Design Principles for Examining Student Practices in a Technology-Mediated Environment

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In this article, we present a set of design principles to guide the development of instructional materials aimed to support preservice secondary mathematics teachers (PSMTs) examining student practices in technology-mediated environments. To develop design principles, we drew on the literature related to technological pedagogical content knowledge (TPACK; Niess, 2005), video cases as learning objects (Sherin & van Es, 2005), and professional noticing (Jacobs, et al., 2010). After presenting the design principles, we share a task created using these design principles. Finally, we share PSMTs' reflections about changes in their own understanding after examining students' practices. Their responses provide insights into the usefulness of the design principles for deepening PSMTs' mathematical knowledge and knowledge of students' understanding, thinking, and learning with technology.

Keywords: technology; function; preservice secondary mathematics teachers

The National Council of Teachers of Mathematics (NCTM, 2000) has long advocated that "technology is essential in teaching and learning mathematics; it influences what is taught and enhances students' learning" (p. 24), and research has backed this claim (e.g., Arzarello et al., 2002; Ronau et al., 2011). Given the impact meaningful incorporation of technology tools can have on students' understanding of mathematics, it is important that prospective teachers learn to make informed decisions about appropriate uses of technology to develop mathematically proficient students. This was most recently articulated in the Association of Mathematics Teacher Educators' (AMTE) *Standards for Preparing Teachers of Mathematics* (2017), which states that "Well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics" (C.1.6, p. 11). This requires teachers to not only be proficient users of technologies but also to understand how to use technology in meaningful ways to support students' thinking about mathematics. Whether or not the use of technology will enhance students' learning depends on decisions teachers make when using technology tools to design and implement meaningful tasks. These decisions are informed by teachers' knowledge of mathematics, technology, and pedagogy, which has been identified as Technological Pedagogical and Content Knowledge (TPACK; Niess, 2005).

Niess (2005) articulated four components of TPACK: (a) "an overarching conception of what it means to teach a particular subject integrating technology in the learning"; (b) "knowledge of instructional strategies and representations for teaching particular topics with technology"; (c) "knowledge of students' understanding, thinking, and learning with technology in a particular subject"; and (d) "knowledge of curriculum and curriculum materials that integrate technology with learning in the subject area" (p. 511). It is the third component, i.e., knowledge of students' understanding, thinking, and learning with technology in a specific subject, that is the focus of this article. Specifically, drawing on the extant literature related to TPACK, video case instruction, and professional noticing, we propose a set of design principles for the development of technology mediated and video-enhanced modules for preservice secondary mathematics teachers (PSMTs) with an eye toward the development of their *knowledge of students' understanding, thinking, and learning with*

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technology in mathematics. The purpose of this article is to discuss the design principles broadly, share how they were used to develop the task of study in this article—which is focused on the function concept—and to report research considering PSMTs' reflections on their learning as a result of engaging with the task.

Technological Pedagogical Function Knowledge

Building off Niess's (2005) components of TPACK, we believe that when designing learning experiences intended to develop PSMTs' TPACK related to a specific concept that mathematics teacher educators (MTEs) need to consider how the concept can be operationalized within the TPACK framework (Figure 1a). In this framework, there are components that are not content specific (i.e., pedagogical knowledge and technological knowledge). Because our work as MTEs is in the discipline of mathematics, we view mathematics content knowledge as foundational. With this in mind, similar to Lee and Hollebrands (2011) who developed specific constructs for technological pedagogical statistical knowledge, we identified constructs specific to teaching the concept of function, referred to as technological pedagogical function knowledge. The three components of technological pedagogical function knowledge are shown in Figure 1b, and the knowledge of the function concept is the foundation to develop technological function knowledge and then ultimately technological pedagogical function knowledge. Note that we do not intend for these constructs to capture all areas of functional understanding and the teaching of functions, and these are specific to the *concept* of function (as defined by Cooney et al., 2010).

To develop knowledge of the function concept, technology tools can be used to engage PSMTs in tasks that simultaneously develop their understanding of function and their technology skills. For example, this may include visualization of function concepts through intentional technology engagement (e.g., dynamic dragging) (e.g., Arzarello et al., 2002; Trouche & Drijvers, 2010); exploration, coordination, and creation of linked function representations (e.g., numeric, symbolic, graphic, and geometric representations) (e.g., Doerr & Zangor, 2000); and creation and interaction with models of function relationships in context (e.g., Rochelle et al., 2012). With this in mind, it is important to provide PSMTs with opportunities to explore a variety of dynamic (and linked when appropriate) representations of functions.

