+N_n)}-\frac{2\eta^2N_s^2[(1-\delta)(\delta''(0)+\delta''(s))+\delta'(s)^2]}{(2N_n+1)(2\eta N_s(1-\delta)+2N_n+1)-1}. \tag{B18}$$

Let us find the optimal measurement that gives the classical Fisher information equal to quantum Fisher information. The optimal measurement can be found by diagonalizing the matrix G [29]. Let us first consider the symmetric mode. The matrix G_+ can be diagonalized as

$$G_+=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\otimes\mathbb{1}_2=O_+^{\mathrm{T}}\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix}O_+\otimes\mathbb{1}_2,\qquad \text{where}\quad O_+=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{B19}$$

Thus, G_+ can be decoupled into two-mode by a beam splitter corresponding to the symplectic matrix $O_+\otimes\mathbb{1}_2$. To be more specific, the beam splitter angle θ with the transmittance and reflectance being $\cos\theta$ and $\sin\theta$ is given by $\theta=1/2\tan^{-1}(2g_{12}/g_{11})$. Similarly, G_- for anti-symmetric modes can also be decoupled by a beam splitter represented by $O_-\otimes\mathbb{1}_2$, which can be obtained in the same way.

Note that the symmetric logarithmic derivative operator for Gaussian states can be written as [29]

$$\hat{L}\propto\hat{Q}^{\mathrm{T}}G\hat{Q} \tag{B20}$$

with $\hat{Q}=(\hat{x}_1,\hat{p}_1,\hat{x}_2,\hat{p}_2,\hat{x}_3,\hat{p}_3,\hat{x}_4,\hat{p}_4)$. Here, each quadrature operator corresponds to the mode $\hat{a}_+,\hat{b}_+,\hat{a}_-$, and $\hat{b}_-$. In this case,

$$\hat{L}\propto\hat{Q}^{\mathrm{T}}G\hat{Q}=(O\hat{Q})^{\mathrm{T}}(\mathrm{diag}(g_1,g_2,g_3,g_4)\otimes\mathbb{1}_2)(O\hat{Q})\propto g_1\hat{n}_1'+g_2\hat{n}_2'+g_3\hat{n}_3'+g_4\hat{n}_4', \tag{B21}$$

where $\hat{Q}'=O\hat{Q}$, $\hat{n}_i=(\hat{x}_i^2+\hat{p}_i^2-1)/2$, and

$$O=(O_+\otimes\mathbb{1}_2)\oplus(O_-\otimes\mathbb{1}_2). \tag{B22}$$

Thus, the photon-number resolving detection after the beam splitters for each two-mode is optimal.

APPENDIX C: LOWER BOUND OF CLASSICAL FISHER INFORMATION OF FIN-SPADE

We calculate the lower bound of classical Fisher information of fin-SPADE method with thermal noise, following the procedure employed in Ref. [4]. Let us recall that the lower bound of classical Fisher information for an unknown parameter θ is given by $F(\theta)\geq\dot{\vec{\mu}}^{\mathrm{T}}C^{-1}\dot{\vec{\mu}}$, where $\vec{\mu}$ is the mean vector of the measurement outcome, and C is the covariance matrix of the outcome. Thus, in the section, we find the mean and the covariance matrix of the measurement outcome from fin-SPADE.

We assume a Gaussian point spread function,

$$\psi(x)=\frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left[-\frac{x^2}{4\sigma^2}\right]. \tag{C1}$$

Let h_q be a Hermite-Gaussian spatial mode,

$$h_q(x)=\left(\frac{1}{2\pi\sigma^2}\right)^{1/4}\frac{1}{\sqrt{2^q q!}}H_q\left(\frac{x}{\sqrt{2}\sigma}\right)\exp\left(-\frac{x^2}{4\sigma^2}\right). \tag{C2}$$

The quantum state of light in thermal states on the image plane can be written as

$$\hat{\rho}=\int d^2A_1d^2A_2p_{N_s}(A)|\psi_{A,s}\rangle\langle\psi_{A,s}| \tag{C3}$$

where

$$p_{N_s}(A)=\left(\frac{1}{\pi\eta N_s}\right)^2\exp\left(-\frac{|A_1|^2+|A_2|^2}{\eta N_s}\right) \tag{C4}$$

is the probability density of the source field amplitudes $A = (A_1, A_2)$, and the conditional state $|\psi_{A,s}\rangle$ represents a coherent state with an amplitude