The ultimate goal of this framework is to identify what it would mean for PSMTs to develop the specialized knowledge needed to teach the function concept using technology. To do so, we operationalize what technological pedagogical function knowledge would look like in practice. We aim for PSMTs to (a) understand students' learning and thinking about function ideas with technology; (b) conceive of how technology tools and representations support function thinking; (c) develop a repertoire of instructional strategies for designing function lessons with technology; and (d) take a critical stance toward evaluation and use of curricula materials for teaching function ideas with technology. With these aims in mind, it is imperative that PSMTs have opportunities to examine and analyze students' practices related to the function concept in a technological environment.

Analyzing Student Work

Research has highlighted the important role that students' mathematical thinking plays in high-quality instruction (e.g., Jacobs & Spangler, 2017). This points to the need for PSMTs to have opportunities to grapple with and make sense of how students think about mathematics. *Principles to Actions: Ensuring Mathematical Success for*

Figure 1a

Components of TPACK

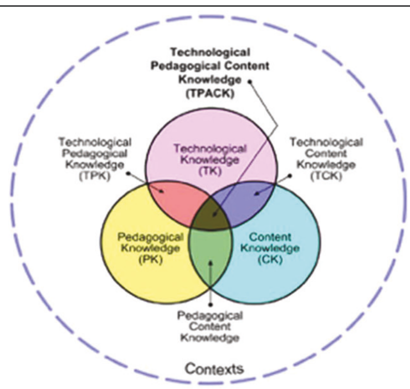
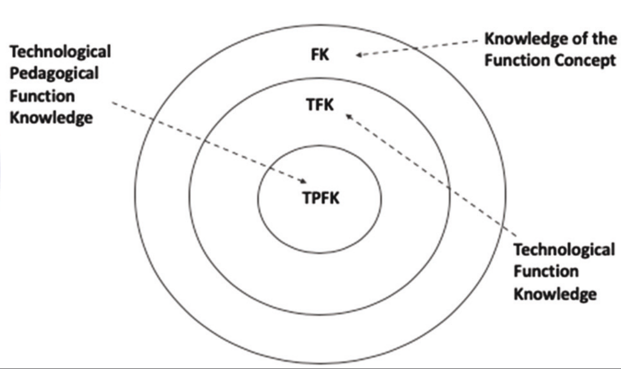


Figure 1b

Components of Technological Pedagogical Function Knowledge



All (2014), NCTM's publication, identified "elicit and use evidence of student thinking" (p. 10) as one of the eight Mathematics Teaching Practices. For PSMTs, this skill must be purposefully developed via teaching practice. One method that has been shown to help PSMTs develop an understanding of student thinking is the analysis of authentic student work (e.g., Casey et al., 2018; Jansen & Spitzer, 2009; Philipp, 2008).

Authentic student work can come in the form of written artifacts or video cases. Here, we focus on video cases and their corresponding written artifacts because together they provide insights into student thinking because they are engaged in mathematical work. Video cases have been shown to improve PSMTs' ability to critically observe classroom practice and attend to teacher choices and student thinking rather than merely content delivery (e.g., Santagata & Taylor, 2018; Sherin & van Es, 2005; Star & Strickland, 2008). Additionally, a focus on student thinking through video case analysis has been shown to improve PSMTs' abilities to draw attention to and describe teachers' instructional moves to make student thinking visible, to reason about the impact of teachers' decisions related to student learning, and to propose alternatives to what was observed in the video (Santagata & Guarino, 2011). When students are working in groups, Wells (2017) highlighted the importance of "paying attention to student posture and gesture, [because] there is much more that the teacher can learn about students' shared understanding" (p. 201), and thus, in a technological environment, videos should show both the students' work and the students' interactions.

Although video case instruction has been shown to benefit PSMTs, researchers caution that the *selection* of video clips (e.g., Kurz et al., 2005; Sherin et al., 2009; and *how* video cases are used are critical to promoting teacher learning (e.g., Brophy, 2004; Seago et al., 2018). To this end, Sherin et al. (2009) articulated a framework with three dimensions for selecting video: clips should provide a *window* into student thinking (i.e., are there multiple entry points to consider student thinking?), the *depth* of student thinking (i.e., does the mathematical work reflect depth and rigor?), and the *clarity* of student thinking (i.e., is the student thinking obvious or does it require sense making on the part of the viewer?) (Kang & van Es, 2019). They also suggest that cases be designed so that they focus on aspects of student work in which there are elements of confusion or surprise (Sherin et al. 2009; Shulman, 1996).

Once video clips are selected, the activities that surround their use must be carefully designed, articulating clear goals to focus the analysis of the video (Borko et al., 2008). A method often used to guide PSMTs' analysis of student work in video cases is the *professional noticing* construct developed by Jacobs et al. (2010). The three components of the professional noticing construct are attending to students' strategies, interpreting students' mathematical thinking, and deciding how to respond on the basis of students' understandings.

Much of the research on PSMTs' analysis of student work has been completed through the lens of professional noticing. Within professional noticing research, more attention has been paid to developing the skill of professional noticing of the whole class video (e.g., Krupa et al., 2017; McDuffie et al., 2014), with less on prospective teachers' noticing of student written work (e.g., Dick, 2017; Goldsmith & Seago, 2011). In terms of PSMTs' professional noticing of students' mathematical thinking in technological environments, very few studies have been conducted (e.g., Chandler, 2017; Dick et al., Under Review; Wilson et al., 2011).

The initial work on PSMTs' professional noticing of students' understanding, thinking, and learning of mathematics with technology comes from Wilson et al. (2011). They engaged preservice teachers in analyzing video cases of students' technological mathematical work, and their research resulted in identifying four categories related to the way PSMTs construct models of students' thinking with a technological mathematical task: describing, comparing, inferring, and restructuring. They describe restructuring as reflecting on student understandings and "incorporating them into the reorganization of their own knowledge" (Wilson et al., 2011, p. 58). Restructuring is crucial for the development of PSMTs' TPACK because it is through this process that PSMTs make the critical connections between students' technological mathematical thinking and their own understandings. Since 2011, very little work has been completed in this area. Thus, to assist MTEs in supporting PSMTs in their development of TPACK and ultimately AMTE indicator C.1.6, we propose a set of design principles for engaging PSMTs in professional noticing of students' mathematical technological practices.

Design Principles for Examining Students' Technological Mathematical Work

The design principles we propose draw on the integration of our review of literature surrounding TPACK, video case pedagogies, and the construct of professional noticing. Specifically, we propose the following design principles for examining students' technological mathematical work:

1. PSMTs need to observe secondary students engaged in technology-based tasks. The selected tasks must

be of high cognitive demand (Smith & Stein, 1998) and position the use of technology to develop mathematical or statistical understanding (Dick & Hollebrands, 2011).

2. Selected video clips (and associated written artifacts) must focus on aspects of student work in which there are elements of confusion or surprise, as suggested by Sherin et al. (2009) and Shulman (1996). Furthermore, they are selected on the basis of Sherin et al.'s (2009) recommendations for dimensions of video clips that support teacher discussion of students' mathematical thinking (i.e., *window* into student thinking, *depth* of student thinking, and *clarity* of student thinking). Finally, to capture the interaction among students and any gestures made in relation to their technological work (e.g., pointing), video clips must simultaneously include students' interactions, their gestures, and their technological work (Wells, 2017).
3. PSMTs must complete the technology-based task first as learners and discuss the mathematics and technology involved (McCulloch et al., 2015; Olive & Leatham, 2000; Zbiek & Hollebrands, 2008) before analyzing students' engagement in technology-based tasks (Wilson et al., 2011).
4. Guiding questions that accompany video clips are designed on the basis of the framework by Jacobs et al. (2010) for professional noticing. This means specifically asking PSMTs to attend, interpret, and predict students' mathematical technological understanding and their engagement with the technology (e.g., dragging, clicking, and creating representations) (Dick et al., Under Review). Based on their noticing, PSMTs are asked to make pedagogical decisions that incorporate both the students' understanding and their engagement.
5. PSMTs must reflect on their own understanding of TPACK in light of their examinations of student practices. This allows for opportunities to reconcile their observations and inferences of students' practices with their own understandings (Wilson et al., 2011).

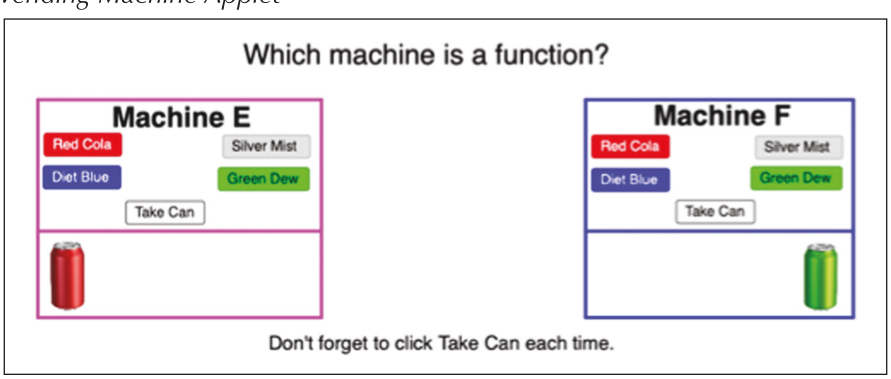
We conjecture that careful selection of technology-mediated tasks, video case clips, and carefully selected questions work together to provide PSMTs an opportunity to develop their knowledge of students' understanding, thinking, and learning with technology in mathematics and to conceive of how the technology tools and representations support students' mathematical thinking in technology-mediated learning environments.