$$\psi_A(x) = A_1\psi(x - s/2) + A_2\psi(x + s/2). \tag{C5}$$

When thermal noise occurs, the quantum state conditioned on A is changed to

$$\psi_{A,\xi}(x) = A_1\psi(x - s/2) + A_2\psi(x + s/2) + \xi(x), \tag{C6}$$

where $\xi(x)$ is a random variable satisfying $\langle\xi(x)\rangle = 0$, and $\langle\xi^*(x_1)\xi(x_2)\rangle = N_n\delta(x_1 - x_2)$ which describes a random Gaussian displacement noise. Conditioned on A, the amplitude in the q-mode can be written as

$$B_{q|A,\xi} = \int_{-\infty}^{\infty} dx h_q^*(x)\psi_{A,\xi}(x) = \int_{-\infty}^{\infty} dx h_q^*(x)[A_1\psi(x - s/2) + A_2\psi(x + s/2) + \xi(x)] \tag{C7}$$

$$= R_q \exp(-Q/2)\frac{Q^{q/2}}{\sqrt{q!}} + \int_{-\infty}^{\infty} dx h_q^*(x)\xi(x). \tag{C8}$$

where

$$\int_{-\infty}^{\infty} dx h_q^*(x)\psi(x + s/2) = (-1)^q \int_{-\infty}^{\infty} dx h_q^*(x)\psi(x - s/2) = (-1)^q \exp(-Q/2)\frac{Q^{q/2}}{\sqrt{q!}} \tag{C9}$$

with $Q = s^2/16\sigma^2$ and $R_q = A_1 + A_2$ when q is even, $R_q = A_1 - A_2$ otherwise. Thus, the photocounts $N_{q|A,\xi}$ in each mode are the independent Poisson random variable with the mean

$$\mu_{q|A,\xi} = |R_q|^2 f_q + \sqrt{f_q}\left(R_q \int_{-\infty}^{\infty} dx h_q^*(x)\xi^*(x) + R_q^* \int_{-\infty}^{\infty} dx h_q(x)\xi(x)\right) + \int_{-\infty}^{\infty} dx_1 dx_2 h^*(x_1)h(x_2)\xi^*(x_1)\xi(x_2), \tag{C10}$$

where $f_q = \exp(-Q)\frac{Q^q}{q!}$, and the unconditional photocurrent on each mode is written as

$$\mu_q = \langle|B_{q|A,\xi}|^2\rangle_{A,\xi} = 2\eta N_s f_q + N_n. \tag{C11}$$

Thus, the derivative of the mean photocurrent is given by

$$\frac{\partial\mu_q}{\partial s} = \frac{\eta N_s s}{4\sigma^2}(f_{q-1} - f_q), \tag{C12}$$

with $f_{-1} \equiv 0$. For the second moments, for $q = q'$, we obtain

$$\mathbb{E}[N_q^2] = \langle\mathbb{E}[N_{q|A,\xi}^2]\rangle_{A,\xi} = \langle\mu_{q|A,\xi}^2 + \mu_{q|A,\xi}\rangle_{A,\xi} \tag{C13}$$

$$= \langle|R_q|^4 f_q^2 + 4|R_q|^2 f_q \int_{-\infty}^{\infty} dx_1 dx_2 h_q^*(x_1)h_q(x_2)\xi^*(x_1)\xi(x_2) + \left(\int_{-\infty}^{\infty} dx_1 dx_2 h_q^*(x_1)h_q(x_2)\xi^*(x_1)\xi(x_2)\right)^2\rangle_{A,\xi} + \mu_q \tag{C14}$$

$$= 8\eta^2 N_s^2 f_q^2 + 8\eta N_s N_n f_q + 2N_n^2 + 2\eta N_s f_q + N_n. \tag{C15}$$

When $q \neq q'$ and $q - q'$ is even, we get

$$\mathbb{E}[N_q N_q'] = \langle\mathbb{E}[N_{q|A,\xi}N_{q'|A,\xi}]\rangle_{A,\xi} = \langle|B_{q|A,\xi}|^2|B_{q'|A,\xi}|^2\rangle_{A,\xi} \tag{C16}$$

$$= \langle|R_q|^4 f_q f_q' + |R_q|^2 f_q \int_{-\infty}^{\infty} dx_1 dx_2 h_{q'}^*(x_1)h_{q'}(x_2)\xi^*(x_1)\xi(x_2) + |R_{q'}|^2 f_{q'} \int_{-\infty}^{\infty} dx_1 dx_2 h_q^*(x_1)h_q(x_2)\xi^*(x_1)\xi(x_2) \tag{C17}$$