An Example: The Function Concept

To provide opportunities for PSMTs to grow in their development of technological pedagogical function knowledge, a module including a set of tasks targeting the components of technological pedagogical function knowledge was developed. The tasks were designed around a preconstructed GeoGebra applet, called *Vending Machine—Introduction to Function* (see Lovett et al., 2019 for more details). PSMTs first engage with the applet and answer questions related to their own understanding of function and representations of function (Design Principle 3). Then, using carefully selected video recordings of eighth graders working in pairs on the applet (Design Principle 2), PSMTs watch video clips of two pairs of students engaging with the applet (Design Principle 1) and are asked to respond to questions based on the professional noticing construct by Jacobs et al. (2010) (Design Principle 4). Finally, PSMTs are asked to reflect on their own understanding after engaging with the various experiences examining students' mathematical technological work (Design Principle 5). Each of these components is described in detail in the following sections.

Figure 2

Vending Machine Applet



Vending Machine Applet

PSMTs engaged with the *Vending Machine—Introduction to Function* applet (**https://www.geogebra.org/m/uR9uFreX**), which is focused on developing a definition of function based on students' exploration of the applet machines. The applet is a GeoGebra book that uses the metaphor of a Vending Machine and asks the user to identify whether each Vending Machine is a function or nonfunction (Figure 2). The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed, it produces none, one, or more than one of the four different colored cans (red, blue, silver, and green).

By removing numeric and algebraic representations, students and PSMTs attend to the nature of and the relationship between the input and outputs. The machines were designed to address misconceptions from the literature on distinguishing functions and nonfunctions. For example, research shows difficulties identifying constant functions as functions (e.g., Carlson, 1998). Thus, there is a machine that acts as a constant function, in that every button produces the same color can. A description of each machine and the directions for each page of the applet are provided in Figure 3. See **Appendix A** for the student handout.

Noticing Assignment

After engaging with the applet as a learner, PSMTs engaged in a noticing assignment specifically focusing on the first two aspects of the noticing construct by Jacobs et al. (2010), attend, and interpret (See **Appendix B**). The decision was made to focus on these aspects and not the decision aspect because research has shown that PSMTs struggle making next-step decisions, especially when they are new to noticing (e.g., Gupta et al., 2018; Jacobs et al., 2010).

To provide an opportunity for PSMTs to examine student practices, videos of pairs of eighth-grade students engaging with the applet were collected when the task was implemented in a whole class setting. From this collection of videos, we identified clips that contained moments of surprise or confusion. To narrow down the collection of videos to the final two chosen for the assignment, we used Sherin et al.'s (2009) recommendations to choose two clips that clearly highlighted two different conceptions of function on the same two sets of machines.

After analyzing the video recordings, PSMTs completed a written reflection in which they were asked to explain how each group determined whether the pairs of machines were a function or not (attend) and to discuss the students'

Figure 3

Description of Each Machine for the *Introduction to Function* applet

Page 1				Page 2			
This One is a Function		This One is Not a Function		This One is a Function		This One is Not a Function	
A	Red Cola → red Diet Blue → blue Silver Mist → silver Green Dew → green	B	Red Cola → red Diet Blue → blue Silver Mist → random color Green Dew → green	C	Red Cola → blue Diet Blue → red Silver Mist → silver Green Dew → green	D	Red Cola → red Diet Blue → random color Silver Mist → silver Green Dew→ green
Page 3				Page 4			
Which One is a Function?				Which One is a Function?			
E	Red Cola→ red Diet Blue→ blue Silver Mist→ silver Green Dew → random color	F	Red Cola → silver Diet Blue → green Silver Mist → silver Green Dew → blue	G	Red Cola → random color Diet Blue → random color Silver Mist → random color Green Dew → random color	H	Red Cola → blue Diet Blue→ silver Silver Mist → green Green Dew → red
Page 5				Page 6			
Which One is a Function?				Which One is a Function?			
I	Red Cola → 2 silvers Diet Blue → green Silver Mist → red Green Dew → blue	J	Red Cola → red Diet Blue → blue & random color Silver Mist → silver Green Dew → green	K	Red Cola → random pair Diet Blue → blue Silver Mist → silver Green Dew → green	L	Red Cola → green Diet Blue → green Silver Mist → green Green Dew → green
		Page 7					
		Could these be functions?					
		M	Red Cola → red Diet Blue → red Silver Mist → silver Green Dew → silver	N	Red Cola → red Diet Blue → blue Silver Mist → silver Green Dew → red & green		

understanding of function based on their analysis of the video clips (interpret). Next, they predicted how the pair of students would respond to the remaining applet machines based on their noticing analysis. This component was included because PSMTs need to be able to predict and anticipate different strategies students might employ to solve mathematical tasks (Hiebert et al., 2007; Smith & Stein, 1998) prior to making decisions about what to do next. Finally, the PSMTs reflected on their own understanding of function after examining the students' thinking and engagement with the applet. It is this final component that provided insights to PSMTs' perceptions of their learning and provides the data for this study.

PSMTs' Reflections of Their Learning

As previously explained, the design principles we developed were based on our conjecture that by carefully selecting tasks, technology, video case clips, and questions posed to the PSMTs using the professional noticing construct, the module would provide PSMTs with an opportunity to explicitly develop the four components of technological pedagogical function knowledge. In previous papers, we have reported how the PSMTs engaged with the applet as learners (McCulloch et al., 2019; McCulloch et al., Under Review) and as how the PSMTs noticed the students' understanding of function and engagement with the applet (Dick et al., Under Review). Given the importance of PSMTs being provided opportunities to reconcile students' work with their own understanding (Wilson et al., 2011), in this paper, we aimed to examine our design principles by answering the following research question: In what ways do PSMTs articulate their developing technological pedagogical function knowledge when asked to reflect on their learning as a result of completing the tasks?

Methodology

After engaging with the Vending Machine task, participating in a class discussion on the definition of function and completing the noticing task, we asked PSMTs to reflect on their own learning. This written reflection was prompted by the following question: "How has your own understanding of function been influenced by thinking about how middle school students develop the concept of function?" Participants for this study were 45 PSMTs from four public universities in the U.S. who were enrolled in a secondary mathematics education method course. At each university, the secondary mathematics education method course focused on developing mathematical knowledge for teaching secondary school mathematics. In each course, PSMTs were exposed to students' mathematical thinking both with and without the use of technology.

Table 1

PSMTs' Reported Knowledge Gains

Themes	Exemplar
Understand student thinking with technology	It has helped me to understand better how others think of functions by allowing me to see where people [students] tend to get confused and what reasoning they use when deciding whether something is a function.
Conceive of technology support	Sometimes function can get confusing to students with just a simple definition and plugging in numbers because that is what they are told. This activity is great for visual learners too, so they can really see and visualize what a function is. Students get to see first-hand what a function means in a different way.
Develop a repertoire of strategies	I was never exposed to functions in anyway except graphs, tables, and expressions and occasionally built myself from word problems. This applet gave me a new way to visualize the input/output relationship.
Take a critical stance	I still think this machine activity is hard to interpret though because two cans of the same color and/or different color could be interpreted as a function to some and not to others.
Develop knowledge of the function concept	My own understanding of function has been shifted from one input has one unique output to functions have specific rules and patterns that each input must follow.

Two of the researchers worked together to analyze the 45 written reflections and brought questions to the team for discussion. Analysis began by following DeCuir-Gunby et al.'s (2011) recommendations for the development of a codebook. We first mapped PSMTs' reflection responses onto the four components of the technological pedagogical function knowledge framework, but an important theme emerged related to the PSMTs' perceived development of their knowledge of the function concept. Thus, the mapping process resulted in five themes of PSMTs' reported knowledge gains as seen in Table 1.

Results and Discussion

Our ultimate goal for developing the tasks and the rationale behind the design principles was related to developing PSMTs' technological pedagogical function knowledge, which includes Knowledge of the Function Concept. Implementing Design Principle 3, all PSMTs first engaged with the applet as learners, and thus, it is not surprising that 37 of the 45 (82%) PSMTs reported developing their Knowledge of the Function Concept as a result of completing the module. Of these, 26 (70%) discussed their increased understanding related to the definition of function. A perception of a deeper understanding of function related to consistency was very common. For example, PSMT 9 stated, "Making sure to include that the results must be consistent in my definition is the biggest change to my understanding of functions." Similarly, PSMT 6 elaborated,

> **I have always (or at least as far as I can remember anyway) thought of a function as there being some elements that were your inputs, and something was done to them so that you got the outputs. Whether or not two inputs mapped to the same output or one of the outputs looked different than the other never really mattered to me. However, I never really thought about functions as being 'consistent.' I knew they were, because I knew if something gave you a bunch of different outputs, it was not a function, but I never used the word consistent when describing a function.**