$$+ \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 dx_4 h_q^*(x_1)h_q(x_2)h_{q'}^*(x_3)h_{q'}(x_4)\xi^*(x_1)\xi(x_2)\xi^*(x_3)\xi(x_4)\rangle_{A,\xi} \tag{C18}$$

$$= 8\eta^2 N_s^2 f_q f_{q'} + 2\eta N_s N_n(f_q + f_{q'}) + N_n^2. \tag{C19}$$

Finally, when $q \neq q'$ and $q - q'$ is odd, we obtain

$$\mathbb{E}[N_q N_q'] = \langle \mathbb{E}[N_{q|A,\xi} N_{q'|A,\xi}] \rangle_{A,\xi} = \langle \mu_{q|A,\xi} \mu_{q'|A,\xi} \rangle_{A,\xi} \tag{C20}$$

$$= \langle |R_q|^2 |R_{q'}|^2 f_q f_q' + |R_q|^2 f_q \int_{-\infty}^{\infty} dx_1 dx_2 h_{q'}^*(x_1) h_{q'}(x_2) \xi^*(x_1) \xi(x_2) + |R_{q'}|^2 f_{q'} \int_{-\infty}^{\infty} dx_1 dx_2 h_q^*(x_1) h_q(x_2) \xi^*(x_1) \xi(x_2) \tag{C21}$$

$$+ \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 dx_4 h_q^*(x_1) h_q(x_2) h_{q'}^*(x_3) h_{q'}(x_4) \xi^*(x_1) \xi(x_2) \xi^*(x_3) \xi(x_4) \rangle_{A,\xi} \tag{C22}$$

$$= 4\eta^2 N_s^2 f_q f_{q'} + 2\eta N_s N_n (f_q + f_{q'}) + N_n^2. \tag{C23}$$

Thus, the covariance matrix is written as

$$C_{qq'} = \begin{cases} 4\eta^2 N_s^2 f_q^2 + 4\eta N_s N_n f_q + 2\eta N_s f_q + N_n^2 + N_n & q = q' \\ 4\eta^2 N_s^2 f_q f_{q'} & q \neq q' \text{ and } q - q' \text{ is even} \\ 0 & q \neq q' \text{ and } q - q' \text{ is odd} \end{cases} \tag{C24}$$

The covariance matrix and the derivative of the first moment give the lower bound of classical Fisher information as stated in the main text.