This student discussed not only the ideas of consistency related to their understanding, but also discussed their new understanding of using the term consistent when describing a function. This perception that PSMTs' newfound Knowledge of the Function Concept included new terminology for understanding the definition of function was seen in other PSMT responses such as PSMT 34 who stated that instead of describing a function as an equation, "I would say a function is a relationship where every input has one output (and the output never changes)." Not all students were as specific about their changes in their knowledge related to the definition of function but referred to it nonetheless, such as PSMT 29 who expounded, "This made me notice inconsistencies among definitions and how important it is to be precise and careful in how a function is defined."

Although the previous examples related new understandings to terminology for the definition of function, other PSMTs discussed different elements of their increased understanding of their Knowledge of the Function Concept. For example, some PSMTs shared their realization that functions apply to a wide range of situations, one of Cooney et al.'s (2010) Essential Understandings. PSMT 2 explained a new understanding, "A function does not need a formula or equation to guide it. And while a function can be determined by looking at a graph, it is not always necessary." Similarly, PSMT 19 said that thinking throughout the tasks "gave me an increased understanding of the wide variety of functions beyond simply applying a rule to an input." Perhaps PSMT 20 best explains their developing understanding compared to the students'; PSMT 20 says,

> **I visualize a function as a graph and I can guarantee middle schoolers don't. I think it's interesting and it has broadened my definition of a function in many ways. Lots of other words can be used to describe functions besides 'equation,' 'graph,' and 'x and y.'**

It is promising that the design of the tasks resulted in many PSMTs considering their own understanding of the concept of function to have increased as a result of their engagement with both the applet themselves and their examination of the students' practices.

Because our design was focused on developing PSMTs' technological pedagogical function knowledge regarding the function concept, it is not surprising that evidence for all four components were present. Information regarding the PSMTs' articulation of their understandings related to each of the four components is displayed in Table 2. The most common component the PSMTs discussed was related to their repertoire of strategies for teaching function (76%). Close to half of the PSMTs discussed an increased understanding of students' function thinking and/or their conceptions relating to how the Vending Machine applet supported the students' function thinking. There was less discussion of the PSMTs' perceptions of their understanding related to taking a critical stance evaluating the Vending Machine technology itself; only six PSMT responses showed such a stance. In the following paragraphs, exemplars (segments of full responses) for each component will be presented and discussed.

Understand student thinking with technology. Over half (53%) of the PSMTs' discussions about their perceived understanding, stemming from the module tasks, are related to their understanding of students' function thinking in relation to the technology. Many of the responses included "getting into students' heads" such as PSMT 11 who explained, "Putting myself into the

Table 2

Frequency of PSMTs' Reflections of Their Understanding Mapped onto the Components of Technological Pedagogical Function Knowledge

Technological pedagogical function knowledge component	*n* (%)
Understand student thinking with technology	24 (53)
Conceive of technology support	19 (42)
Develop a repertoire of strategies	34 (76)
Take a critical stance	6 (13)

mindset of a younger student made me examine this [the function concept] from a more simplistic perspective . . . and how a student might process it." In addition, many PSMTs shared new insights into the ways students might think about the concept of function. For example, PSMT 44 stated, "I believe that this activity has brought to my attention and helped me understand many possible ideas that could make students think that the machines may or may not be functions and why."

Another common theme was the PSMTs' increased understanding of student misconceptions related to function. PSMT 41 explained, "The activities allowed me to understand their thought process, so that I would know which mistakes to be aware of when teaching these concepts." Other PSMTs also discussed a new understanding of potential difficulties students may encounter related to function. PSMT 38 conjectured, "I think that the misconceptions of the students will help [me] be able to teach this subject better in the future." These examples provide evidence of the successful implementation of Design Principle 2, because the video clips were selected specifically to focus on student confusion and therefore highlighted some potential misconceptions for the PSMTs to consider.

Conceive of technology support. All five design principles were developed with the explicit conjecture that tasks that examine student practices would provide PSMTs with opportunities to explicitly develop conceptions of how technology supports students' thinking. Thus, we were encouraged that almost half (42%) of the PSMTs' responses at the end of the module included discussions related to how the technology supported students' thinking. For example, PSMT 35 described how seeing the students' think within the applet environment would help with introducing the function concept; PSMT 35 stated,

> **Seeing the way the students explain a function from the Vending Machines gave me a visual way of seeing how the students think through a function and what all goes into a function. I feel like I could explain a function better now that I see the way they think about it when they are first introduced to the concept.**

Another PSMT specifically envisioned a class discussion following engagement with the applet; PSMT 26 described, "I realize that this activity with subsequent discussion will illustrate that the expectation of allowed function outputs will result in different opinions of whether it is a function or not." These examples show how the PSMTs were coordinating their developing knowledge about the students' thinking specifically related to the technology.

Many of the responses conceiving how the applet supported thinking were related to the PSMTs' own engagement with the applet. For example, PSMT 4 shared, "If I had been introduced to functions this way, I probably wouldn't forget the definition and have to remind myself what they are every time I take a new math class," which illustrates their thoughts on the impact of the applet for developing a solid definition of function. PSMT 36 specifically discussed how the design of the applet supported students' thinking, "They [students] got to evolve their definitions and conceptions of a function and were introduced to challenging cases unlike textbook examples."

Develop a repertoire of strategies. All responses that included pedagogical insights related to teaching function with the applet were considered as evidence for PSMTs' development of strategies for function lessons with technology. Because the module was based around a new technology, it is not surprising that the majority of PSMTs (76%) discussed their newfound understandings related to teaching function within the applet. Even some PSMTs who indicated no changes to their own understanding of function claimed increased pedagogical understanding such as PSMT 27 who said, "How I would describe a function to others has [changed]." Many PSMTs discussed thinking about using the applet with students. PSMT 30 admitted, "I actually was surprised that the middle school students understood as much as they did about functions." This PSMT recognized that the applet environment provided a means for developing a function understanding.

Other PSMTs' responses were more explicit with descriptions about how they would use the applet to teach. PSMT 3 discussed how they might use the applet in conjunction with other examples to help students build their understanding,

> **I need to make sure that my students understand the many concepts of what a function is…I would provide many examples about how to look at a function, like the vertical line test and use this applet. Giving the students multiple examples of how to look at a function will help them to build on their understanding.**

PSMT 41 shared how the tasks helped form the realization that a common definition might not be sufficient for students' understanding. They stated, "The way that I would introduce and teach this concept to middle schoolers has been highly impacted through these activities. I realize by stating that each input only has one output is not enough."

Although not referencing the applet, one PSMT discussed another technology strategy introduced for their repertoire for teaching, that of screencasts. PSMT 11 conjectured, “With that technology, or something similar, I could identify moments when students get the right answer without fully understanding it. Then, I could address students’ uncertainties about functions, or any other concept, before it caused them trouble later.” This discussion specifically related to the PSMTs’ thoughts about teaching not only function but other concepts as well.

Take a critical stance. Perhaps because the prompt the PSMTs were asked to answer did not explicitly have them evaluate the applet itself, only six PSMTs’ discussions of their understanding included evidence of taking a critical stance about teaching the function concept using the applet. Two of these reflections included a discussion about specific machines and how the machines affected student learning. PSMT 43 reflected on the use of two cans as an output, “Also the idea of cans can skew one’s perspective of a function. Group 2 struggled with the idea of quantity.” PSMT 43 continued to discuss the possibility of labeling two can outputs in a different manner to assist with this misconception. PSMT 5 also critiqued the applet’s use of two cans, “I still think this machine activity is hard to interpret though because two cans of the same color and/or different color could be interpreted as a function to some and not to others.” Both PSMTs took a stance and decided there were elements of the applet design they did not like and were able to articulate their reasons related to students’ understanding of the function concept.

Although the previous two examples included negative stances related to the applet, some were positive. PSMT 10 shared,

> **I enjoyed using this applet as an exploratory activity to define function. The soda machines were simple and engaging to middle school students. The evolution of complexity and variation is important when designing activities like this. It was necessary to progress from a simple version to a more expansive double can output, or all green output.**

This example highlights the PSMTs’ thinking about the design of the applet and how each machine was ordered to assist the students in developing their understanding of function.

The percentages of the number of components presented in individual PSMT responses are shown in Table 3. The fact that 69% of the PSMTs’ responses included evidence of two or more components of technological pedagogical function knowledge provides strong evidence that the design principles used to develop the tasks supported the development of PSMTs’ technological pedagogical function knowledge, a subset of PSMTs’ TPACK.