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[1] L. Rayleigh, XXXI. Investigations in Optics, with Special Reference to the Spectroscope, Philos. Mag. Ser. 5 **8,** 261 (1879).
[2] M. Born and E.Wolf, Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light (Cambridge University Press, Cambridge, England, 1999).
[3] Quantum Theory of Superresolution for Two Incoherent Optical Point Sources, M. Tsang, R. Nair, and X.-M. Lu, Phys. Rev. X **6,** 031033 (2016).
[4] R. Nair and M. Tsang, Far-Field Superresolution of Thermal Electromagnetic Sources at the Quantum Limit, Phys. Rev. Lett. **117,** 190801 (2016).
[5] C. Lupo and S. Pirandola, Ultimate Precision Bound of Quantum and Subwavelength Imaging, Phys. Rev. Lett. **117,** 190802 (2016).
[6] S. Z. Ang, R. Nair, and M. Tsang, Quantum limit for two-dimensional resolution of two incoherent optical point sources, Phys. Rev. A **95,** 063847 (2017).
[7] Z. Yu and S. Prasad, Quantum Limited Superresolution of an Incoherent Source Pair in Three Dimensions, Phys. Rev. Lett. **121,** 180504 (2018).
[8] C. Napoli, S. Piano, R. Leach, G. Adesso, and T. Tufarelli, Towards Superresolution Surface Metrology: Quantum Estimation of Angular and Axial Separations, Phys. Rev. Lett. **122,** 140505 (2019).
[9] S. Zhou and L. Jiang, Modern description of Rayleigh's criterion, Phys. Rev. A **99,** 013808 (2019).
[10] M. Tsang, Quantum limit to subdiffraction incoherent optical imaging, Phys. Rev. A **99,** 012305 (2019).
[11] C. Lupo, Z. Huang, and P. Kok, Quantum Limits to Incoherent Imaging are Achieved by Linear Interferometry, Phys. Rev. Lett. **124,** 080503 (2020).
[12] Achieving the ultimate optical resolution, M. Par, B. Stoklasa, Z. Hradil, L. L. Snchez-Soto, and J. Rehacek, Optica, **3**, 2334 (2016).
[13] Z. S. Tang, K. Durak, and A. Ling, Fault-tolerant and finite-error localization for point emitters within in the diffraction limit, Opt. Express **24,** 22004 (2016).
[14] F. Yang, A. Taschilina, E. S. Moiseev, C. Simon, and A. I. Lvovsky, Far-field linear optical superresolution via heterodyne detection in a higher-order local oscillator mode, Optica **3,** 1148 (2016).
[15] W. K. Than, H. Ferretti, and A. M. Steinberg, Beating Rayleigh's Curse by Imaging Using Phase Information, Phys. Rev. Lett. **118,** 070801 (2017).
[16] M. Parniak, S. Borowka, K. Boroszko, W. Wasilewski, K. Bonaszek, and R. Demkowicz-Dobrzanski, Beating the Rayleigh Limit Using Two-Photon Interference," Phys. Rev. Lett. **121,** 250503 (2018).
[17] Y. L. Len, C. Datta, M. Parniak, and K. Banaszek, Resolution limits of spatial mode demultiplexing with noisy detection, Int. J. Quantum Inf. **18,** 1941015 (2020).
[18] C. Lupo, Subwavelength quantum imaging with noisy detectors, Phys. Rev. A **101,** 022323 (2020).
[19] M. Gessner, C. Fabre, and N. Treps, Superresolution limits from measurement crosstalk, arXiv: 2004.07228 (2020).
[20] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976).
[21] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72,** 3439 (1994).
[22] M. G. A. Paris, Quantum estimation for quantum technology. International Journal of Quantum Information, Int. J. Quant. Inf. **7,** 125 (2009).
[23] O. Pinel, P. Jian, N. Treps, C. Fabre, and D. Braun, Quantum parameter estimation using general single-mode Gaussian states, Phys. Rev. A **88,**, 040102 (R) (2013).
[24] Z. Jiang, Quantum Fisher information for states in exponential form, Phys. Rev. A **89,** 032128 (2014).
[25] D. afrnek,, A. R. Lee and I. Fuentes, Quantum parameter estimation using multi-mode Gaussian states, New. J. Phys. **17,** 073016 (2015).

[26] D. afrnek and I. Fuentes, Optimal probe states for the estimation of Gaussian unitary channels, Phys. Rev. A **94,** 062313 (2016).

[27] R. Nichols, P. Liuzzo-Scorpo, P. A. Knott, and G. Adesso, Multiparameter Gaussian quantum metrology, Phys. Rev. A **98,** 012114 (2018).

[28] D. afrnek, Estimation of Gaussian quantum states, J. Phys. A **52,** 035304 (2019).

[29] C. Oh, C. Lee, L. Banchi, S.-Y. Lee, C. Rockstuhl, and H. Jeong, Optimal measurements for quantum fidelity between Gaussian states and its relevance to quantum metrology, Phys. Rev. A **100,** 012323 (2019).

[30] S. Zhou and L. Jiang, An exact correspondence between the quantum Fisher information and the Bures metric, arXiv:1910.08473 (2019).

[31] C. R. Rao, Information and Accuracy Attainable in the Estimation of Statistical Parameters. Bulletin of the Calcutta Mathematical Society 37 8191 (1945).

[32] H. Cramér, Methods of Mathematical Statistics. Princeton University Press 23 (1946).

[33] O. E. Barndorff-Nielsen and R. D. Gill. Fisher information in quantum statistics. Journal of Physics A: Mathematical and General **30,** 4481 (2000).

[34] M. Stein, A. Mezghani, and J. A. Nossek, IEEE Signal Process. Lett. **21,** 796 (2014).

[35] J. ehek, Z. Hradil, B. Stoklasa, M. Par, J. Grover, A. Krzic, and L. L. Snchez-Soto, Multiparameter quantum metrology of incoherent point sources: Towards realistic superresolution, Phys. Rev. A **96,** 062107 (2017).

[36] J. ehek, Z. Hradil, D. Koutn, J. Grover, A. Krzic, and L. L. Snchez-Soto, Optimal measurements for quantum spatial superresolution, Phys. Rev. A **98,** 012103 (2018).

[37] K. A. G. Bonsma-Fisher, W.-K. Tham, H. Ferretti and A. M. Steinberg, Realistic sub-Rayleigh imaging with phase-sensitive measurements, New J. Phys. **21,** 093010 (2019).

[38] S. Prasad, Quantum limited super-resolution of an unequal-brightness source pair in three dimensions, Phys. Scr. **95,** 054004 (2020).