Table 3

Frequency of the Number of Components Present in Individual PSMT Responses

Exact number of components	*n* (%) of individual PSMT responses
None	4 (9)
One	6 (13)
Two	21 (47)
Three	8 (18)
Four	6 (13)

Implications for Teacher Education

The AMTE indicator (C.1.6) regarding the use of mathematical tools and technology highlights the importance that all beginning teachers have experiences with technology as a learner of mathematics themselves and that they consider how technology can support students’ learning of mathematics (AMTE, 2017). With a focus on supporting PSMTs’ development of skills related to noticing students’ thinking in technology-mediated environments, we proposed a set of design principles to guide the development of instruction for those working with PSMTs in this realm. Our design principles are grounded in the literature related to TPACK (Niess, 2005), professional noticing of students’ mathematical thinking (Jacobs et al., 2010; Wilson et al., 2011), and video case instruction (e.g., Sherin & van Es, 2005; Star & Strickland, 2008). In this article, we shared a particular set of tasks that were designed on the basis of these principles and results of a study of PSMTs that engaged with the tasks.

Attending to PSMTs’ reflections of their learning provided insights into the effectiveness of the design principles as a framework for MTEs in their design of curricular materials to support PSMTs’ development with respect to the aspirational goals set out by AMTE (2017). Specifically, our findings indicate that many PSMTs who have opportunities to engage with technology tasks as learners and examine carefully chosen video clips through a lens of professional noticing are able to articulate changes in their own understanding, an understanding of students’ thinking, the ways the technology might have supported students’ thinking, and a change in their repertoire of pedagogical strategies. Recall that the design principles were written to specifically focus on two components of technological pedagogical function knowledge (knowledge of students’ understanding, thinking, and learning with technology in mathematics and to conceive of how the technology tools and representations support students' mathematical thinking in technology-mediated learning environments), even so the PSMTs perceived deepening their knowledge in the other components as well. For example, though they were not explicitly prompted to do so, all PSMTs perceived they had developed a repertoire of instructional strategies for designing function lessons with technology. Some PSMTs took a critical stance in their reflections on the relationship between the technology and students' mathematical thinking. Even though these critical stances focused on mathematical conceptions that students might develop through use of the applet, examining students' mathematical thinking as they engage with a technology tool has the potential for PSMTs to also critically evaluate the tools that students are engaging with (Yeo & Webel, 2019).

Conclusion

Teacher noticing is a "hidden core practice of teaching" (Jacobs & Spangler, 2017, p. 771) that can be developed through engagement with student artifacts. Teacher noticing of students' mathematical thinking has been understudied in technological environments partially due to a lack of authentic artifacts illustrating students' mathematical work within technology-mediated environments. We have presented evidence that learning experiences developed with the Design Principles for Examining Students' Technological Mathematical Work are effective in eliciting PSMTs' articulation of restructuring their function knowledge and technological pedagogical function knowledge. There is a need for mathematics teacher educators to create more artifacts, yet we know that this is not a trivial task (Sherin et al., 2009). The design principles we offer aim to help support this important work.

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Appendix A: Function Machines

Instructions: As you explore the machines, make sure you test each button multiple times, removing the soda can from the machine after each trial.

Machines	Function or not a function?	How do you know?
E		
F		
G		
H		
I		
J		
K		
L		

Using the terms "input" and "output" write a definition for function based on your exploration of the machines.

Machines	Function or not a function?	How do you know?
M		
N		

Appendix B: Examining Student Responses—Vending Machine

Watch the video I and J Student Responses, **https://youtu.be/nQ7bp6lntso**, of two groups engaging with the middle school student version of the applet, **https://www.geogebra.org/m/wcuPt43b**. While watching the video, focus on the students' language. Based on their responses, predict the students' responses for the rest of the machines in the student version of the applet. Provide evidence from the video to justify your responses.

1. How did Group 1 engage with the applet to decide which machine was or was not a function?
2. Explain Group 1's understanding of function. Use examples from the screencast as evidence to show how you know what they do or do not fully understand.
3. Predict (using language you believe the students will use) how Group 1 will answer each machine from the middle school student version of the applet and how Group 1 will engage with each machine. Your justification should include evidence from the video.

Machine	**Prediction (function/non-function)**	**Justification (Include BOTH a discussion of how the students will engage with the applet AND how their understanding of function will lead them to this conclusion)**
E		
F		
G		
H		
K		
L		
M		
N		

4. How did Group 2 engage with the applet to decide which machine was or was not a function?
5. Explain Group 2's understanding of function. Use examples from the screencast as evidence to show how you know what they do or do not fully understand.
6. Predict (using language you believe the students will use) how Group 2 will answer each machine and how Group 2 will engage with each machine. Your justification should include evidence from the video.

Machine	**Prediction (function/non-function)**	**Justification (Include BOTH a discussion of how the students will engage with the applet AND how their understanding of function will lead them to this conclusion)**
E		
F		
G		
H		
K		
L		
M		
N		

7. How has your own understanding of function been influenced by thinking about how middle school students develop the concept of function?